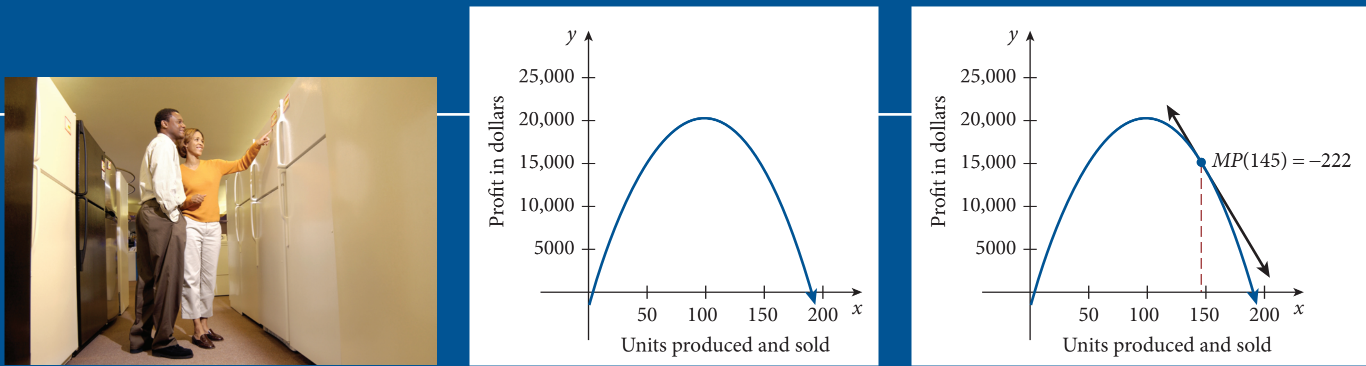
CHAPTER 3

Differentiation Techniques, the Differential, and Marginal Analysis



 (a) (b) (c)

One of the factors that must be taken into consideration by business managers when determining whether or not to increase production of an item is that of marginal profit. While the goal for any successful business manager is for profit to increase, there is a point at which profit is maximized and the production of a single additional unit will result in a loss of gain in profits. Consider the case of a refrigerator manufacturer. The graph of the profit function in Figure (c) illustrates the profit lost by producing and selling the 146th refrigerator.

What We Know

In Chapter 2, we found that the instantaneous rate of change gave us the slope of the line tangent to the graph of a function. The way we found this new rate of change was through the derivative.

Where Do We Go

In this chapter, we will learn rules to allow us to compute derivatives more efficiently. We will also use the derivative to study a new quantity called the differential. We will use the differential to approximate solutions and to study some functions of business.

Chapter Sections

3.1 Derivatives of Constants, Powers, and Sums
3.2 Derivatives of Products and Quotients
3.3 The Chain Rule
3.4 The Differential and Linear Approximations
3.5 Marginal Analysis
Chapter Review Exercises

SECTION OBJECTIVES

1. Determine the derivative of a constant.
2. Determine the derivative of a power function.
3. Differentiate a function that is multiplied by a constant.
4. Differentiate the sum and/or difference of functions.
5. Apply the differentiation rules to environmental science, physics, and business.

3.1 Derivatives of Constants, Powers, and Sums

In Section 2.4 we defined the derivative of a function and used it to find slopes of tangent lines as well as instantaneous rates of change. However, computing the derivative of a function from the definition can be quite involved. Calculus would not be very useful if all derivatives had to be calculated from the definition.

In the first three sections of this chapter, we present several **rules of differentiation**, or shortcuts if you like, that will greatly simplify differentiation. So why did we painstakingly compute derivatives via the definition if these shortcuts exist? Quite simply, these rules are *derived* from the definition! Also, we want to make sure that the *concept* of instantaneous rate of change/tangent line slope was understood for what it is … a limit.

Before we delve into differentiation rules, we need to present alternative ways that can be used to represent the derivative.

Alternative Notation for the Derivative

For $y = f(x)$, the following may be used to represent the derivative:

$$f'(x), y', \frac{dy}{dx}, \frac{d}{dx}[f(x)]$$

Each notation has its own advantage, and we will use each of these where appropriate.

Rule 1: Constant Function Rule

For any constant k, if $f(x) = k$, then $f'(x) = 0$.

NOTE: Rule 1 can be summarized by saying that the derivative of a constant is zero.

Derivative of a Constant Function

The first rule that we present shows how to differentiate a constant function. Geometrically, this rule is fairly obvious. The graph of a constant function, $f(x) = k$, is a horizontal line like the one shown in **Figure 3.1.1**. We know that the slope of a horizontal line is 0. This leads us to believe that the instantaneous rate of change at any point on the graph of a horizontal line is 0 or, in other words, for $f(x) = k$ in Figure 3.1.1, we believe that $f'(x) = 0$.

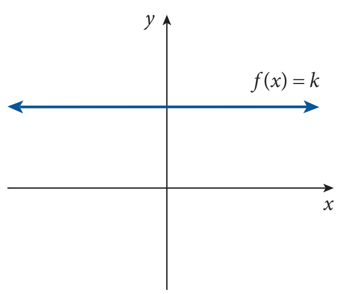


Figure 3.1.1

To show that the **Constant Function Rule** is true, we simply apply the definition of the derivative as follows. If $f(x) = k$ then we know that $f(x + h) = k$ as well. From the definition of the derivative we have

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} \frac{0}{h} = \lim_{h \to 0} 0 = 0$$

Example 1: Differentiating Constant Functions

Differentiate the following.

a. $f(x) = 7$ **b.** $g(x) = \sqrt[3]{2}$

Perform the Mathematics

a. Since 7 is a constant, we have $f'(x) = 0$.

b. Since $\sqrt[3]{2}$ is a constant, we have $g'(x) = 0$. ■

OBJECTIVE 1

Determine the derivative of a constant.

Power Rule

We now present one of the most useful differentiation rules in calculus. It is used to differentiate functions of the form $f(x) = x^n$, where n is any real number. Functions of this form are collectively called **power functions**. The definition of the derivative can be used to determine derivatives for power functions, although the algebra may become cumbersome quickly. For example, the definition of the derivative can be used to derive the following:

$$\text{For } f(x) = x^3, \text{ we have } f'(x) = 3x^2.$$
$$\text{For } f(x) = x^6, \text{ we have } f'(x) = 6x^5.$$
$$\text{For } f(x) = x^7, \text{ we have } f'(x) = 7x^6.$$

There appears to be a pattern with the derivatives of power functions. It appears that the power of $f(x)$ becomes the coefficient of $f'(x)$, and the power of $f'(x)$ is 1 less than the power of $f(x)$. This observation leads us to the **Power Rule**. The proof of the Power Rule is in Appendix C.

Rule 2: Power Rule

If $f(x) = x^n$, where n is any real number, then $f'(x) = nx^{n-1}$.

NOTES:

1. Rule 2 is equivalent to writing $\frac{d}{dx}[x^n] = nx^{n-1}$.
2. Rule 2 can be summarized by saying that to differentiate x^n, simply bring the exponent out front as a coefficient and then decrease the exponent by 1.

Example 2: Differentiating Power Functions

Differentiate the following functions.

a. $f(x) = x^4$ **b.** $g(x) = x^{1.32}$ **c.** $y = \sqrt{x}$ **d.** $g(x) = \frac{1}{x^3}$

OBJECTIVE 2

Determine the derivative of a power function.

Perform the Mathematics

a. Thanks to the Power Rule, we quickly compute the derivative to be

$$f'(x) = 4x^{4-1} = 4x^3$$

b. Again, using the Power Rule, we have

$$g'(x) = 1.32x^{1.32-1} = 1.32x^{0.32}$$

c. Here we need to use a little algebra before differentiating. We need to recall that $\sqrt[n]{x} = x^{1/n}$. This allows us to rewrite $y = \sqrt{x}$ as $y = x^{1/2}$. Now we can apply the Power Rule to get

$$y' = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2}$$

Utilizing the algebraic fact that $x^{-n} = \frac{1}{x^n}$ allows us to write the simplified version of the derivative as

$$y' = \frac{1}{2x^{1/2}} = \frac{1}{2\sqrt{x}}$$

d. We begin by using a little algebra to rewrite the function. That is, by utilizing the algebraic fact that $x^{-n} = \dfrac{1}{x^n}$, the function can be rewritten as

$$g(x) = \frac{1}{x^3} = x^{-3}$$

Now we can apply the Power Rule and get

$$g'(x) = -3x^{-3-1} = -3x^{-4} = -\frac{3}{x^4}$$

Rule 3: Constant Multiple Rule

If $f(x) = k \cdot g(x)$, where k is any real number, then $f'(x) = k \cdot g'(x)$, assuming that g is differentiable.

NOTES:

1. Rule 3 is equivalent to writing $\dfrac{d}{dx}[k \cdot g(x)] = k \cdot \dfrac{d}{dx}[g(x)]$.
2. In words, the Constant Multiple Rule states that the derivative of a constant times a function is simply the constant times the derivative of the function.

Try It Yourself

Some related Exercises are 13 and 14.

Constant Multiple Rule

The Constant Multiple Rule extends the Power Rule to differentiating functions that are of the form $k \cdot g(x)$, a constant times a function. The definition of the derivative can be used to derive the following:

For $f(x) = 2x^3$, we have $f'(x) = 6x^2$.

For $f(x) = 3x^5$, we have $f'(x) = 15x^4$.

For $f(x) = -2x^4$, we have $f'(x) = -8x^3$.

Again, there appears to be a pattern in these derivatives. It appears that the power of $f(x)$ gets multiplied by the coefficient of $f(x)$, and this product becomes the coefficient of $f'(x)$, while the power of $f'(x)$ is 1 less than the power of $f(x)$. This observation leads to the **Constant Multiple Rule**.

The proof of this rule is in Appendix C.

OBJECTIVE 3

Differentiate a function that is multiplied by a constant.

Example 3: Differentiating a Constant Times a Function

Differentiate the following.

a. $g(x) = 1.2x^5$ **b.** $y = \dfrac{1}{7x^3}$ **c.** $f(x) = \dfrac{2}{3}\sqrt[5]{x}$

Perform the Mathematics

a. Applying the Constant Multiple Rule yields

$$g'(x) = 1.2 \cdot (5x^{5-1}) = 6x^4$$

b. Rewriting the function via algebra gives

$$y = \frac{1}{7}x^{-3}$$

Now we apply the Constant Multiple Rule to get

$$y' = \frac{1}{7}(-3x^{-3-1}) = \frac{1}{7}(-3x^{-4})$$

$$= -\frac{3}{7}x^{-4} = -\frac{3}{7x^4}$$

c. We begin by rewriting the function as

$$f(x) = \frac{2}{3}x^{1/5}$$

Now we apply the Constant Multiple Rule, which gives

$$f'(x) = \frac{2}{3}\left(\frac{1}{5}x^{1/5-1}\right) = \frac{2}{15}x^{-4/5}$$

Simplifying the derivative yields

$$f'(x) = \frac{2}{15x^{4/5}} = \frac{2}{15\sqrt[5]{x^4}}$$

Try It Yourself

Some related Exercises are 19 and 23.

Sum and Difference Rule

Consider two functions $f(x) = 2x^2$ and $g(x) = 3x + 1$. We can now define a new function, S, as follows.

$$S(x) = f(x) + g(x) = 2x^2 + 3x + 1$$

From the differentiation rules developed so far in this section, we know that

$$f'(x) = 4x \text{ and } g'(x) = 3$$

However, we have no rule to help us determine $S'(x)$. We can appeal to the definition of the derivative to help us compute $S'(x)$:

$$\begin{aligned}
S'(x) &= \lim_{h\to 0}\frac{S(x+h) - S(x)}{h} \\
&= \lim_{h\to 0}\frac{[2(x+h)^2 + 3(x+h) + 1] - [2x^2 + 3x + 1]}{h} \\
&= \lim_{h\to 0}\frac{[2(x^2 + 2xh + h^2) + 3x + 3h + 1] - 2x^2 - 3x - 1}{h} \\
&= \lim_{h\to 0}\frac{2x^2 + 4xh + 2h^2 + 3x + 3h + 1 - 2x^2 - 3x - 1}{h} \\
&= \lim_{h\to 0}\frac{4xh + 2h^2 + 3h}{h} \\
&= \lim_{h\to 0}\frac{h(4x + 2h + 3)}{h} \\
&= \lim_{h\to 0}(4x + 2h + 3) \\
&= 4x + 3
\end{aligned}$$

So we have determined that $S'(x) = 4x + 3$. Recall that we defined $S(x)$ as

$$S(x) = f(x) + g(x) = 2x^2 + 3x + 1$$

Also recall that we knew that $f'(x) = 4x$ and $g'(x) = 3$. It appears that we have discovered that

$$S'(x) = 4x + 3 = f'(x) + g'(x)$$

This is not a coincidence. This illustrates a rule that handles functions that are sums or differences of other functions and is called the **Sum and Difference Rule**.

Rule 4: Sum and Difference Rule

If $h(x) = f(x) \pm g(x)$, where f and g are both differentiable functions, then $h'(x) = f'(x) \pm g'(x)$.

NOTES:

1. Rule 4 is equivalent to writing $\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)]$.
2. In words, the Sum and Difference rule states that to differentiate a sum/difference of two (or more) functions, just differentiate the functions separately and add/subtract the results.

OBJECTIVE 4

Differentiate the sum and/or difference of functions.

Example 4: Differentiating Sums and Differences

Differentiate the following functions.

a. $f(x) = 3x^2 + 2x - 1$ **b.** $g(x) = \frac{1}{2}x^3 - \frac{3}{2}x^{-1}$ **c.** $y = 3x^4 + 2\sqrt{x} - \frac{2}{x^2}$

Perform the Mathematics

a. According to the Sum and Difference Rule,

$$f'(x) = \frac{d}{dx}[3x^2 + 2x - 1] = \frac{d}{dx}[3x^2] + \frac{d}{dx}[2x] - \frac{d}{dx}[1]$$
$$= 6x + 2 - 0 = 6x + 2$$

b. Applying the Sum and Difference Rule yields

$$g'(x) = \frac{d}{dx}\left[\frac{1}{2}x^3 - \frac{3}{2}x^{-1}\right] = \frac{d}{dx}\left[\frac{1}{2}x^3\right] - \frac{d}{dx}\left[\frac{3}{2}x^{-1}\right]$$
$$= \frac{3}{2}x^2 - \left(-\frac{3}{2}x^{-2}\right) = \frac{3}{2}x^2 + \frac{3}{2}x^{-2}$$

c. First we rewrite the function as

$$y = 3x^4 + 2x^{1/2} - 2x^{-2}$$

Now we differentiate to get

$$\frac{dy}{dx} = \frac{d}{dx}[3x^4 + 2x^{1/2} - 2x^{-2}]$$
$$= \frac{d}{dx}[3x^4] + \frac{d}{dx}[2x^{1/2}] - \frac{d}{dx}[2x^{-2}]$$
$$= 12x^3 + x^{-1/2} + 4x^{-3}$$

Try It Yourself

Some related Exercises are 31 and 43.

Notice in Example 4 that we not only used the Sum and Difference Rule, but we also used the Constant Function Rule, the Power Rule, and the Constant Multiple Rule. In short, we used all the rules presented in this section.

Applications

Example 5: Analyzing Greenhouse Gas Emissions

The majority of scientists support the notion that the release of greenhouse gasses since the beginning of the industrial age has had an effect on the climate and will continue to have an effect on the climate. Carbon dioxide is one such gas that is attributed to warming due to the greenhouse effect. The amount of carbon dioxide emissions by the U.S. can be modeled by

$$f(x) = 0.61x^3 - 22.57x^2 + 276.36x + 4767.7 \qquad 1 \le x \le 19$$

where x represents the number of years since 1989 and $f(x)$ represents the total carbon dioxide emissions in millions of metric tons. (*Source:* U.S. Census Bureau.) Evaluate $f'(18)$ and interpret.

Understand the Situation

We can use the differentiation techniques learned in this section to quickly determine the derivative, $f'(x)$. To evaluate $f'(18)$ we will simply substitute $x = 18$ into the derivative, $f'(x)$, for each occurrence of x. Note that $x = 18$ corresponds to the year 2007.

Perform the Mathematics

Using the techniques learned in this section we get

$$f'(x) = 0.61(3x^2) - 22.57(2x) + 276.36(1) + 0$$
$$f'(x) = 1.83x^2 - 45.14x + 276.36$$

To determine $f'(18)$, we substitute $x = 18$ into the derivative. This gives us

$$\begin{aligned} f'(18) &= 1.83(18)^2 - 45.14(18) + 276.36 \\ &= 56.76 \end{aligned}$$

Interpret the Result

This means that in 2007, U.S. emissions of carbon dioxide into the atmosphere was increasing at a rate of 56.76 million metric tons per year. See **Figure 3.1.2**.

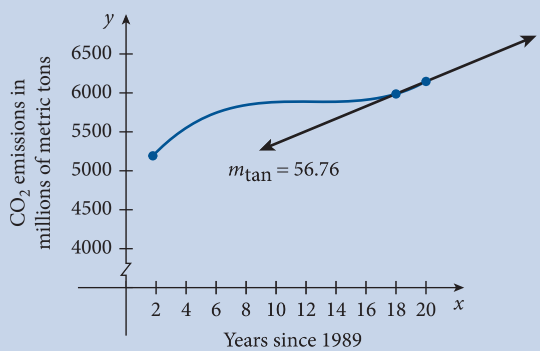


Figure 3.1.2

Try It Yourself

Some related Exercises are 81 and 83. ■

OBJECTIVE 5

Apply the differentiation rules to environmental science, physics, and business.

The next example deals with what is known as a **position function** and a **velocity function**. These functions are commonly used in physics.

Example 6: Analyzing Velocity

If a coconut falls from a tree that is 75 feet tall, its height above the ground after t seconds is given by

$$s(t) = 75 - 16t^2$$

where $s(t)$ is measured in feet. This function is called a **position function** since it gives the position of the coconut above the ground as a function of time.

a. The derivative of a position function, $s'(t)$, is called the **velocity function** and is denoted by $v(t)$. Determine $v(t)$.

b. Compute $s(2)$ and $v(2)$, and interpret each.

c. When does the coconut hit the ground?

Perform the Mathematics

a. Since we have been told that $s'(t) = v(t)$, we determine $v(t)$ as follows:

$$v(t) = s'(t) = -32t$$

b. Evaluating $s(2)$ and $v(2)$, we get

$$s(2) = 75 - 16(2)^2 = 11 \text{ feet}$$
$$v(2) = -32(2) = -64 \text{ feet/sec}$$

This means that after 2 seconds, the coconut is 11 feet above the ground and is falling at a rate of 64 feet per second. Notice that the negative value of $v(2)$ indicates that the coconut is falling toward the ground.

c. The coconut hits the ground when it is 0 feet above the ground. Hence, we need to solve $s(t) = 0$. Setting $s(t) = 0$ and solving yields

$$\begin{aligned} 75 - 16t^2 &= 0 \\ 75 &= 16t^2 \\ \frac{75}{16} &= t^2 \\ \pm\sqrt{\frac{75}{16}} &= t \\ t &\approx \pm 2.17 \end{aligned}$$

Since -2.17 seconds does not make sense (we cannot yet travel back in time), we conclude that the coconut hits the ground in about 2.17 seconds. ■

Technology Option

We can graphically support the result of Example 6c by graphing the position function and utilizing the `Zero` command. See **Figure 3.1.3**.

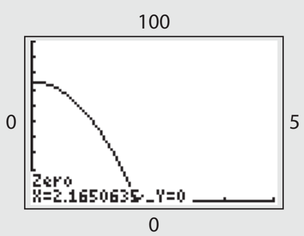


Figure 3.1.3 Result of `Zero` command.

Our final example of this section looks at how we can apply the derivative to a profit function.

Example 7: Analyzing a Profit Function

The Cool Air Company has determined that the total cost function for the production of its refrigerators can be described by

$$C(x) = 2x^2 + 15x + 1500$$

where x is the weekly production of refrigerators and $C(x)$ is the total cost in dollars. The revenue function for these refrigerators is given by

$$R(x) = -0.3x^2 + 460x$$

where x is the number of refrigerators sold and $R(x)$ is in dollars. Determine $P(20)$ and $P'(20)$ and interpret each.

Understand the Situation

We begin by determining the profit function, P, and evaluating the profit function when $x = 20$. Once we have the profit function, P, we can determine the derivative, P', and then evaluate it at $x = 20$.

Perform the Mathematics

In Chapter 1 we learned that profit function is the revenue function minus the cost function. This gives us

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (-0.3x^2 + 460x) - (2x^2 + 15x + 1500) \\ &= -2.3x^2 + 445x - 1500 \end{aligned}$$

Evaluating the profit function when $x = 20$ gives

$$\begin{aligned} P(20) &= -2.3(20)^2 + 445(20) - 1500 \\ &= 6480 \end{aligned}$$

Since $P(x) = -2.3x^2 + 445x - 1500$, the derivative of P is

$$P'(x) = \frac{d}{dx}[-2.3x^2] + \frac{d}{dx}[445x] - \frac{d}{dx}[1500] = -4.6x + 445$$

Evaluating $P'(20)$ yields

$$\begin{aligned} P'(20) &= -4.6(20) + 445 \\ &= 353 \end{aligned}$$

Interpret the Results

This means that when the Cool Air Company makes and sells 20 refrigerators, a profit of \$6480 is realized and the profit is increasing at a rate of \$353 per refrigerator.

Try It Yourself

Some related Exercises are 73 and 74. ■

Summary

In this section we introduced four differentiation rules to allow us to compute derivatives more quickly:

Name	*Rule*	*Example*
Constant Function Rule	If $f(x) = k$, then $f'(x) = 0$.	If $f(x) = 7$, then $f'(x) = 0$.
Power Rule	If $f(x) = x^n$, then $f'(x) = nx^{n-1}$.	If $f(x) = x^8$, then $f'(x) = 8x^7$.
Constant Multiple Rule	If $f(x) = k \cdot g(x)$, then $f'(x) = k \cdot g'(x)$.	If $f(x) = 3x^4$, then $f'(x) = 12x^3$.
Sum and Difference Rule	If $h(x) = f(x) \pm g(x)$, then $h'(x) = f'(x) \pm g'(x)$.	If $h(x) = x^3 + x^2$, then $h'(x) = 3x^2 + 2x$.

Even though we now have some tools that allow us to calculate a derivative more quickly than by using the definition of the derivative, do not lose sight of what the derivative tells us: It is a limit that gives an instantaneous rate of change as well as the slope of a tangent line.

Section 3.1 Exercises

Vocabulary Exercises

1. The process of finding the derivative of a function is known as __________________.
2. For a derivative f', the symbol above the function notation is read __________________.
3. The derivative of a constant is __________________.
4. To differentiate a power term x^n, we bring the exponent n out front as a __________________ and then subtract __________________ from the exponent.
5. The derivative of a constant times a function is the constant times the __________________ of the function.
6. The rule that essentially states that we can differentiate functions term by term is the __________________ rule.
7. A function used in physics that has the form $s(t) = -16t^2 + v_0t + h_0$ is known as a __________________ function.
8. A function used in physics that has the form $v(t) = -32t + v_0$ is known as a __________________ function.

Skill Exercises

In Exercises 9–24, determine the derivative for the given single-term function. When appropriate, simplify the derivative so that there are no negative or fractional exponents. A few helpful rules from algebra are:

(i) $x^{-n} = \dfrac{1}{x^n}$

(ii) $x^{m/n} = \sqrt[n]{x^m}$

9. $f(x) = 3$
10. $f(x) = 5$
11. $f(x) = -2$
12. $f(x) = -7$
13. $f(x) = x^6$
14. $f(x) = x^{10}$
15. $f(x) = -3x^4$
16. $f(x) = -5x^7$
17. $f(x) = 2x^{2/3}$
18. $f(x) = 3x^{4/5}$
19. $f(x) = -3x^{-1/3}$
20. $f(x) = -2x^{-2/5}$
21. $f(x) = \dfrac{2}{3}x^4$
22. $f(x) = \dfrac{3}{4}x^8$
23. $f(x) = -\dfrac{2}{5}x^{5/3}$
24. $f(x) = -\dfrac{5}{6}x^{6/5}$

In Exercises 25–56, determine the derivative for the given function. When appropriate, simplify the derivative so that there are no negative or fractional exponents.

25. $f(x) = 2x^3 + 4x^2 - 7x + 1$
26. $f(x) = 4x^3 - 3x^2 + 5x - 3$
27. $f(x) = 3x^2 - 2x + 6$
28. $f(x) = 3x^2 + 5x - 2$
29. $f(x) = -5x^2 - 6x + 2$
30. $f(x) = -3x^2 + 9x - 1$
31. $f(x) = -5x^3 + 7x - 5$
32. $f(x) = -8x^3 - 8x + 3$
33. $f(x) = \dfrac{1}{2}x^3 + \dfrac{3}{5}x^2 - \dfrac{2}{3}x + \dfrac{2}{5}$
34. $f(x) = \dfrac{2}{5}x^3 - \dfrac{2}{3}x^2 + \dfrac{1}{2}x - 5$

35. $f(x) = 1.31x^2 + 2.05x - 3.9$

36. $f(x) = 3.15x^2 - 1.13x + 5.2$

37. $f(x) = -0.2x^2 + 3.5x^3 - 0.4x^4$

38. $f(x) = 0.3x^2 - 0.67x^3 + 0.8x^4$

39. $f(x) = 1.15x^3 - 2.3x^2 + 2.53x - 7.1$

40. $f(x) = 2.35x^3 + 3.56x^2 - 63.25x + 365.3$

41. $f(x) = 3\sqrt{x} + \frac{1}{2}x - 5x^2$

42. $f(x) = 5\sqrt{x} - \frac{1}{2}x + 7x^2$

43. $f(x) = \sqrt[3]{x} + x^2 - 3x^3$

44. $f(x) = \sqrt[3]{x} - 2x^3 + 5x^4$

45. $f(x) = \sqrt[3]{x^2} - \frac{4}{\sqrt{x}}$

46. $f(x) = \sqrt[3]{x^2} + \frac{3}{\sqrt{x}}$

47. $f(x) = 2.35x^{1.35}$

48. $f(x) = 3.2x^{1.14}$

49. $f(x) = 2000 + \frac{5}{x^2}$

50. $f(x) = 3300 + \frac{13}{x^2}$

51. $f(x) = 2x^2 + \frac{1}{x}$

52. $f(x) = -3x^2 - \frac{1}{x}$

53. $f(x) = 3x^{-3/2} - 4x^{-1/2} + 5$

54. $f(x) = 4x^{-3/2} - 2x^{-1/2} + x^{-1/2}$

55. $f(x) = 2.35x^{-1/2} - 2.3x^{-2/3}$

56. $f(x) = 3.52x^{-2/5} + 3.2x^{-1/2}$

For Exercises 57–64, an algebraic function in the form $f(x) = \dfrac{g(x)}{h(x)}$ *is given. Complete the following:*

(a) Write the domain of the f using interval notation.

(b) Use the addition rule for fractions, $\frac{a+b}{c} = \frac{a}{c} + \frac{b}{c}$, to rewrite f. For example,

$$f(x) = \frac{x^3 + 3x^2 - x + 2}{x} = \frac{x^3}{x} + \frac{3x^2}{x} - \frac{x}{x} + \frac{2}{x} = x^2 + 3x - 1 + 2x^{-1}.$$

(c) Compute f' using the rules from this section.

(d) Write the domain of f' using interval notation.

57. $f(x) = \dfrac{3x^3 - 9x^2 + 4}{12}$

58. $f(x) = \dfrac{x^2 + 4x + 3}{2}$

59. $f(x) = \dfrac{2x^3 + 3x^2 - x + 3}{x}$

60. $f(x) = \dfrac{-3x^3 - 4x^2 + 2x - 7}{x}$

61. $f(x) = \dfrac{7x^4 - 50x^2 + x}{x^2}$

62. $f(x) = \dfrac{-5x^4 + 25x^2 - x}{x^2}$

63. $f(x) = \dfrac{4x^3 - 14x^2 + 3}{2\sqrt{x}}$

64. $f(x) = \dfrac{x^3 + 2x^2}{\sqrt[3]{x^2}}$

For Exercises 65–70, complete the following.

(a) Use the differentiation rules from this section to determine f'.

(b) Find the slope of the tangent line to the graph of f at the indicated x-value.

(c) Use the result of part (b) to write an equation of the line tangent to the graph of f at the indicated x-value.

(d) Graph the function and the tangent line in the same viewing window.

65. $f(x) = x^3$ at $x = -1$

66. $f(x) = \sqrt{x}$ at $x = 4$

67. $f(x) = \frac{1}{x}$ at $x = 3$

68. $f(x) = \frac{1}{x^2}$ at $x = 3$

69. $f(x) = x^{2/3}$ at $x = 8$

70. $f(x) = x^{-1/3}$ at $x = 1$

Application Exercises

71. Marketing—Advertising Budget: The number of dollars spent on advertising for a product influences the number of items of the product that will be purchased by customers. The number of Krypto-Cases sold for a new smartphone is a function of the amount spent on advertising and can be modeled by

$$N(x) = 2000 - \frac{520}{x} \qquad x \geq 0.208$$

where x represents the amount spent on advertising, measured in thousands of dollars, and $N(x)$ represents the number of cases purchased.

(a) Determine $N'(x)$.

(b) Compute $N(10)$ and $N'(10)$ and interpret each.

72. Marketing—Advertising Budget: The number of dollars spent on advertising for a product influences the number of items of the product that will be purchased by customers. The number of Find-it-Now GPS devices is a function of the amount spent on advertising and can be modeled by

$$N(x) = 3600 - \frac{700}{x} \qquad x \geq 0.195$$

where x represents the amount spent on advertising, measured in thousands of dollars, and $N(x)$ represents the number of devices purchased.

(a) Determine $N'(x)$.

(b) Compute $N(5)$ and $N'(5)$ and interpret each.

73. Microeconomics—Manufacturing Cost: From past data analysis, the LaRoche Raincoat Company finds that the manufacturing cost for their Jeffery coat can be modeled by

$$C(x) = 3000 + 11x - 7\sqrt{x} + 0.03x^{3/2}$$

where x represents the number of coats produced and $C(x)$ represents the cost in dollars.

(a) Determine $C'(x)$.

(b) Evaluate $C(300)$ and $C'(300)$ and interpret each.

74. Microeconomics—Manufacturing Cost: (*continuation of Exercise 73*) Recall that for a cost function, the average cost $AC(x)$ is given by $AC(x) = \frac{C(x)}{x}$ and gives the per-unit cost of making a product at production level x.

(a) For the cost function in Exercise 73, write and simplify the average cost function $AC(x)$.

(b) Determine $AC'(x)$.

(c) Evaluate $AC(300)$ and $AC'(300)$ and interpret each.

75. Ecology—Sulfur Dioxide Levels: The Sperry Company's Cantor Ridge coal-burning power plant collects air samples of its emissions at various points downwind to determine the concentration of sulfur dioxide at different distances from the plant. The amount of sulfur dioxide measured can be modeled by the function

$$f(x) = \frac{93.21}{x^2} \qquad x > 0$$

where x represents the distance downwind in miles and $f(x)$ represents the sulfur dioxide concentration in parts per million (ppm).

(a) Determine $f'(x)$.

(b) Evaluate $f(1)$ and $f'(1)$ and interpret each.

76. Ecology—Sulfur Dioxide Levels: The Sperry Company's Rangel Valley coal-burning power plant collects air samples of its emissions at various points downwind to determine the concentration of sulfur dioxide at different distances from the plant. The amount of sulfur dioxide measured can be modeled by the function

$$f(x) = \frac{78.35}{x^2} \qquad x > 0$$

where x represents the distance downwind in miles and $f(x)$ represents the sulfur dioxide concentration in parts per million (ppm).

(a) Determine $f'(x)$.

(b) Evaluate $f(2)$ and $f'(2)$ and interpret each.

In Exercises 77–80, a supply function s is given. A ***supply function*** *gives the price $s(x)$ at which exactly x units of a product is supplied. Its derivative $s'(x)$ gives the instantaneous rate of change in the price of the product at supply level x.*

77. Macroeconomics—Supply Function: Balata Incorporated produces golf balls and finds that its supply function for the new Xtrah golf ball is given by

$$s(x) = 0.24x + 3.70$$

where x represents the number of dozens of golf balls supplied each month and $s(x)$ is the price per dozen, measured in dollars. Evaluate $s'(65)$ and interpret.

78. Macroeconomics—Supply Function: The Farver Company has determined that the supply function for their new Healthy Day breakfast bar is given by

$$s(x) = \frac{1}{2}x + 40$$

where x is the number of cases of breakfast bars supplied each month and $s(x)$ represents the price per case. Evaluate $s'(70)$ and interpret.

79. Macroeconomics—Supply Function: Baker's Bake Shoppe makes specialty baking pans and finds that the supply function for the candle-shaped birthday cake pan is given by

$$s(x) = 3.75x^{0.22}$$

where x is the number of baking pans supplied and $s(x)$ represents the price per pan. Evaluate $s'(31)$ and interpret.

80. Macroeconomics—Supply Function: The U-Build-It home improvement corporation determines that the supply function for their Haul-Out disposal bags is given by

$$s(x) = 6.81x^{0.33}$$

where x is the number of boxes of bags supplied and $s(x)$ represents the price per box of bags. Evaluate $s'(5)$ and interpret.

81. Political Science—Hispanic Voters: One group of voters that has received attention recently is Hispanic Americans. The number of Hispanic voters from 1995 to 2010 can be modeled by

$$f(x) = 1.08x + 11.7 \qquad 0 \le x \le 15$$

where x represents the number of years since 1995 and $f(x)$ represents the number of Hispanic voters in millions. (*Source:* U.S. Census Bureau.) Determine $f(9)$ and $f'(9)$ and interpret each.

82. Political Science—African American Voters: A group that political analysis has traditionally been interested in is African American voters. The number of African American voters from 1995 to 2010 can be modeled by

$$f(x) = 3.10x + 20.67 \qquad 0 \le x \le 15$$

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where x represents the number of years since 1995 and $f(x)$ represents the number of African American voters in millions. (*Source:* U.S. Census Bureau.) Determine $f(9)$ and $f'(9)$ and interpret each.

83. **Macroeconomics—European Unemployment:** Suppose an economist wishes to examine the unemployment rate in the middle of the first decade of the 21st century. The unemployment rate in Bulgaria from 2000 to 2010 can be modeled by

$$f(x) = 0.08x^3 - 1.24x^2 + 3.34x + 15.37 \qquad 0 \le x \le 10$$

where x represents the number of years since 2000 and $f(x)$ represents the unemployment rate as a percentage. (*Source:* Google Public Data.) Use the model to determine the rate of change in the Bulgarian unemployment rate in 2005.

84. **Macroeconomics—European Unemployment:** Suppose an economist wishes to examine the unemployment rate in the middle of the first decade of the 21st century. The unemployment rate in Latvia from 2000 to 2010 can be modeled by

$$f(x) = 0.09x^3 - 0.98x^2 + 1.63x + 13.21 \qquad 0 \le x \le 10$$

where x represents the number of years since 2000 and $f(x)$ represents the unemployment rate as a percentage. (*Source:* Google Public Data.) Use the model to determine the rate of change in the Latvian unemployment rate in 2008.

Concept and Writing Exercises

85. The constant function rule states that for any constant k, if $f(x) = k$, then $f'(x) = 0$. Sketch a graph of the constant function f and explain using the properties of slope that the derivative is zero.

*For a function f at a value $x = a$, the **normal line** of f at a is the equation of the line that is perpendicular to the tangent line at $x = a$. In Exercises 86, 87, and 88, a function f and a value a are given. Complete the following.*

(a) *Determine the equation of the tangent line at $x = a$. Call this line y_1.*

(b) *Determine the equation of the normal line at a. Recall from algebra that perpendicular lines have negative reciprocal slopes. Call this line y_2.*

(c) *Graph f along with the lines y_1 and y_2 in the same viewing window.*

86. $f(x) = 2x + 3 \quad x = 4$

87. $f(x) = \sqrt{x} \quad x = 9$

88. $f(x) = x^2 - 5x + 6 \quad x = 1$

89. The Sum Rule states that $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$, and the Constant Multiple Rule states that, for a real number constant k, that $\frac{d}{dx}[k \cdot f(x)] = k \cdot \frac{d}{dx}f(x)$. Use these rules to prove the Difference Rule, which states that for differentiable functions f and g, $\frac{d}{dx}[f(x) - g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$.

90. Determine the derivative for the function $f(x) = (3x^2 + 4)^2$ by squaring the binomial $3x^2 + 4$ and differentiating the resulting trinomial.

*A function f is said to be **differentiable from the left** if $\lim_{h \to 0^-} \frac{f(x+h) - f(x)}{h}$ exists, and a function f is said to be **differentiable from the right** if $\lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$ exists. In Exercises 91 and 92, a piecewise-defined function is given. Complete the following.*

(a) Determine if f is differentiable from the left at $x = a$. If so, state the derivative's value.

(b) Determine if f is differentiable from the right at $x = a$. If so, state the derivative's value.

(c) Determine if $f'(a)$ exists.

91. $f(x) = \begin{cases} 2x^2 - 3, & x < 1 \\ 4, & x \geq 1 \end{cases}; \quad x = 1$

92. $f(x) = \begin{cases} x^2, & x \leq 1 \\ \sqrt{x}, & x > 1 \end{cases}; \quad x = 1$

93. Recall from algebra that a function is *even* if $f(-x) = f(x)$ for all values of x in the domain of f, and a function is *odd* if $f(-x) = -f(x)$ for all values of x in the domain of f. Show that the derivative of an even function is an odd function.

Section Project

Turbochargers were used to increase the horsepower output of cars in the Indianapolis 500 from 1952 until 1996, when they were banned from competition through 2011. The first driver to complete a lap at over 200 mph was Tom Sneva in 1978. (*Source:* Indycar.com.) The pole position speed for the Indianapolis 500 from 1978 to 2011 can be modeled by the piecewise-defined function

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$$f(x) = \begin{cases} 404.23 - \dfrac{16{,}329.3}{x}, & 78 \leq x \leq 96 \\ -0.092x^2 + 19.54x - 805, & 97 \leq x \leq 111 \end{cases}$$

where x represents the number of years since 1900 and $f(x)$ represents the pole position speed in miles per hour. (*Source:* Indy500.com.) Use the model to answer the following.

(a) Evaluate $f(96)$ and $f(97)$ and interpret each value.

(b) Evaluate $f(97) - f(96)$ and interpret in terms of the turbocharger technology.

(c) Compute the rate of change in the pole position speed at the Indianapolis 500 in 2011.

(d) With the return of turbocharged engines to the Indianapolis 500 in 2012, would we expect the value of $f'(112)$ to be positive or negative? Explain your answer in a brief sentence.

3.2 Derivatives of Products and Quotients

SECTION OBJECTIVES

1. Differentiate products using the Product Rule.
2. Apply the Product Rule to business functions.
3. Differentiate quotients using the Quotient Rule.
4. Apply Quotient Rule to medication concentration.

In Section 3.1 we learned four useful rules for computing derivatives. In this section, we illustrate how to differentiate the **product** and **quotient** of two functions. These two rules are not as simple as the ones presented in Section 3.1.

Product Rule

To differentiate the product of two functions, we use the **Product Rule**.

Rule 5: Product Rule

If $h(x) = f(x) \cdot g(x)$, where f and g are differentiable functions, then the derivative is $h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x)$.

NOTE: In words, the Product Rule states that we can take the derivative of the first function times the second function plus the first function times the derivative of the second function.

A proof of the Product Rule is in Appendix C.

OBJECTIVE 1

Differentiate products using the Product Rule.

Example 1: Differentiating Products

Differentiate $h(x) = 3x^3(x^4 + 2)$ by using the Product Rule.

Perform the Mathematics

Let $f(x) = 3x^3$ and $g(x) = x^4 + 2$. By the Product Rule, we compute the derivative to be

$$\begin{aligned} h'(x) &= f'(x) \cdot g(x) + f(x) \cdot g'(x) \\ &= 9x^2(x^4 + 2) + 3x^3(4x^3) \\ &= 9x^6 + 18x^2 + 12x^6 \\ &= 21x^6 + 18x^2 \\ &= 3x^2(7x^4 + 6) \end{aligned}$$

■

Example 2: Differentiating Products

Differentiate the following by using the Product Rule.

a. $f(x) = (2x^2 + 4x + 5)(5x - 4)$

b. $y = \sqrt{x}(3x^3 - 4x^2 + 8x)$

Perform the Mathematics

a. Applying the Product Rule yields

$$\begin{aligned} f'(x) &= (4x + 4)(5x - 4) + (2x^2 + 4x + 5)(5) \\ &= (20x^2 - 16x + 20x - 16) + (10x^2 + 20x + 25) \\ &= 30x^2 + 24x + 9 \end{aligned}$$

b. Before differentiating, we rewrite the function as

$$y = x^{1/2}(3x^3 - 4x^2 + 8x)$$

Now using the Product Rule we get

$$y' = \frac{1}{2}x^{-1/2}(3x^3 - 4x^2 + 8x) + x^{1/2}(9x^2 - 8x + 8)$$

Writing the derivative without negative or fractional exponents yields

$$y' = \frac{3x^3 - 4x^2 + 8x}{2\sqrt{x}} + \sqrt{x} \cdot (9x^2 - 8x + 8)$$

■

Try It Yourself

Some related Exercises are 15 and 19.

Example 3: Differentiating Products

Differentiate $f(x) = (6x^{4/3} + 2x)(3x^{5/3} + 4x - 1)$.

Perform the Mathematics

By the Product Rule, we immediately compute the derivative to be

$$f'(x) = (8x^{1/3} + 2)(3x^{5/3} + 4x - 1) + (6x^{4/3} + 2x)(5x^{2/3} + 4)$$

■

Example 4: Determining an Equation for a Tangent Line

Determine an equation for the line tangent to $y = (x^4 - x^2)(x^3 - x + 2)$ at $x = 1$.

Perform the Mathematics

Since the derivative gives the slope of a tangent line at any point, our first task is to determine the derivative of the function by using the Product Rule. This gives

$$\frac{dy}{dx} = (4x^3 - 2x)(x^3 - x + 2) + (x^4 - x^2)(3x^2 - 1)$$

We now evaluate $\frac{dy}{dx}$ at $x = 1$ to get the slope of the tangent line. Substituting $x = 1$ gives

$$\frac{dy}{dx} = (4 \cdot 1^3 - 2 \cdot 1)(1^3 - 1 + 2) + (1^4 - 1^2)(3 \cdot 1^2 - 1) = 4$$

So, at $x = 1$, $m_{tan} = 4$. Also, when $x = 1$, we determine the point of tangency by substituting 1 for x in the original function. This gives

$$y = (1^4 - 1^2)(1^3 - 1 + 2) = 0$$

So our point of tangency is $(1, 0)$. Since we know a point on the tangent line and we know that the slope of the tangent line is 4, we use the point-slope form of a line to determine an equation for the tangent line. This results in

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 0 &= 4(x - 1) \\ y &= 4x - 4 \end{aligned}$$

Figure 3.2.1 shows a graph of the function along with the tangent line.

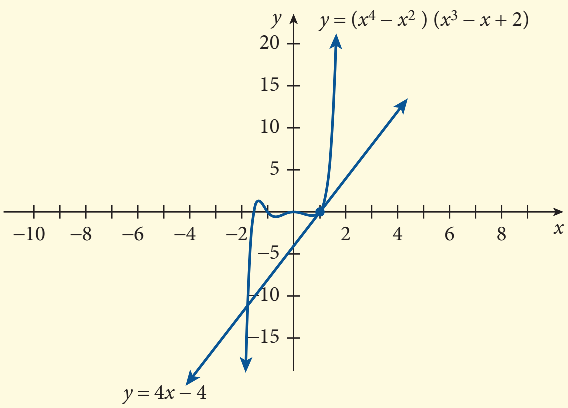

Figure 3.2.1

Try It Yourself

Some related Exercises are 41 and 43.

Technology Option

In **Figure 3.2.2**, we have used a graphing calculator to graph the function in Example 4. We have also used the `Draw Tangent` command. Notice that the `Draw Tangent` command does not give the same equation for the tangent line as we found in Example 4. This is a limitation of the calculator. Even though it is very close, we needed calculus for the **exact** answer.

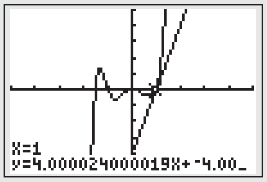

Figure 3.2.2

OBJECTIVE 2

Apply the Product Rule to business functions.

Example 5: Applying the Product Rule

Extensive market research has determined that for the next five years the price of a certain mountain bike is predicted to vary according to $p(t) = 300 - 30t + 7.5t^2$, where t is time in years and $p(t)$ is the price in dollars. The number of mountain bikes sold annually by Skinner's Bikes is expected to follow $q(t) = 3000 + 90t - 15t^2$, where $q(t)$ is the number sold and t is time in years.

a. Determine $R(t)$ and $R'(t)$.

b. Compute $R'(1)$ and interpret.

c. Compute $R'(4)$ and interpret.

Perform the Mathematics

a. Because revenue equals price times quantity, we have

$$R(t) = p(t) \cdot q(t) = (300 - 30t + 7.5t^2) \cdot (3000 + 90t - 15t^2)$$

By the Product Rule, we compute the derivative to be

$$R'(t) = \underbrace{(-30 + 15t)}_{p'(t)} \cdot \underbrace{(3000 + 90t - 15t^2)}_{q(t)} + \underbrace{(300 - 30t + 7.5t^2)}_{p(t)}\underbrace{(90 - 30t)}_{q'(t)}$$

We will not simplify $R'(t)$ so that we may see separately the effects on $R(t)$ caused by changing prices and sales. Notice in the derivative that the first term is $p'(t) \cdot q(t)$. This term describes the rate at which $R(t)$ is changing as price changes. The second term in the derivative is $p(t) \cdot q'(t)$. This term describes the rate at which $R(t)$ is changing as the number of bikes sold changes.

b. We compute $R'(1)$ to be

$$\begin{aligned} R'(1) &= (-30 + 15)(3000 + 90 - 15) + (300 - 30 + 7.5)(90 - 30) \\ &= (-15)(3075) + (277.5)(60) \\ &= -46{,}125 + 16{,}650 = -29{,}475 \end{aligned}$$

The negative sign indicates that after one year, the revenue is decreasing at a rate of \$29,475 per year. Notice that the effect of falling prices represented by the first term, $p'(1) \cdot q(1) = -46{,}125$, is greater than the effect of rising sales given by the second term, $p(1) \cdot q'(1) = 16{,}650$.

c. We have

$$\begin{aligned} R'(4) &= (-30 + 60)(3000 + 360 - 240) + (300 - 120 + 120)(90 - 120) \\ &= (30)(3120) + (300)(-30) \\ &= 93{,}600 - 9000 = 84{,}600 \end{aligned}$$

Since this result is positive, we claim that after four years the revenue is increasing at a rate of \$84,600 per year. Notice that the effect of rising prices represented by the first term, $p'(4) \cdot q(4) = 93{,}600$, is greater than the effect of falling sales given by the second term, $p(4) \cdot q'(4) = -9000$. ■

Quotient Rule

The final differentiation rule of this section, the **Quotient Rule**, shows how to differentiate the quotient of two functions.

Rule 6: Quotient Rule

If $h(x) = \dfrac{f(x)}{g(x)}$, where f and g are differentiable functions, then $h'(x) = \dfrac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}$, where $g(x) \neq 0$.

NOTE: In words, the Quotient Rule states that we can take the derivative of the numerator times the denominator minus the numerator times the derivative of the denominator, all of this over the denominator squared.

Example 6: Differentiating Quotients

OBJECTIVE 3

Differentiate Quotients using the Quotient Rule.

Differentiate the following using the Quotient Rule.

a. $h(x) = \dfrac{x + 3}{x - 2}$ **b.** $y = \dfrac{x^4 - 3x}{x^2 + 1}$

Perform the Mathematics

a. Using the Quotient Rule, we have

$$\begin{aligned} h'(x) &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{1 \cdot (x - 2) - (x + 3) \cdot 1}{(x - 2)^2} \\ &= \frac{x - 2 - x - 3}{(x - 2)^2} = \frac{-5}{(x - 2)^2} \end{aligned}$$

b. Again, by the Quotient Rule, the derivative is

$$\begin{aligned} y' &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \\ &= \frac{(4x^3 - 3)(x^2 + 1) - (x^4 - 3x)(2x)}{(x^2 + 1)^2} \\ &= \frac{(4x^5 + 4x^3 - 3x^2 - 3) - 2x^5 + 6x^2}{(x^2 + 1)^2} \\ &= \frac{2x^5 + 4x^3 + 3x^2 - 3}{(x^2 + 1)^2} \end{aligned}$$

■

Try It Yourself

Some related Exercises are 29 and 31.

The next example requires us to use both the Product Rule and the Quotient Rule.

Example 7: Differentiating Using the Product Rule and Quotient Rule

Differentiate $y = \dfrac{(2x + 1)(3x - 2)}{x + 1}$.

Perform the Mathematics

Notice that when applying the Quotient Rule, we will use the Product Rule when differentiating the numerator. To aid us in this computation, let $f(x) = 2x + 1$ and $g(x) = 3x - 2$. We then have

$$f'(x) = 2 \text{ and } g'(x) = 3$$

Now when applying the Quotient Rule, the derivative of the numerator is

$$\begin{aligned}\frac{d}{dx}[(2x + 1)(3x - 2)] &= 2(3x - 2) + (2x + 1)3 \\ &= 6x - 4 + 6x + 3 = 12x - 1\end{aligned}$$

Putting this all together using the Quotient Rule gives

$$\begin{aligned}\frac{dy}{dx} &= \frac{(12x - 1)(x + 1) - (2x + 1)(3x - 2) \cdot 1}{(x + 1)^2} \\ &= \frac{12x^2 + 11x - 1 - (6x^2 - x - 2)}{(x + 1)^2} \\ &= \frac{12x^2 + 11x - 1 - 6x^2 + x + 2}{(x + 1)^2} \\ &= \frac{6x^2 + 12x + 1}{(x + 1)^2}\end{aligned}$$

■

Try It Yourself

Some related Exercises are 35 and 37.

OBJECTIVE 4

Apply the Quotient Rule to medication concentration.

Example 8: Applying the Quotient Rule

Researchers have determined through experimentation that the percent concentration of a certain medication can be approximated by

$$p(t) = \frac{200t}{2t^2 + 5} - 4 \qquad [0.25, 20]$$

where t is the time in hours after administering the medication and $p(t)$ is the percent concentration. Evaluate $p'(1)$ and $p'(6)$ and interpret each.

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Understand the Situation

In order to evaluate $p'(1)$ and $p'(6)$, we first need to determine $p'(t)$. Once we have $p'(t)$, we will substitute $t = 1$ and $t = 6$ to determine the values of $p'(1)$ and $p'(6)$, respectively.

Perform the Mathematics

Utilizing the Quotient Rule, along with the Sum and Difference Rules, we compute the derivative to be

$$p'(t) = \frac{200(2t^2 + 5) - (200t)(4t)}{(2t^2 + 5)^2} = \frac{-200(2t^2 - 5)}{(2t^2 + 5)}$$

Evaluating $p'(t)$ for $t = 1$ gives

$$p'(1) = \frac{-200(2(1)^2 - 5)}{(2(1)^2 + 5)^2} \approx 12.24$$

Evaluating $p'(t)$ for $t = 6$ gives

$$p'(6) = \frac{-200(2(6)^2 - 5)}{(2(6)^2 + 5)^2} \approx -2.26$$

Interpret the Results

For $t = 1$, $p'(1) \approx 12.24$. This means that at the end of one hour, the concentration of medication is increasing at a rate of about 12.24% per hour. For $t = 6$, $p'(6) \approx -2.26$. This means that at the end of six hours, the concentration of medication was decreasing at a rate of about 2.26% per hour. See **Figure 3.2.3**.

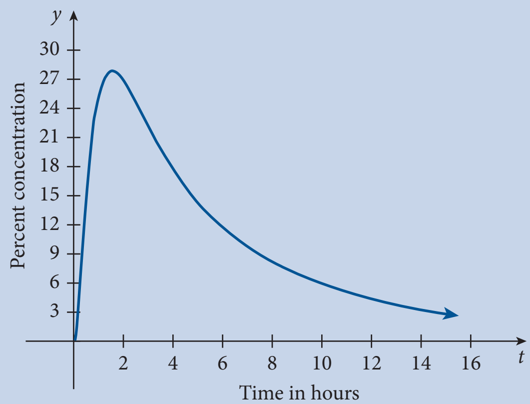

Figure 3.2.3

Try It Yourself

Some related Exercises are 65 and 71. ■

Summary

In this section we presented the **Product Rule** and **Quotient Rule** for computing derivatives. The differentiation rules learned so far are listed in **Table 3.2.1**. The letters k and n represent constants, whereas f and g represent differentiable functions of x.

Table 3.2.1

Name	*Rule*
Constant Function	$\frac{d}{dx}[k] = 0$
Power	$\frac{d}{dx}[x^n] = nx^{n-1}$
Constant Multiple	$\frac{d}{dx}[k \cdot f(x)] = k \cdot f'(x)$
Sum and Difference	$\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$
Product	$\frac{d}{dx}[f(x) \cdot g(x)] = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
Quotient	$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{[g(x)]^2}, g(x) \neq 0$

Section 3.2 Exercises

Vocabulary Exercises

1. When we use the term *product*, we mean the result when two functions are being __________________.
2. The quotient of two functions implies that the functions are being __________________.
3. When using the Quotient Rule, we take the derivative of the numerator times the denominator, minus the numerator times the derivative of the denominator, divided by the __________________ of the denominator.
4. The Product Rule states that we take the derivative of the first function times the second function __________________ the first function times the derivative of the second function.

Skill Exercises

In Exercises 5–26, use the Product Rule to determine the derivative of the function.

5. $f(x) = x^2(2x + 1)$
6. $f(x) = x^2(3x - 5)$
7. $f(x) = x^3(3x^2 + 2x - 5)$
8. $f(x) = x^3(5x^2 - 6x + 3)$
9. $f(x) = 3x^4(2x^2 - 9x + 1)$
10. $f(x) = 5x^3(3x^2 - 6x + 2)$
11. $f(x) = -5x^2(3x^3 + 5x - 7)$
12. $f(x) = -7x^3(2x^3 - 3x^2 + 8)$
13. $f(x) = (3x + 4)(2x - 1)$
14. $f(x) = (4x - 1)(x + 6)$
15. $f(x) = (5x + 3)(3x^3 + 2x^2 + 1)$
16. $f(x) = (2x - 1)(x^2 - 2x + 3)$
17. $f(x) = (3x^2 - 2x + 1)(2x^2 + 5x - 7)$
18. $f(x) = (2x^2 + 5x - 1)(3x^2 - 7x + 3)$
19. $f(x) = (2\sqrt{x} + 4x - 3)(3x - 4)$
20. $f(x) = (3\sqrt{x} - 2x + 1)(5x + 2)$
21. $f(x) = (3x^{6/5} - 5x)(4x^{5/3} + 2x - 5)$
22. $f(x) = (2x^{4/3} + 3x)(-2x^{7/3} + 2x - 5)$
23. $f(x) = (3\sqrt{x} - 5)\left(2\sqrt{x} - \frac{1}{x^3}\right)$
24. $f(x) = (4\sqrt{x} + 2x - 6)(3\sqrt{x} + 6x)$
25. $f(x) = (x^{2/3} + x + 1)(x^{-1} + x^{-2})$
26. $f(x) = (6x^{4/3} - 2x + 3)(3x^{-1} - 4x^{-2})$

In Exercises 27–40, determine the derivative for the given function.

27. $f(x) = \frac{x + 2}{x + 1}$
28. $f(x) = \frac{3x - 4}{x - 1}$

29. $f(x) = \dfrac{4x - 3}{2x + 1}$

30. $f(x) = \dfrac{5x - 11}{3x - 4}$

31. $f(x) = \dfrac{3x^2 - 5x + 1}{5x^2 + 3x + 2}$

32. $f(x) = \dfrac{-2x^2 + 6x - 5}{3x^2 + 5x + 2}$

33. $f(x) = \dfrac{3\sqrt{x} - 5}{6x - 1}$

34. $f(x) = \dfrac{4\sqrt{x} + 3}{2x + 7}$

35. $f(x) = \dfrac{(x^2 + 2)(x - 3)}{x - 1}$

36. $f(x) = \dfrac{(x + 2)(x^3 - 3x^2 + 1)}{x - 2}$

37. $f(x) = \dfrac{(5x^4 + 2)(x^2 + 3)}{x - 4}$

38. $f(x) = \dfrac{(6x^3 - 2x^2 + 1)(3x - 5)}{2x + 1}$

39. $f(x) = \dfrac{4x^3 + 2x^2 - 3x - 5}{2}$

40. $f(x) = \dfrac{2x^3 + 3x^2 - 7x + 1}{5}$

For Exercises 41–48, complete the following:

(a) *Determine the derivative.*

(b) *Write the equation of the line tangent to the graph of the function at the indicated x-value.*

41. $f(x) = x^2(x^2 - 5)$ at $x = 1$

42. $f(x) = -3x^2(2x + 3)$ at $x = 3$

43. $f(x) = (x^2 + 1)(x^3 + 1)$ at $x = 1$

44. $f(x) = (2x - 3)(x^3 + 3)$ at $x = 5$

45. $f(x) = \dfrac{x + 2}{x - 1}$ at $x = 2$

46. $f(x) = \dfrac{x^2 + 1}{x}$ at $x = -1$

47. $f(x) = \dfrac{3x^2 - 2x}{-2x + 3}$ at $x = -1$

48. $f(x) = \dfrac{-2x^2 + 3x}{3x - 5}$ at $x = 3$

For Exercises 49–54, complete the following:

(a) *Determine the equation of the line tangent to the graph of the function at the indicated x-value.*

(b) *Graph the function and the tangent line in the same viewing window.*

(c) *Use the* `dy/dx` *command or the* `Draw Tangent` *command on your calculator to verify the result in part (a).*

49. $f(x) = x^2(x^2 - 3)$ at $x = 2$

50. $f(x) = -2x^2(3x + 1)$ at $x = 1$

51. $f(x) = (x^3 + 2)(x^2 + 2)$ at $x = -1$

52. $f(x) = (3x - 1)(x^2 + 1)$ at $x = -2$

53. $f(x) = \dfrac{4x^3 - 3x}{2x + 1}$ at $x = 2$

54. $f(x) = \dfrac{-2^2 + 5x}{4x - 1}$ at $x = -1$

In this section, the Product Rule is shown using the product of two functions. The rule can be extended to differentiate the product of any finite number of differentiable functions. For example, if $k(x) = f(x) \cdot g(x) \cdot h(x)$, then the derivative of k is given by $k'(x) = f'(x) \cdot g(x) \cdot h(x) + f(x) \cdot g'(x) \cdot h(x) + f(x) \cdot g(x) \cdot h'(x)$. Use this form of the Product Rule to differentiate the functions in Exercises 55–60.

55. $f(x) = (x + 1)(x - 2)(x + 5)$

56. $f(x) = x^2(x^3 - 3x^2 + 1)(x - 4)$

57. $f(x) = (x + 1)(2x^2 - 3)(3x + 4)$

58. $f(x) = (x - 4)(3x^2 - 5)(2x - 9)$

59. $f(x) = \sqrt{x}(2x - 1)(3x^2 + 2)$

60. $f(x) = \sqrt[3]{x}(3x + 1)(2x^3 - 3)$

Application Exercises

In Exercises 61–64, recall that for varying quantities produced and sold over time period t, the revenue function is $R(t) = p(t) \cdot q(t)$.

61. Finance—Market Analysis: Market analysts at the Maxwell Company have estimated that the monthly sales during the first seven months of its new MePad tablet computer can be modeled by

$$q(t) = 30t - 0.5t^2 \qquad 0 \le t \le 7$$

where t represents the number of months since the MePad became available and $q(t)$ represents the number of computers sold in hundreds. The analysts also determine that the price of the MePad will vary and sets the price using the model

$$p(t) = 2200 - 34t^2 \qquad 0 \le t \le 7$$

where t represents the number of months since the MePad became available and $p(t)$ represents the price of the MePad in dollars.

(a) Compute $q(3)$ and $q'(3)$ and interpret each.

(b) Compute $p(3)$ and $p'(3)$ and interpret each.

(c) Write the revenue function $R(t)$. Do not algebraically simplify the function.

(d) Compute $R(3)$ and $R'(3)$ and interpret each.

62. Finance—Market Analysis (*continuation of Exercise 61*): Use the revenue function from Exercise 61 to complete the following.

(a) Verify that after multiplying and simplifying $R'(t)$ in Exercise 61, we get the function $R'(t) = 68t^3 - 3060t^2 - 2200t + 66{,}000$.

(b) Graph R', and use the `Zero` command on your calculator to determine the t-value so that $R'(t) = 0$ on the interval $0 \le t \le 7$.

(c) Graph R and use the `dy/dx` or `Draw Tangent` command on your calculator to calculate the slope of the line tangent to the graph of R at the value you found in part (b). (Hint: The slope should be close to zero.)

(d) Is the revenue maximized or minimized at the t-value you found in part (b)? Explain your answer.

63. Finance—Market Analysis: The PlayPro Company has projected that the monthly sales of their Bluetooth headphones can be modeled by

$$q(t) = 30t - \frac{t^2}{2} \qquad 0 \le t \le 5$$

where t represents the number of months since the headphones were initially sold and $q(t)$ represents the number of units sold in hundreds. They also project that the retail price of the headphones can be modeled by

$$p(t) = 220 - t^2 \qquad 0 \le t \le 5$$

where t represents the number of months since the headphones were initially sold and $p(t)$ represents the retail price in dollars.

(a) Evaluate $q(3)$ and $q'(3)$ and interpret each.

(b) Evaluate $p(3)$ and $p'(3)$ and interpret each.

64. Finance—Market Analysis (*continuation of Exercise 63*): Use the sales and price functions from Exercise 63 to complete the following.

(a) Write the revenue function $R(t)$. Do not algebraically simplify the function.

(b) Compute $R(3)$ and $R'(3)$ and interpret each.

65. Ecology—Removing Pollutants: In planning its future operating budget, the city of Rockton hires a consulting firm to determine the cost of removing the pollutants discharged into Lake Watson. They find the cost can be projected by the model

$$C(x) = \frac{113x}{100 - x} \qquad 0 \le x < 100$$

where x represents the percentage of the pollutants removed and $C(x)$ represents the cost in thousands of dollars.

(a) Use the Quotient Rule to differentiate $C(x)$.

(b) Evaluate $C(50)$ and $C'(50)$ and interpret each.

66. Ecology—Removing Pollutants: In planning its future operating budget, the city of Utica hires a consulting firm to determine the cost of removing the pollutants discharged into Lake Chevelle. They find the cost can be projected by the model

$$C(x) = \frac{50x}{100 - x} \qquad 0 \le x < 100$$

where x represents the percentage of the pollutants removed and $C(x)$ represents the cost in thousands of dollars.

(a) Use the Quotient Rule to differentiate $C(x)$.

(b) Evaluate $C(50)$ and $C'(50)$ and interpret each.

67. Conservation Science—Fish Propagation: At the request of local anglers, the Graysville game commission decides to stock Lake Mumford with bass. From past data collection, conservation scientists determine that the population of the stocked lake can be modeled by the function

$$P(t) = \frac{10(10 + 7t)}{1 + 0.02t} \qquad t \ge 0$$

where t represents the number of months since the lake was initially stocked and $P(t)$ represents the population size.

(a) How many bass were initially stocked in the lake?

(b) Determine $P'(t)$.

(c) Evaluate and interpret $P(5)$ and $P'(5)$.

68. Conservation Science—Fish Propagation: In order to generate income from public resources, the Crogan city council decides to stock Lake Simpson with commercial catfish. From past data collection, conservation scientists determine that the population of the stocked lake can be modeled by the function

$$P(t) = \frac{20(10 + 7t)}{1 + 0.02t} \qquad t \ge 0$$

where t represents the number of months since the lake was initially stocked and $P(t)$ represents the population size.

(a) How many catfish were initially stocked in Lake Simpson?

(b) Determine $P'(t)$.

(c) Evaluate and interpret $P(12)$ and $P'(12)$.

69. Recreation Studies—Pleasure Craft Purchasing: Suppose an investment firm is trying to determine which type of recreational activity shows a high rate of growth so that it can make a profit on its investment. The amount Americans spent on boats and pleasure craft annually from 1990 to 2010 can be modeled by

$$f(x) = 0.16(\sqrt{x} + 13.19)(x + 12.6) \qquad 0 \le x \le 20$$

where x represents the number of years since 1990, and $f(x)$ represents the annual amount spent on boats and pleasure craft in billions of dollars. (*Source:* U.S. Bureau of Economic Analysis.)

(a) Determine $f'(x)$.

(b) Use part (a) to determine the rate at which the spending on boats and pleasure craft was increasing in 2008.

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70. Public Policy—Patents Issued: A patent is a set of exclusive rights granted by the U.S. government to an inventor for a limited period of time in exchange for a public disclosure of an invention and is often used as a yardstick to measure both the innovativeness and entrepreneurial climate of the country. The number of patents issued in the United States annually from 1990 to 2010 can be modeled by

$$f(x) = 0.2(\sqrt{x} + 7.78)(x + 60.86) \qquad 0 \le x \le 20$$

where x represents the number of years since 1990 and $f(x)$ represents the number of patents issued, measured in thousands. (*Source:* U.S. Patent and Trademark Office.)

(a) Determine $f'(x)$.

(b) Use part (a) to determine the rate that the issuance of patents was increasing in 2007.

71. International Economics—Unemployment in Ireland: Suppose a social worker wants to explore the number of unemployed in Ireland to see how rapidly the unemployment rate has been increasing. The number unemployed in Ireland annually from 2000 to 2010 can be modeled by

$$f(x) = \frac{2.03x - 26.34}{0.03x - 0.35} \qquad 0 \le x \le 10$$

where x represents the number of years since 2000, and $f(x)$ represents the number of unemployed in Ireland, measured in thousands. (*Source:* Google Public Data.)

(a) Determine $f'(x)$.

(b) Use part (a) to determine the rate at which the number of unemployed in Ireland was increasing in 2009.

72. Economics—Consumer Spending: Social networking and Internet commerce have contributed to the amount of spending on video and audio products during the past two decades. The amount spent on audio and video products in the United States from 1990 to 2010 can be modeled by

$$f(x) = \frac{3.39x + 44.41}{-0.01x + 0.82} \qquad 0 \le x \le 20$$

where x represents the number of years since 1990 and $f(x)$ represents the amount spent on audio and video products, measured in billions of dollars. (*Source:* U.S. Bureau of Economic Analysis).

(a) Determine $f'(x)$.

(b) Use the results of part (a) to determine the rate at which spending on video and audio products was increasing in 2008.

73. Macroeconomics—Corporate Tax Rates: In a July 22, 2010, Associated Press interview, Treasury Secretary Tim Geithner said, "We are likely to have to take a broader look at corporate tax reform next year," adding that it was likely to be one of the areas the fiscal commission appointed by President Obama would examine to make recommendations on deficit reduction. The amount collected in U.S. corporate taxes annually from 1990 to 2010 can be modeled by

$$f(x) = 1.65x + 13.13 \qquad 0 \le x \le 20$$

where x represents the number of years since 1990 and $f(x)$ represents the amount of corporate taxes collected in billions of dollars. The amount in corporate profits of U.S. corporations from 1990 to 2010 can be modeled by

$$g(x) = 2.45x^2 + 26.52x + 419.01 \qquad 0 \le x \le 20$$

where x represents the number of years since 1990 and $g(x)$ represents the corporate profits in billions of dollars. (*Sources:* Associated Press, U.S. Bureau of Economic Analysis.)

(a) Form the function $h(x) = \dfrac{f(x)}{g(x)} \cdot 100$, $0 \le x \le 20$, to determine the ratio of taxes per profit as a percentage. Then graph h in the viewing window [0, 20] by [0, 4].

(b) Determine $h'(x)$.

(c) Evaluate $h'(3)$ and $h'(18)$ and interpret each.

74. Demographics—Immigration and Greenhouse Gases: In his book *Bleeding Hearts and Empty Promises: A Liberal Rethinks Immigration,* Philip Cafaro, associate professor of philosophy at Colorado State University, claims that immigration to the United States has contributed to higher levels of greenhouse gas emissions. (*Source:* Center for Immigration Studies.) The amount of greenhouse gas emissions annually in the United States from 1990 to 2010 can be modeled by

$$f(x) = -3.39x^2 + 122.44x + 5977 \qquad 0 \le x \le 20$$

where x represents the number of years since 1990 and $f(x)$ represents the amount of greenhouse gas emissions in millions of metric tons. (*Source:* Google Public Data.) The number of immigrants to the United States from 1990 to 2010 can be modeled by

$$g(x) = -0.03x + 1.42 \qquad 0 \le x \le 20$$

where x represents the number of years since 1990, and $g(x)$ represent the number of immigrants coming to the United States annually in millions. (*Source:* U.S. Census Bureau.)

(a) Form the function $h(x) = \dfrac{f(x)}{g(x)}$, $0 \le x \le 20$, to measure the number of tons of greenhouse gasses in the United States per immigrant.

(b) Determine $h'(x)$.

(c) Evaluate $h'(2)$ and $h'(17)$ and interpret each.

(d) Do the results in part (c) support the professor's premise? Explain.

Concept and Writing Exercises

In Exercises 75 and 76, assume that f, g, and h are differentiable functions.

75. If $k(x) = f(x) \cdot g(x) \cdot h(x)$, determine the derivative $k'(x)$ by differentiating $f(x)[g(x) \cdot h(x)]$ and apply the Product Rule twice.

76. If $k(x) = \dfrac{f(x) \cdot g(x)}{h(x)}$, use the Quotient and Product Rules to determine a general formula for $k'(x)$.

77. Let $h(x) = c \cdot f(x)$, where c represents a real number constant, and let f represent a differentiable function. Use the Product Rule to verify the Constant Multiple Rule by showing that $\frac{d}{dx}[c \cdot f(x)] = c \cdot f'(x)$.

78. Prove the Quotient Rule by showing that the difference quotient for $\dfrac{f(x)}{g(x)}$ is $\dfrac{1}{h}\left[\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}\right] = \dfrac{g(x)f(x+h) - f(x)g(x+h)}{g(x+h)g(x)h}$. (Hint: Before computing the limit, add and subtract $g(x)f(x)$ in the numerator.)

79. Apply the Quotient Rule to the function $f(x) = \dfrac{1}{x^n}$ to show that the Power Rule applies if n is a negative integer.

80. Suppose we have a differentiable function g and know that $g'(x) = \frac{1}{x}$. If $h(x) = x \cdot (g(x) - 1)$, determine an expression for $h'(x)$.

81. Let $f(x) = \frac{1}{x}$ and $g(x) = x^4$. Use the Product Rule to determine h', where $h(x) = f(x) \cdot g(x)$. Compute the product $f'(x) \cdot g'(x)$ to show that the result is not the same as h'.

82. A form of rational function that is commonly used is the ratio of two linear functions $f(x) = \dfrac{ax + b}{cx + d}$ where a, b, c and d are real number constants. Use the Quotient Rule to determine the general form for the derivative f'.

83. Let $h(x) = (f(x))^2$. Use the Product Rule to show that $h'(x) = 2f(x)f'(x)$.

84. Let $h(x) = (f(x))^3$. Show that $h'(x) = 3(f(x))^2 f'(x)$.

Section Project

Hauser's Law is a dictum from San Francisco investment economist W. Kurt Hauser, who claims that since the Second World War, federal tax revenues have been equal to about 19.5% of the U.S. gross domestic product (GDP). (*Source:* The Hoover Institute.) The annual amount of tax receipts from 1990 to 2010 can be modeled by

$$f(x) = 1.29x^3 - 39.1x^2 + 342.99x + 1036.16 \qquad 0 \le x \le 20$$

where x represents the number of years since 1990 and $f(x)$ represents the annual federal tax receipts in billions of dollars. The annual U.S. GDP from 1990 to 2010 can be modeled by

$$g(x) = 11.73x^2 + 263.06x + 5815.13 \qquad 0 \le x \le 20$$

where x represents the number of years since 1990 and $g(x)$ represents the annual GDP in billions of dollars. (*Source:* U.S. Bureau of Economic Analysis).

(a) Form the function $h(x) = \dfrac{f(x)}{g(x)} \cdot 100$, $0 \le x \le 20$, to determine the ratio of federal tax receipts to GDP as a percentage.

(b) Graph $h(x)$ in the viewing window [0, 20] by [0, 30].

(c) During what years is the percentage less than 19.5%? During what years is the percentage greater than 19.5%?

(d) Use the Quotient Rule to determine $h'(x)$.

(e) Evaluate $h'(7)$ and $h'(19)$ and interpret each rate of change.

SECTION OBJECTIVES

1. Differentiate power functions using the Chain Rule.
2. Differentiate radical functions using the Chain Rule.
3. Differentiate rational functions using the Chain Rule.
4. Apply the Chain Rule to health science and business functions.

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3.3 The Chain Rule

So far, the types of functions that we have differentiated are polynomial functions, rational functions, and power functions. But one family of functions that we have not differentiated is the **composite** function family. If the composite function $f(x) = 1000\sqrt{180 - 2x}$ models the number of college graduates surviving to x years of age, we must find a way to compute the derivative of f in order to determine the rate of change of this function. In this section, we introduce the **Chain Rule**, a powerful technique used to differentiate composite functions.

Chain Rule

We will discuss the differentiation of composite functions that have the form $h(x) = f(g(x))$. A reasonable question to ask at this point is "Can we even determine the derivative of a composite function in the first place?" To answer that question, let's take a Flashback to an example first presented in Chapter 1.

Flashback: Oil Spill Revisited

In Section 1.4 we saw that a refinery's underwater supply line ruptures, resulting in a fairly circular oil spill. The radius was modeled by

$$r(t) = 0.7t$$

where t represents the number of seconds since the spill occurred and $r(t)$ represents the radius of the oil slick in feet. The area of the spill was given by

$$A(r) = \pi r^2$$

where r is the radius of the slick in feet and $A(r)$ is the area of the slick in square feet. Simplify the composition $A(r(t))$, which yields $A(t)$, and find the derivative $\frac{d}{dt}[A(t)]$. Interpret the resulting function.

Perform the Mathematics

Simplifying, we get

$$A(r(t)) = A(0.7t)$$

Substituting $0.7t$ for r in the function $A(r) = \pi r^2$, we get the area of the oil slick as a function of time:

$$A(r(t)) = \pi(0.7t)^2 = 0.49\pi t^2 = A(t)$$

Differentiating this result yields

$$\frac{d}{dt}[A(t)] = \frac{d}{dt}[0.49\pi t^2] = 0.98\pi t$$

The derivative, $\frac{d}{dt}[A(t)] = 0.98\pi t$, represents the instantaneous rate of change of the area of the oil slick with respect to time. ■

The Flashback shows that we can differentiate a composite function, yet it does not explicitly show how this is done. Let's examine the functions in the Flashback again and see whether there is another way to find the derivative of $A(t) = A(r(t))$. If we compute the derivatives of the original functions $r(t) = 0.7t$ and $A(r) = \pi r^2$, with respect to t and r respectively, we have

$$r'(t) = 0.7 \quad \text{and} \quad A'(r) = 2\pi r$$

Since we know that $A'(r(t)) = 2\pi(0.7t) = 1.4\pi t$ and $r'(t) = 0.7$, it appears that the derivative $A'(t) = \frac{d}{dt}[A(t)]$ is equivalent to

$$A'(t) = A'(r(t)) \cdot r'(t) = (1.4\pi t) \cdot (0.7) = 0.98\pi t$$

This illustrates exactly how to differentiate a composite function using the **Chain Rule**. We now state this rule for functions made up of a composition of functions f and g.

Chain Rule

If $y = f(u)$ and $u = g(x)$ are used to define $h(x)$, where $h(x) = f(g(x))$, then

$$h'(x) = f'(g(x)) \cdot g'(x)$$

provided $f'(g(x))$ and $g'(x)$ exist.

NOTE: Using Leibniz notation, the Chain Rule is equivalent to $h'(x) = \frac{dy}{du} \cdot \frac{du}{dx}$ provided that $\frac{dy}{du}$ and $\frac{du}{dx}$ exist.

To differentiate $h(x) = f(g(x))$, it appears that we can differentiate the "outside" function f, and then chain it to the derivative of the "inside" function g. Many functions that we will study have the form $h(x) = (\textit{function})^{\textit{power}}$, where the power is some real number. For these types of functions, we can rely on an extension, or corollary, of this rule, called the **Generalized Power Rule**. Observe that this rule looks much like our Power Rule presented in Section 3.1.

Generalized Power Rule

If u is a differentiable function of x and n is any real number with $f(x) = [u(x)]^n$, then

$$f'(x) = n[u(x)]^{n-1} \cdot u'(x)$$

OBJECTIVE 1

Differentiate power functions using the Chain Rule.

Example 1: Using the Generalized Power Rule

Use the Generalized Power Rule to determine derivatives for the following.

a. $f(x) = (5x^3 + 3x)^4$ **b.** $g(x) = (x^2 + 1)^{15}$

Perform the Mathematics

a. For the function $f(x) = (5x^3 + 3x)^4$, consider $u(x) = 5x^3 + 3x$ and $n = 4$. Applying the Generalized Power Rule gives

$$\begin{aligned} f'(x) &= \frac{d}{dx}[(5x^3 + 3x)^4] \\ &= 4(5x^3 + 3x)^{4-1} \cdot \frac{d}{dx}(5x^3 + 3x) \\ &= 4(5x^3 + 3x)^3(15x^2 + 3) \end{aligned}$$

b. Here, we can consider $u(x) = x^2 + 1$ and $n = 15$ and apply the generalized Power Rule to get

$$\begin{aligned} g'(x) &= \frac{d}{dx}[(x^2 + 1)^{15}] \\ &= 15(x^2 + 1)^{14} \cdot \frac{d}{dx}(x^2 + 1) \\ &= 15(x^2 + 1)^{14}(2x) = 30x(x^2 + 1)^{14} \end{aligned}$$

■

Try It Yourself

Some related Exercises are 15 and 17.

Notice the power of the Generalized Power Rule in part (b) of Example 1. It would have been possible, yet not at all practical, to algebraically expand the binomial $(x^2 + 1)^{15}$ in order to apply the differentiation techniques from Section 3.1.

We can use this new technique of differentiation to readily determine derivatives of new families of functions, including the radical and rational exponent functions. To use the Generalized Power Rule with these functions, let's review how these functions can be rewritten as shown in the Toolbox.

From Your Toolbox: Radical and Rational Exponent Functions

A function of the form $f(x) = \sqrt[b]{[g(x)]^a}$ is called a *radical function*. Rewriting $f(x) = \sqrt[b]{[g(x)]^a}$ as $f(x) = [g(x)]^{a/b}$ produces what we call the *rational exponent function*.

The key in differentiating radical functions is to rewrite them in the rational exponent form so that we can apply the Generalized Power Rule. We illustrate this in Example 2.

OBJECTIVE 2

Differentiate radical functions using the Chain Rule.

Example 2: Determining Derivatives of Radical Functions

a. Use the Generalized Power Rule to determine $f'(x)$ for $f(x) = \sqrt[3]{2x - 4}$.

b. Find an equation of the line tangent to the graph of f at the point $(6, 2)$.

Perform the Mathematics

a. Before we can differentiate, we rewrite $f(x) = \sqrt[3]{2x-4}$ using rational exponents.

$$f(x) = \sqrt[3]{2x-4} = (2x-4)^{1/3}$$

Using the Generalized Power Rule with $u(x) = (2x-4)$ and $n = \frac{1}{3}$ gives us

$$\begin{aligned} f'(x) &= \frac{d}{dx}[(2x-4)^{1/3}] \\ &= \frac{1}{3}(2x-4)^{1/3-1} \cdot \frac{d}{dx}(2x-4) = \frac{1}{3}(2x-4)^{-2/3}(2) = \frac{2}{3}(2x-4)^{-2/3} \end{aligned}$$

Writing the derivative without negative or rational exponents, this simplifies to

$$f'(x) = \frac{2}{3}(2x-4)^{-2/3} = \frac{2}{3(2x-4)^{2/3}} = \frac{2}{3\sqrt[3]{(2x-4)^2}}$$

b. Since $f'(6)$ gives the slope of the tangent line at $x = 6$, we have

$$f'(6) = \frac{2}{3\sqrt[3]{(2(6)-4)^2}} = \frac{2}{3\sqrt[3]{64}} = \frac{2}{3 \cdot 4} = \frac{1}{6}$$

With a slope of $\frac{1}{6}$ and point $(6, 2)$, we use the point-slope form of a line to get the tangent line equation:

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= \frac{1}{6}(x-6) \\ y &= \frac{1}{6}(x-6) + 2 \end{aligned}$$

The graphs of $f(x) = \sqrt[3]{2x-4}$ and $y = \frac{1}{6}(x-6) + 2$ are shown in **Figure 3.3.1**.

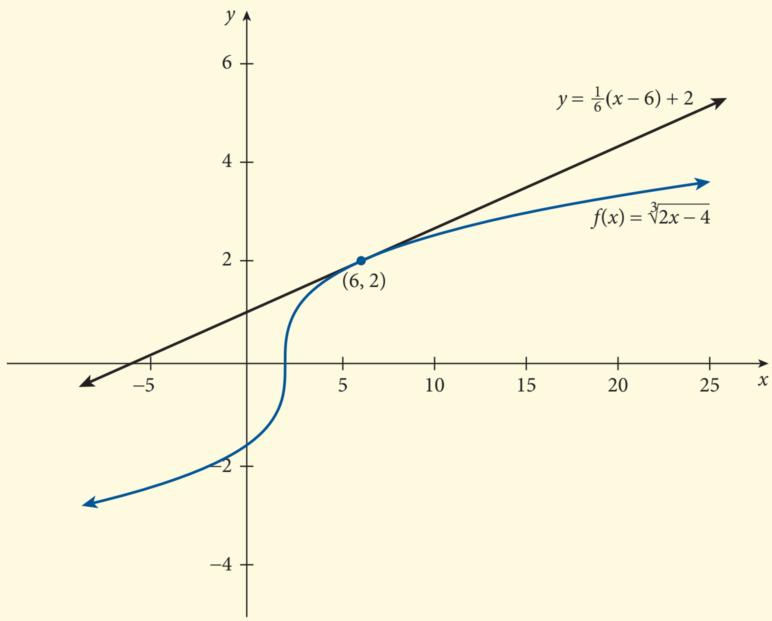


Figure 3.3.1

Try It Yourself

Some related Exercises are 41 and 47.

Another family of functions that can be differentiated using the Generalized Power Rule are rational functions. Even though the Generalized Power Rule can be used on any rational function, it is particularly useful on rational functions that have constants in their numerators.

OBJECTIVE 3

Differentiate rational functions using the Chain Rule.

Example 3: Determining Derivatives of Rational Functions Two Ways

For the function $h(x) = \dfrac{5}{(2x-3)^2}$, determine $h'(x)$ using the Quotient Rule and then determine $h'(x)$ using the Generalized Power Rule.

Perform the Mathematics

Using the Quotient Rule, we get

$$\begin{aligned} h'(x) &= \frac{\frac{d}{dx}[5] \cdot (2x-3)^2 - (5) \cdot \frac{d}{dx}[(2x-3)^2]}{[(2x-3)^2]^2} \\ &= \frac{0 \cdot (2x-3)^2 - 5 \cdot [2(2x-3)(2)]}{(2x-3)^4} \\ &= \frac{-20(2x-3)}{(2x-3)^4} = \frac{-20}{(2x-3)^3} \end{aligned}$$

When using the Generalized Power Rule, we rewrite the rational function as $h(x) = 5(2x-3)^{-2}$. With help from the Constant Multiple Rule, we determine

$$\begin{aligned} h'(x) &= 5 \cdot \frac{d}{dx}[(2x-3)^{-2}] \\ &= 5 \cdot (-2)(2x-3)^{-3} \cdot (2) \\ &= -20(2x-3)^{-3} = \frac{-20}{(2x-3)^3} \end{aligned}$$

■

Try It Yourself

Some related Exercises are 35 and 39.

Example 4: Applying the Generalized Power Rule

Even though the number of deaths due to heart disease has been declining over the past couple of decades, heart disease is the leading cause of death in the United States. About every 25 seconds, an American will have a coronary event, and about once every minute an American will die from one. The death rate caused by heart disease in the United States can be modeled by

$$f(x) = 369.2(x+1)^{-0.17} \qquad 0 \le x \le 17$$

where x represents the number of years since 1990 and $f(x)$ represents the death rate caused by heart disease measured in deaths per 100,000 people. Evaluate $f'(12)$ and interpret. (*Source:* Centers for Disease Control and Prevention.)

Understand the Situation

In order to evaluate $f'(12)$, we need to determine the derivative, $f'(x)$, using the Generalized Power Rule. Once we have the derivative, we will substitute 12 for each occurrence of x. Note that $x = 12$ corresponds to 2002.

Perform the Mathematics

Using the Generalized Power Rule, we get

$$\begin{aligned} f'(x) &= \frac{d}{dx}[369.2(x+1)^{-0.17}] \\ &= (369.2)\cdot(-0.17)(x+1)^{-0.17-1}\cdot(1) \\ &= -62.764(x+1)^{-1.17} \end{aligned}$$

We now evaluate $f'(12)$ as follows:

$$f'(12) = -62.764(12+1)^{-1.17} \approx -3.12$$

Interpret the Result

This means that in 2002, the death rate caused by heart disease was decreasing at a rate of about 3.12 deaths per 100,000 people per year. See **Figure 3.3.2**.

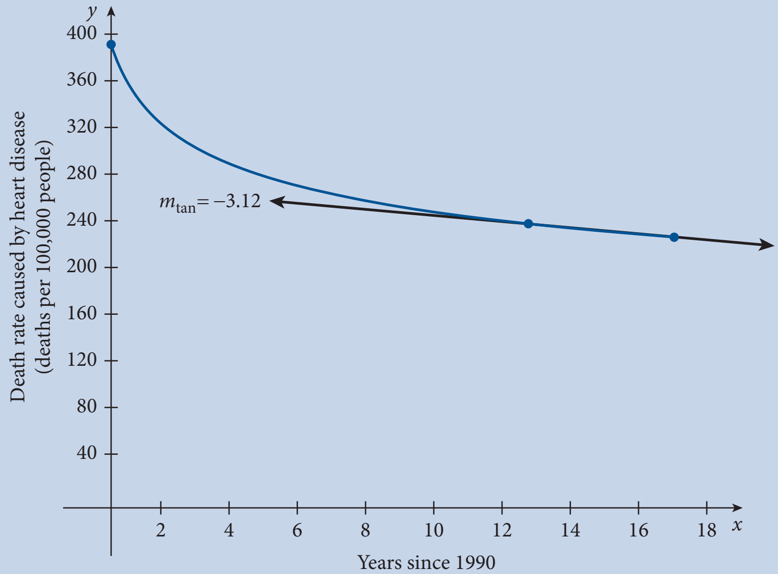
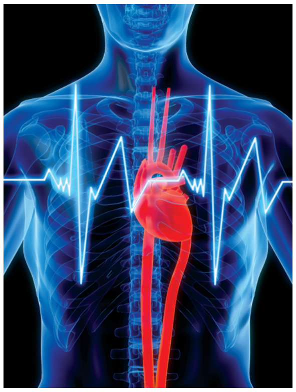


Figure 3.3.2

Try It Yourself

Some related Exercises are 81 and 83.



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OBJECTIVE 4

Apply the Chain Rule to health science and business functions.

Our final example in this section demonstrates how the Generalized Power Rule can be used with business functions.

Example 5: Applying the Generalized Power Rule to Business Functions

The RadRadio Company produces and sells personal stereo devices. Market research has found that the price-demand function is

$$p(x) = 100 - \sqrt{x^2 + 20} \qquad 0 \le x \le 35$$

where x is the number of devices demanded in thousands and $p(x)$ represents the unit price in dollars.

a. Evaluate $p'(30)$ and interpret. **b.** Evaluate $R'(30)$ and interpret.

Perform the Mathematics

a. To aid us in determining $p'(x)$, we write this radical function as a rational exponent function. The rational exponent form of the price-demand function is

$$p(x) = 100 - (x^2 + 20)^{1/2} \qquad 0 \le x \le 35$$

Using the Generalized Power Rule, we get the derivative

$$\begin{aligned} p'(x) &= -\frac{1}{2}(x^2 + 20)^{-1/2} \cdot \frac{d}{dx}(x^2 + 20) \\ &= -\frac{1}{2}(x^2 + 20)^{-1/2} \cdot (2x) \\ &= -x(x^2 + 20)^{-1/2} \\ &= \frac{-x}{(x^2 + 20)^{1/2}} = \frac{-x}{\sqrt{x^2 + 20}} \end{aligned}$$

Evaluating $p'(30)$ yields

$$p'(30) = \frac{-30}{\sqrt{(30)^2 + 20}} = \frac{-30}{\sqrt{920}} \approx -0.99$$

This means that at a demand level of 30,000 personal stereo devices, the price is decreasing at a rate of about \$0.99 (or 99 cents) per 1000 devices sold.

b. We need to first determine the revenue function. Knowing that the revenue function is $R(x) = x \cdot p(x)$, we use the price-demand function to get

$$\begin{aligned} R(x) &= x \cdot p(x) = x \cdot (100 - \sqrt{x^2 + 20}) \\ &= 100x - x\sqrt{x^2 + 20} = 100x - x(x^2 + 20)^{1/2} \end{aligned}$$

Notice that when we determine the derivative $R'(x)$, we must apply the Product Rule to the second term of $R(x)$. This gives us

$$R'(x) = \frac{d}{dx}[100x - x(x^2 + 20)^{1/2}]$$

Sum and Difference Rule
$$= \frac{d}{dx}[100x] - \frac{d}{dx}[x(x^2 + 20)^{1/2}]$$

Product Rule
$$= \frac{d}{dx}[100x] - \left[\frac{d}{dx}(x) \cdot (x^2 + 20)^{1/2} + x \cdot \frac{d}{dx}(x^2 + 20)^{1/2}\right]$$

Chain Rule
$$= 100 - (1)(x^2 + 20)^{1/2} - x\left(\frac{1}{2}\right)((x^2 + 20)^{-1/2}(2x))$$

Simplify
$$= 100 - (x^2 + 20)^{1/2} - x^2(x^2 + 20)^{-1/2}$$

Rewrite
$$= 100 - \sqrt{x^2 + 20} - \frac{x^2}{\sqrt{x^2 + 20}}$$

Evaluating $R'(x)$ at $x = 30$ produces

$$R'(30) = 100 - \sqrt{(30)^2 + 20} - \frac{(30)^2}{\sqrt{(30)^2 + 20}}$$
$$= 100 - \sqrt{920} - \frac{900}{\sqrt{920}} \approx 40.0$$

This means that when the production level is 30,000 personal stereo devices, revenue is increasing at a rate of about \$40 per thousand devices sold. ■

Summary

In this section, we reviewed composite functions and then introduced the Chain Rule, which is a technique used to differentiate composite functions. We then focused on the Generalized Power Rule, which is an extension of the Chain Rule.

- **Chain Rule:** If $y = f(u)$ and $u = g(x)$ are used to define $h(x)$, where $h(x) = f(g(x))$, then $h'(x) = f'(g(x)) \cdot g'(x)$ provided $f'(g(x))$ and $g'(x)$ exist.
- **Generalized Power Rule:** If u is a differentiable function of x and n is any real number with $f(x) = [u(x)]^n$, then $f'(x) = n[u(x)]^{n-1} \cdot u'(x)$.

Section 3.3 Exercises

Vocabulary Exercises

1. The Chain Rule is used to differentiate ________________ functions.

2. A function of the form $f(x) = \sqrt[b]{[g(x)]^a}$ is called a ________________ function.

3. If n represents a real number and $f(x) = [g(x)]^n$, then we can differentiate f using the ________________ ________________ Rule.

4. A function of the form $f(x) = [g(x)]^{a/b}$ produces what we call a ________________ exponent function.

Skill Exercises

For Exercises 5–10, differentiate using the Generalized Power Rule.

5. $f(x) = (x + 1)^2$

6. $f(x) = (x + 3)^2$

7. $f(x) = (x - 5)^3$

8. $f(x) = (x - 2)^3$

9. $f(x) = (2 - x)^2$

10. $f(x) = (5 - x)^2$

For Exercises 11–34, differentiate using the Generalized Power Rule.

11. $f(x) = (2x + 4)^3$

12. $f(x) = (3x + 3)^3$

13. $f(x) = (5 - 2x)^5$

14. $f(x) = (10 - 5x)^4$

15. $f(x) = (3x^2 + 7)^5$

16. $f(x) = (4x^2 - 3)^3$

17. $f(x) = (x^3 - 2x^2 + x)^2$

18. $f(x) = (2x^3 + 4x + 3)^3$

19. $f(x) = 3(x^3 - 4)^3$

20. $f(x) = 5(4x^2 - 10)^6$

21. $f(x) = 5(5x^2 - 3x - 1)^{10}$

22. $f(x) = 10(10x^3 + x - 9)^8$

23. $f(x) = (4x^2 - x - 4)^{55}$

24. $f(x) = (8x^2 - 2x + 5)^{94}$

25. $f(x) = (2x - 4)^{1/2}$
26. $f(x) = (7x + 6)^{1/2}$
27. $f(x) = (x^2 + 2x)^{1/3}$
28. $f(x) = (x^3 + 5x)^{1/3}$
29. $f(x) = (5x - 2)^{-2}$
30. $f(x) = (4x + 3)^{-3}$
31. $f(x) = (x^2 + 2x + 4)^{-1/2}$
32. $f(x) = (3x^2 + 5x + 6)^{-1/2}$
33. $f(x) = (3x^3 - x)^{-1/4}$
34. $f(x) = (4x^5 + 5x^3)^{-1/3}$

For the rational functions in Exercises 35–40, complete the following:

(a) *Determine the derivative using the Quotient Rule.*

(b) *Determine the derivative using the Generalized Power Rule.*

35. $f(x) = \dfrac{1}{3x + 4}$
36. $f(x) = \dfrac{1}{7x - 5}$
37. $f(x) = \dfrac{5}{(x - 2)^2}$
38. $f(x) = \dfrac{10}{(2x - 1)^3}$
39. $f(x) = \dfrac{2}{x^2 + 2x + 3}$
40. $f(x) = \dfrac{9}{x^3 + 2x + 10}$

In Exercises 41–48, determine an equation of the line tangent to the graph of f at the indicated point.

41. $f(x) = (2x - 1)^3$; $(1, 1)$
42. $f(x) = (3x - 4)^3$; $(1, -1)$
43. $f(x) = (2 - x)^4$; $(1, 0)$
44. $f(x) = (x^2 - 1)^4$; $(1, 0)$
45. $f(x) = (x^3 - 4x + 2)^4$; $(2, 16)$
46. $f(x) = (4x - 3)^{1/2}$; $(3, 3)$
47. $f(x) = (2x - 4)^{1/2}$; $(2, 0)$
48. $f(x) = (2x + 8)^{1/2}$; $(4, 4)$

Use the Generalized Power Rule to differentiate the functions in Exercises 49–56.

49. $f(x) = \sqrt{x^2 + 5}$
50. $f(x) = \sqrt{3x + 6}$
51. $f(x) = \sqrt[3]{2x - 1}$
52. $f(x) = \sqrt[3]{4x - 3}$
53. $f(x) = \dfrac{5}{\sqrt{2x - 8}}$
54. $f(x) = \dfrac{10}{\sqrt{5x + 8}}$
55. $f(x) = \dfrac{64}{\sqrt[3]{5x^2 - 6x + 3}}$
56. $f(x) = \dfrac{27}{\sqrt[3]{3x^3 + x}}$

In Exercises 57–66, use the Generalized Power Rule, along with the Product and Quotient Rules, to determine the derivative of the given functions.

57. $f(x) = x(x - 4)^3$
58. $f(x) = x(10 - x)^3$
59. $f(x) = x\sqrt{x^2 + 3x}$
60. $f(x) = x^2\sqrt{2x^2 - 11}$
61. $f(x) = \dfrac{x^3}{(3x - 8)^2}$
62. $f(x) = \dfrac{x^2}{(4x^2 - x + 5)^3}$
63. $f(x) = (x + 3)^3(2x - 1)^2$
64. $f(x) = (3x - 3)^4(2x - 2)^3$
65. $f(x) = \sqrt{\dfrac{x + 3}{x - 3}}$
66. $f(x) = \sqrt{\dfrac{2x + 1}{2x - 1}}$

In Exercises 67–72, find the derivatives using the Generalized Power Rule.

67. $f(x) = (4x^2 + 5x + 6)^{0.23}$
68. $f(x) = 3(0.7x^3 - 0.02x^2)^{0.09}$
69. $f(x) = \left(\dfrac{1}{x + 3}\right)^{-1.03}$
70. $f(x) = \left(\dfrac{1}{0.2x + 1.7}\right)^{-1.1}$
71. $f(x) = 1.44(x + 1)^{1.22}$
72. $f(x) = 67.41(x + 1)^{0.97}$

Application Exercises

73. Actuarial Science—Survival Rates: The Hogan Actuary Firm has determine that for residents who were born and raised in Buchannan County, the number of people surviving since World War II can be modeled by

$$f(x) = 400\sqrt{100 - x} \qquad 0 \le x \le 100$$

where x represents the age of the Buchannan County resident and $f(x)$ represents the number of residents surviving. Evaluate and interpret $f'(70)$.

74. Market Research—TV Ratings: During its first season, the number of viewers who watched the new television series "It Ain't Me!" can be modeled by

$$f(x) = \sqrt[3]{(50 + 2x)^2} \qquad 1 \le x \le 26$$

where x represents the number of weeks the series has been airing and $f(x)$ represents the number of viewers in millions. Evaluate and interpret $f'(13)$.

75. Education—Enrollment Trends: A study by the Dean of Clarksman University determines that the annual student enrollment in the Arts and Science College from 2001 to 2011 can be modeled by

$$f(x) = -\frac{10{,}000}{\sqrt{1 + 0.18x}} + 10{,}000 \qquad 1 \le x \le 11$$

where x represents the number of years since 2000 and $f(x)$ represents the student enrollment. Use the model to determine the rate of change in enrollment in the Arts and Science College in 2010.

76. Medicine—Arteriosclerosis: Medical researchers studying arteriosclerosis at the Giuliani Institute have found that if the radius of an examined patient's artery is currently one centimeter, the amount of fatty tissue called plaque that will build up in the artery can be modeled by the function

$$f(x) = 0.5x^2(x^2 + 10)^{-1} \qquad 0 \le x \le 10$$

where x represents the number of years since the initial examination and $f(x)$ represents the thickness of the artery wall in centimeters. Use the model to determine the rate of change in plaque thickness seven years after a patient was initially examined.

In Exercises 77 and 78, consider the following. In the early 1930s, psychologist L. L. Thurstone determined that the time needed to memorize a list of random words is given by the model

$$f(x) = ax\sqrt{x - b} \qquad x \ge b$$

where x represents the number of words on the list and $f(x)$ represents the time needed to memorize the list, measured in minutes. The constants a and b are different for each subject and are determined by pretesting.

77. Psychology—Memorization Rates: Erica is a freshman subject in a psychology class and learns the items on the memorization list according to the model

$$f(x) = \frac{5}{2}x\sqrt{x - 6} \qquad x \ge 6$$

(a) Evaluate $f(20)$ and interpret.

(b) Determine $f'(x)$.

(c) Evaluate $f'(20)$ and interpret.

78. Psychology—Memorization Rates: José has completed his pretesting, and it has been determined that he learns the items on the memorization list according to the model

$$f(x) = 2x\sqrt{x - 3} \qquad x \ge 3$$

(a) Evaluate $f(12)$ and interpret.

(b) Determine $f'(x)$.

(c) Evaluate $f'(12)$ and interpret.

79. Ecology—Chemical Run-Off: The amount of toxic material entering Lake Formica is related to the number of years that the Bristine Chemical Company has been operating and can be modeled by

$$f(x) = \left(\frac{4}{5}x^{1/5} + 2\right)^4 \qquad 0 \le x \le 30$$

where x represents the number of years that the company has been operating and $f(x)$ represents the amount of toxic material entering the lake, measured in gallons.

(a) Evaluate $f(15)$ and interpret.

(b) Determine $f'(x)$.

(c) Evaluate $f'(15)$ and interpret.

80. Accounting—Tax Revenue: Suppose that the rural town of Rufusville decides to relax its zoning laws so that more land can be made eligible for commercial use. They estimate that the town's annual tax revenue after the zoning changes can be modeled by

$$f(x) = 5x\sqrt{2x + 2} \qquad x \ge 0$$

where x represents the number of years since the zoning laws were changed, and $f(x)$ represents the annual tax revenue in thousands of dollars.

(a) Evaluate $f(15)$ and interpret.

(b) Determine $f'(x)$.

(c) Evaluate $f'(15)$ and interpret.

81. Macroeconomics—Minimum Wage: Suppose an economist wants to examine the rate of growth in minimum wage in certain European countries. The monthly minimum wage in Romania from 2000 to 2010 can be modeled by

$$f(x) = 2.27(2.66x + 8.11)^{1.19} \qquad 0 \le x \le 10$$

where x represents the number of years since 2000 and $f(x)$ represents the Romanian national monthly minimum wage, measured in euros per month. (*Source:* EuroStat.) Evaluate $f'(8)$ and interpret.

82. Macroeconomics—Minimum Wage: Suppose an economist wants to examine the rate of growth in minimum wage in certain European countries. The monthly minimum wage in Greece from 2000 to 2010 can be modeled by

$$f(x) = 2.46(0.83x + 23.98)^{1.68} \qquad 0 \le x \le 10$$

where x represents the number of years since 2000 and $f(x)$ represents the Greek national monthly minimum wage, measured in euros per month. (*Source:* EuroStat.) Evaluate $f'(4)$ and interpret.

83. Women's Studies—Unmarried Labor Force: A sociological topic that has been discussed since the end of World War II is the status of women in the labor force. The number of unmarried women in the U.S. labor force from 1970 to 2010 can be modeled by

$$f(x) = 1.11(1.57x + 17.27)^{0.66} \qquad 0 \le x \le 40$$

where x represents the number of years since 1970 and $f(x)$ represents the number of unmarried women in the U.S. labor force, measured in millions. (*Source:* U.S. Bureau of Labor Statistics.)

(a) Determine $f'(x)$.

(b) Evaluate $f(35)$ and $f'(35)$ and interpret each.

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84. **Women's Studies—Married Labor Force**: Over the past half century, increasing numbers of married women have gone to work. The number of married women in the U.S. labor force from 1970 to 2010 can be modeled by

$$f(x) = 2.57(18.33x + 139.79)^{0.4} \qquad 0 \le x \le 40$$

where x represents the number of years since 1970 and $f(x)$ represents the number of married women in the U.S. labor force, measured in millions. (*Source:* U.S. Bureau of Labor Statistics.)

(a) Determine $f'(x)$.

(b) Evaluate $f(35)$ and $f'(35)$ and interpret each.

Concept and Writing Exercises

For Exercises 85–88, suppose that g is a differentiable function and the derivative of g is $g'(x) = \frac{1}{x}$. Use the Chain Rule to differentiate the following functions. Algebraically simplify your answers.

85. $f(x) = g(x^3)$

86. $f(x) = g\left(\dfrac{1}{x}\right)$

87. $f(x) = g\left(\dfrac{3}{4\sqrt{x}}\right)$

88. $f(x) = g\left(\dfrac{1 + 2x}{1 - x}\right)$

89. Consider the function $f(x) = (3x + 5)^4(2x - 8)^3$. Write a few complete sentences that describe the strategy you would use to differentiate the function f. Do not determine the derivative f'; just explain how you would do it.

90. Let $h(x) = \dfrac{f(x)}{g(x)}$. Use the Chain Rule to derive the Quotient Rule by first rewriting h as $h(x) = f(x) \cdot g(x)^{-1}$.

91. Determine the derivative of the function $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$ by using the Chain Rule two times.

In Exercises 92–94, consider the following. Suppose that the value of x and y are determined by another value, or parameter t. That is, $x = x(t)$ and $y = y(t)$. Then x and y are called ***parametric equations****. We can show that the derivative of y with respect to x for parametric equations is given by $\dfrac{dy}{dt} = \dfrac{y'(t)}{x'(t)} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}$. Determine the derivative $\frac{dy}{dx}$ for each of the following and write your answer in terms of the independent variable t.*

92. $y = y(t) = t^2 + 1, x = x(t) = 3t^3$

93. $y = y(t) = \sqrt{t + 1}; x = x(t) = \sqrt{2t}$

94. $y = y(t) = \dfrac{1}{1 - t}; x = x(t) = \dfrac{1}{t}$

Section Project

In November 2006, Mitra Toossi, an economist at the Office of Occupational Statistics and Employment Projections, stated that "peaking at 2.6 percent during the 1970s, the growth rate of the labor force has been decreasing with the passage of each decade and is expected to continue to do so in the future." (*Source:* Monthly Labor Review Online). To test the economist's remarks, we can derive a model for the size of the labor force, and then inspect its relative growth. The civilian labor force since 1980 and projected into 2016 can be modeled by the function

$$f(x) = \sqrt{ax + b} \qquad 0 \le x \le 36$$

where x represents the number of years since 1980 and $f(x)$ represents the civilian labor force, measured in millions. (*Source:* U.S. Bureau of Labor Statistics.) The values a and b represent whole number constants.

(a) In 1980 the civilian labor force was 107.35 million. For the model this means that $f(0) = 107.35$. Use this information to solve for the constant b.

(b) We also know that in 2000, the civilian labor force was 140.9 million. Use this information to determine the constant a in the model $f(x) = \sqrt{ax + b}$.

(c) Use the techniques of this section to determine $f'(x)$.

(d) Use your calculator to make a table of values for f and f' where $Y_1 = f(x)$ and $Y_2 = f'(x)$. What observation can you make about the values of the derivative?

(e) Assign `Y3 = Y2/Y1*100` to form the percentage rate of change function $g(x) = \frac{f'(x)}{f(x)} \cdot 100$. Evaluate this percentage rate of change function at $x = 0$, $x = 10$, $x = 20$, $x = 30$, and $x = 36$. Do these percentages support the economist's claim?

SECTION OBJECTIVES

1. Compute a differential.
2. Apply the differential to a business function.
3. Compute a linear approximation.

3.4 The Differential and Linear Approximations

How often have you heard on TV or read in a newspaper or online a statement such as, "If crime keeps growing at this rate. . . ." The problem with this statement is that rates do indeed change. This is the reason that we study calculus—to see the impact of these fluctuating rates of change. In this section, we will examine the mathematics of the statement made above, and how to use the **differential** to make short-term conclusions about applications. Then we will study the idea of **linear approximations**, which can be used to approximate the change in the dependent variable as changes in the independent variable are made.

The Differential

The differential is a tool used to study the relationship between changes in the independent and dependent variable values. So let's begin by revisiting some tools that we introduced with the difference quotient.

From Your Toolbox: The Difference Quotient

Recall that for a function f on a closed interval $[x_1, x_2]$:

- $\Delta x = x_2 - x_1$ is called the increment in x. In Chapter 2 we called this quantity h.
- $\Delta y = f(x_2) - f(x_1)$ is called the increment in y.
- The slope of a secant line, m_{sec}, is given by $m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$. Recall that $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ is called the difference quotient.

Now we can establish a context for our discussion through a Flashback.

Flashback: Prison Inmate Population Revisited


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In Section 3.1, we modeled the total U.S. Federal Bureau of Prisons inmate population with the function

$$f(x) = -\frac{3}{50}x^2 + \frac{361}{40}x + 56.21 \qquad 1 \le x \le 19$$

where x represents the number of years since 1989, and $f(x)$ represents the total inmate population in thousands. (*Source*: Federal Bureau of Prisons.)

a. Evaluate $f(17) - f(16)$ and interpret this result.

b. Determine the difference quotient with $x = 16$ and $\Delta x = 1$ and interpret. Compare to part (a).

Perform the Mathematics

a. First note that the values $x = 16$ and $x = 17$ correspond to the years 2005 and 2006, respectively. Also note that when computing $f(17) - f(16)$, we are simply finding a change in y, that is, Δy.

$$f(17) - f(16) = \left[-\frac{3}{50}(17)^2 + \frac{361}{40}(17) + 56.21\right] - \left[-\frac{3}{50}(16)^2 + \frac{361}{40}(16) + 56.21\right]$$
$$= 192.295 - 185.25 = 7.045$$

So from 2005 to 2006, the U.S. federal prison inmate population increased by 7.045 thousand, or 7045 inmates.

b. Computing the difference quotient with $x = 16$ and $\Delta x = 1$ gives us

$$m_{\text{sec}} = \frac{f(x + \Delta x) - f(x)}{\Delta x} = \frac{f(16 + 1) - f(16)}{1}$$
$$= \frac{f(17) - f(16)}{1} = \frac{192.295 - 185.25}{1} = 7.045$$

This means that from 2005 to 2006, the U.S. federal prison inmate population increased at an average rate of 7045 inmates per year. Notice that this result is the same as in part (a). In other words, **when** $\Delta x = 1$, **then** $\Delta y = 3.61 = m_{\text{sec}}$, where m_{sec} is the slope of the secant line. ■

Now let's mathematically try the scenario that we suggested at the beginning of this section. What if the prison population continued to grow at a rate constant with that of 2005? Will this assumption give an accurate prediction of the 2006 U.S. federal prison population? Using the derivative and the tangent line, we can find out. First, we find the growth rate in 2005 using the derivative f', since we know that $f'(16)$ is the instantaneous rate of change at $x = 16$:

$$f'(x) = \frac{d}{dx}\left(-\frac{3}{50}x^2 + \frac{361}{40}x + 56.21\right) = -\frac{3}{25}x + \frac{361}{40}$$
$$f'(16) = -\frac{3}{25}(16) + \frac{361}{40} = \frac{1421}{200} = 7.105$$

Thus, the U.S. federal prison inmate population was growing at a rate of 7105 inmates per year in 2005. Assuming that the rate of change is constant during 2005, we use the equation of the line tangent to the curve at $x = 16$ to predict the 2006 federal inmate population. Example 1 illustrates the process.

Example 1: Using the Tangent Line Equation

Determine an equation of the line tangent to the graph of the model $f(x) = -\frac{3}{50}x^2 + \frac{361}{40}x + 56.21$ when $x = 16$. On the tangent line, determine y when $x = 17$ and interpret what this means.

Perform the Mathematics

We will use the derivative to determine the slope of the tangent line. Specifically, $f'(16)$ gives us the slope of the line tangent to the graph of f at $x = 16$. The point of tangency is $(16, f(16)$. Once we have the slope of the tangent line and the point of tangency, we will use the point-slope form to determine an equation of the tangent line.

Knowing that $f(16) = -\frac{3}{50}(16)^2 + \frac{361}{40}(16) + 56.21 = 185.25$ and $f'(16) = 7.105$, we use the point-slope form of a line to determine an equation of the tangent line at $x = 16$:

$$y - y_1 = m(x - x_1)$$
$$y - 185.25 = 7.105(x - 16)$$
$$y = 7.105(x - 16) + 185.25$$

Evaluating the tangent line equation at $x = 17$ gives us

$$y = 7.105(17 - 16) + 185.25 = 7.105(1) + 185.25 = 192.355$$

This means that if the U.S. federal prison inmate population continued growing at the 2005 rate, the 2006 inmate population would be about 192.355 thousand. Using the model, the number of inmates in 2006 was $f(17) = 192.295$, meaning that there was a difference of only about 0.06 thousand, or 60 inmates. See **Figure 3.4.1**.

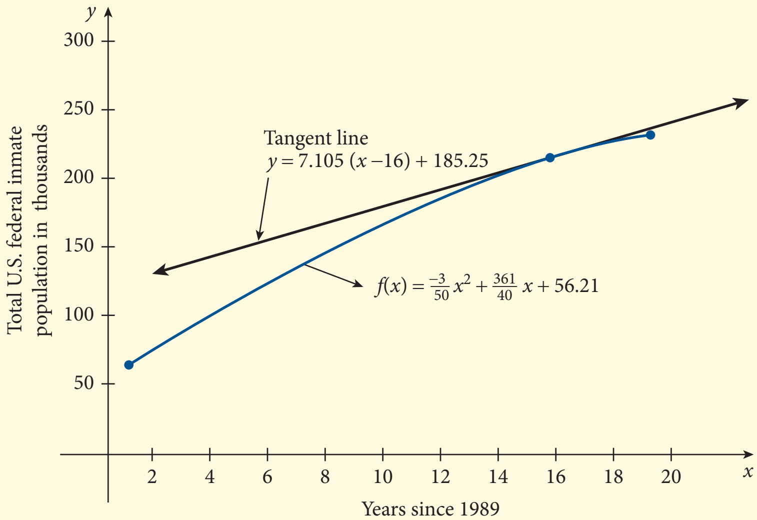

Figure 3.4.1

DEFINITION

The Differential

If $y = f(x)$, where f is a differentiable function, then we define the following:

- The **differential in x,** denoted by **dx,** of the independent variable is given by

$$dx = \Delta x$$

- The **differential in y,** denoted by **dy,** of the dependent variable is given by

$$dy = f'(x)dx$$

NOTE: Δy and dy are not the same. Δy is the *actual* change in the dependent variable values, whereas dy is an *approximation* of this change. If dx is small, then $dy \approx \Delta y$.

We can make two important observations from the results of the Flashback and from Example 1:

1. From part (a) of the Flashback, we found that the actual change as x changed from 16 to 17 ($\Delta x = 1$) was $\Delta y = 7.045$ thousand prisoners. We also found that $f'(16)$, the instantaneous rate of change at $x = 16$ was 7.105 thousand inmates per year. Thus, for $\Delta x = 1$, Δy is approximately equal to $f'(16)$.
2. Example 1 showed that $f(17) = 192.295$ and the y-value on the tangent line at $x = 17$ ($y = 192.355$) are very close.

The concept of the **differential** is embedded in observation 1, while the concept of a **linear approximation** is embedded in observation 2. We will discuss the linear approximation later in this section, but before that, let's examine the **differential** and its applications.

Many disciplines think of dy as the change in y on the tangent line. We show the difference between Δy and dy in **Figure 3.4.2**.

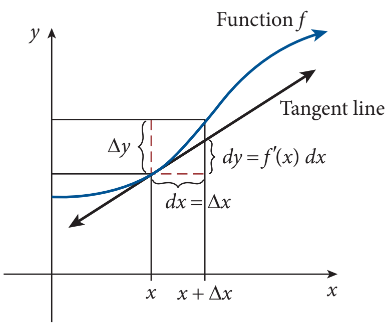

Figure 3.4.2

Example 2: Computing the Differential

OBJECTIVE 1

Compute a differential.

For the function $f(x) = x^2 + 3x - 8$, evaluate Δy and dy for $x = 2$ and $\Delta x = dx = 0.1$.

Perform the Mathematics

We can compute the change in y, Δy, by evaluating $f(x + \Delta x) - f(x)$:

$$\begin{aligned} y &= f(2 + 0.1) - f(2) \\ &= f(2.1) - f(2) \\ &= ((2.1)^2 + 3(2.1) - 8) - ((2)^2 + 3(2) - 8) \\ &= 2.71 - 2 = 0.71 \end{aligned}$$

Since $f'(x) = 2x + 3$, the differential in y for the function is

$$\begin{aligned} dy &= f'(x)dx \\ dy &= (2x + 3)dx \end{aligned}$$

So at $x = 2$ and $\Delta x = dx = 0.1$, we get

$$dy = (2(2) + 3)(0.1) = 7(0.1) = 0.7$$

Notice that $dy \approx \Delta y$ in this example. Again, this is true when dx is small. ■

Try It Yourself

Some related Exercises are 23 and 25.

Our next example illustrates how differentials can be used in business applications.

Example 3: Using Differentials in Applications

OBJECTIVE 2

Apply the differential to a business function.

The Garland Toddler Company determined that the price-demand function for their pacifier/thermometer is given by

$$p(x) = 15 - 0.2\sqrt{x}$$

where x represents the quantity demanded and $p(x)$ represents the unit price in dollars.

a. Compute Δp, the actual change in price, for $x = 100$ and $\Delta x = dx = 1$.

b. Determine the differential dp for the price-demand function. Use the differential dp to approximate the change in price that would cause the quantity demanded to increase from 100 to 101 units.

Perform the Mathematics

a. Using $x = 100$ and $\Delta x = dx = 1$, we get the increment in p as

$$\begin{aligned} \Delta p &= p(x + \Delta x) - p(x) \\ &= p(100 + 1) - p(100) \\ &= p(101) - p(100) \\ &\approx 12.99002 - 13 \\ &= -0.00998 \end{aligned}$$

b. For the function $p(x) = 15 - 0.2\sqrt{x}$, the differential in the dependent variable p is

$$\begin{aligned} dp = p'(x)dx &= \frac{d}{dx}[15 - 0.2\sqrt{x}]dx \\ &= -0.2\left(\frac{1}{2}x^{-1/2}\right)dx \\ &= (-0.1x^{-1/2})dx \end{aligned}$$

Evaluating dp when $x = 100$ and $\Delta x = dx = 101 - 100 = 1$, we get

$$\begin{aligned} dp &= (-0.1(100)^{-1/2})(1) \\ &= (-0.1)(0.1)(1) = -0.001 \end{aligned}$$

This means that as the quantity demanded changes from 100 to 101 units, the unit price of the pacifier/thermometer would decrease by about 1 cent. Notice how close the numerical solutions are in parts (a) and (b). This is further evidence that **if dx is small, then $dy \approx \Delta y$.**

Linear Approximations

To get a better picture of why linear approximations are used, let's return to the model for the U.S. federal prison inmate population:

$$f(x) = -\frac{3}{50}x^2 + \frac{361}{40}x + 56.21 \qquad 1 \le x \le 19$$

Let's investigate the graph of f around $x = 18$. We graph f in smaller and smaller intervals as shown in **Figure 3.4.3**. Notice that as we zoom in on f for values close to $x = 18$, the graph appears to straighten out and look like a line. Let's see if we can take advantage of this characteristic to approximate function values.

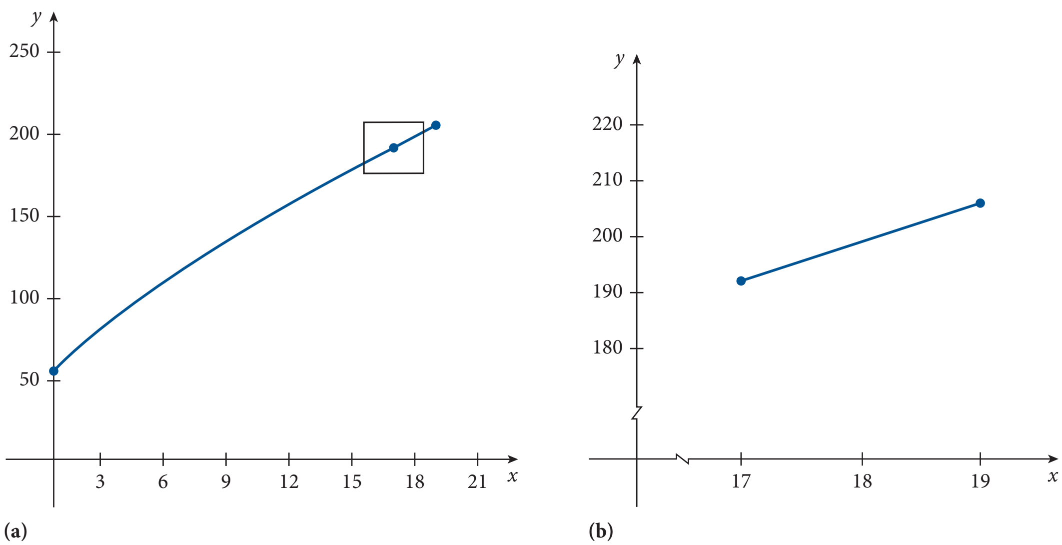

Figure 3.4.3 **(a)** We zoom in on that portion of the graph shown in the rectangle and redraw the graph shown in Figure 3.4.3(b). **(b)** On this smaller interval, the graph of f appears to be nearly linear.

Example 4: Computing a Linear Approximation

For the model of the total U.S. federal prison inmate population:

a. Determine dy and evaluate when $x = 18$ and $\Delta x = dx = 1$.

b. Evaluate $f(19)$ and interpret.

c. Add $f(18)$ to the result from part (a) and compare to part (b).

Perform the Mathematics

a. For the model $f(x) = -\frac{3}{50}x^2 + \frac{361}{40}x + 56.21$, the derivative is $f'(x) = -\frac{3}{25}x + \frac{361}{40}$. The differential in y for the model is

$$dy = f'(x)dx = \left(-\frac{3}{25}x + \frac{361}{40}\right)dx$$

Evaluating for $x = 18$ and $\Delta x = dx = 1$, the differential in y is

$$\begin{aligned} dy &= f'(18) \cdot 1 \\ &= 6.865 \end{aligned}$$

b. We evaluate $f(19)$ by substituting 19 for every occurrence of x in the function $f(x) = -\frac{3}{50}x^2 + \frac{361}{40}x + 56.21$. Note that $x = 19$ corresponds to the year 2008. Evaluating we get

$$\begin{aligned} f(x) &= -\frac{3}{50}x^2 + \frac{361}{40}x + 56.21 \\ &= -\frac{3}{50}(19)^2 + \frac{361}{40}(19) + 56.21 = 206.025 \end{aligned}$$

This means that in 2008, there were about 206,025 inmates in U.S. federal prisons.

c. Evaluating the model at $x = 18$ gives

$$f(18) = -\frac{3}{50}(18)^2 + \frac{361}{40}(18) + 56.21 = 199.22$$

Adding this result to part (a) gives us

$$f(18) + dy = f(18) + f'(18) \cdot 1 = 199.22 + 6.865 = 206.085$$

Notice how close this result is to part (b). In other words, $f(19) \approx f(18) + f'(18) \cdot 1$. ■

OBJECTIVE 3

Compute a linear approximation.

Notice in Example 4 that we can rewrite $f(19)$ as $f(18 + 1)$. So it appears, for a small value of dx, that $f(18 + 1) \approx f(18) + f'(18)(1)$. This is exactly what a **linear approximation** is. We generalize this new tool in the following definition.

DEFINITION

Linear Approximation

For a differentiable function f, where $y = f(x)$, the **linear approximation** of f is given by

$$f(x + dx) \approx f(x) + dy = f(x) + f'(x)dx$$

when the value of dx is small.

Both the linear approximation and the differential will be the cornerstone when we study Marginal Analysis in Section 3.5. Let's now try a classic application of linear approximations.

Example 5: Using Linear Approximations to Make Estimates

Use a linear approximation to estimate $\sqrt[3]{63}$ and compare with the calculator value of 3.9791 (rounded to four decimal places).

Perform the Mathematics

A first, it appears that we have nothing to work with to get the differential. But, on closer inspection, we see that a number very close to $\sqrt[3]{63}$ has an integer cube root, that is, $\sqrt[3]{64} = 4$. Thus, we can use the linear approximation to make an estimate of $\sqrt[3]{63}$ by letting $f(x) = \sqrt[3]{x}$, $x = 64$, and $dx = 63 - 64 = -1$. This means that we want to determine

$$f(x + dx) \approx f(x) + f'(x)dx$$
$$f(64 + (-1)) \approx f(64) + f'(64)(-1)$$

Knowing that $f(x) = \sqrt[3]{x} = x^{1/3}$, we get $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$. So the linear approximation is

$$\begin{aligned}\sqrt[3]{63} &= f(64 + (-1)) \approx f(64) + f'(64)(-1)\\ &= \sqrt[3]{64} + \frac{1}{3\sqrt[3]{(64)^2}}(-1)\\ &= 4 + \frac{1}{3(16)}(-1)\\ &= 4 - \frac{1}{48} = 3\frac{47}{48}\end{aligned}$$

As a decimal, $3\frac{47}{48} \approx 3.9792$, whereas $\sqrt[3]{63} \approx 3.9791$. The difference using a linear approximation is only about one ten-thousandth. ■

Summary

This section began with a discussion of the **differential in *x*,** denoted by dx, and the **differential in *y***, where $dy = f'(x)dx$. We said that if dx is small in value, then $dy \approx \Delta y$. Then we introduced the **linear approximation** and stated that, when dx is small, $f(x + dx) \approx f(x) + dy = f(x) + f'(x)dx$.

Section 3.4 Exercises

Vocabulary Exercises

1. Recall that the expression $\frac{f(x + \Delta x) - f(x)}{\Delta x}$ is called the ________________ quotient.
2. The ________________ in x is given by $dx = \Delta x$.
3. The value Δy represents the ________________ change in the dependent variable values.
4. The differential in y, denoted by dy, gives an ________________ change in the dependent variable values.
5. We know that dy is approximately equal to Δy if the value of dx is relatively ________________.
6. If dx is relatively small, then $f(x) + dy$ gives a ________________ approximation of $f(x + dx)$.

Skill Exercises

In Exercises 7–22, find dy for the given function and write your answer in the form $dy = f'(x)dx$.

7. $f(x) = 6x$

8. $f(x) = -3x$

9. $f(x) = -3x^2 + 2x$

10. $f(x) = 7x^3 + 3x^2 - 13$

11. $f(x) = \dfrac{5}{x - 1}$

12. $f(x) = \dfrac{-2}{x + 3}$

13. $f(x) = \dfrac{x}{x + 1}$

14. $f(x) = \dfrac{2x}{x - 3}$

15. $f(x) = \sqrt{x} + \dfrac{2}{x}$

16. $f(x) = \sqrt[3]{x} - \dfrac{3}{x^2}$

17. $f(x) = \dfrac{1}{\sqrt{x}} + \sqrt[3]{x^2}$

18. $f(x) = \dfrac{1}{2\sqrt{x}} - \sqrt{x}$

19. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}$

20. $f(x) = \dfrac{x^2 + 3}{x^2 - 3}$

21. $f(x) = 3x^{1.7} + 7x^{0.8} + 3$

22. $f(x) = 4x^{2.2} - 6x^{0.7} + 7$

In Exercises 23–34, evaluate Δy and dy for each function using the given x and dx values.

23. $f(x) = x^2 - 2x - 1, \quad x = 2, \quad \Delta x = dx = 0.1$

24. $f(x) = x^2 + 5x, \quad x = 1, \quad \Delta x = dx = 0.2$

25. $f(x) = 750 + 5x - 2x^3, \quad x = 50, \quad \Delta x = dx = 2$

26. $f(x) = 1000 + 2x - 3x^2, \quad x = 100, \quad \Delta x = dx = 1$

27. $f(x) = 100 - \dfrac{270}{x}, \quad x = 9, \quad \Delta x = dx = 0.5$

28. $f(x) = 75 - \dfrac{150}{x}, \quad x = 5, \quad \Delta x = dx = 0.5$

29. $f(x) = \sqrt{x}, \quad x = 2, \quad \Delta x = dx = 0.1$

30. $f(x) = 3\sqrt{x}, \quad x = 1.5, \quad \Delta x = dx = 0.1$

31. $f(x) = \dfrac{x^2 + 1}{x^2 - 1}, \quad x = 2, \quad \Delta x = dx = 0.1$

32. $f(x) = \dfrac{x^2 - 5}{2x^2 + 1}, \quad x = 3, \quad \Delta x = dx = 0.1$

33. $f(x) = 2x^2(3x^2 - 2x), \quad x = 1, \quad \Delta x = dx = 0.1$

34. $f(x) = x^3(x^2 - 1), \quad x = 2, \quad \Delta x = dx = 0.1$

For Exercises 35–42, use the linear approximation to estimate the value of the given number. Compare to the calculator value when rounded to four decimal places.

35. $\sqrt{26}$

36. $\sqrt{8.9}$

37. $\sqrt[3]{26}$

38. $\sqrt[3]{124}$

39. $\sqrt[4]{15.8}$

40. $\sqrt[4]{15}$

41. $\dfrac{2}{\sqrt{50}}$

42. $\dfrac{3}{\sqrt[3]{7}}$

Application Exercises

43. Geometry—Area Increase: The radius of a circle increases from an initial value of $r = 5$ inches by an amount $\Delta r = dr = 0.2$ inch. Estimate the corresponding increase in the circle's area by evaluating dA. The area of a circle is given by $A(r) = \pi r^2$.

44. Geometry—Area Increase: The radius of a circle increases from an initial value of $r = 8$ inches by an amount $\Delta r = dr = 0.1$ inch. Estimate the corresponding increase in the circle's area by evaluating dA. The area of a circle is given by $A(r) = \pi r^2$.

45. Business Management—Insurance Costs: The Dakorn Company determines that the annual cost of covering its employees' vision and dental insurance can be modeled by the function

$$f(x) = 1000 + 110\sqrt{x} \qquad x \geq 0$$

where x represents the number of employees covered and $f(x)$ represents the annual insurance cost in dollars.

(a) Determine the differential dy for the model.

(b) Suppose that the company's CEO has decided to hire four new employees in the Human Resources Department, so that the number of employees increases from 250 to 254. Use the differential to approximate the increase in cost in dental and vision insurance.

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46. Demographics—Poverty: A study conducted by the Northern Aid Organization shows that the number of people identified as having incomes below poverty level in the northern provinces of a certain country can be modeled by

$$f(x) = 10 + 707\sqrt{x} \qquad x \geq 0$$

where x represents the population in thousands and $f(x)$ represents the number of people identified as having incomes below poverty level, measured in thousands.

(a) Determine the differential dy for the model.

(b) Suppose the population of the Prudential Province has increased from 20 thousand to 22 thousand. Use the differential to approximate the increase in number of people below poverty level.

47. Business Management—Production Costs: The Paulson Motor Company estimates that the weekly cost of producing its new car, the Evadour, can be modeled by the cost function

$$C(x) = 0.22x^3 - 2.35x^2 + 14.32x + 10.22 \qquad 0 \leq x \leq 50$$

where x represents the number of Evadours produced each week and $C(x)$ represents the weekly manufacturing cost, measured in thousands of dollars.

(a) Determine the differential $dy = C'(x)dx$ for the model.

(b) Increased demand of the Evadour has resulted in a weekly increase in production from 30 to 33 cars. Use the differential to approximate the change in production cost.

48. Business Management—Production Costs: The A&D Publishing Company has determined that the printing cost of its new self-help book *Me and You Too* can be modeled by the cost function

$$C(x) = 0.02x^3 - 0.6x^2 + 9.15x + 98.43 \qquad 0 \leq x \leq 40$$

where x represents the number of books produced each day and $C(x)$ represents the daily production cost in thousands of dollars.

(a) Determine the differential $dy = C'(x)dx$ for the model.

(b) Favorable reviews of the book have resulted in an increase in daily production from 19 to 20 books. Use the differential to approximate the change in production cost.

49. Marketing—Advertising Budget: Using past records, managers at the PowerSet Company have estimated that the monthly amount spent on advertising and the sales of its Spiker volleyballs can be modeled by

$$f(x) = 120x - 2.4x^2 \qquad 0 \leq x \leq 25$$

where x represents the amount spent on advertising in thousands of dollars and $f(x)$ represents the number of volleyballs sold, in hundreds.

(a) Determine the differential dy for the model.

(b) The company's director of advertising has increased the advertising budget from \$10,000 to \$11,000. Use the differential to approximate the increase in sales by increasing the amount spent on advertising from $x = 10$ to $x = 11$.

50. Marketing—Advertising Budget: The PowerSet Company has estimated that the monthly amount spent on advertising and the sales of its Side-Out volleyballs can be modeled by

$$f(x) = 189.24x - 3.5x^2 \qquad 0 \le x \le 25$$

where x represents the amount spent on advertising in thousands of dollars and $f(x)$ represents the number of volleyballs sold, in hundreds.

(a) Determine the differential dy for the model.

(b) The company's director of advertising has increased the advertising budget from \$10,000 to \$11,000. Use the differential to approximate the increase in sales by increasing the amount spent on advertising from $x = 10$ to $x = 11$.

(c) Compute the actual change in sales $\Delta y = f(11) - f(10)$ and compare Δy to the approximation dy found in part (b).

51. Nursing—Inpatient Healthcare: Countries have different procedures and philosophies in the way they approach acute inpatient healthcare. The country with the longest average inpatient stay is Japan. The annual average number of days an acute inpatient stays in the hospital in Japan from 2000 to 2010 can be modeled by

$$f(x) = 0.07x^2 - 1.33x + 24.8 \qquad 0 \le x \le 10$$

where x represents the number of years since 2000 and $f(x)$ represents the average number of days an acute inpatient stays in the hospital. (*Source:* Organization for Economic Cooperation and Development.)

(a) Evaluate dy for $x = 9$ and $\Delta x = dx = 1$ and interpret.

(b) Write the equation of the tangent line of f at $x = 9$ and evaluate the equation for the year 2014. Interpret this result.

52. Nursing—Inpatient Healthcare: The country with the shortest average inpatient stay is Denmark. The annual average number of days an acute inpatient stays in the hospital in Denmark from 2000 to 2010 can be modeled by

$$f(x) = 0.01x^2 - 0.11x + 3.8 \qquad 0 \le x \le 10$$

where x represents the number of years since 2000, and $f(x)$ represents the average number of days an acute inpatient stays in the hospital. (*Source*: Organization for Economic Cooperation and Development.)

(a) Evaluate dy for $x = 10$ and $\Delta x = dx = 1$ and interpret.

(b) Write the equation of the tangent line of f at $x = 10$ and evaluate the equation for the year 2014. Interpret this result.

53. Personal Finance—Bank Credit Cards: A topic that has gained attention since the mid-2000s is the growing private credit card debt in the United States. The number of bank-issued credit cards in circulation annually in the United States can be modeled by

$$f(x) = -10.8x^2 + 82.8x + 455 \qquad 0 \le x \le 10$$

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where x represents the number of years since 2000 and $f(x)$ represents the number of bank-issued credit cards in circulation annually, measured in millions. (*Source:* The Nilson Report.)

(a) Evaluate dy for $x = 8$ and $\Delta x = dx = 1$ and interpret.

(b) Write the equation of the tangent line of f at $x = 8$ and evaluate the equation for the year 2012. Interpret this result.

54. Personal Finance—Oil Company Credit Cards: Many consumers use credit cards issued by gas stations in association with oil companies. The number of oil company–issued credit cards in circulation annually in the United States can be modeled by

$$f(x) = 0.67x^2 + 8.33x + 98 \qquad 0 \le x \le 10$$

where x represents the number of years since 2000 and $f(x)$ represents the number of oil company–issued credit cards in circulation annually, measured in millions. (*Source:* The Nilson Report.)

(a) Evaluate dy for $x = 9$ and $\Delta x = dx = 1$ and interpret.

(b) Write the equation of the tangent line of f at $x = 10$ and evaluate the equation for the year 2016. Interpret this result.

Concept and Writing Exercises

55. In a few sentences, explain the difference between the values of Δy and dy.

56. By using differentials, explain why 1.06 is a reasonable approximation for $(1.01)^6$.

In Exercises 57 and 58, suppose we do not know the formula for the function f, but know $f(2) = -4$ *and* $f'(x) = \sqrt{x^2 + 5}$. *Complete the following.*

57. Use linear approximation to estimate $f(1.96)$ and $f(2.04)$.

58. In a few sentences, explain whether the estimates in Exercise 57 are too large or too small.

In Exercises 59–64, let u and v represent differentiable functions and let k and n represent real number constants. Establish the following rules for working with differentials.

59. Constant Rule for Differentials: $dk = 0$.

60. Power Rule for Differentials: $du^n = nu^{n-1} \cdot du$

61. Sum Rule for Differentials: $d(u + v) = du + dv$

62. Difference Rule for Differentials: $d(u - v) = du - dv$

63. Product Rule for Differentials: $d(uv) = du \cdot v + u \cdot dv$

64. Quotient Rule for Differentials: $d\left(\dfrac{u}{v}\right) = \dfrac{du \cdot v - u \cdot dv}{v^2}$

Section Project

One way we can use differentials is to approximate the volume of Earth and its atmosphere. We know that the radius of the earth is about 3963 miles and that the formula for the volume of a sphere is given by $V(r) = \frac{4}{3}\pi r^3$. In this situation, we let r represent the radius in miles and $V(r)$ represent the volume in cubic miles.

(a) Determine an expression for dV the differential in the volume with respect to the radius r.

(b) There is no definitive border between the end of the atmosphere and the beginning of space, but many meteorologists consider the thickness of Earth's atmosphere to be about 62.1 miles. In the expression for dV, what does this value represent?

(c) Use the expression found in part (a) and the value in part (b) to determine a value for dV. Interpret the result.

3.5 Marginal Analysis

SECTION OBJECTIVES

1. Determine a marginal cost function.
2. Determine a marginal profit function.
3. Determine a linear approximation.
4. Determine an average cost function.
5. Compute a marginal average cost function.

Many decisions made by managers in business involve analyzing the effect on the dependent variable when a small change is made to a specific independent variable value. For example, a company may wish to consider changing the price of an item and examining how this change affects the revenue or profit of the product. **Marginal analysis** can be defined as the study of the amount of change in the dependent variable that results from a single unit change in an independent variable. A **unit change** means a change of a single unit. This change in the dependent variable is a direct application of our now-familiar tool—the derivative.

Marginal Analysis

Let's start this discussion of business functions by reviewing their definitions.

From Your Toolbox: Business Functions

1. The *price-demand function* p gives us the price $p(x)$ at which people buy exactly x units of product.
2. The cost $C(x)$ of producing x units of a product is given by the *cost function*

$$C(x) = (\text{variable costs}) \cdot (\text{units produced}) + (\text{fixed cost})$$

   Note that since variable costs are often expressed as a function, $C(x)$ may be a higher-order polynomial function.
3. The total revenue R generated by producing and selling x units of product at price $p(x)$ is given by the *revenue function*

$$R(x) = (\text{quantity sold}) \cdot (\text{unit price}) = x \cdot p(x)$$

4. The profit P generated after producing and selling x units of a product is given by the *profit function*

$$P(x) = \text{revenue} - \text{cost} = R(x) - C(x)$$

We referred to marginal analysis as the study of the dependent variable if the independent variable had a single unit change. Let's say that we want to study the marginal cost at a production level x, given a cost function C. We can start by determining the *actual change* in cost, denoted by ΔC, when the number of units produced is increased by 1.

$$(\text{actual change in cost}) = (\text{cost to produce } x + 1 \text{ units}) - (\text{cost to produce } x \text{ units})$$

In terms of the cost function C, this is the same as

$$\Delta C = C(x + 1) - C(x)$$

This relationship is illustrated in **Figure 3.5.1**.

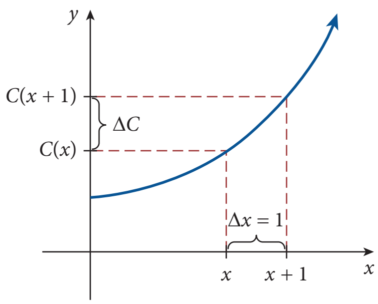

Figure 3.5.1 ΔC is the actual change in cost.

Let's try an example of computing this actual change. Suppose that the cost of producing x units of a recreation vehicle can be modeled by

$$C(x) = 100 + 60x + 3x^2$$

where x represents the number of vehicles produced and $C(x)$ is the cost in hundreds of dollars. To find ΔC for $x = 5$ and $\Delta x = 1$, we seek the difference in cost where

$$\begin{aligned}\Delta C &= C(x + \Delta x) - C(x)\\ &= C(5 + 1) - C(5)\\ &= C(6) - C(5)\\ &= 568 - 475 = 93\end{aligned}$$

Since $C(6)$ is the cost of producing six vehicles and $C(5)$ is the cost of producing the first five vehicles, then $\Delta C = C(6) - C(5)$ must represent the cost of producing the sixth vehicle. Thus, the cost of producing the sixth vehicle is $9300.

An exact value can be computed by evaluating $\Delta C = C(x + \Delta x) - C(x)$, but in Section 3.4 we found that, when dx is small, $\Delta C \approx dy$, where dy is the differential in y given by $dy = C'(x)dx$. So for dx being small, we have

$$(\text{actual change in cost, } \Delta C) \approx (\text{differential in } C, dC)$$

But for marginal analysis, $dx = 1$. This gives us

$$(\text{marginal cost at production level } x) \approx (\text{differential in } C, \text{ where } dx = 1) = C'(x) \cdot 1 = C'(x)$$

Consequently, the **marginal cost function**, denoted by MC, is simply the derivative of the cost function. Also note that MC is the differential in the cost function where $dx = 1$. This is sometimes called a *unit differential*. The relationship of $C(x)$, ΔC, and $MC(x)$ is shown in **Figure 3.5.2**.

DEFINITION

Marginal Cost Function

The **marginal cost function**, MC, given by $MC(x) = C'(x)$, is the approximate cost of producing one additional unit at a production level x.

NOTE: Normally, when we write the units for a rate of change, we write them as dependent units per independent units. But since $dx = 1$ in these cases, we just use the dependent units, which are usually measured in dollars.

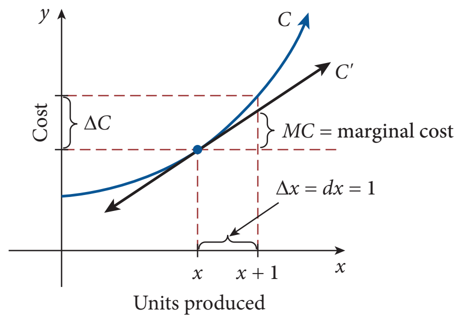

Figure 3.5.2

OBJECTIVE 1

Determine a marginal cost function.

Example 1: Computing a Marginal Cost Function

The cost of producing x units of a certain recreational vehicle can be modeled by

$$C(x) = 100 + 60x + 3x^2$$

where x represents the number of vehicles produced and $C(x)$ is the cost in hundreds of dollars. Compute the marginal cost $MC(x) = C'(x)$. Evaluate $MC(5)$ and interpret.

Perform the Mathematics

Computing the derivative of $C(x)$ with respect to x, we get

$$MC(x) = C'(x) = \frac{d}{dx}(100 + 60x + 3x^2) = 60 + 6x$$

Evaluating the marginal cost function at $x = 5$ yields

$$MC(5) = 60 + 6(5) = 90$$

This means that the approximate cost of producing the next, or sixth, vehicle is 90 hundred, or $9000. Earlier in this section, we showed that the *exact cost* of producing the sixth vehicle is $9300, so the error of the marginal cost approximation is $300. ■

Try It Yourself

Some related Exercises are 7 and 9.

Other Marginal Business Functions

At the beginning of this section, we stated that marginal analysis focused on the change in cost, revenue, and profit with a single unit change of the independent variable. This small change can have a significant impact on profit. This is because a change in price can cause changes in the quantity produced and sold, the cost and revenue, and the profit. See **Figure 3.5.3**.

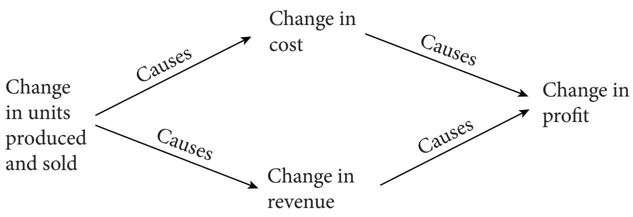

Figure 3.5.3

So it seems logical that we need to examine the marginal functions that are associated with the remainder of the business functions.

These definitions show that differentiating the business functions gives us the **marginal business functions**. Let's see how these functions can be used to affect profit.

DEFINITION

Marginal Business Functions

- The **marginal revenue function** MR, given by $MR(x) = R'(x)$, is the approximate loss or gain in revenue by producing and selling one additional unit at a production level x.
- The **marginal profit function** MP, given by $MP(x) = P'(x)$, is the approximate loss or gain in profit by producing and selling one additional unit at a production level x.

Example 2: Using the Marginal Profit Function

OBJECTIVE 2

Determine a marginal profit function.

The FrezMore Company has determined that its cost of producing x refrigerators can be modeled by

$$C(x) = 2x^2 + 15x + 1500 \qquad 0 \le x \le 200$$

where x is the number of refrigerators produced each week and $C(x)$ represents the weekly cost in dollars. The company also determines that the price-demand function for the refrigerators is

$$p(x) = -0.3x + 460$$

a. Determine the profit function, P, for the refrigerators.

b. Determine the marginal profit function.

c. Compute $MP(60)$ and $MP(145)$ and interpret the results.

Perform the Mathematics

a. In order to determine the profit function, we must first find the revenue function. Since the price-demand function for producing and selling x refrigerators is $p(x) = -0.3x + 460$, the revenue function from selling x refrigerators is

$$R(x) = x \cdot p(x) = x(-0.3x + 460) = -0.3x^2 + 460x$$

The profit function can now be determined as

$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= (-0.3x^2 + 460x) - (2x^2 + 15x + 1500) \\ &= -0.3x^2 + 460x - 2x^2 - 15x - 1500 \\ &= -2.3x^2 + 445x - 1500 \end{aligned}$$

b. Differentiating the profit function to determine the marginal profit function yields

$$MP(x) = \frac{d}{dx}(-2.3x^2 + 445x - 1500) = -4.6x + 445$$

c. Evaluating the marginal profit function at $x = 60$ gives

$$MP(60) = -4.6(60) + 445 = 169$$

This means that at a production level of $x = 60$ refrigerators each week, there is about \$169 profit for making and selling the 61st refrigerator.

Evaluating the marginal profit function at $x = 145$ gives

$$MP(145) = -4.6(145) + 445 = -222$$

This means that at a production level of $x = 145$ refrigerators each week, there is about \$222 loss in profit for making and selling the 146th refrigerator. **Figure 3.5.4** shows that the profit at $x = 145$ is decreasing at a rate of \$222 per refrigerator because the slope of the tangent line at $x = 145$ is -222.

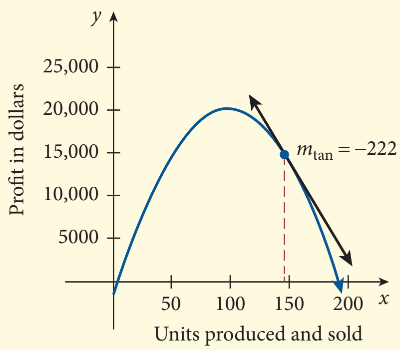

Figure 3.5.4

An extension of the marginal concept is the use of linear approximation for business functions. Recall from Section 3.4 that for a differentiable function f where $y = f(x)$, the **linear approximation** of f is given by

$$f(x + dx) \approx f(x) + dy = f(x) + f'(x)dx$$

when the value of dx is small.

In the case of marginal analysis, $dx = 1$. This means that we can write a linear approximation for a cost function with $f(x) = C(x)$ as

$$C(x + 1) \approx C(x) + C'(x)$$

or, in terms of the marginal cost function, as

$$C(x + 1) \approx C(x) + MC(x)$$

Consequently, we can apply the linear approximation concept to marginal analysis to easily make approximations for cost, revenue, and profit functions with information that is already known. This is particularly useful if the data given are tabular or some partial results are already available. Let's apply this new piece of information to determine a linear approximation for a revenue function.

OBJECTIVE 3

Determine a linear approximation.

Example 3: Computing Linear Approximations for a Revenue Function

The revenue function for the production of x refrigerators per week in Example 2 was given as $R(x) = -0.3x^2 + 460x$.

a. Compute $R(110)$ and interpret.

b. Compute $MR(110)$ and interpret.

c. Use the solution from parts (a) and (b) to get a linear approximation for $R(111)$. Compare the result to the exact value of $R(111)$.

Perform the Mathematics

a. For the revenue function $R(x) = -0.3x^2 + 460x$, we substitute $x = 110$ to get

$$\begin{aligned} R(110) &= -0.3(110)^2 + 460(110) = -0.3(12{,}100) + 50{,}600 \\ &= -3630 + 50{,}600 = 46{,}970 \end{aligned}$$

So when producing and selling 110 refrigerators per week, the revenue realized is \$46,970.

b. We differentiate the revenue function to determine the marginal revenue function as follows:

$$\begin{aligned} MR(x) = R'(x) &= \frac{d}{dx}(-0.3x^2 + 460x) \\ &= -0.6x + 460 \end{aligned}$$

The marginal revenue at a production level of $x = 110$ is

$$\begin{aligned} MR(110) &= -0.6(110) + 460 \\ &= -66 + 460 = 394 \end{aligned}$$

This means that the revenue gained from producing and selling the 111th refrigerator is about \$394.

c. To determine an approximation for $R(111)$, we use the linear approximation for the revenue function

$$R(x + 1) \approx R(x) + MR(x)$$

Using this, along with the results from parts (a) and (b), we get an approximation for $R(111)$ as follows:

$$\begin{aligned} R(111) = R(110 + 1) &\approx R(110) + MR(110) \\ &= 46{,}970 + 394 = 47{,}364 \end{aligned}$$

So our approximation for $R(111)$ is $R(111) \approx 47{,}364$. The exact revenue using the given revenue function is $R(111) = 47{,}363.70$, so the error from using the linear approximation is only \$0.30! The key to the closeness of this linear approximation is that the change in the independent variable is relatively small. ■

Average Business Functions

Many times in business situations, financial reports are simplified so that the numerical results can be easily understood. Often managers are interested in the *per unit* cost of a product, which is usually easier to work with than the total cost for the production of x units of a product. For example, it is easier for a manager to think of the production costs at a video production company to be \$8.25 per DVD or for a cycling company to think of it costing \$215 to produce each bike.

To find this per unit, or **average**, for the cost function, we take the total cost and divide it by the number of items produced. In words,

Per unit cost = (total cost to produce x items)/(number of items produced, x)

Let's see how this works through a Flashback.

Flashback: Fashion Mystique Revisited

In Section 1.2 we found that the cost function of copying promotional flyers for The Fashion Mystique was given by

$$C(x) = 0.10x + 5$$

where x represents the number of copies produced and $C(x)$ represents the cost of producing x flyers.

a. Compute $C(100)$ and interpret.

b. Evaluate $\dfrac{C(100)}{100}$ and interpret.

Perform the Mathematics

a. Evaluating the cost function at $x = 100$, we get

$$C(100) = 0.10(100) + 5 = 10 + 5 = 15$$

This means that the cost of producing 100 promotional flyers is $15.

b. We use the result from part (a) to evaluate the expression $\dfrac{C(100)}{100}$.

$$\frac{C(100)}{100} = \frac{15}{100} = 0.15$$

This tells us that at a production level of $x = 100$ flyers, the average cost is 15 cents per flyer. ■

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DEFINITION

Average Business Functions

The **average cost function**, AC, which gives the per unit cost of producing x items, is given by

$$AC(x) = \frac{C(x)}{x}$$

The **average profit function**, AP, which gives the per unit profit of producing and selling x items, is given by

$$AP(x) = \frac{P(x)}{x}$$

NOTE: Statistics students may recall that the average of a set of data is given by $\bar{x}$. This is why some textbooks denote the average cost function as $\bar{C}$.

The Flashback leads us to the definition of the **average business functions**.

Notice that the average revenue function is omitted from the average business functions definition. This is because the average revenue function is $AR(x) = \dfrac{R(x)}{x} = \dfrac{x \cdot p(x)}{x} = p(x)$. So we see that the average revenue function is just another name for the price function.

OBJECTIVE 4

Determine an average cost function.

Example 4: Analyzing an Average Cost Function

The Ventoux Athletic Shoe Company knows that for its Stampeder model basketball shoes, the daily cost function can be modeled by

$$C(x) = 700\sqrt{x} + 5000 \qquad 0 \le x \le 500$$

where x is the number of pairs of shoes produced daily and $C(x)$ is the daily cost in dollars.

a. Determine AC, the average cost function.

b. Evaluate and interpret $C(400)$ and $AC(400)$.

Perform the Mathematics

a. Using the definition of average cost function, we get

$$AC(x) = \frac{C(x)}{x} = \frac{700\sqrt{x} + 5000}{x}$$

b. Evaluating the cost function when $x = 400$ yields

$$\begin{aligned} C(400) &= 700\sqrt{400} + 5000 \\ &= 700(20) + 5000 \\ &= 19{,}000 \end{aligned}$$

This means that the total cost of producing 400 pairs of shoes a day is \$19,000.

Evaluating $AC(x)$ when $x = 400$, we get

$$AC(400) = \frac{700\sqrt{400} + 5000}{400} = 47.5$$

This means that the average cost of producing 400 pairs of shoes a day is \$47.50 for each pair. ■

Try It Yourself

Some related Exercises are 25a and 26a.

A manager may also wish to apply marginal analysis techniques to the average business functions. The result is the **marginal average business functions**. These are found by computing the derivative of each of the average business functions, respectively. The result is a function that approximates the per unit cost (or profit) of producing one more item. Now let's define these **marginal average business functions**.

DEFINITION

Marginal Average Business Functions

The **marginal average cost function** approximates the per unit cost for producing an additional item of a product and is given by

$$MAC(x) = \frac{d}{dx}[AC(x)] = \frac{d}{dx}\left[\frac{C(x)}{x}\right]$$

The **marginal average profit function** approximates the per unit profit for producing and selling an additional item of a product and is given by

$$MAP(x) = \frac{d}{dx}[AP(x)] = \frac{d}{dx}\left[\frac{P(x)}{x}\right]$$

NOTE: To determine these marginal average business functions, we first determine the average business function and then compute the derivative, *in that order*.

Example 5: Computing a Marginal Average Cost Function

OBJECTIVE 5

Compute a marginal average cost function.

In Example 4, we found that the cost function for producing x pairs of Stampeder model basketball shoes was

$$C(x) = 700\sqrt{x} + 5000 \qquad 0 \le x \le 500$$

a. Compute the marginal average cost function MAC.

b. Evaluate $MAC(400)$. Round the result to the nearest hundredth and interpret.

Perform the Mathematics

a. In Example 4 we computed the average cost function as $AC(x) = \frac{700\sqrt{x} + 5000}{x}$. Before differentiating, we simplify to get

$$AC(x) = \frac{700\sqrt{x}}{x} + \frac{5000}{x} = 700x^{-1/2} + 5000x^{-1}$$

Now, differentiating this average cost function with respect to x gives us

$$\begin{aligned} MAC(x) &= \frac{d}{dx}(700x^{-1/2} + 5000x^{-1}) \\ &= -350x^{-3/2} - 5000x^{-2} \end{aligned}$$

Simplifying the rational and negative exponents, we get the marginal average cost function

$$MAC(x) = -\frac{350}{\sqrt{x^3}} - \frac{5000}{x^2}$$

b. Evaluating the result of part (a) at $x = 400$, we get

$$MAC(400) = -\frac{350}{\sqrt{(400)^3}} - \frac{5000}{(400)^2} = -0.075 \approx -0.08$$

This means that when 400 pairs of basketball shoes have been produced, the average cost per pair decreases by about \$0.08 for an additional pair produced. Note that the negative sign tells us that the per unit cost is decreasing. ■

Summary

In this section, we revisited the business functions and from them derived the marginal business functions. These functions are found by differentiation, and they approximate the cost, revenue, or profit for producing one more item of a product. Then we discussed the average business functions, which were found by taking the business function and dividing by the independent variable. Finally, we discussed the marginal average business functions, which were the derivatives of the average business functions.

Important Functions

- **Marginal cost function:** $MC(x) = C'(x)$
- **Marginal revenue function:** $MR(x) = R'(x)$
- **Marginal profit function:** $MP(x) = P'(x)$
- **Average cost function:** $AC(x) = \frac{C(x)}{x}$
- **Average profit function:** $AP(x) = \frac{P(x)}{x}$
- **Marginal average cost function:** $MAC(x) = \frac{d}{dx}[AC(x)]$
- **Marginal average profit function:** $MAP(x) = \frac{d}{dx}[AP(x)]$

Section 3.5 Exercises

Vocabulary Exercises

1. The study of the amount of change in the dependent variable that results from a single unit change in the independent variable is called ________________ analysis.
2. A change of a single unit is called a ________________ change.
3. The approximate cost of producing one additional unit of a product at production level x is given by the ________________ ________________ function.
4. The total cost divided by the number of units produced gives the ________________ cost.
5. The average revenue function is another name for the ________________ function.
6. To get the marginal profit function, we begin by ________________ the profit function.

Skill Exercises

In Exercises 7–12, assume the cost function $C(x)$ is measured in dollars. Complete the following:

(a) *Determine the marginal cost function MC.*

(b) *Evaluate and interpret $MC(x)$ for the given production level x.*

(c) *Evaluate the actual change in cost by evaluating $C(x + 1) - C(x)$ and compare with the answer in part (b).*

7. $C(x) = 23x + 5200; x = 10$
8. $C(x) = 14x + 870; x = 12$
9. $C(x) = \frac{1}{2}x^2 + 12.7x + 2100; x = 11$
10. $C(x) = \frac{1}{2}x^2 + 27x + 1200; x = 20$
11. $C(x) = 0.2x^3 - 3x^2 + 50x + 20; x = 30$
12. $C(x) = 0.08x^3 - 2x^2 + 10x + 70; x = 90$

In Exercises 13–18, the cost function C and the price-demand function p are given. Assume that the value of $C(x)$ and $p(x)$ are in dollars. Complete the following.

(a) *Determine the revenue function R and the profit function P.*

(b) *Determine the marginal cost function MC and the marginal profit function MP.*

13. $C(x) = 5x + 500; p(x) = 6$
14. $C(x) = 12x + 4500; p(x) = 15$
15. $C(x) = \frac{x^2}{100} + 7x + 1000; p(x) = -\frac{x}{20} + 15$
16. $C(x) = \frac{1}{100}x^2 + \frac{1}{2}x + 8, p(x) = -\frac{x}{200} + 1$
17. $C(x) = -.0001x^3 + 4x + 100,\ 0 \le x \le 70; p(x) = -0.005x + 7$
18. $C(x) = -0.002x^3 + 0.01x^2 + 2x + 50,\ 0 \le x \le 40; p(x) = -x^{-1/2} + 5$

Application Exercises

19. **Economics—Marginal Cost:** The Country Day Company determines that the daily cost of producing its Garden King lawn tractor tires can be modeled by

$$C(x) = 100 + 40x - 0.001x^2 \qquad 0 \le x \le 300$$

where x represents the number of tires produced each day, and $C(x)$ represents the cost in dollars. Determine MC, the marginal cost function. Evaluate $MC(200)$ and interpret.

20. Economics—Marginal Cost: The Kelomata Company determined that the monthly cost of producing its Sun Stopper patio swings can be modeled by

$$C(x) = 15{,}000 + 100x - 0.001x^2 \qquad 0 \le x \le 200$$

where x represents the number of patio swings manufactured monthly and $C(x)$ represents the production cost in dollars. Determine MC, the marginal cost function. Evaluate $MC(100)$ and interpret.

21. Manufacturing—Raincoat Production: Using historical data, the Seas Beginning Corporation determines that the daily cost of producing its Rain Forrest Ultra raincoat can be modeled by

$$C(x) = 1000 + 35x - 0.01x^2 \qquad 0 \le x \le 300$$

where x represents the number of raincoats produced daily, and $C(x)$ represents the daily cost in dollars. Determine MC, the marginal cost function. Evaluate $MC(200)$ and interpret.

22. Manufacturing—Transmission Production: The accounting division of the Kranky Krank Company determines that its weekly cost of producing its continuous variable transmissions (CVTs) can be modeled by the cost function

$$C(x) = 55{,}000 + 600x - 1.25x^2 \qquad 0 \le x \le 230$$

where x represents the number of CVTs produced annually and $C(x)$ represents the manufacturing costs in dollars. Determine MC, the marginal cost function. Evaluate $MC(210)$ and interpret.

23. Manufacturing—Hard Drive Production: The Memory Master Company makes portable hard drives for laptop computers, and its managers have determined that the cost of producing its 3-terabyte hard drive can be modeled by

$$C(x) = 10{,}000 + 200x - 0.2x^2 \qquad 0 \le x \le 650$$

where x represents the number of hard drives produced each week and $C(x)$ represents the manufacturing cost in dollars.

(a) Determine the marginal cost function $MC(x)$ and evaluate $MC(500)$.

(b) If the Memory Master Company sells 500 hard drives weekly to a computer chain store for \$120 each, should production be increased? Explain your answer.

24. Economics—Marginal Analysis: For the cost function in Exercise 19, complete the following.

(a) Determine the average cost function AC. Evaluate and interpret $AC(200)$.

(b) Determine MAC, the marginal average cost function. Evaluate and interpret $MAC(200)$.

25. For the cost function in Exercise 20, complete the following.

(a) Determine the average cost function AC. Evaluate and interpret $AC(100)$.

(b) Determine MAC, the marginal average cost function. Evaluate and interpret $MAC(100)$.

26. Profit Analysis—Bobbleheads: The financial planning team at the Tesch Company determines that the profit function for producing and selling its Captain U.S. bobbleheads can be modeled by

$$P(x) = -0.001x^2 + 8x - 4000 \qquad 0 \le x \le 7000$$

where x represent the number of bobbleheads produced and sold and $P(x)$ represents the monthly profit in dollars.

(a) Determine MP, the marginal profit function. Evaluate $MP(3000)$ and interpret.

(b) If the Tesch Company is producing and selling 3000 bobbleheads per month, is profit increasing or decreasing?

27. Profit Analysis—Radio Production: The Hanash Corporation determines that the weekly profit from producing and selling its Jog-R-Radios can be modeled by

$$P(x) = -0.01x^2 + 12x - 2000 \qquad 0 \le x \le 1000$$

where x represents the number of radios produced and sold weekly and $P(x)$ represents the weekly profit in dollars.

(a) Determine MP, the marginal profit function. Evaluate $MP(700)$ and interpret.

(b) If the Hanash Corporation is producing and selling 700 radios per week, is profit increasing or decreasing?

28. Publishing—Magazine Publications: By inspecting their tracking records, the telemarketing company Calls-R-Us has concluded that the monthly profit from selling magazine subscriptions can be modeled by

$$P(x) = 5x + \sqrt{x} \qquad 0 \leq x \leq 100$$

where x represents the number of subscriptions sold per month, and $P(x)$ represents profit in dollars. Determine MP, the marginal profit function. Evaluate $MP(55)$ and interpret.

29. Publishing—Newspaper Profit: Ashton is a newspaper motor courier and determines that the monthly profit from his current newspaper route can be modeled by

$$P(x) = 2x - \sqrt{x} \qquad 0 \leq x \leq 200$$

where x represents the number of subscribers, and $P(x)$ represents the monthly profit in dollars. Determine MP, the marginal profit function. Evaluate $MP(110)$ and interpret.

30. Economics—Marginal Analysis: For the profit function in Exercise 28, complete the following.

(a) Determine AP, the average profit function. Evaluate and interpret $AP(55)$.

(b) Determine MAP, the marginal profit function. Evaluate $MAP(55)$ and interpret.

31. Economics—Marginal Analysis: For the profit function in Exercise 29, complete the following.

(a) Determine AP, the average profit function. Evaluate and interpret $AP(110)$.

(b) Determine MAP, the marginal profit function. Evaluate $MAP(110)$ and interpret.

32. Economics—Marginal Analysis: The GlobalText Company makes social media–oriented mobile phones apps and determines that the fixed and variable costs to produce x apps are \$1200 and \$12 per app, respectively.

(a) Write the cost function C in the form $C(x) = mx + b$.

(b) Determine the marginal cost function MC and then evaluate $MC(100)$ and $MC(150)$ and interpret these results.

(c) Why are the marginal costs in part (b) equal?

33. Economics—Marginal Analysis: The NewJoy toy company has just produced a new Street Kings action figure set that it sells to wholesalers for \$20 each. From collecting data, they determine that the cost to produce x action figures can be modeled by the function $C(x) = 0.001x^2 + 4x + 5000$.

(a) Derive and algebraically simplify the profit function $P(x)$.

(b) Evaluate $P(1000)$ and interpret.

(c) Evaluate $MP(1000)$ and interpret.

34. Economics—Marginal Analysis: The NewJoy toy company hires an accounting firm to audit their books and revises their price-demand and cost functions to $p(x) = 23$ and $C(x) = \frac{x^2}{95} + \frac{7}{2}x + 5500$, respectively.

(a) Derive and algebraically simplify the profit function $P(x)$.

(b) Evaluate $P(500)$ and interpret.

(c) Evaluate $MP(500)$ and interpret.

35. Economics—Marginal Analysis: The EZ-Craft Company determines that the price-demand function, in dollars, for their new U-Make-It picture frame is $p(x) = -\frac{x}{30} + 200$, with a cost function of $C(x) = 60x + 72{,}000$, where x represents the number of frames produced and $C(x)$ represents the cost in dollars.

(a) Determine the revenue function R.

(b) Determine the profit function P. Find the smallest and largest production levels x so that the company realizes a profit. Do this by determining the smallest and largest x-values so that $R(x) > C(x)$.

(c) Evaluate $P'(3000)$ and interpret.

36. Economics—Marginal Analysis: Vroncom Incorporated determines that the price-demand function for their TruTouch tablet device can be modeled by $p(x) = -\frac{x}{30} + 300$. They also have determined that their fixed costs are \$150,000 and variable costs are 30 dollars per device.

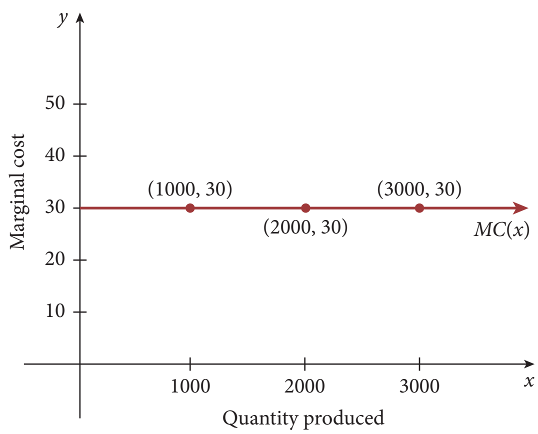
(a) Determine the revenue function R and then write the cost function in the form $C(x) = mx + b$.

(b) Determine the profit function P. Find the smallest and largest production levels x so that the company realizes a profit. Do this by determining the smallest and largest x-values so that $R(x) > C(x)$.

(c) Evaluate $P'(1000)$ and interpret.

37. Economics—Marginal Analysis: The stockholder's report for the Step-Up Company lists the following information and graph for its recently released W-Racer walking shoes.

x	$C(x)$
1000	42,500
2000	57,070



(a) Use the table along with the graph of $MC(x)$ to get a linear approximation for $C(1001)$. Use the approximation $C(x+1) \approx C(x) + MC(x)$.

(b) Repeat the procedure in part (a) to get a linear approximation for $C(2001)$.

Concept and Writing Exercises

38. Show that for a linear cost function, the marginal cost is constant.

39. Write a few sentences to explain the relationship between the marginal revenue and the marginal cost at the point where the marginal profit is zero.

In Exercises 40–43, let $P(x)$ represent the profit for producing x units of a product and let k and c represent constants. Answer the following.

40. How do we determine the profit from producing k units of a product?

41. How do we determine the level of production so that the marginal profit will be c dollars?

42. How do we determine the marginal profit from producing k units of a product?

43. How do we find the production level so that the profit will be c dollars?

For Exercises 44–47, consider the general cost function $C(x) = -ax^2 + bx + c$ and let $R(x) = x \cdot k$ where k represents a real number constant.

44. Determine the marginal cost function MC.

45. Determine the marginal revenue function MR.

46. Write the profit function P.

47. Determine the marginal profit function MP.

Section Project

Consider the following data table for the cost and revenues at various production levels for a new brand of handheld television.

Number of Handheld TVs produced, x	*Cost, C(x)*	*Revenue, R(x)*
100	$5500	$3,050
200	$9600	$24,100
300	$13,500	$81,200
400	$17,600	$192,200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Use the tabular information to complete the following.

(a) Use the regression capabilities of your graphing calculator to get a linear regression model for the cost of producing the TVs in the form

$$C(x) = ax + b \qquad 100 \le x \le 400$$

where x represents the number of TVs produced and $C(x)$ represents the cost in dollars.

(b) Use the regression capabilities of your graphing calculator to get a cubic regression model for the revenue of producing and selling the TVs in the form

$$R(x) = ax^3 + bx^2 + cx + d \qquad 100 \le x \le 400$$

where x represents the number of TVs produced and $R(x)$ represents the revenue in dollars. Round the values of a, b, c, and d to four decimal places if necessary.

(c) Determine the average cost function AC and simplify the result.

(d) Determine the marginal average cost function MAC. At a production level of $x = 150$, is the average cost increasing or decreasing?

(e) Using the results of parts (a) and (b), write and simplify the profit function $P(x)$.

(f) Determine the marginal average profit function MAP. At a production level of $x = 250$, is the average profit increasing or decreasing?

Chapter 3 Review Exercises

Section 3.1 Review Exercises

In Exercises 1–12, determine the derivative of the given function. When appropriate, simplify the derivative so that there are no negative or fractional exponents.

1. $f(x) = x^8$

2. $f(x) = 3x^{1/5}$

3. $f(x) = -4x^{-3/7}$

4. $f(x) = -\dfrac{4}{5}x^{7/6}$

5. $f(x) = 6x^2 + 5x - 11$

6. $f(x) = -9x^3 + x + 12$

7. $f(x) = \dfrac{1}{2}x^3 - 3x^2 + \dfrac{2}{3}x + 5$

8. $f(x) = 6.23x^2 + 1.98x - 3.34$

9. $f(x) = -7.10x^3 - 5.02x^2 + 11.19x - 16.07$

10. $f(x) = 2\sqrt{x} + 7x^2 - \dfrac{1}{2}x^3$

11. $f(x) = \sqrt[5]{x} - \sqrt{x} + \dfrac{1}{x}$

12. $f(x) = 2.08x^{3.79}$

For the functions in Exercises 13 and 14, complete the following.

(a) Write the domain of f in interval notation.

(b) Use the addition rule for fractions to rewrite and simplify f.

(c) Find $f'(x)$ using the differentiation rules.

(d) Write the domain of f' using interval notation.

13. $f(x) = \dfrac{3x^2 - 2x^2 + 6x + 2}{x}$

14. $f(x) = \dfrac{6x^4 + 25x^3 - 9x + 11}{x^2}$

For Exercises 15 and 16, complete the following.

(a) Determine the derivative for the given function using the differentiation rules.

(b) Determine the slope of the line tangent to the graph of the function at the indicated x-value.

(c) Determine an equation for the line tangent to the graph of the function at the indicated x-value.

(d) Graph the function and the tangent line in the same viewing window.

15. $f(x) = \dfrac{1}{x^2}$; $\quad x = 4$

16. $f(x) = x^{3/4}$; $\quad x = 16$

17. Energy Consumption—Dishwashers: The total amount of energy that the Kitchen Master dishwasher uses can be modeled by the function

$$f(t) = 6t + 9\sqrt{t} + 0.02t^{3/2} \qquad t \geq 0$$

where t represents the number of hours the dishwasher has been running and $f(t)$ represents the amount of energy spent in kilowatt-hours.

(a) Compute and simplify $f'(t)$.

(b) Suppose that an electrician wants to know how fast the dishwasher is using energy as it is turned on. Evaluate $f'\left(\dfrac{1}{3600}\right)$ and interpret this value.

18. Rocket Science—Ballistics: A rocket taking off has an altitude from the ground of $h(t) = 200t^2$, where $h(t)$ is in feet and t is in seconds. Find and interpret $h'(4)$.

19. Real Estate—Home Prices: Over the course of one year, the average price of homes for sale in a certain area can be modeled by the function

$$P(x) = -30x^2 + 2100x + 163{,}250$$

where x represents the number of weeks after January 1, and $P(x)$ represents the average home price, measured in dollars.

(a) Find the average rate of change of the price from $x = 10$ weeks to $x = 30$ weeks.

(b) Graph the function in (a) in the viewing window [0, 52] by [0, 250,000] and determine the maximum price and the week at which that price is at its maximum.

20. Energy Consumption Strategic Oil Reserve: Throughout recent U.S. history, the U.S. Department of Energy has periodically released oil in its strategic oil reserves to help ease gasoline prices. In August 2011, the release had negligible effect on gas prices. "Although it helped initially to pull down prices it was probably too little," said American Automobile Administration spokesman John Townsend. (*Source: The Washington Times.*) The amount of oil in the U.S. strategic oil reserve from 1980 to 2011 can be modeled by the function

$$f(x) = 0.1x^3 - 5.24x^2 + 88.1x + 128 \qquad 0 \le x \le 31$$

where x represents the number of years since 1980, and $f(x)$ represents the amount of oil in reserve in millions of barrels. (*Source:* U.S. Energy Information Administration.)

(a) Determine $f'(x)$.

(b) How fast was the strategic oil reserve increasing in 1984?

(c) Evaluate $f'(20)$ and interpret the result.

Section 3.2 Review Exercises

In Exercises 21–28, determine the derivative of the given function.

21. $f(x) = x^3(6x^2 - 3x + 8)$

22. $f(x) = -3x^2(6x^3 - 2x^2 + 9x + 10)$

23. $f(x) = (2x^2 - 5x + 1)(3x^2 + 4x - 1)$

24. $f(x) = (3x^{1/2} + 5x)(-2x^{2/5} + x - 9)$

25. $f(x) = (5\sqrt{x} - 3x - 1)(4\sqrt{x} - 7x)$

26. $f(x) = \dfrac{3x + 2}{x - 1}$

27. $f(x) = \dfrac{2x^2 + 2x - 5}{3x^2 - x + 9}$

28. $f(x) = \dfrac{3\sqrt{x} - 2}{3x + 1}$

For Exercises 29 and 30, complete the following.

(a) Determine the derivative.

(b) Determine an equation of the line tangent to the graph of the function at the indicated x-value.

(c) Graph the function and the tangent line in the same viewing window.

(d) Use the `dy/dx` or `Draw Tangent` command on your calculator to verify the solution to part (b).

29. $f(x) = -3x^2(3x + 5);\ x = 2$

30. $f(x) = \dfrac{2x^2 - 3}{x};\ x = 2$

Section 3.3 Review Exercises

In Exercises 31–40, differentiate using the Generalized Power Rule.

31. $f(x) = (x + 2)^3$

32. $f(x) = (x - 5)^2$

33. $f(x) = (8 - x)^3$

34. $f(x) = (4x - 3)^3$

35. $f(x) = (2x + 5)^4$

36. $f(x) = (4x^2 + 7)^5$

37. $f(x) = 3(x^2 - 5x + 3)^2$

38. $f(x) = (3x^2 + 7x - 2)^{63}$

39. $f(x) = (2x^2 - 5x + 7)^{1/3}$

40. $f(x) = (3x^2 - 9x - 4)^{-6}$

For the rational functions in Exercises 41–44, complete the following.

(a) Determine the derivative using the Quotient Rule.

(b) Determine the derivative using the Generalized Power Rule.

41. $f(x) = \dfrac{3}{2x + 9}$

42. $f(x) = \dfrac{7}{(x - 7)^2}$

43. $f(x) = \dfrac{2}{(3x + 5)^3}$

44. $f(x) = \dfrac{9}{x^2 - 6x + 18}$

In Exercises 45–48, determine an equation of the line tangent to the graph of f at the indicated ordered pair.

45. $f(x) = (5x + 3)^5$; $(-1, -32)$

46. $f(x) = (7x - 6)^3$; $(1, 1)$

47. $f(x) = (x^2 - 5x + 8)^4$; $(3, 16)$

48. $f(x) = (6x - 11)^{1/2}$; $(6, 5)$

In Exercises 49–52, use the Generalized Power Rule to differentiate the functions.

49. $f(x) = \sqrt{7x - 12}$

50. $f(x) = \sqrt[3]{8x + 1}$

51. $f(x) = \dfrac{3}{\sqrt{4x + 5}}$

52. $f(x) = \dfrac{8}{\sqrt[3]{2x^3 - 5x + 4}}$

For Exercises 53–56, use the Generalized Power Rule, along with the Product and Quotient Rules, to find the derivatives of the given functions.

53. $f(x) = x(x^2 + 5)^3$

54. $f(x) = 4x\sqrt{x^2 - 2x}$

55. $f(x) = \dfrac{3x - 7}{\sqrt{5x - 6}}$

56. $f(x) = (6x - 5)^7(3x + 2)^4$

In Exercises 57–60, determine the derivatives using the Generalized Power Rule.

57. $f(x) = (3x^2 - x + 1)^{0.67}$

58. $f(x) = \left(\dfrac{1}{8.3x - 5.7}\right)^{-2.4}$

59. $f(x) = (x^3 + x^2 + 5x + 1)^{-0.7}$

60. $f(x) = 4.96(x + 1)^{2.78}$

61. Corporate Finance—Oil Profits: An economic sector that has done well in the past few decades is petroleum- and coal-based energy companies. “The earnings reflect continued leadership in operational performance during a period of strong commodity prices,” said Exxon’s chairman, Rex W. Tillerson. (*Source: The New York Times*). The net profits for petroleum and coal corporations from 1990 to 2011 can be modeled by

$$f(x) = (1.8x + 6.3)^{1.26} \quad 0 \le x \le 21$$

where x represents the number of years since 1990 and $f(x)$ represents the net profits for petroleum and coal corporations, measured in billions of dollars. (*Source:* U.S. Census Bureau.)

(a) Use the Chain Rule to determine $f'(x)$.

(b) Evaluate and interpret $f'(8)$.

(c) Graph f' in the viewing window [0, 21] by [0, 7].

(d) Use the value command to verify your answer to part (b).

62. Corporate Finance—Renewable Energy: Renewable energy companies have recently attempted to benefit from the same tax advantages oil and coal companies receive. “It would be a real boon to the renewable energy industry,” said John McKenna, of Hamilton Clark Securities. (*Source: The San Francisco Chronicle.*) The net profits for renewable energy companies from 1990 to 2011 can be modeled by

$$f(x) = (1.16x + 4.2)^{1.37} \quad 0 \le x \le 21$$

where x represents the number of years since 1990 and $f(x)$ represents the net profits for renewable energy companies, measured in billions of dollars. (*Source:* U.S. Census Bureau.)

(a) Use the Chain Rule to determine $f'(x)$.

(b) Evaluate and interpret $f'(8)$.

(c) Graph f' in the viewing window [0, 21] by [0, 6].

(d) Compare the solution to part (b) of Exercise 61. Which model was growing at a faster rate?

Section 3.4 Review Exercises

In Exercises 63–74, determine dy for the given function.

63. $f(x) = 4x + 2$

64. $f(x) = 5x^2 - 3x + 2$

65. $f(x) = \dfrac{x+3}{x-5}$

66. $f(x) = \dfrac{7}{x+5}$

67. $f(x) = \sqrt{x} - \dfrac{3}{x^4}$

68. $f(x) = \dfrac{8}{x^2} + \sqrt[3]{x}$

69. $f(x) = x^4 - 2x + \sqrt[3]{x^2}$

70. $f(x) = x^3 - 5x^2 + 2x + 3$

71. $f(x) = 4x^5 - 21x + 4$

72. $f(x) = \dfrac{x^2-5}{x^2+5}$

73. $f(x) = 2x^{1.7} - 5x^{0.8} + 4$

74. $f(x) = 3x^{4.1} + 7x^{0.6} - 12$

For Exercises 75–80, evaluate Δy *and dy for the given function and indicated values.*

75. $f(x) = 4x^2 - x + 6;\ x = 3, \Delta x = dx = 0.1$

76. $f(x) = \dfrac{18}{x} + 5;\ x = 2, \Delta x = dx = 0.5$

77. $f(x) = \sqrt[3]{x};\ x = 8, \Delta x = dx = 1.261$

78. $f(x) = 2x^3 - 7x^2 + 2x;\ x = 4, \Delta x = dx = 0.2$

79. $f(x) = \dfrac{x^2+2}{x^2-2};\ x = 2, \Delta x = dx = 0.1$

80. $f(x) = 4x(2x+5);\ x = 1, \Delta x = dx = 0.1$

For Exercises 81–84, use the linear approximation to estimate the values of the given numbers. Compare to the calculator value when rounded to four decimal places. Recall that $f(x + dx) \approx f(x) + dy$.

81. $\sqrt{65}$

82. $\sqrt{24.6}$

83. $\sqrt[4]{16.3}$

84. $\sqrt[3]{62}$

85. Advertising—T-Shirt Sales: The Wild & Wacky T-shirt company has estimated that the association between its monthly T-shirt sales and its advertising can be modeled by

$$f(x) = 90x - 2.7x^2 \qquad 0 \le x \le 8$$

where x represents the amount spent on advertising in hundreds of dollars and $f(x)$ is the number of T-shirts sold in hundreds.

(a) Determine dy.

(b) Approximate the increase in sales if the advertising is increased from \$400 to \$500.

86. Repeat Exercise 85 using the model $f(x) = 82.76x - 1.87x^2$.

87. For Exercises 85 and 86, compute the actual change Δy in sales and compare to the approximation.

88. **International Ecology—India CO_2 Emissions:** One country that has been aware of its increasing emission of fossil fuel emissions is India. "India's carbon dioxide emissions will increase by nearly three-fold to 3200 million metric tons by 2030," according to an economic survey that has been tabled in Parliament. (*Source: The Hindu News.*) The amount of carbon dioxide emissions from fossil fuels in India from 1990 to 2011 can be modeled by

$$f(x) = 0.91x^2 + 28x + 625.6 \qquad 0 \le x \le 21$$

where x represents the number of years since 1990 and $f(x)$ represents the amount of carbon dioxide emissions emitted annually, measured in millions of metric tons. (*Source:* The International Energy Statistics Database.)

(a) According to the model, is the 2030 estimate for CO_2 emissions accurate?

(b) Determine the derivative $f'(x)$.

(c) Write the equation of the tangent line of f when $x = 20$.

(d) Find the y-value on the tangent line when $x = 40$. Explain what this value means in terms of the CO_2 emissions.

Section 3.5 Review Exercises

For Exercises 89–94, complete the following.

(a) Determine the marginal cost function $MC(x)$.

(b) For the given production level x, evaluate $MC(x)$ and interpret.

(c) Determine the actual change in cost by evaluating $C(x + 1) - C(x)$ and compare with the answer to part (b).

89. $C(x) = 18x + 642;\ x = 2$

90. $C(x) = 9x + 1460;\ x = 27$

91. $C(x) = 26.7x + 87.4;\ x = 8$

92. $C(x) = \frac{1}{2}x^2 + 3x + 16;\ x = 15$

93. $C(x) = \frac{1}{4}x^2 + 12x + 47;\ x = 31$

94. $C(x) = \frac{1}{3}x^2 + 318x + 1783;\ x = 23$

In Exercises 95–100, the cost function C and the price–demand function p are given.

(a) Determine the revenue function R.

(b) Determine the profit function P.

(c) Differentiate P in part (b) to get the marginal profit function MP.

(d) Determine the marginal cost function MC and the marginal revenue function MR.

(e) Subtract the solutions found in part (d) to get $MR - MC$ and simplify. Compare with the result of part (c).

95. $C(x) = 7x + 250;\ p(x) = 11$

96. $C(x) = 14x + 1380;\ p(x) = 21$

97. $C(x) = \frac{1}{10}x^2 + 3x + 850;\ p(x) = -\frac{x}{15} + 50$

98. $C(x) = \frac{1}{50}x^2 + \frac{1}{4}x + 70;\ p(x) = -\frac{x}{50} + 5$

99. $C(x) = -0.01x^3 + 8x^2 + 100;\ p(x) = -0.005x + 10$

100. $C(x) = -0.01x^3 + 0.1x^2 + 4x + 18;\ p(x) = -0.6x + 15$

Manufacturing—Bicycles: *For Exercises 101–105, the Wheelex Company manufactures bicycles and finds the price function for the bicycles to be*

$$p(x) = -0.02x + 150$$

where x represents the number of bicycles produced and sold and p(x) is the price of the bicycle. Furthermore, the fixed and variable costs to produce x bicycles are \$5600 and \$85 per bicycle, respectively.

101. The cost function follows the linear form $C(x) = mx + b$. Answer the following.

(a) Write the cost C in the linear form $C(x) = mx + b$.

(b) Use calculus to compute the marginal cost function $MC(x) = \frac{dC}{dx}$.

102. Using the information found in Exercise 101, complete parts (a) through (d).

(a) Evaluate $MC(250)$ and $MC(500)$ and interpret these answers.

(b) Why are the answers in part (a) equal?

(c) Algebraically find $AC(x)$ and simplify.

(d) Evaluate $AC(250)$ and interpret.

103. Use the solution from part (c) of Exercise 102 to answer parts (a) and (b).

(a) Use calculus to compute the marginal average cost function $MAC(x)$.

(b) Evaluate $MAC(250)$ and interpret.

104. Use $p(x)$ and the cost function information given in Exercise 101 to complete parts (a) through (c).

(a) Derive the revenue function $R(x)$.

(b) Use calculus to compute $MR(x) = \frac{dR}{dx}$.

(c) Evaluate $MR\,(500)$ and interpret.

105. Use the solution from part (a) of Exercise 101 with the solution to part (a) of Exercise 104 to complete parts (a) and (b).

(a) Derive the profit function $P(x)$.

(b) If the bicycles must be manufactured in lots of 250, how many bicycles should be manufactured so that the profit is as large as possible? Verify your answer by completing the table.

Produced, x	*Total profit, P(x)*
0	
250	
500	
750	
1000	
1250	
1500	
1750	
2000	

106. Sales Analysis—Watercolors: The Colorama Company, a paint manufacturer, has just produced a watercolor set that it sells to wholesalers for \$6 each. The cost $C(x)$ to produce x watercolor sets is given by the function

$$C(x) = 0.0002x^2 + 2x + 1250$$

(a) Algebraically derive the profit function $P(x)$ and simplify it.

(b) Evaluate $P(3000)$ and interpret.

(c) Differentiate to compute the marginal profit function.

(d) Evaluate $MP(3000)$ and interpret.

(e) Use the solutions from parts (b) and (d) to get a linear approximation for the value of $P(3001)$.

(f) Compute the error of the approximation for $P(3001)$ in part (e).

107. **Sales Analysis—Watercolors:** The Colorama Company hires a consulting firm to assess its work and consequently revises its price and cost functions to

$$p(x) = 6.50 \text{ and } C(x) = \frac{x^2}{5500} + \frac{7}{3}x + 1500$$

Redo parts (a) to (f) in Exercise 106 using these revisions.

108. Knowing that $AC(x) = \frac{C(x)}{x}$, use the quotient rule to show that the marginal average cost function can be written as $MAC(x) = \frac{MC(x) - AC(x)}{x}$.

109. **Publishing—Price/Demand:** The Between the Lines Publishing Company determines that the price-demand function for a new book is $p(x) = \frac{-x}{500} + 20$, with fixed costs of \$12,000 and variable costs of 4.5 dollars per book.

(a) Determine the cost function C and the revenue function R.

(b) Find the profit function P.

(c) Find the smallest and largest production levels x so that the company realizes a profit. (That is, find the smallest and largest independent values so that the revenue is greater than the cost.)

(d) Compute the marginal profit function $P'(x)$.

(e) Evaluate $P'(2500)$ and interpret the result.

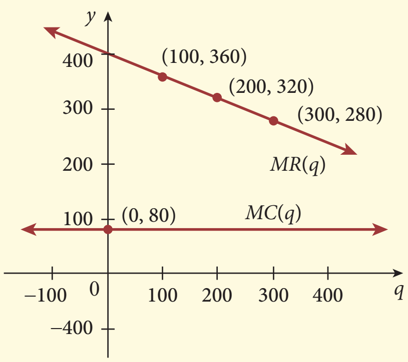
110. **Manufacturing—Price/Demand:** The Deluxe Furniture Company determines that the price–demand function for its new bookshelf is

$$p(x) = -\frac{x}{75} + 250$$

The fixed costs are \$10,000 and variable costs are 150 dollars per unit. Redo parts (a) through (e) in Exercise 109.

Finance—Stockholder Analysis: *For Exercises 111–116, consider a stockholder's report that lists the information shown in the table.*

q	$C(q)$	$R(q)$
100	22,830	38,000
200	30,830	72,000
300	38,830	102,000



Use the table and the graphs of $MC(q)$ and $MR(q)$ to get a linear approximation for the given cost and revenue functions.

111. $C(101)$

112. $C(201)$

113. $C(301)$

114. $R(101)$

115. $R(201)$

116. $R(301)$

117. **Manufacturing—Average Costs:** Consider the following data table for the cost and revenues at various production levels for a new brand of computer printer.

Number of printers produced, x	*Cost, C(x)*	*Revenue, R(x)*
1000	243,600	296,950
2000	363,000	575,800
3000	482,800	818,550
4000	603,300	1,007,200

(a) Use your calculator to determine a linear regression model for the cost of producing the printer in the form

$$C(x) = ax + b \qquad 1 \le x \le 4$$

where x represents the number of printers produced in thousands and $C(x)$ represents the cost of production.

(b) Use your calculator to determine a cubic regression model for the revenue of producing and selling the printers in the form

$$R(x) = ax^3 + bx^2 + cx + d \qquad 1 \le x \le 4$$

where x represents the number of printers produced and sold in thousands and $R(x)$ represents the resulting revenue.

(c) Compute $MAC(x)$ and simplify the result.

(d) Evaluate $MAC(1.5)$ and interpret.

118. Use the models found in Exercise 117 parts (a) and (b), to answer parts (a) and (b).

(a) Compute $MAP(x)$ and simplify the result.

(b) Evaluate $MAP(1.5)$ and interpret the answer.