Chapter Objectives

check off when you've completed an objective

- Know the definition of a graph and how to construct one as a model.
- Find paths and circuits in a graph.
- Determine when two graphs are isomorphic.
- Compute the number of edges in a complete graph.
- Know and apply Euler's theorem.
- Find an Eulerization of a graph.
- Learn about the lives of Leonhard Euler and Rowan Hamilton.
- Find a Hamilton circuit in a graph.
- Compute the number of Hamilton circuits in a complete graph.
- Know and apply the brute-force and nearest-neighbor algorithms.

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chapter

5 Graph Theory

Many tasks of the modern marketplace require traveling among a large number of places that have a variety of paths connecting them. Delivery of mail and packages, removal of snow from streets, and traveling by bus, car, or airplane are but a few of the examples that involve orchestrating the visitation of many places in a manner that is time- and cost-efficient. Many of you, for instance, probably took one or more campus tours when visiting colleges as part of your decision-making process for your plans after graduation from high school. Tours of this sort are mapped out by workers in admissions offices and are constructed to allow the potential student to see as many important campus sights as possible within a particular time frame. The problems involved in constructing such a tour are representative of several types of similar issues that face workers who plan the schedules and routes of everything from airplane flights to telephone traffic. The solution to a problem of this type consists of a path that satisfies the needs of the traveler and is optimal in some way, usually meaning that it minimizes time or cost. This chapter studies some of the basic tools used in finding these solutions, all of which lie in the broad area of mathematics known as **graph theory**.

graph theory The study of the properties and applications of graphs.

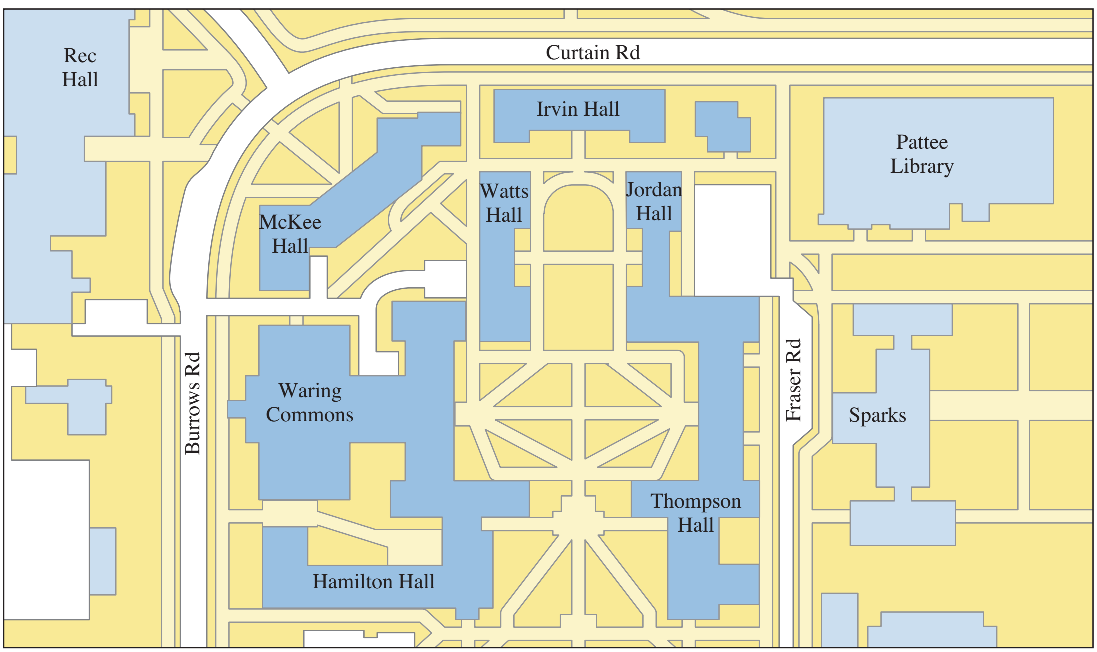

FIGURE 5.1.1 Part of a state university campus.

5.1 Graphs and Paths

An aerial photograph of a portion of the main campus of a state university is shown in Figure 5.1.1. Look at a small subset of sidewalks in the upper left of this picture (Figure 5.1.2), and suppose an historical plaque is located at each intersection. We wish to view each plaque without walking on any sidewalk more than once. To think about this problem, it is useful to draw a diagram that represents the possible paths to be taken. Because we are concerned not with distances or orientation but only with the sequence of connecting sidewalks to be traversed, Figure 5.1.3(a) or (b) will serve our purpose. The capital letters represent the plaques, and the lines connecting them represent the sidewalks.

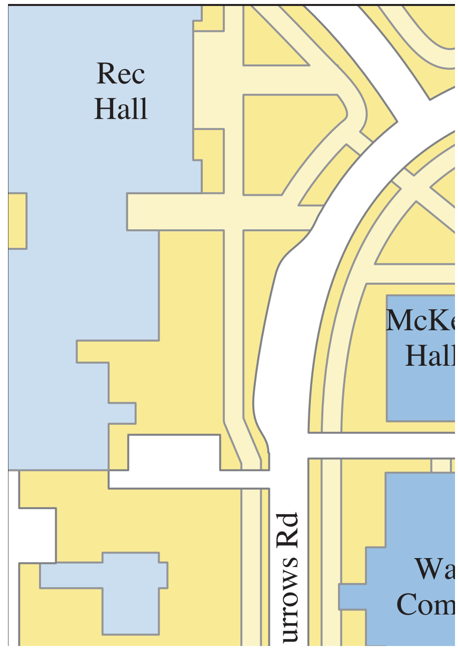

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FIGURE 5.1.2

graph A collection of vertices and edges.

network Another name for a graph.

vertices One of a set of points that all represent a similar type of object. The vertex set of a graph G is denoted by $V(G)$.

nodes Another name for a vertex.

edges A line segment drawn between two vertices that represents some relationship between them.

isomorphism A one-to-one correspondence between the vertices and edges of two graphs. The degrees of corresponding vertices must be equal.

Each of these diagrams is called a **graph** (or **network**), and every graph consists of two essential features:

- A set of points, known as the **vertices** or **nodes**
- A set of **edges**, each of which joins a pair of vertices

The lengths and positioning of the edges differ in the two diagrams, but the essential connecting features of each layout are the same. Therefore, either graph can be used for our purposes. Two graphs are said to be *isomorphic* if their vertices and edges match up in a one-to-one correspondence in such a way that an edge joining two vertices in one graph always corresponds to an edge joining the corresponding vertices in the other graph. When the actual correspondence is constructed as a function between the graphs, it is called an **isomorphism**. If graphs G and G_1 are isomorphic, we will denote this by $G \cong G_1$. This is the case for the graphs in Figure 5.1.3.

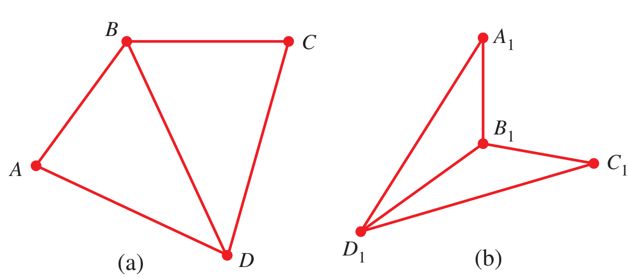


FIGURE 5.1.3 (a) Graph G. (b) Graph $G_1 \cong G$.

What other situation might be represented by either of the graphs in Figure 5.1.3? Perhaps A, B, C, and D represent the basketball teams at Alabama, Buffalo, Creighton, and Drake universities. During the season, suppose team A played B and D, team B played C and D (in addition to A), and team C also played D (in addition to B). Then the graph in Figure 5.1.3 could be used as a model of that schedule, in which each vertex represents a team and each edge represents a game played between the two teams it connects. So we see that a graph can be useful in representing any construct in which there are a discrete set of objects (vertices) and a set of relationships (edges) that connect them.

Example 1

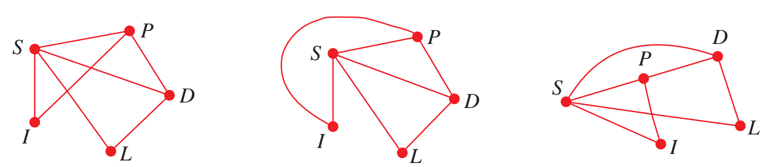
Sashif and Morah live in Seattle and wish to map out an itinerary to visit Portland, Oregon; Denver, Colorado; Lincoln, Nebraska; and Iowa City, Iowa, by plane. They wish to take only direct flights and have discovered that direct flights exist between:

- Seattle and each of the other cities.
- Portland and Denver as well as Portland and Iowa City.
- Denver and Lincoln.

Draw a graph that represents the possible routes taken by Sashif and Morah.

Solution

In this case, the set of vertices stands for the cities and can be denoted by $\{S, P, D, L, I\}$. A suitable graph does not have to correspond to the actual geography of the cities involved. Any one of the following graphs can be used, and each pair of them is isomorphic.



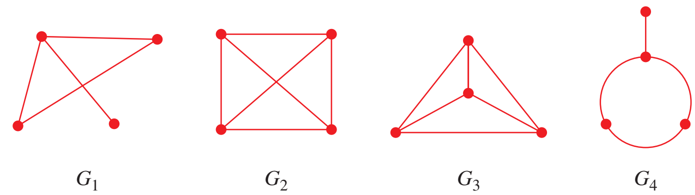
The vertex set of a given graph G is denoted by $V(G)$. In this example, $V(G) = \{S, P, D, L, I\}$. We also see that it is not required that a vertex be located at the intersection of two edges. It is called a **crossing** if no vertex is located at the intersection of two edges (e.g., the point where SD and PI intersect in the right-hand graph). When you analyze a graph to solve a problem, it is usually beneficial to use one with no crossings. If a graph is isomorphic to one with no crossings and is contained in a plane surface, it is known as a **planar graph**. In general, the graph that you use to model a problem should be one that is most visually convenient for you. ♦

crossing An intersection of two edges that is not considered to be a vertex.

planar graph A graph that is isomorphic to a graph without crossings that is contained in a plane surface.

Example 2

Which of the following graphs are isomorphic?



Solution

When you are exploring the possibility of an isomorphism, it is helpful to think intuitively of the edges of a graph as being "made of rubber." It is permissible to stretch or distort any edge as long as it is not broken or torn. Similarly, the vertices can be moved in any direction as long as they remain in the plane of the graph and no vertices are added or removed. Then a graph G will be isomorphic to another graph H if you can deform the edges and vertices of G until its shape resembles H. In this manner, you can conclude for the graphs here in Example 2 that $G_1 \cong G_4$ and $G_2 \cong G_3$. Note that this implies that both G_1 and G_2 are planar. ♦

degree of a vertex
The number of edges that meet at vertex v.

An important characteristic of a vertex in any graph is the number of edges that are adjoined to the vertex. The **degree of a vertex** v, denoted by $\delta(v)$, is defined to be equal to the number of edges that meet at v. Clearly, δ is a function whose domain is the vertex set of any graph and whose range is the set of nonnegative integers. In the graph in **Figure 5.1.4**, $\delta(a) = 2$, $\delta(b) = 3$, $\delta(c) = 1$, $\delta(d) = 2$, and $\delta(e) = 0$.

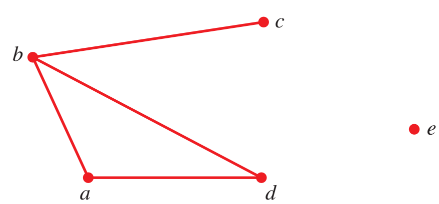


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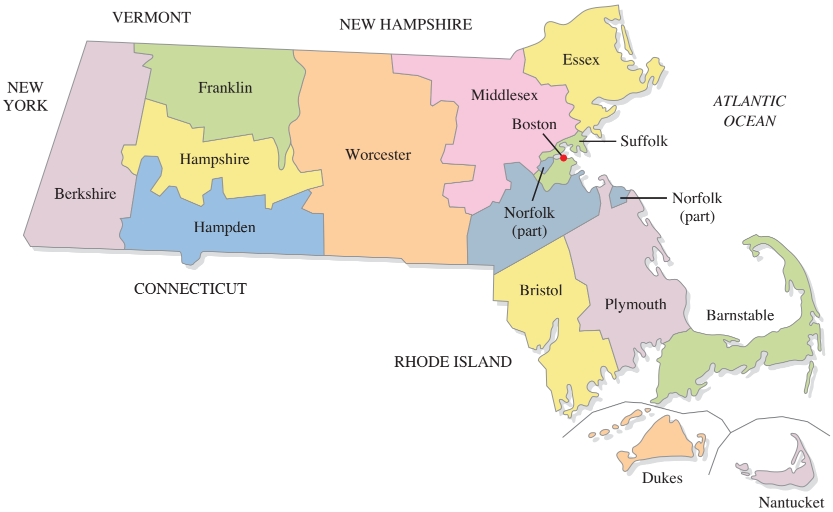
FIGURE 5.1.4

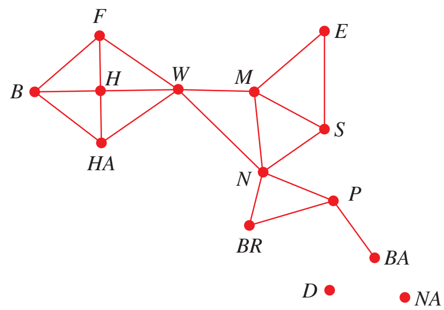
Example 3

The map here features most of the counties of the state of Massachusetts. Create a graph G in which each vertex represents a county and two vertices are joined by an edge if the corresponding counties share a border. (Ignore the eastern part of Norfolk County and the two small counties by Boston.) Define $V(G)$ for this graph. What are the degrees of the vertices? What does it mean for a vertex to have degree 0?

Solution

In the graph G that corresponds to the map, the vertex set is $V(G) = \{B, F, H, HA, W, M, E, S, N, BR, P, BA, D, NA\}$, if we denote each county by the first letter of its name (or the first two letters when necessary to avoid duplication). The degrees of the vertices are as follows: $\delta(B) = 3$, $\delta(F) = 3$, $\delta(H) = 4$, $\delta(HA) = 3$, $\delta(W) = 5$, $\delta(M) = 4$, $\delta(E) = 2$,





$\delta(S) = 3$, $\delta(N) = 5$, $\delta(BR) = 2$, $\delta(P) = 3$, $\delta(BA) = 1$, $\delta(D) = 0$, $\delta(NA) = 0$. If the degree of a vertex is 0, it means that the county does not share a border with any other county. In this particular case, this is because Dukes and Nantucket are islands. ♦

odd vertex A vertex whose degree is an odd integer.

even vertex A vertex whose degree is an even integer.

adjacent vertices Two vertices joined by an edge.

adjacent edges Two edges that meet at a common vertex.

path A sequence of consecutive adjacent vertices.

simple path A path in which no vertex occurs more than once.

circuit A path in a graph whose terminal vertex is the same as its initial vertex.

base The initial and terminal vertex of a circuit.

simple circuit A circuit in which the base is the only vertex used more than once.

A vertex v in a graph is called an **odd vertex** if $\delta(v)$ is an odd integer and an **even vertex** if $\delta(v)$ is an even integer. Two vertices are called **adjacent vertices** if they are joined by an edge, and two edges are called **adjacent edges** if they meet at a common vertex. Any finite sequence of edges joining two given vertices is a called a **path**. A path from vertex a to vertex b is typically represented by a finite, ordered sequence of vertices beginning with a and ending with b such that every pair of consecutive vertices is adjacent. For example, three paths from a to c in Figure 5.1.4 are given by the sequences *abc*, *adbc*, and *abdabc*. If no vertices are repeated in a path, it is called a **simple path**. Therefore, *abc* and *adbc* in Figure 5.1.4 are simple paths. A **circuit** is a path that begins and ends at the same vertex, called the **base**. If the base is the only repeated vertex in a circuit, the circuit is also referred to as a **simple circuit**. (Two sequences consisting of the same list of vertices but in reverse order will be considered to represent the same circuit.)

If graphs G and G' are isomorphic, then any pair of corresponding vertices must have the same degree. Similarly, isomorphic graphs must have equal numbers of paths, circuits, and odd and even vertices. However, no converse to any of these statements is true. Two graphs can have the same number of vertices and equal numbers of vertices with the same degrees but still not be isomorphic. An example of this is given in **Figure 5.1.5**.

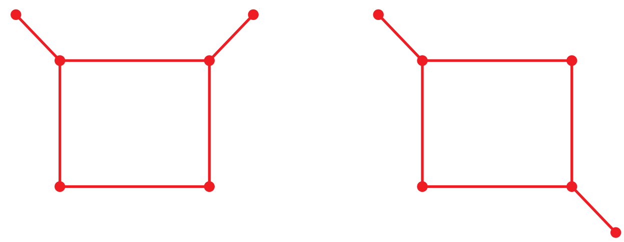


FIGURE 5.1.5 Non-isomorphic graphs with vertices of equal degrees.

Example 4

In the graph in **Figure 5.1.6**, list several:

(a) simple paths from vertex s to vertex x.
(b) simple paths from vertex t to vertex w.
(c) simple circuits based at t.
(d) simple circuits based at x.

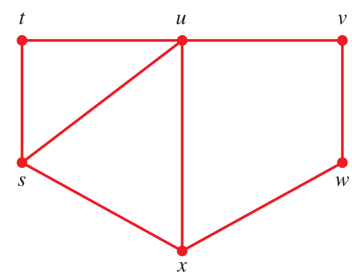


FIGURE 5.1.6

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Solution

(a) Five simple paths from s to x are sx, sux, $stux$, $suvwx$, and $stuvwx$.
(b) Six simple paths from t to w are $tuvw$, $tuxw$, $tusxw$, $tsxw$, $tsuvw$, and $tsuxw$.
(c) Three simple circuits based at t are $tust$, $tuxst$, and $tuvwxst$.
(d) Four simple circuits based at x are $xsux$, $xstux$, $xuvwx$, and $xstuvwx$. ♦

connected graph A graph in which every vertex is connected to at least one other vertex.

disconnected graph A graph that is not connected.

bridge An edge whose removal changes a connected graph into a disconnected graph.

If every vertex in a graph is connected by a path to every other vertex, then the graph is said to be a **connected graph**. A graph that is not connected is called a **disconnected graph**. Any graph containing a vertex of degree 0 must be disconnected. If the removal of a single edge changes a connected graph to a disconnected one, then that edge is called a **bridge**. Examples of these are shown in **Figure 5.1.7**. Clearly, a connected graph and a disconnected graph cannot be isomorphic to each other. Most of the applications we will examine primarily involve connected graphs.

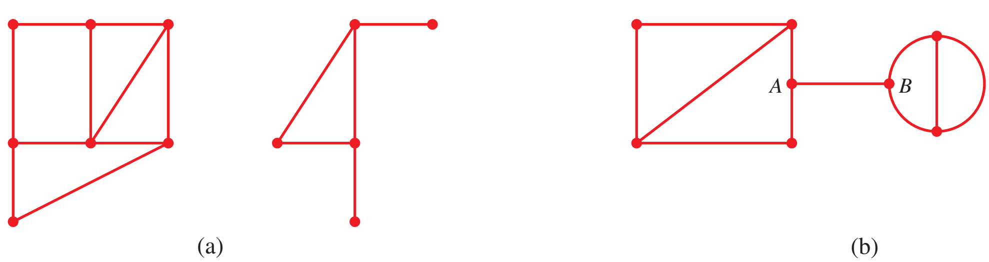

FIGURE 5.1.7 (a) Disconnected graph. (b) Graph with bridge AB

complete graph
A graph in which every pair of vertices is joined by an edge. The complete graph on n vertices is denoted by K_n.

A graph in which every vertex is joined to every other vertex by an edge is called a **complete graph**. The complete graph on n vertices is usually denoted by K_n. The K here is used in honor of the brilliant Polish mathematician **Kazimierz Kuratowski** (1896–1980). Kuratowski was instrumental in maintaining and nurturing Polish mathematics during and following World War II. He made fundamental contributions to graph theory, set theory, and topology.

Complete graphs can arise as an appropriate model in many contexts involving communication and transportation. Consider, for example, a set of towns in which each pair has a road connecting them or a set of cities in which an airplane is flown daily between each pair of cities. Either situation can be effectively modeled by a complete graph. Several examples of complete graphs are shown in **Figure 5.1.8**. Some exploration of these graphs may convince you that K_n cannot be planar if $n > 4$.

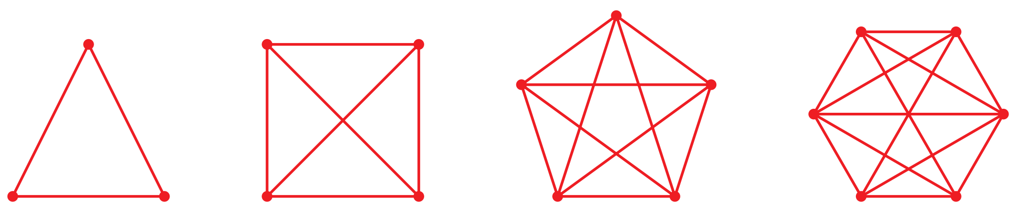

FIGURE 5.1.8 K_n for $n = 3, 4, 5$, and 6.

The graphs in Figure 5.1.8 suggest that we could devise a formula for the number of edges in a complete graph if we knew the number of vertices. Because $n - 1$ edges are connected to each of n vertices in K_n and each edge joins two vertices, the total number of edges must be one-half of $n(n - 1)$. For example, K_4 has $(4 \cdot 3)/2 = 6$ edges and K_5 has $(5 \cdot 4)/2 = 10$ edges, as you can check for yourself. In general, the total number of edges in K_n is given by $\frac{n(n-1)}{2}$.

This section has introduced some of the terminology and several of the main concepts associated with graph theory. The next two sections will explore two particular types of problems that are best analyzed by graph-theoretic techniques. They involve searching for certain types of circuits, known as Euler and Hamilton circuits, that provide optimal paths through a network representing some real-life situation. The term *optimal circuit* here means that traversing this circuit requires a minimum amount of time or cost or reduces replication of a task. Such circuits are excellent examples of practical applications of an area of mathematics that consists of quite abstract and sophisticated ideas.

Name ____________________

Exercise Set 5.1

1. Betty, Gretchen, and Harold are college students who all took History 101 at the same time. Furthermore, Betty, Harold, and Perry all took Math 120 together. Draw a graph in which vertices represent students and an edge represents a shared class. Define the vertex set for your graph.

2. You go to a meeting and shake hands with each of the other five people in attendance. If everyone has shaken hands with everyone else, draw a graph that represents this situation. How many handshakes occurred?

3. The mayors from Los Angeles, Chicago, Boston, Philadelphia, and Miami all have direct-line phone hookups with one another. Draw a graph that shows these connections, and define the associated vertex set $V(G)$. Is this a complete graph? How many edges are there?

4. Let the vertices of a graph G represent a set of five movie actors: A, B, C, D, and E. Two vertices are joined by an edge if the two corresponding actors have appeared in the same movie. Draw the appropriate graph if actor A has appeared with B and D; actor B has appeared with C, D, and E; actor C has appeared with D; and actor D has appeared with E. What are the degrees of each vertex of this graph?

5. The map of Eastern Europe features several countries. Create a graph in which each vertex represents a labeled country and two vertices are joined by an edge if the corresponding countries share a border. Define $V(G)$ for your graph.

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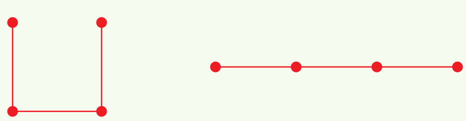
Which of the pairs of graphs in Exercises 6–11 are isomorphic to each other? Which graphs contain bridges?

6.



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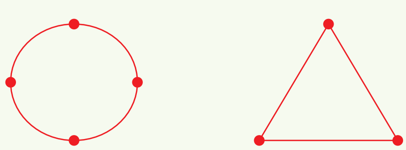

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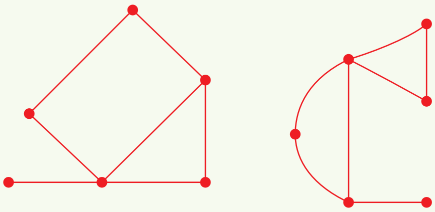

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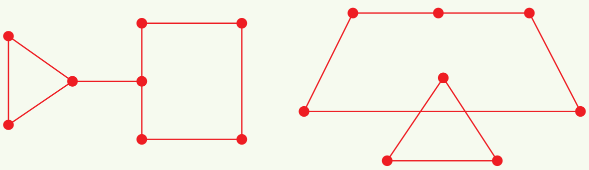

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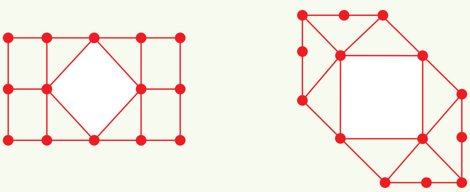

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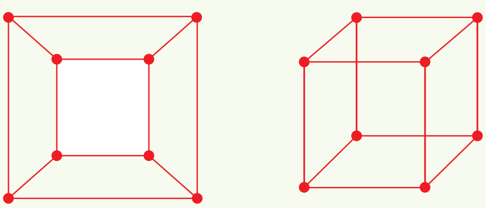
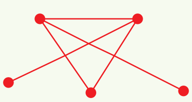
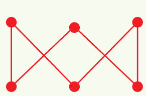


12. Draw a graph that is isomorphic to the given graph but does not contain any crossings.

(a)

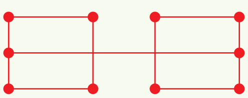


(b)

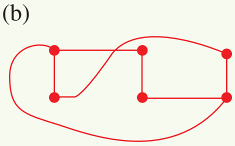


13. Draw a graph that is isomorphic to the given graph but does not contain any crossings.

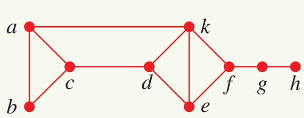
(a)



(b)

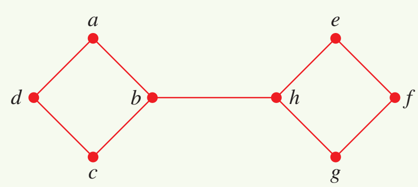


14. In the graph shown here, find five paths starting at *a* and ending at *e*. Find five paths starting at *c* and ending at *g*. Find four circuits based at *b* and at *k*.

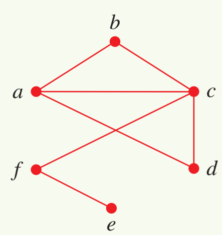


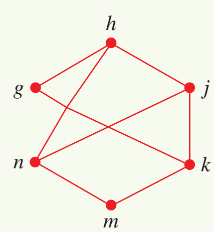
15. What are the degrees of the vertices in the graph in Exercise 14? Which vertices are odd? Which are even?

16. Which edge in the graph shown here is a bridge? Find three paths starting at vertex a and ending at g. Any path beginning at a, b, c, or d and ending at e, f, g, or h must contain what edge? Find three circuits based at a, all of which contain g. Find three circuits based at e, all of which contain d. Are these simple circuits?



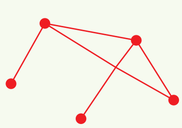
17. Identify two simple circuits based at a in the graph on the left and two simple paths from d to f. Identify four simple circuits based at h in the graph on the right and five simple paths from n to k.

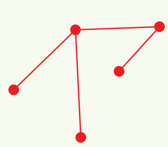


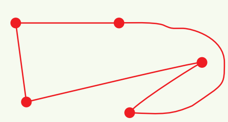


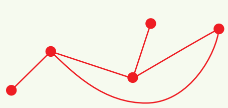
18. What are the degrees of the vertices in both graphs in Exercise 17? Which vertices are odd? Which are even? Are these graphs isomorphic?

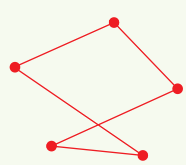
19. Three of the graphs shown here are isomorphic to one another, and two different ones are also isomorphic to each other. Identify them.

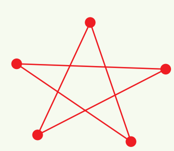

G_1


G_2

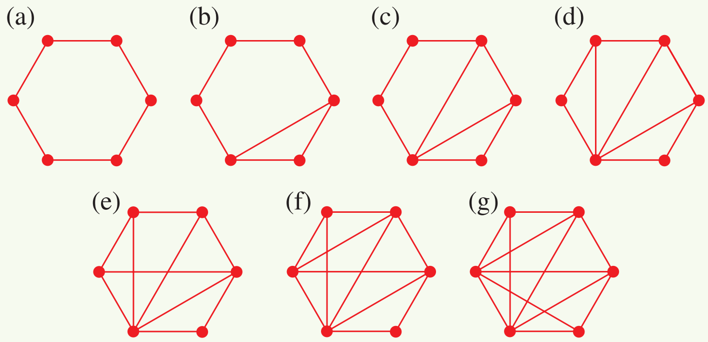

G_3


G_4

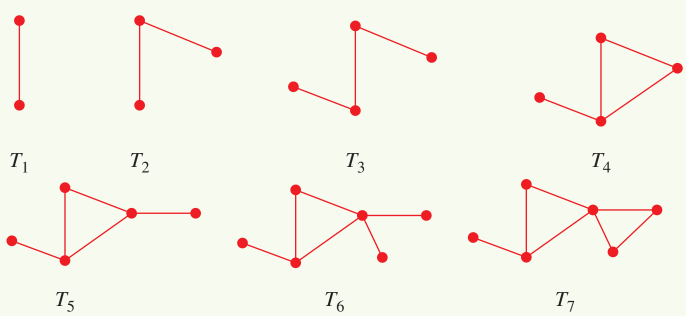

G_5


G_6

20. How many odd vertices are contained in each of the graphs in Exercise 19? How many odd vertices are contained in each of the following graphs? Postulate a property about the number of odd vertices contained in any graph.



21. Complete the table for the following graphs. The degree sum of a graph is the sum of the degrees of its vertices. Postulate a relationship between the number of edges of any graph and its degree sum.



Graph	T_1	T_2	T_3	T_4	T_5	T_6	T_7
Edges							
Degree Sum							

22. Based on the formula developed in Exercise 21, how many edges would be contained in a graph that has:

(a) 10 vertices, each of degree 3.
(b) 7 vertices, each of degree 2.
(c) 8 vertices, of which four have degree 5 and four have degree 2.

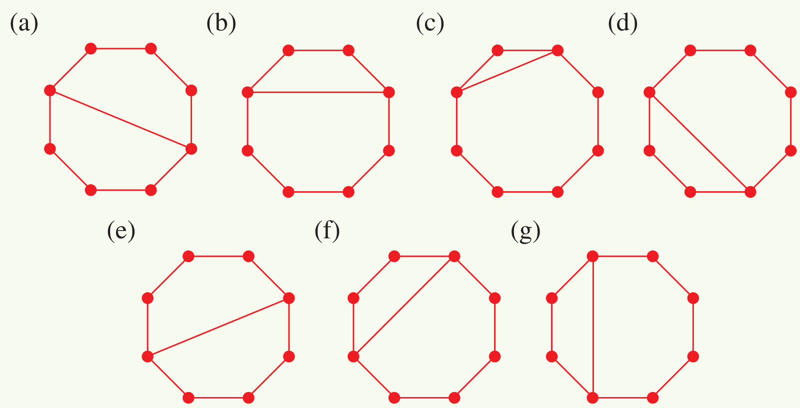
23. Based on the formula developed in the previous exercise, how many edges would be contained in a graph that has:

(a) 4 vertices, each of degree 3.
(b) 6 vertices, each of degree 4.
(c) 7 vertices of degrees 2, 2, 3, 3, 4, 5, and 5.

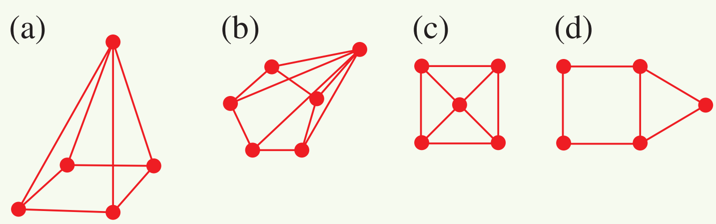
24. What is the degree sum of a graph with 35 edges? 186 edges? 250,000 edges?

25. How many edges are contained in the complete graphs K_7, K_{11}, K_{20}?

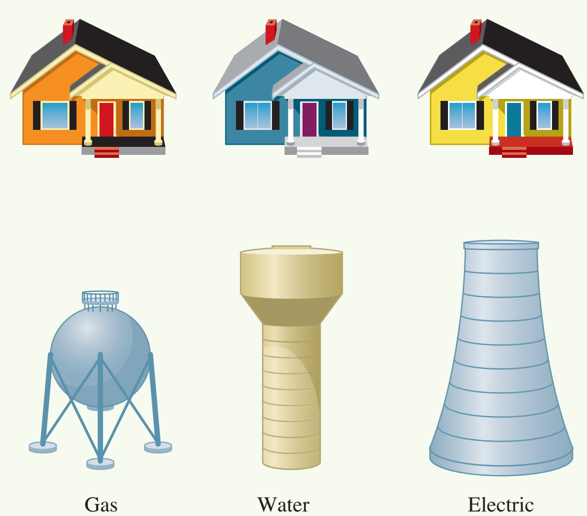
26. Graph (a) is isomorphic to which of the six graphs, (b) to (g)?



27. The graph on the left, (a), is isomorphic to which of the three graphs on the right, (b) to (d)?



28. Three houses are to be hooked up to three utilities. Gas, water, and electricity lines all need to be established, and no one of these lines can cross any of the others. Draw a graph that shows all nine of these connections. Is it possible to draw a representative graph without any crossings? Is this a planar graph?



29. A **bipartite graph** is one in which all the edges join a vertex in one group R to a vertex in a second group S. These graphs are useful in modeling what are called *matching problems*. For example, suppose three people are applying for one of a set of four jobs (W, X, Y, and Z) at a particular business. Arthur is qualified for jobs W, X, and Z; Bridget is qualified for jobs X and Y; and Chin is qualified for jobs W and Y. Draw a bipartite graph that represents this situation. Is it possible for all three of the applicants to each obtain a different job?

bipartite graph
A graph whose only edges consist of those that join every vertex in one group of vertices to every vertex in another distinct group.

30. Four people are applying for four jobs at Franklin Premium Insurance Company. Arin is qualified to be a programmer and an actuary, Brett is qualified to be an accountant and an actuary, Casey is qualified to be a hardware technician and a programmer, and Darcy is qualified to be an accountant. Draw a bipartite graph (defined in the preceding exercise) that represents this situation. Is it possible for all four of the applicants to each obtain a different job?

31. A *digraph* is a graph whose edges also have an indicated direction associated with them. For instance, Web page A that contains a link from itself to site B does not imply that site B has a link back to A. The direction of an edge in a graph is usually denoted by drawing an arrowhead in the middle of the edge. Suppose Tom has links on his homepage to the Web pages of his friends Omar, Beatrice, and Maggie. Likewise, Omar has a link on his Web page to that of Beatrice, and Beatrice has links to Tom and Maggie. Maggie has no links to anyone. Draw a digraph that represents this network of links.

32. Define the vertices of $V(G)$ to correspond to 10 different people who each have a homepage on Facebook. Define an edge between two vertices to indicate that the corresponding people are in each other's friends' lists. If each member of $V(G)$ is friends with every other member of $V(G)$, then what is the total number of edges of G?

Identify each statement as true or false. If the statement is false, give an example that demonstrates it is false. Let G and G′ be graphs.

33. If G and G' are connected, then they must be isomorphic to each other.

34. If a graph has a vertex of degree 0, then it cannot be connected.

35. If a graph has a bridge, it must be connected.

36. If $G \cong G'$, then G and G' must have the same number of vertices.

37. If G and G' have the same number of vertices, then $G \cong G'$.

38. If $G \cong G'$, then the degree of each vertex of G is equal to the degree of the corresponding vertex of G'.

39. If G and G' have the same number of vertices and equal numbers of vertices with the same degree, then $G \cong G'$.

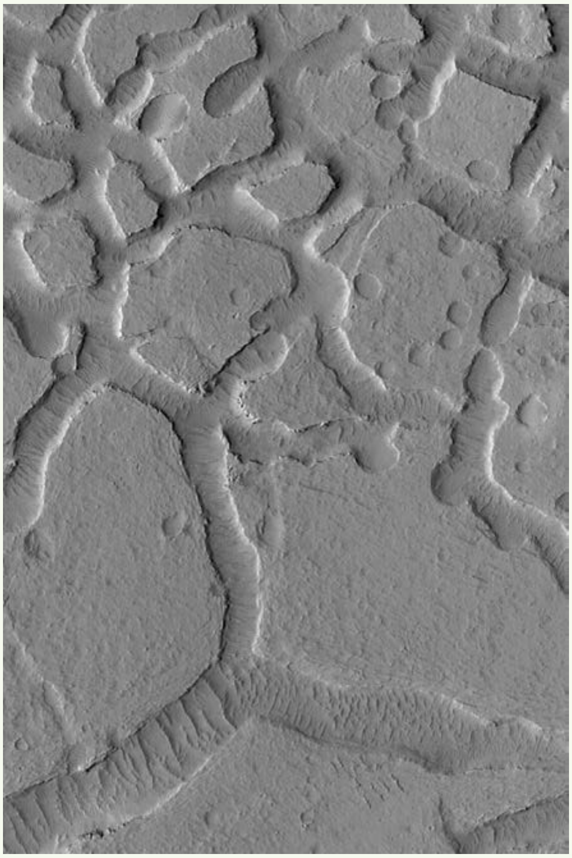
40. If neither G nor G' has a circuit, then $G \cong G'$.

41. If the number of odd vertices of G is not equal to the number of odd vertices of G', then G cannot be isomorphic to G'.

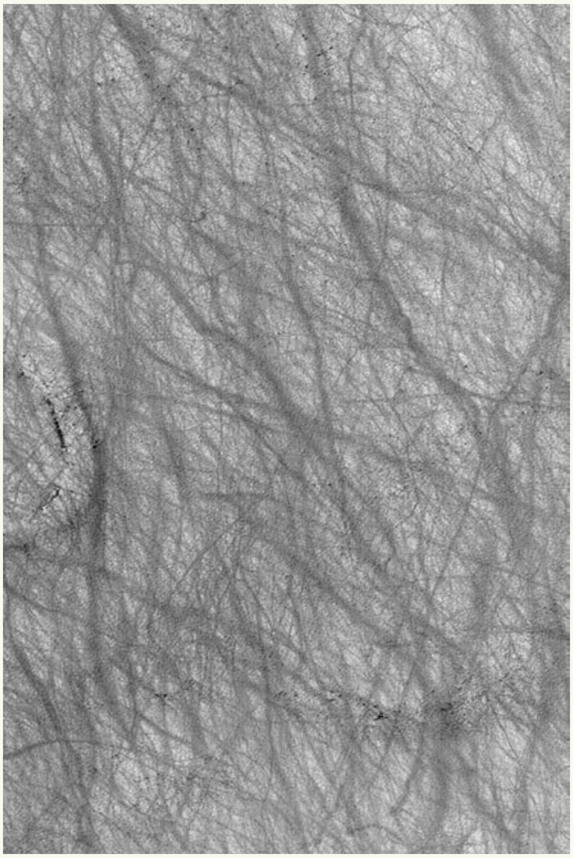
42. If G and G' have the same number of paths, then $G \cong G'$.

43. The degree sum of any graph is always even.

44. The Orbiter Camera on the Mars Global Surveyor took the pictures shown here of the surface of Mars in 2003. Similar surface features can be found on Earth. In what ways do you think graph theory might be helpful in comparing them to the Martian structures shown here?



Mesas and troughs.



Dust devils.

5.2 Euler Circuits

One of the primary modern-day uses of graph theory is for the analysis of routing problems. A *routing problem* is concerned with choosing an efficient path that takes a traveler along a required route or that leads him or her to visit a specific set of destinations. Examples of such problems include finding an optimal way to plow snow without repetition of streets, visiting a list of cities with a minimum of travel cost, or channeling a telephone call through a chain of relays in the shortest amount of time. The use of graph theory as a tool to solve such problems traces its beginnings to a recreational puzzle involving Sunday strollers in the city of Koenigsburg in eighteenth-century Prussia (currently Kalingrad, Russia). A river with several islands flowed through the center of this town, and seven bridges connected two of the islands with each other and with opposite banks of the river. A picture of the situation is given in **Figure 5.2.1**. Through the years, people had become convinced that it was not possible to walk a path that traveled across each of the seven bridges and returned you to your starting point unless you had crossed at least one bridge more than once. In order to find a definitive proof that such a circuit was indeed impossible, Leonhard Euler (1707–1783) created what was perhaps the first graph as we have defined it. He thought of each landmass (separated by water from other land) as a single destination, and so it could be represented by a single point or vertex. In this way, each bridge between two landmasses could correspond to an edge joining two vertices. The graph representing this clever idea is shown in Figure 5.2.1.

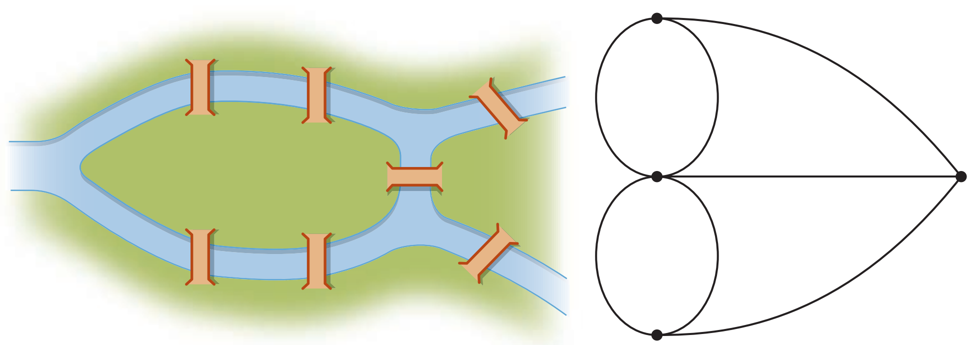


FIGURE 5.2.1 The bridges of Koenigsburg and the associated graph.

You may have heard of the influence of Leonhard Euler (**Figure 5.2.2**) on other areas of mathematics, such as the development of the definition of a function. Euler was a truly remarkable individual who made many deep and lasting contributions to mathematics and science in more than 500 published papers. His memory and computing skills were unmatched. A friend once remarked that Euler could "calculate as easily as other men breathe." It is said that Euler, a caring father, was able to work even with small children seated in his lap. Astonishingly, he continued to work and produce deep results for many years, even after he was overtaken by complete blindness. In fact, when he first became blind in his right eye, his comment was, "Now I will have less distraction." His numerous mathematical accomplishments provided a critical step forward in providing clarity and precision to the concept of proof and to the construction of abstract mathematical models that could be used to attempt to solve difficult practical problems.

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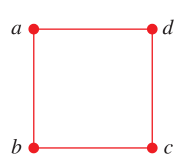
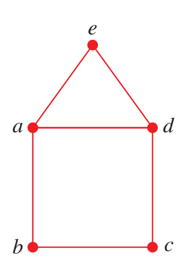
FIGURE 5.2.2 Portrait of Euler.

Euler circuit A closed path through a graph that contains every edge exactly once.

Named in honor of Leonhard Euler, an **Euler circuit** is defined to be a path through a graph that traverses every edge exactly one time and terminates at the same vertex from which it began. As you might expect, not every graph will possess an Euler circuit. However, by examining a sequence of graphs, we will develop a condition for determining precisely when an Euler circuit exists. We begin with the simple observation that each edge in the rectangular graph shown here is used once in forming a circuit. Each vertex has one edge that can be used to approach the vertex and another one to leave it. In other words, the degree of each vertex is 2.



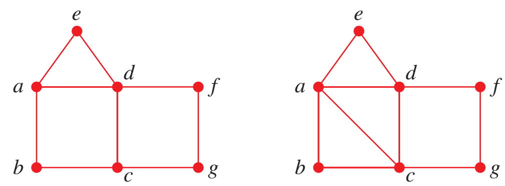
Now note that if any corner has an additional edge added to it, it becomes impossible to begin a circuit at that point and return to it more than once. This is the case at vertex a or d in the graph shown here. Likewise, a circuit beginning elsewhere can approach such a vertex twice but only leave it once. So no Euler circuit can be found in this situation. We note that the degree of vertex a (and d also) is 3. In fact, we observe that whenever the degree of a vertex is odd, the number of edges in any circuit that approach the vertex will differ by 1 from the number of edges that leave the vertex. So no Euler circuit can exist in a graph that has one or more vertices of odd degree. This fact is stated in the following rule.



Every vertex of a connected graph G has even degree if and only if G possesses an Euler circuit.

Example 1

Do either of these graphs contain an Euler circuit? If so, find one.

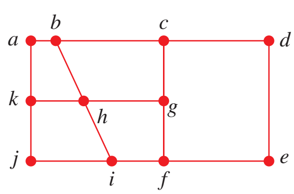


Solution

In the graph on the left, vertices *a* and *c* both have odd degree. So no Euler circuit can be found. All the vertices in the graph on the right have even degree, and so the graph must have an Euler circuit. While many such circuits could be found, two examples based at *a* are *aedfgcdacba* and *abcdeacgfda*. ♦

Graph theory is useful in the efficient operation of city services. Optimal routes for such tasks as the removal of snow, delivery of mail, and checking of telephone and electric lines can be found by searching for Euler circuits in graphs that represent networks of city streets. Clearly, the mileage accumulated by a snowplow is minimized by driving a route that traverses each street in a particular network with as little duplication as possible. Consider the map of a sector of Manhattan, New York, shown in **Figure 5.2.3**. Suppose a driver needs to clear snow along each street between and including 5th and 7th avenues and 40th and 42nd streets. If we assume that only one trip down each street is needed to clear away the snow, can the driver do this without having to travel a street twice? When we examine the representative graph that corresponds to the map, we see that vertices *b*, *c*, *g*, *f*, *i*, and *k* all have degree 3. Because this means no Euler circuit can be found, the next question would be, What circuit could be taken that reuses a minimum number of edges? Solving this related problem uses a technique known as the **Eulerization** of a graph. This is a procedure in which we add edges in order to create a new graph with no vertex of odd degree. The idea is to duplicate existing edges in a manner that adds as few new edges as possible.

Eulerization
Procedure in which edges are duplicated to create a new graph containing no odd vertices.



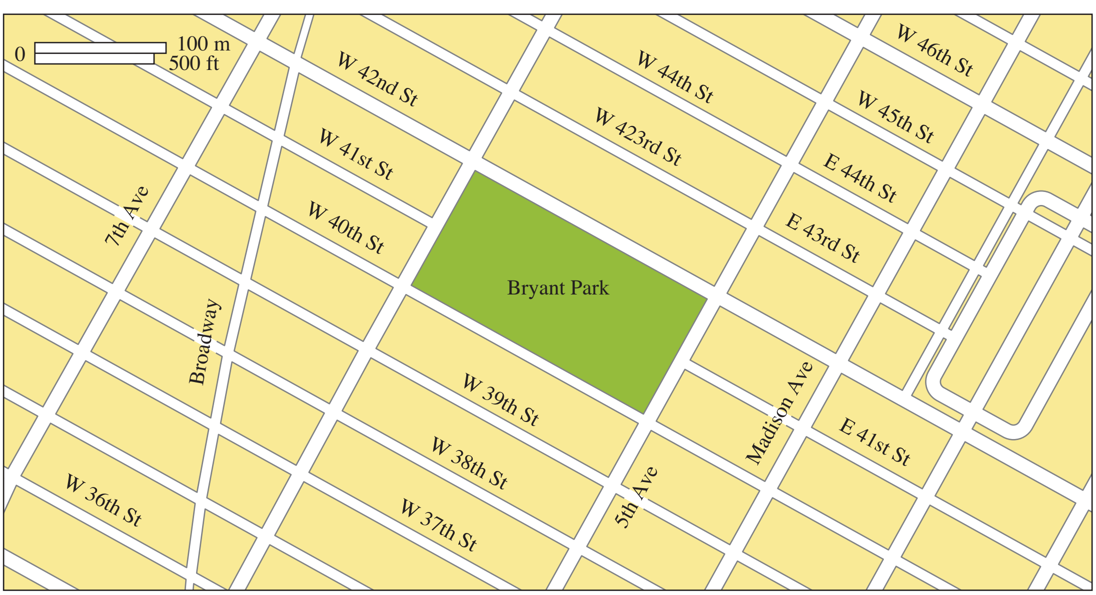


FIGURE 5.2.3 A Manhattan neighborhood.

For example, in the Eulerization of the Manhattan graph shown in **Figure 5.2.4**, we note that each vertex originally of odd degree has had at least one edge joined to it in order to change the degree to an even number. Observe also that although both vertices *h* and *j* initially had even degree, edges were attached to them to alter the degree of adjacent vertices. Moreover, we must be careful to duplicate only existing edges and not create any new connections, because creating a new edge would correspond to a street that doesn't actually exist. For example, a new edge cannot be added between vertices *b* and *g*, or between *c* and *e*.

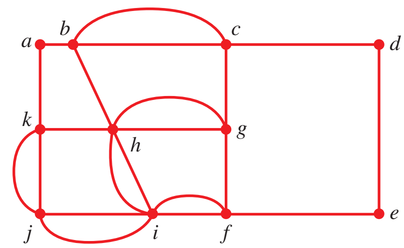


FIGURE 5.2.4 An Eulerization of a graph.

Now consider what the addition of an edge represents to the snowplow's task. It implies that any street corresponding to a duplicated edge will be traveled twice by the truck along any possible circuit constructed from the new graph. Because we are not currently accounting for the lengths of the streets represented by the edges, the main criterion for an optimal Eulerization is the addition of a minimum number of duplications. A graph satisfying this criterion is called a **minimum Eulerization**. Consider the two examples in **Figure 5.2.5**. Both of these required only four duplicated edges, as compared to the six edges added in Figure 5.2.4. A quick bit of experimentation will endorse the fact that no other modification exists that requires fewer than four new edges, and so both of these are minimum Eulerizations.

minimum Eulerization
Eulerization of a graph that uses a minimum number of duplicated edges.

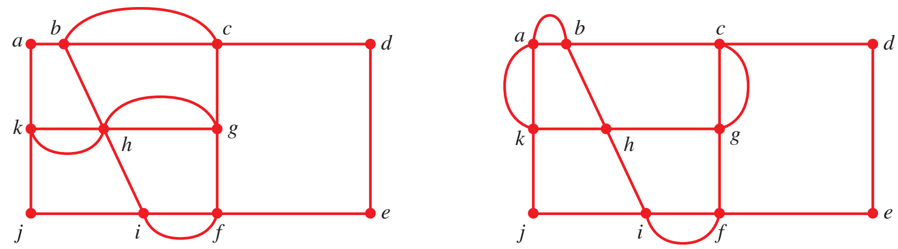


FIGURE 5.2.5 Two minimum Eulerizations.

Because the degrees of all the vertices in each of these graphs are now even, each graph possesses an Euler circuit. Once the circuit is identified, it is then squeezed back onto the original Manhattan graph to obtain a circuit that has twice-used edges. For example, an Euler circuit based at *a* in the left-hand graph of Figure 5.2.5 would be *abcbhgcdefghifijkhka*. The edges *bc*, *hg*, *fi*, and *hk* are all used twice when this circuit is traversed in the original graph. (See **Figure 5.2.6**.)

Incidentally, note that the number of times each vertex (except the base) appears in the sequence for labeling the circuit must necessarily be equal to one-half its degree. For example, the vertex *h* has degree 6 in Figure 5.2.6 and is listed three times in our circuit.

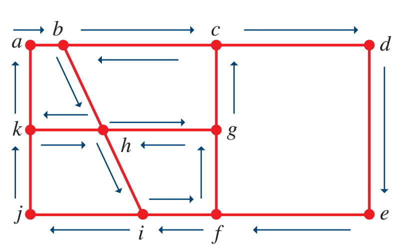
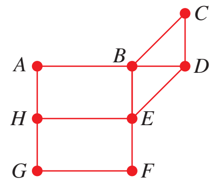


FIGURE 5.2.6 A circuit with reused edges.

Example 2

Find a minimum Eulerization for the following graph.



Solution

The degrees of all the vertices are even, with the exceptions of H and D, and therefore an edge must be added to each to Eulerize the graph. If H and D were adjacent, this could be done with a single edge, but this is not the case. We can see that the next best solution is to duplicate the edge connecting H to E and then another one connecting E to D.

The resulting minimum Eulerization is shown on the left in **Figure 5.2.7**. One Euler circuit in this graph would be *ABCDBEDEFGHEHA*, and squeezing it onto the original graph results in the circuit shown on the right side of Figure 5.2.7. Again we note the 2-to-1 ratio between the degree of a vertex and the number of times it appears in the sequence for the circuit. Vertices C, F, and G each have degree 2 and appear once, while B, D, and H have degree 4 and appear twice, and E has degree 6 and appears three times. ♦

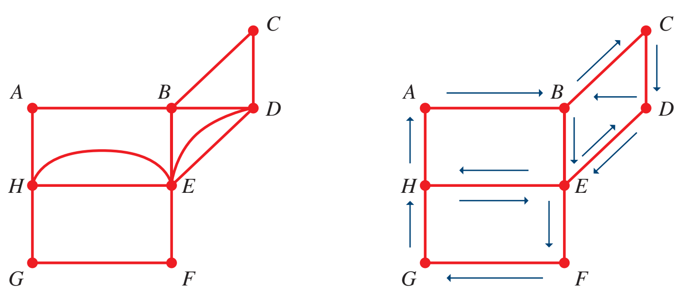


FIGURE 5.2.7 A minimum Eulerization.

In some situations, it is desirable to be able to construct a connected graph that possesses an Euler circuit that corresponds to a given sequence of vertices. For instance, one consideration in the planning of a new housing subdivision might be cost-efficient street maintenance. Therefore, it makes sense to design the street network such that it contains an Euler circuit even when faced with strange surface contours in the building site. The 2-to-1 relationship that we have observed in the circuits earlier can guide us in this construction. We conclude this section with an example.

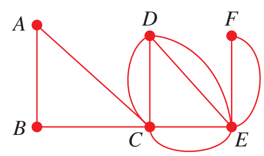
Example 3

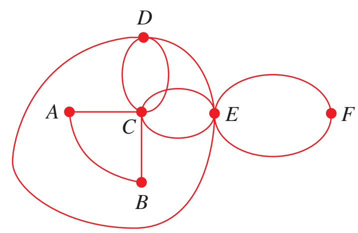
Suppose that the path *ABCDCEFEDECA* is an Euler circuit in a particular graph.

(a) What are the degrees of the vertices in this graph?
(b) Draw a graph for which this path could be an Euler circuit.

Solution

(a) With the exception of the starting point, the degree of each vertex must equal twice the number of times the vertex occurs in the circuit. Therefore, $\delta(A) = 2$, $\delta(B) = 2$, $\delta(C) = 6$, $\delta(D) = 4$, $\delta(E) = 6$, and $\delta(F) = 2$.
(b) All graphs that have *ABCDCEFEDECA* as an Euler circuit must be isomorphic. (Why?) Both of the following graphs are examples. ♦



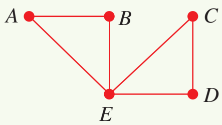


Name ______________________

Exercise Set **5.2**

Determine the degree of the vertices in each of these graphs. If the graph has an Euler circuit, find it.

1.



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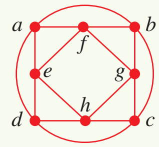

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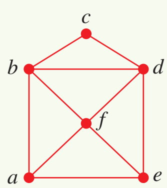

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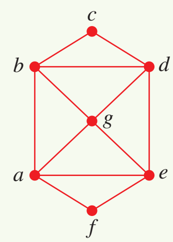

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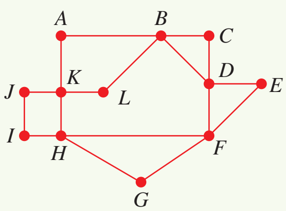

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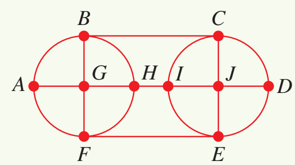

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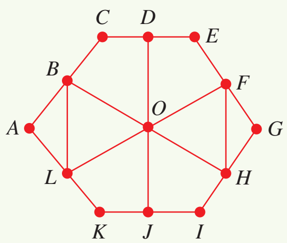

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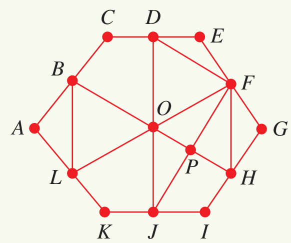
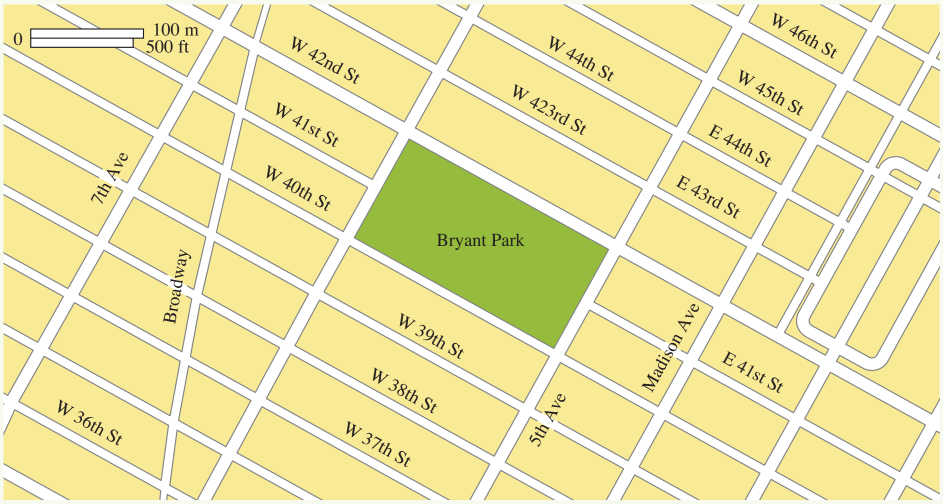


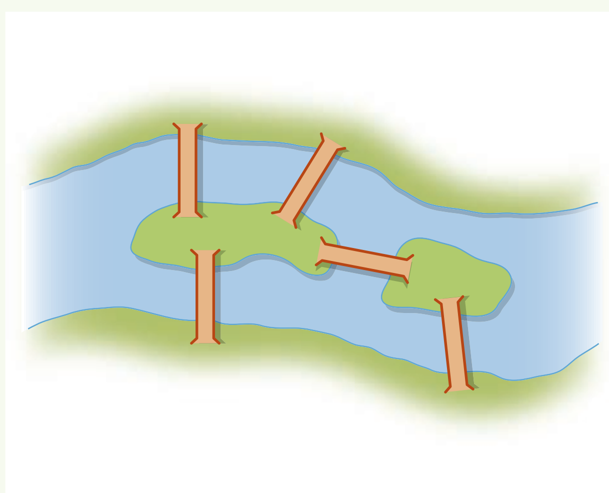
9. The Manhattan street maintenance crew wishes to check the no-passing lines for possible repainting. Suppose they wish to find the most efficient route by which to check all the streets north of 39th and south of 42nd streets and east of 7th and west of 5th avenues. Draw a graph that represents this region (including the boundary streets).



10. Does the graph in Exercise 9 have an Euler circuit? If so, find one.

11. Now draw a graph that extends the south boundary of the region in Exercise 1 to 37th Street. Does this graph have an Euler circuit?

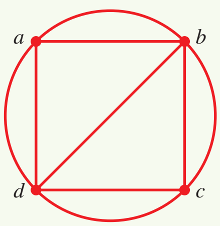
12. This map shows a river, two islands, and five bridges. A jogger wishes to cross each of the bridges just one time as she runs a route that begins and ends on the north side of the river. Draw a graph that models the situation.



13. Does your graph in the previous exercise have an Euler circuit? If so, find one that begins on the north (top) side of the river.

Find a minimum Eulerization for each of the following graphs, and identify a circuit in that Eulerization.

14.



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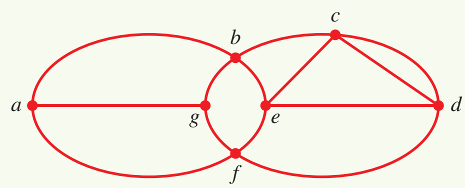

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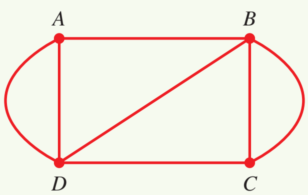

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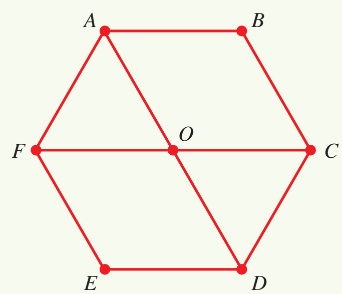

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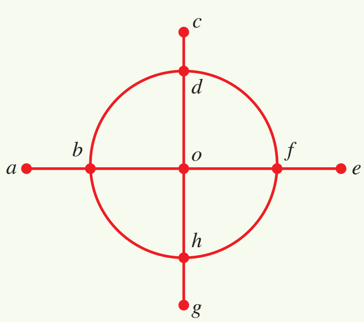

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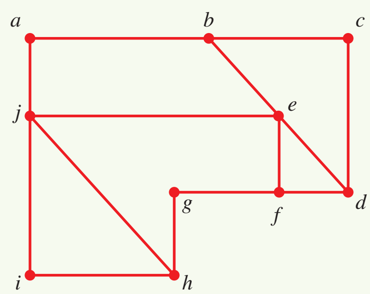

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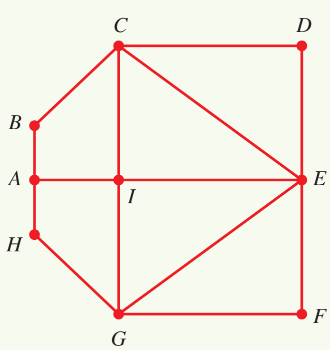

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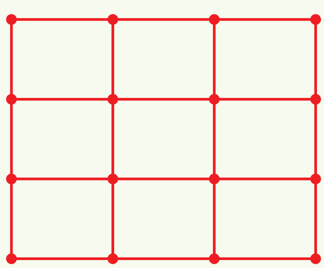

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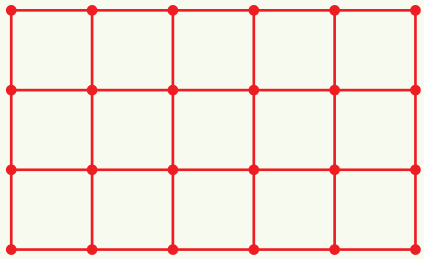


23. The map shown here displays a portion of Washington, D.C. A street inspector wants to check for holes and cracks on the streets between and including L and M streets and 17th and 19th streets. Form a graph that represents this network of streets, and find an optimal circuit.

24. Draw a graph of the street network of the region between and including M and O streets and 15th and 17th streets. How many vertices of odd degree does this graph contain?

25. Draw a graph that represents the street network of the region between and including M and N streets and 17th and 19th streets. (Use a single edge to represent Connecticut Avenue.) How many vertices of odd degree does this graph contain?

26. Suppose that the path *ABCDCEBA* is an Euler circuit in a particular graph.

(a) What are the degrees of the vertices in this graph?
(b) Draw a graph for which this path would be an Euler circuit.

27. Suppose that the path *ABDEBECA* is an Euler circuit in a particular graph.

(a) What are the degrees of the vertices in this graph?
(b) Draw a graph for which this path would be an Euler circuit.

28. Suppose that the path *ABDCDEFEDFBA* is an Euler circuit in a particular graph.

(a) What are the degrees of the vertices in this graph?
(b) Draw a graph for which this path would be an Euler circuit.

29. Suppose that the path *ABCDCEFEDECA* is an Euler circuit in a particular graph.

(a) What are the degrees of the vertices in this graph?
(b) Draw a graph for which this path would be an Euler circuit.

Identify each statement as true or false. If the statement is false, give an example that demonstrates it is false. Let G and G′ be graphs.

30. If G is isomorphic to G' and G has an Euler circuit, then G' also has an Euler circuit.

31. If G and G' both have Euler circuits, then G is isomorphic to G'.

32. If an even number of vertices of G have odd degree, then G has an Euler circuit.

33. A graph cannot have an Euler circuit if it is not connected.

34. A graph with exactly two vertices of odd degree has a minimum Eulerization with just one additional edge.

35. A graph with exactly two vertices of odd degree, in which the vertices are also adjacent, has a minimum Eulerization with just one additional edge.

36. A graph has only one minimum Eulerization.

37. In an Eulerized graph, the number of times a vertex (other than the base vertex) appears in any Euler circuit must equal to one-half of its degree.

5.3 Hamilton Circuits

In this section, we shift our attention in the use of graphs from a search for a closed path that uses every edge once to looking for one that visits every vertex once. A vertex-oriented circuit does not have to use every edge in a graph and is the solution to a different kind of problem in graph theory. If the task is to plow a network of streets, then every road must be included in the chosen route, and an Euler circuit is appropriate. However, if the task is to visit just the intersections of the streets in order to check traffic patterns, then the most efficient path to travel would probably not use every street. For instance, one can envision an abundance of problems in the delivery of packages by companies such as Federal Express, UPS, and the U.S. Postal Service that require efficient paths that stop once at each of many destinations. These companies' economic survival depends on rapidly finding these paths every day. Our first example displays an elementary but representative situation in which the solution is a path that visits every desirable stop exactly once.

Example 1

The students in a middle school class are going to visit the Lincoln Park Zoo in Chicago, and they wish to find an efficient route to walk in order to see their favorite animal exhibits. A map of the zoo is shown in **Figure 5.3.1**. Create a graph in which the edges represent walking paths and the vertices consist of the following:

- Penguin house (N)
- Bears (B)
- Primate house (P)
- Petting zoo (Z)
- Elephant exhibit (E)

Adapted from Lincoln Park Zoo

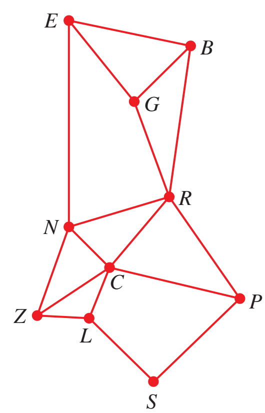
FIGURE 5.3.1 The Lincoln Park Zoo in Chicago.

- Birds of prey (R)
- Giraffes (G)
- Sea lion pool (L)
- Swan pond (S)
- Landmark cafe (C)

Use the graph to find a closed path that visits each of the exhibits.

Solution

Any graph isomorphic to the one displayed here would suffice to model the zoo.



One circuit that visits each vertex once is shown by the darkened edges in **Figure 5.3.2**. If we think of it as beginning and ending at the cafe (*C*), then we could denote it by *CZNEGBRPSLC*, though the same circuit could also be designated by any of several sequences of vertices, including, for example, *ZNEGBRPSLCZ* or *NEGBRPSLCZN*. It is the *ordering* of the vertices that is the identifying characteristic as opposed to the choice of starting point. ♦

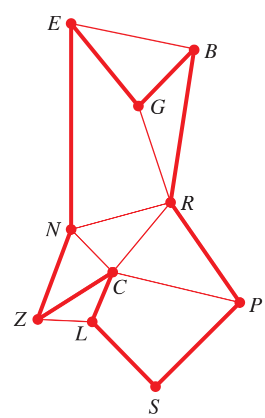
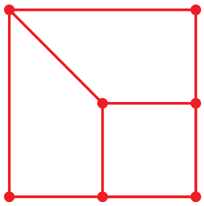


FIGURE 5.3.2 A Hamilton circuit.

Hamilton circuit A closed path through a graph that contains every vertex exactly once.

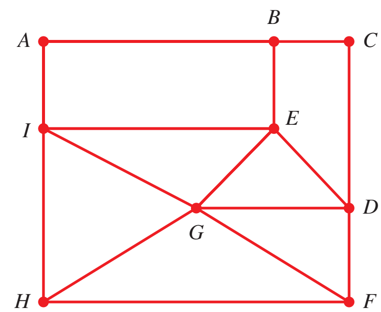
The circuit in Figure 5.3.2 is called a **Hamilton circuit** and is defined as any closed path in a connected graph that includes every vertex exactly once. A given graph may have none, one, or many Hamilton circuits. A graph with no such circuit is shown here.



Unlike the search for an Euler circuit, no precise conditions can be tested (such as the parity of the vertex degrees) in a graph that indicates when a Hamilton circuit can be found. Although exceptions to this statement for certain specialized graphs will be explored in the exercises, in general we can only identify certain helpful clues.

Example 2

Find a Hamilton circuit in the graph shown here.



Solution

We start our path at A and head first to vertex B. At B, we have a choice of continuing on to C or to E. The problem with heading to E is that C could then not be included in any circuit that does not reuse a vertex because any path back through C would visit B a second time. Therefore, from B, we must head next to C and then necessarily on to D, because $\delta(C) = 2$. We now have a choice of E or F, either of which would lead to a Hamilton circuit. Choosing F would result in the circuit *ABCDFHGEIA*, and picking E would give *ABCDEGFHIA* as a solution. These are shown in **Figure 5.3.3**. ♦

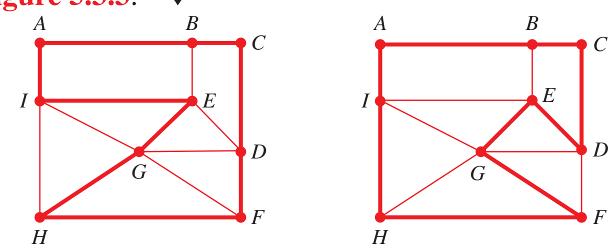


FIGURE 5.3.3 Two Hamilton circuits.

FIGURE 5.3.4 Sir William Rowan Hamilton.

This example illustrates the fact that as we construct our closed path, if we are currently resting at a vertex that is adjacent to a vertex of degree 2, then that vertex must be visited next. Otherwise, no Hamilton circuit could be completed that includes it.

The Hamilton circuit is named for **Sir William Rowan Hamilton** (1805–1865). Born and raised in Dublin, he is one of the most famous mathematicians to ever hail from Ireland. Hamilton was raised and educated by an uncle. His keen mind was apparent as a young child, and by age 11, he had mastered 13 different languages. As a teenager, his attention turned to mathematics, and by the age of 20, he had absorbed Newton's *Principia*, corrected an error in Simon Laplace's difficult *Mecanique Celeste*, and was already engaged in doing original mathematical work. He achieved remarkable success at Trinity College, Dublin, and astonishingly was appointed to the prestigious position of Professor of Astronomy at the unprecedented young age of 21. Fond of writing poetry, Hamilton formed a lifelong friendship with literary master William Wordsworth. The two men spent much time discussing science and poetry, and Wordsworth eventually was forced to advise his friend to stick to science. His poetry was not so good!

Although he accomplished much professionally, Hamilton's personal affairs were fraught with angst and heartbreak. Throughout his life, he was plagued by an unrequited love for a woman he lost because of the interference of her parents. In constant personal turmoil, he succumbed to a lifelong predilection for alcohol that, in his middle forties, led to a deterioration of his health and eventually to his death. If you ever visit Dublin, you can still see the equations he engraved on the stone bridge where they first occurred to him.

weight A numerical value assigned to an edge.

weighted graph A graph in which every edge has a weight.

minimal Hamilton circuit A Hamilton circuit in a weighted graph whose total weight is less than that of any other Hamilton circuit.

Because most graphs that occur in practice contain multiple Hamilton circuits, the next question is, Which of these is the best solution, where *best* usually is defined in terms of some numerical quantity that is a function of the circuit. Often, the relationship represented by each edge of a graph has a number value, or **weight**, associated with it, such as the length of a particular street, a distance between cities, a cost of transmission along a phone line, or a time to complete a certain task. A graph that has a weight associated with each edge is called a **weighted graph**, and the sum of these weights is called its *total weight*. A **minimal Hamilton circuit** is one whose total weight is less than that of any other Hamilton circuit.

As an example, suppose that a school superintendent needs to visit five schools in her district and wishes to do so in the least amount of time. The roads between the schools are located as shown in **Figure 5.3.5**, with the distances (in miles) between the schools given by the indicated values. A weighted graph representing this situation is also shown, and the shaded circuit below it is one Hamilton circuit that the superintendent could use.

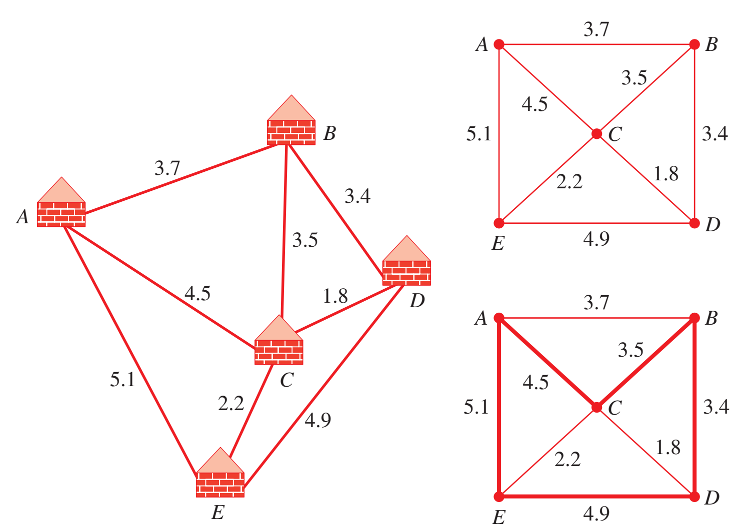


FIGURE 5.3.5 A weighted graph.

If we add up the weights along the edges of the circuit *ACBDEA*, then this total weight corresponds to the mileage associated with this particular route. But do other routes exist that would have a smaller total? Three other Hamilton circuits are shown in **Figure 5.3.6**.

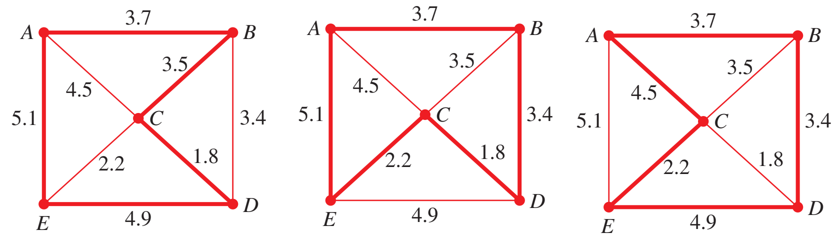


FIGURE 5.3.6 Three more Hamilton circuits.

The total weights of all four circuits are as follows:

$$ACBDEA\text{:}\ 4.5+3.5+3.4+4.9+5.1=21.4$$
$$ABCDEA\text{:}\ 3.7+3.5+1.8+4.9+5.1=19.0$$
$$ABDCEA\text{:}\ 3.7+3.4+1.8+2.2+5.1=\mathbf{16.2}$$
$$ABDECA\text{:}\ 3.7+3.4+4.9+2.2+4.5=18.7$$

We conclude that the path *ABDCEA* has the shortest total weight and therefore provides the optimal solution to this problem. In Figure 5.3.6, *ABDCEA* is a minimal Hamilton circuit.

brute-force algorithm A problem-solving procedure in which the total weight of every Hamilton circuit is computed to determine an optimal circuit.

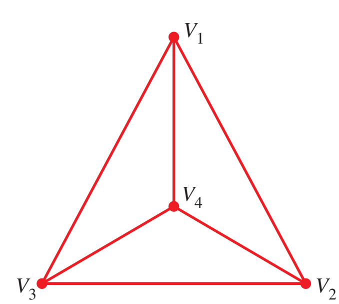
The previous example demonstrates the use of the **brute-force algorithm** to find an optimal Hamilton circuit. According to this method, *every* existing Hamilton circuit is identified, and the total weight is computed. A minimal circuit is then an optimal solution. (There can be more than one.) In practice, however, this method has a serious drawback. Most graphs used in applications consist of a large number of vertices and therefore possess a very large number of Hamilton circuits. Finding all of them and calculating their total weights can be a time-consuming task even in the modern world of powerful high-speed computers. The brute-force method can take hours or days of computer time depending on the size of the graph, and therefore, it is not economically viable. As a result, other algorithms are sought by mathematicians that can determine solutions that may not necessarily be optimum but are still useful and have the advantage that can they can be found quickly. Such procedures are known as **heuristic algorithms** and sometimes are colloquially described as "quick and dirty." They have great value in the world of business, where the perfect solution often may not be worth the cost of finding it.

heuristic algorithm A problem-solving procedure that produces a solution quickly. However, this solution may not be optimal.

A complete graph (on n vertices) was defined in the previous section to be the graph K_n with n vertices, in which every pair of vertices is joined by an edge. We now explore the question of how many Hamilton circuits are contained in K_n by considering the example of K_4.

Example 3

We wish to compute the number of circuits in K_4 that visit every vertex. For convenience, we label the vertices v_1, v_2, v_3, and v_4 and choose one of them, say, v_1, as the starting point. We then have three choices for the second node in the circuit, and after picking one, we would have two choices for the third. After selecting one of those, we have just a single node left. These would give us $3 \cdot 2 \cdot 1 = 6$ circuits. However, because we are not concerned with the



direction traveled along any circuit, we note that *both* the sequences $v_1v_2v_3v_4v_1$ and $v_1v_4v_3v_2v_1$ represent the same circuit, namely, the one in **Figure 5.3.7(a)**. Similarly, the sequences $v_1v_2v_4v_3v_1$ and $v_1v_3v_4v_2v_1$ both represent the path in **Figure 5.3.7(b)**, and **Figure 5.3.7(c)** displays both $v_1v_4v_2v_3v_1$ and $v_1v_3v_2v_4v_1$. Therefore, we must divide our previous result by 2 to get a total of three Hamilton circuits in K_4. ♦

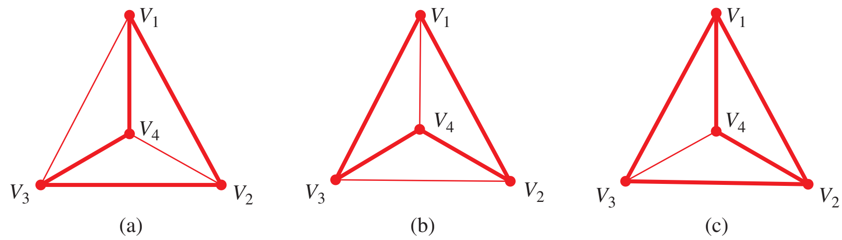


FIGURE 5.3.7 The three Hamilton circuits in K_4.

The process used in Example 3 can be generalized to computing the number of Hamilton circuits in K_n. After picking a starting point for a sequence of vertices, we would have $n - 1$ choices for the second node, $n - 2$ for the third, and so forth. Then, as in Example 3, the product $(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$ gives us the total number of sequences formed. Recalling that every sequence and the corresponding reverse-order sequence represent the same circuit, we then divide this product by 2 to obtain the final number of Hamilton circuits. Factorial notation allows us to summarize this result. The factorial of a positive integer m is denoted by $m!$ and is defined to be the product of m and all the positive integers less than m. For example, $5! = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ and $6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$.

The total number of Hamilton circuits in the complete graph K_n is $(n-1)!/2$.

Traveling Salesperson Problem (TSP) The problem of finding a minimum-weight Hamilton circuit in a complete weighted graph.

One historically well-known problem involving complete weighted graphs is the **Traveling Salesperson Problem** (TSP). This deals with the problem a salesperson faces when required to visit n different cities in the most cost-efficient manner. Assuming that transportation exists between any two cities, we see that the optimal solution to this problem concerns finding a minimal Hamilton circuit in a weighted K_n.

Example 4

A salesperson based in Seattle needs to visit Salt Lake City, Houston, Dallas, and Memphis. What sequence of trips would consist of the least total distance traveled? In other words, find the optimal solution to the TSP for these five cities.

Solution

The distances between the given cities can be easily found on the Internet.

From	To	Distance (miles)
Seattle	Salt Lake City	700
Seattle	Houston	1,891

From	To	Distance (miles)
Seattle	Dallas	1,683
Seattle	Memphis	1,872
Salt Lake City	Houston	1,201
Salt Lake City	Dallas	1,003
Salt Lake City	Memphis	1,256
Houston	Dallas	224
Houston	Memphis	484
Dallas	Memphis	419

The complete graph on five vertices with the appropriate weights is given in **Figure 5.3.8**. There exist $4!/2 = (4 \cdot 3 \cdot 2 \cdot 1)/2 = 12$ Hamilton circuits. Each must be identified and its total weight evaluated. For example, the circuit *SLMDHS* has weight given by $700 + 1{,}256 + 419 + 224 + 1{,}891 = 4{,}490$. Similarly, *SLMHDS* has weight $700 + 1{,}256 + 484 + 224 + 1{,}683 = 4{,}347$.

The weights of all the circuits are shown in the following table.

Circuit	Total Weight
SLMDHS	4,490
SLMHDS	4,347
SLDMHS	4,497
SLDHMS	**4,283**
SLHMDS	4,487
SLHDMS	4,416
SMDLHS	6,386
SMHLDS	6,243
SMLDHS	6,246
SDMLHS	6,450
SDHLMS	6,038
SDLMHS	6,317

The optimal solution is the minimal Hamilton circuit *SLDHMS*. ♦

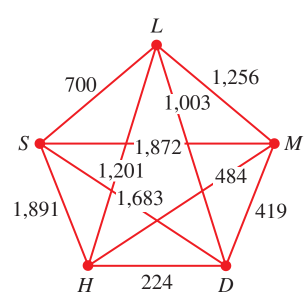


FIGURE 5.3.8 Complete graph for the TSP.

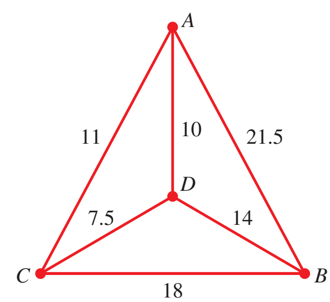
nearest-neighbor algorithm A heuristic algorithm for forming a Hamilton circuit that proceeds from one vertex to an adjacent vertex along the lowest-weight edge that does not complete a circuit until it returns to the initial vertex.

Finding all the Hamilton circuits, even for only five vertices, is an arduous task. You can imagine how much more difficult that process becomes as the number of vertices increases. Therefore, a quicker heuristic algorithm is needed for many practical applications. The **nearest-neighbor algorithm** is just such a method. It is procedure for forming a Hamilton circuit that usually has a smaller weight than most other circuits, although it may not be a minimum. Basically, you travel from vertex to vertex along a low-weight edge, being careful not to form a circuit until you return to your starting point. Specifically, it works like this:

1. Choose a starting vertex. Of the edges that meet at that vertex, choose the one of lowest weight, and travel along it to the next vertex. (In case of a tie, pick either one.)
2. Continue this process, being careful not to add an edge that completes a circuit until you return to your initial vertex.

Example 5

A meat distributor, located at *A,* needs to make a delivery to three different markets every morning before they open for business. He wishes to do this in the shortest time. The time (in minutes) to travel between each pair of stores is given in the accompanying graph. Use the nearest-neighbor algorithm to find a solution.



Solution

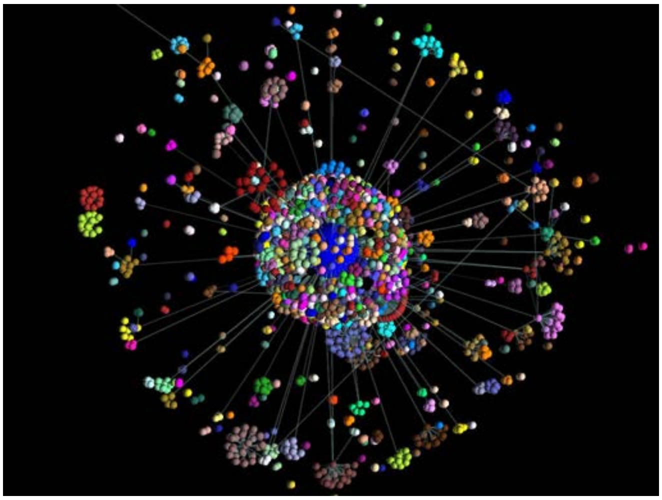
Starting at *A*, we proceed first to *D*, because edge *AD* has the lowest weight (10). From *D*, we have two choices for the next edge, and we choose *DC* because it has a lower weight (7.5) than *DB*. From *C* we must choose *CB* (18) next even though it has a greater weight than *CA* because if we proceed along *CA*, we will complete a circuit that does not visit each vertex. Finally, from *B* we have only *BA* (21.5) to select for the last edge in our circuit. The circuit *ADCBA* has total weight $10 + 7.5 + 18 + 21.5 = 57$. ♦

We make two observations about Example 5:

1. The nearest-neighbor solution depends on the starting point. If we start at *B* instead of *A* in Example 5, then we first travel to *D,* because edge *BD* has a lower weight (14) than either *BA* or *BC.* From *D* we choose edge *DC* (7.5), then *CA* (11)*,* and finally back to *B* via *AB* (21.5). This gives us the circuit *BDCAB*, which, after it is rewritten as *ABDCA,* is a different solution from the one in Example 5. It has a total weight of $14 + 7.5 + 11 + 21.5 = 54$.
2. The brute-force algorithm can produce a different solution from that found with the nearest-neighbor method. The three Hamilton circuits in Example 5 are *ADCBA*, *ADBCA*, and *ABDCA*, having total weights 57, 53, and 54, respectively. Hence, circuit *ADBCA* is the solution found using the brute-force method and indeed is the optimal one. For small problems where the search time is not a factor, the brute-force method is preferred. However, most real-life applications rarely deal with graphs having a small number of circuits.

It is important to observe that it need not be the case that corresponding edges of isomorphic graphs have the same weights. Our definition of isomorphism in the first section did not require that edge weights be preserved. Therefore, even if *G* and *G′* are isomorphic graphs, a minimum-weight Hamilton circuit in *G* would not have to also be an optimal solution in G' (although it is true that a Hamilton circuit in one must certainly correspond to a Hamilton circuit in the other).

Researchers in many fields continue to discover new and powerful ways to employ graph theory in the continuing quest to understand our environment. In the article "100 Trillion Connections" in the January 2011 issue of *Scientific American,* Carl Zimmer explains how networks are used to examine the inner workings of the 100 billion neurons (vertices) and their attendant 100 trillion connections (edges) that compose the architecture of the human brain. Graphs and subgraphs are identified that become active when humans perform a particular task and show how faulty connections can lead to devastating diseases such as schizophrenia or dementia. **Figure 5.3.9** illustrates one such large-scale network.



Courtesy of Olaf Sporns, PhD.

FIGURE 5.3.9 100 trillion connections: Efforts to understand the human brain's architecture require graph theory. *Scientific American*, January 2011.

It is hoped that this chapter has convinced you of the importance of graph theory. From areas such as street maintenance to the complexities of the central nervous system, the use of graphs as a mathematical model provides an effective tool for solving a wide range of difficult and practical problems. As the nature of problems confronting society continues to evolve, the flexibility and power of mathematics arm us with an effective tool for meeting new challenges.

Name ____________________

Exercise Set **5.3**

Identify the statements in Exercises 1–11 as true or false. Let G and G′ be graphs.

1. Every connected graph must have a Hamilton circuit.

2. If G and G' are isomorphic to each other and G has a Hamilton circuit, then so does G'.

3. If G and G' both do not have a Hamilton circuit, then they must be isomorphic to each other.

4. Let G and G' be isomorphic. If G has a minimum-weight Hamilton circuit H, then the corresponding Hamilton circuit H' in G' also has a minimum weight.

5. Let G and G' be isomorphic. If H is a Hamilton circuit in G, then the corresponding closed path H' in G' must also be a Hamilton circuit.

6. The complete graph K_6 contains 60 Hamilton circuits.

7. A heuristic algorithm always finds the optimal solution.

8. The brute-force algorithm is heuristic.

9. The brute-force algorithm always finds an optimal solution.

10. The nearest-neighbor solution depends on the starting point.

11. If $\delta(v) = 2$ for every vertex v in a connected graph G, then G has a Hamilton circuit.

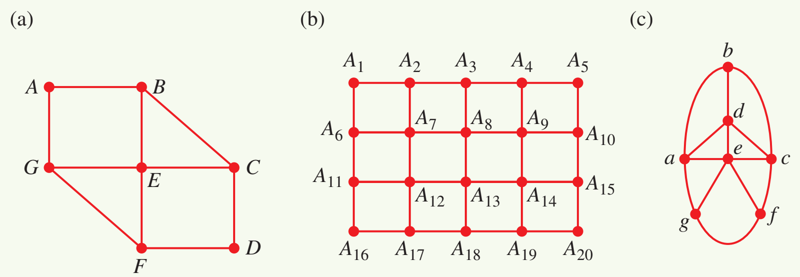
12. Draw the graph from Example 1 whose vertices correspond to the same major exhibits in the Lincoln Park Zoo shown here. Find another Hamilton circuit that is different from the one found in the text.

Adapted from Lincoln Park Zoo

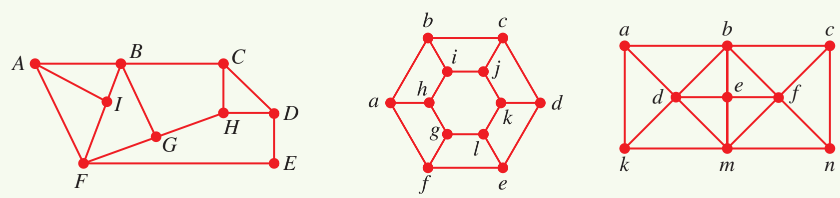
13. Use the graph from the previous exercise to find two closed paths that begin and end at the cafe and that stop once each at the exhibits of bears, giraffes, penguins, sea lions, and primates.

14. Use the graph from Exercise 1 to find two closed paths that begin and end at the cafe and that stop once each at the exhibits of penguins, bears, primates, swans, and elephants.

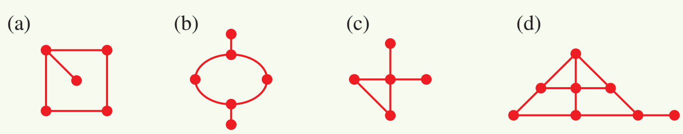
15. Find two Hamilton circuits in each of the following graphs.



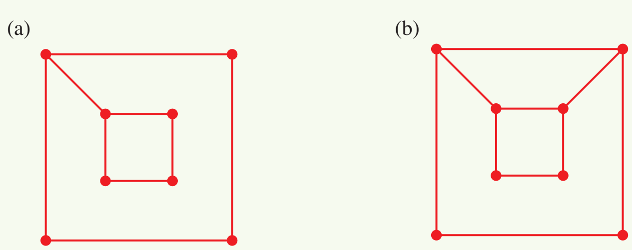
16. Find two Hamilton circuits in each of the following graphs.



17. Do any of the following graphs have a Hamilton circuit? Why not? State a condition that would prevent a graph from having a Hamilton circuit.

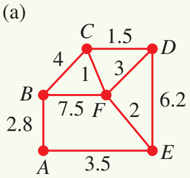


18. Which of the following graphs does not have a Hamilton circuit? State a condition that would prevent a graph from having a Hamilton circuit.

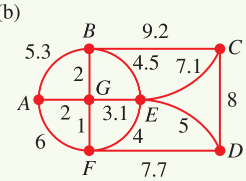


19. Find a Hamilton solution to the following graphs (starting from A), using both the brute-force algorithm and the nearest-neighbor algorithm. Do you get the same solution with both methods?

(a)

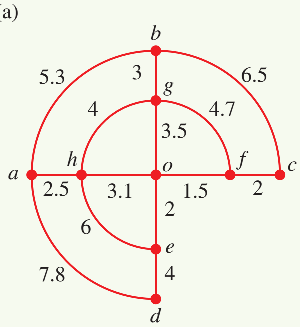


(b)

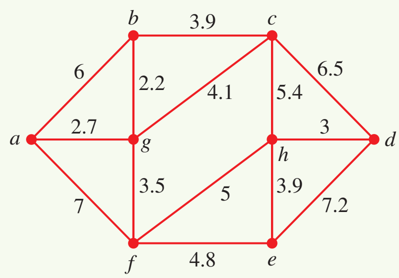


20. Find a Hamilton solution to the following graphs (starting from a), using both the brute-force algorithm and the nearest-neighbor algorithm. Do you get the same solution with both methods?

(a)



(b)



Solve the TSP in each of the following exercises, using the distances given in the following table.

From	To	Distance (miles)
Denver	Boston	1,766
Denver	Atlanta	1,204
Denver	New Orleans	1,078
Denver	Minneapolis	693
Boston	Atlanta	938
Boston	New Orleans	1,348
Boston	Minneapolis	1,126
Atlanta	New Orleans	412
Atlanta	Minneapolis	905
New Orleans	Minneapolis	1,043

21. Starting in Denver, find a solution to the TSP for the cities of Denver, Atlanta, Boston, and New Orleans, using:

(a) the brute-force algorithm.
(b) the nearest-neighbor algorithm.

Are these two solutions the same? Which one is optimal?

22. Starting in Atlanta, find a solution to the TSP for the cities of Atlanta, Boston, Denver, and Minneapolis, using:

(a) the brute-force algorithm.
(b) the nearest-neighbor algorithm.

Are these two solutions the same? Which one is optimal?

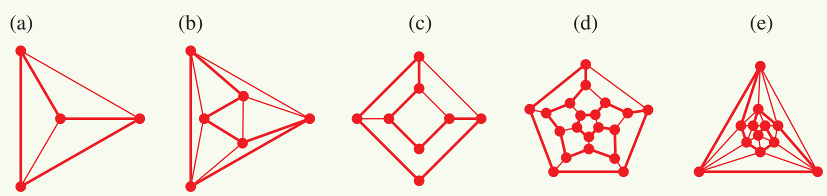
23. Starting in Boston, find the optimal solution to the TSP for the cities of Boston, Atlanta, New Orleans, and Minneapolis.

24. Starting in Denver, find a heuristic solution to the TSP for all five cities: Denver, Boston, Atlanta, New Orleans, and Minneapolis.

25. How many Hamilton circuits are contained in the complete graph K_5? In K_7? In K_{10}?

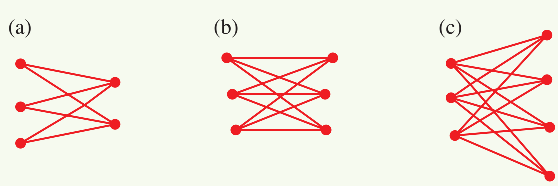
26. What is the maximum whole number n such that $n!$ cannot be computed on your calculator? What type of company might need to use a complete graph K_n for that value of n or greater?

27. A polyhedron is a three-dimensional solid geometrical object, such as a pyramid or soccer ball, that consists of vertices, edges, and plane sides or faces. Only five polyhedrons, known as regular polyhedrons, possess faces that are all congruent to one another. They are the tetrahedron (4 sides), cube (6 sides), octahedron (8 sides), dodecahedron (12 sides), and icosohedron (20 sides). Each of the following graphs represents the projection (compression) of a regular polyhedron onto the plane of the page. Note that each one contains a Hamilton circuit. In each case, find another such circuit in addition to the one indicated. (This was one of Hamilton's original problems.)



28. A **complete bipartite graph** B is one in which the vertices can be sorted into two groups, R and S, such that *every* vertex in R is connected by an edge to *every* vertex in S, and these are the only edges of B. If the number of vertices in R is m and the number of vertices in S is n, then the bipartite graph on R and S is denoted by B_{mn}. The following are graphs for B_{32}, B_{33}, and B_{34}. Which of these contain(s) a Hamilton circuit?

complete bipartite graph A bipartite graph in which every vertex in one vertex group is joined by an edge to every vertex in the second group.



29. Determine a criterion for a complete bipartite graph B_{mn} in terms of m and n that will tell us precisely when B_{mn} contains a Hamilton circuit.

30. In the game of chess, the piece called the knight moves in an L-shaped path. It travels two squares nondiagonally in any direction followed by one square perpendicular to that direction. An interesting problem that dates back more than a century concerns searching for a knight's tour. A *knight's tour* is the name given to a sequence of moves that visits every square on the chessboard exactly once. A tour is also called *closed* or *reentrant* if the terminal square of the tour is one knight's move away from the initial square. The numbered figure here that corresponds to the chessboard depicts one knight's tour on the standard playing board that consists of 64 squares arranged in 8 rows and 8 columns.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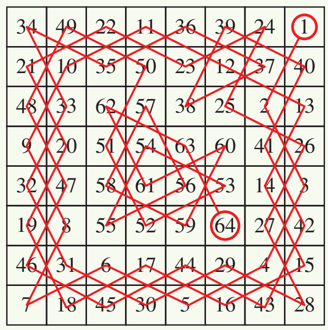
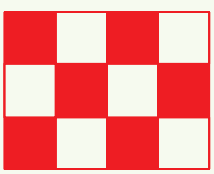
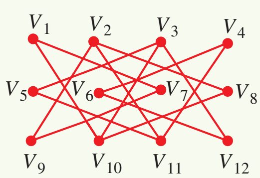


The generalized $m \times n$ knight's tour uses a nonstandard board with m rows (ranks) and n columns (files). A reentrant tour problem can be solved by searching for a Hamilton circuit in a graph in which the vertices $v_1, v_2, \ldots$ are the squares and an edge joins v_i to v_j only if a knight can be moved from v_i to v_j. A 3×4 board and the associated graph are shown here. Can you find a Hamilton circuit?





31. Draw a graph to model the 3×5 knight's tour problem. Does this have a closed tour?

32. Draw a graph to model the 4×5 knight's tour problem. Does this have a closed tour?

33. Schwenk's Theorem states that a closed knight's tour exists on an $m \times n$ graph unless:

(a) m and n are both odd.
(b) $m = 1, 2,$ or 4.
(c) $m = 3$ and $n = 4, 6,$ or 8.

What are the smallest values for m and n for which a closed tour exists on an $m \times n$ graph?

34. Use the map of the southeast United States given here to draw a graph in which each vertex represents a state and two vertices are joined by an edge if the corresponding states share a border. Could you visit every state in this map exactly once and return to where you started? Explain your answer.

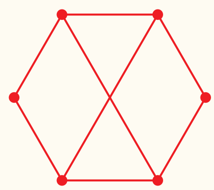
Name ______________________

use the extra space to show your work

Chapter Review Test 5

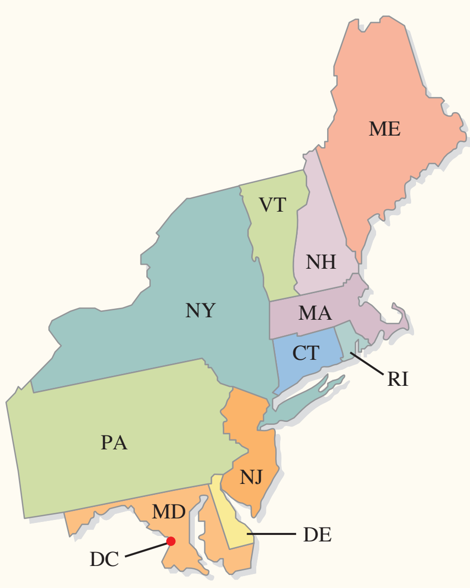
1. True/False. Let G and G' be graphs.

(a) If G is isomorphic to G', then the two graphs must have the same number of edges.
(b) If G and G' have the same number of even vertices, then the two graphs must be isomorphic.
(c) If the number of Euler circuits of G is not equal to the number of Euler circuits of G', then G must be isomorphic to G'.
(d) A graph cannot have an Euler circuit if it is not connected.
(e) A graph cannot have an odd number of odd vertices.
(f) The graph shown here is a planar graph.



(g) The nearest-neighbor algorithm always finds an optimal solution.
(h) The brute-force algorithm is a heuristic method.
(i) A graph may have none, one, or many Hamilton circuits.
(j) A graph has only one minimum Eulerization.

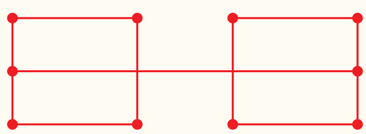
2. Use the map of the northeast United States given here to draw a graph in which each vertex represents a state and two vertices are joined by an edge if the corresponding states share a border. Could you visit every state in this map exactly once and return home? Explain your answer.



3. Fill in the blanks.

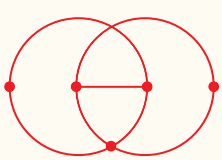
(a) The total number of edges in the complete graph K_{12} is____________.
(b) The total number of Hamilton circuits in K_8 is____________.
(c) The degree sum of a graph with 150 edges is __________.
(d) The number of edges contained in a graph having 500 vertices, each of degree 8, is___________.

4. Draw a graph that is isomorphic to the given graph but does not contain any crossings.

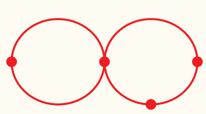


5. Which of the following graphs, (a) to (d), is isomorphic to graph (e)?

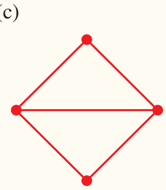
(a)



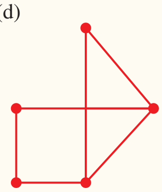
(b)



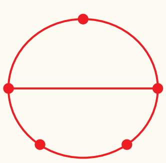
(c)



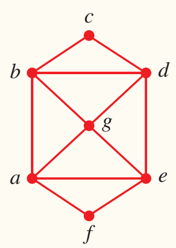
(d)



(e)



6. Does the graph here contain an Euler circuit? If so, find one, and list it as a sequence of vertices.



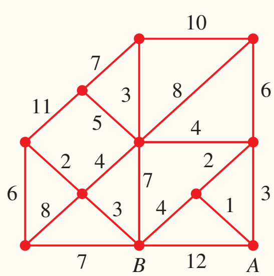
7. Suppose that the path *ABCDCEBA* is an Euler circuit in a particular graph.

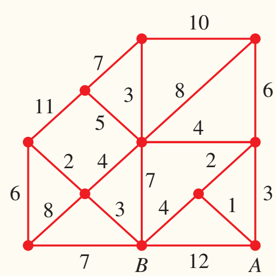
(a) What are the degrees of the vertices in this graph?
(b) Draw a graph for which this path would be an Euler circuit.

8. Consider a trip that starts in Scranton, Pennsylvania. Find the optimal solution to the TSP for the following cities. Be sure to draw the complete graph.

From	**To**	**Distance (miles)**
Scranton	Philadelphia	104
Scranton	Miami	1,111
Scranton	Lexington	522
Philadelphia	Miami	1,015
Philadelphia	Lexington	509
Lexington	Miami	873

9. Use the graph shown here to find the nearest-neighbor solution (by darkening the appropriate edges) and its total weight for a Hamilton circuit starting at vertex *A*. Then do the same, starting at *B*.





10. Define a bipartite graph. Give a problem that might need such a graph for its solution.

11. Draw an appropriate graph to model the 3×4 knight's tour problem. Be sure to give a clear definition of a vertex and an edge.