Chapter Objectives

The material in this chapter is previewed in the following list of objectives. After completing this chapter, review this list again, and complete the self-test.

2.1
- 2.1 In your own words, explain the process of problem solving, and describe mathematical modeling.
- 2.2 State the substitution property.
- 2.3 Define similar triangles.
- 2.4 State the proportional property of similar triangles.
- 2.5 Solve applied problems.

2.2
- 2.6 Define a function. Describe a function by rule, table, equation, graph, mapping, or set of ordered pairs. Use the vertical line test.
- 2.7 Be able to classify examples as one-to-one. Use the horizontal line test.
- 2.8 Distinguish between f and $f(x)$ notation. Use functional notation.
- 2.9 Find the difference quotient for a given function f.
- 2.10 Classify functions.
- 2.11 Use functions to model problems.

2.3
- 2.12 Know the graphs of the standard functions in Table 2.1, page 104.
- 2.13 Find the domain and range for a curve whose equation is given.
- 2.14 Determine when functions are equal.
- 2.15 Be able to find the x- and y-intercepts for a curve whose equation is given.
- 2.16 Find points satisfying specified conditions.
- 2.17 Determine where a function is increasing (graph rising), where it is decreasing (graph falling), where it is constant (graph horizontal), and locate the turning points.
- 2.18 Classify functions as even, odd, or neither.

2.4
- 2.19 Find the shift (h, k) when given an equation $y - k = f(x - h)$.
- 2.20 Given $y = f(x)$ and (h, k), write $y - k = f(x - h)$.
- 2.21 Given a function defined by $y = f(x)$, draw the graph of $y - k = f(x - h)$.
- 2.22 Given a function defined by $y = f(x)$, draw the graph's reflections, compressions, and dilations.

2.5
- 2.23 Define a piecewise function.
- 2.24 Graph a piecewise function.
- 2.25 Graph an absolute value function; translate an absolute value function.
- 2.26 Graph a greatest integer function; translate a greatest integer function. Graph a rounding up function.
- 2.27 Model problems using piecewise functions.

2.6
- 2.28 Find the composition of functions.
- 2.29 Express a given function as the composite of two functions using an inner and an outer function.
- 2.30 Find the sum, difference, product, and quotient functions.
- 2.31 Use composition to write functional iteration.

2.7
- 2.32 Given two functions, decide whether they are inverses.
- 2.33 Given a one-to-one function, find its inverse.
- 2.34 Graph a function, its inverse, and the line $y = x$ on the same coordinate axes.
- 2.35 Evaluate a function and its inverse by looking at a graph.

2.8
- 2.36 State the informal definition of the limit of a function.
- 2.37 Estimate limits by graphing (geometrical method).
- 2.38 Estimate limits by table (numerical method).
- 2.39 Evaluate limits of polynomials (analytic method).
- 2.40 Define continuity, and describe the concept in your own words.
- 2.41 Find suspicious points.
- 2.42 State and use the root location property.

Functions with Problem Solving

2

Solving problems is a practical art, like swimming, or skiing, or playing the piano; you can learn it only by imitation and practice . . . if you wish to learn swimming you have to go into the water, and if you wish to become a problem solver you have to solve problems.

—George Pólya
Mathematical Discovery, Vol. 1
New York: John Wiley and Sons, 1962, p. v

Chapter Sections

CALCULUS PERSPECTIVE

In this chapter, we introduce a concept that is not only basic to this course but also basic to a major portion of a calculus course, namely, the concept of a function. In the calculus perspective from Chapter 1, we mentioned that the concept of a limit is central to the study of calculus. In this chapter, we present an intuitive introduction to limits in a particularly easy-to-understand setting. In this chapter, we introduce the notions of problem solving, functions, properties of functions, as well as limits and continuity.

2.1 Problem Solving

Pólya's Problem-Solving Procedure

We begin this study of problem solving by looking at the *process* of problem solving. As a mathematics teacher, I often hear the comment, "I can do mathematics, but I can't solve word problems." There *is* a great fear and avoidance of "real-life" problems because they do not fit into the same mold as the "examples in the book." It is easier for instructors to teach "word problems" when they fit into some mold, but by definition, that does not constitute genuine problem solving. Few practical problems from everyday life come in the same form as those you study in school. All the meaningful problems in calculus involve problem-solving skills. To compound the difficulty, learning to solve problems takes time. All too often, the mathematics curriculum is so packed with content that the real process of problem solving is slighted and because of time limitations becomes an exercise in mimicking the instructor's steps, instead of developing an approach that can be used long after the final examination is over.

The model for problem solving that we will use was first published in 1945 by the great, charismatic mathematician George Pólya. His book *How to Solve It* (Princeton University Press, 1971) has become a classic and has been reprinted several times since 1945.

In Pólya's book, you will find this problem-solving model as well as a treasure trove of strategy, know-how rules of thumb, good advice, anecdotes, history, and problems at all levels of mathematics. His problem-solving model is as follows:

Historical Note

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George Pólya (1887–1985)

Pólya was born in Hungary, attended the universities of Budapest, Vienna, Göttingen, and Paris, and was a professor of mathematics at Stanford University. Pólya's research and winning personality earned him a place of honor not only among mathematicians but also among students and teachers. His discoveries spanned an impressive range of mathematics: real and complex analysis, probability, combinatorics, number theory, and geometry. His book *How to Solve It* has been translated into 15 languages. His books have a clarity and elegance seldom seen in mathematics, making them a joy to read. For example, here is his explanation of why he became a mathematician: "It is a little shortened but not quite wrong to say: I thought I am not good enough for physics and I am too good for philosophy. Mathematics is in between."

PROBLEM-SOLVING GUIDE

Pólya's problem-solving guideline for problem solving:

Step 1 You have to *understand the problem.*

Step 2 *Devise a plan*. Find the connection between the data and the unknown. Look for patterns, relate to a previously solved problem or a known formula, or simplify the given information to obtain an easier problem.

Step 3 *Carry out the plan.*

Step 4 *Look back.* Examine the solution obtained.

Pólya's original statement of this procedure is reprinted on the inside front cover of this book.

Problem solving is a difficult task to master, and you are not expected to master it after one section of this book (or even after one mathematics course). Don't think you can avoid problem solving by skipping this section. Problem solving is one of the major threads with which this course is woven. You will be challenged to solve problems in nearly every section of this book.

One of the most important aspects of problem solving is to relate new problems to old ones. The problem-solving techniques outlined here should be applied when you are faced with a new problem. When you are faced with a question similar to one you have already worked, you can apply previously developed techniques.

We begin our journey toward problem solving by briefly introducing the two principal types of reasoning used in mathematics.

Inductive Reasoning

The type of reasoning—first observing patterns and then predicting answers for more complicated problems—is called **inductive reasoning**. It is a very important method of reasoning in problem solving and in using Pólya's method. With inductive reasoning, the results are not certain, only probable. These predictions can be checked or otherwise verified. One method of *proving* such conjectures is called **mathematical induction**, which we will discuss in Appendix C.

EXAMPLE 1 **Using patterns** MODELING APPLICATION

What is the sum of the first 100 positive consecutive odd numbers?

Solution

Step 1: *Understand the problem.* Do you know what the words mean?

Odd numbers are $1, 3, 5, \ldots$, and *sum* means to add:

$$1 + 3 + 5 + \cdots + ?$$

The first thing you need to understand is what the last term will be, so you will know when you have reached 100 consecutive odd numbers.

$1 + 3$ *is two terms.*

$1 + 3 + 5$ *is three terms.*

$1 + 3 + 5 + 7$ *is four terms.*

It seems as if the last term is always one less than twice the number of terms. Thus, the sum of the first 100 consecutive terms is

$$1 + 3 + 5 + \cdots + 195 + 197 + 199$$

This is one less than 2(100).

Step 2: *Devise a plan.* The plan we will use is to look for a pattern:

$1 = 1$ One term

$1 + 3 = 4$ Sum of two terms

$1 + 3 + 5 = 9$ Sum of three terms

Do you see a pattern yet? If not, continue:

$1 + 3 + 5 + 7 = 16$

$1 + 3 + 5 + 7 + 9 = 25$

Step 3: *Carry out the plan.* It looks like the sum of two terms is 2^2; of three terms, 3^2; of four terms, 4^2; and so on. The sum of the first 100 consecutive odd numbers therefore seems to be 100^2.

Step 4: *Look back.* Does $100^2 = 10{,}000$ seem correct?

Deductive Reasoning

Another method of reasoning used in mathematics is called **deductive reasoning**. This method of reasoning produces results that are *certain* within the logical system being developed. That is, deductive reasoning involves reaching a conclusion by using a formal structure based on a set of **undefined terms** and a set of accepted, unproved **axioms** or **premises**. The conclusions are said to be proved and are called **theorems**.

The most useful axiom in problem solving is the principle of substitution.

SUBSTITUTION PROPERTY

If $a = b$, then a may be substituted for b in any mathematical statement without affecting the truth or falsity of the given mathematical statement.

» IN OTHER WORDS If two quantities are equal, you can remove one of the quantities and replace it with the other.

The simplest way to illustrate the substitution property is to use it in evaluating a formula.

EXAMPLE 2 Evaluating a formula

© Planetpix/Alamy Images

A water tank is in the shape of an inverted circular cone resting on top of a cylinder. The water is stored in the cylindrical part with base radius 2 m and height 4 m. How much water is in the tank when the water is 3 m deep?

Solution Sometimes some outside information from a previous course is required when solving problems. For this example, you need the formula for the volume of a cone:

$$V = \frac{1}{3}\pi r^2 h$$

Substitute the known values into the formula:

$$V = \frac{1}{3}\pi\left(\overset{r=2}{\overset{\downarrow}{2^2}}\right)\left(\underset{h=3}{\underset{\uparrow}{3}}\right)$$

In this book, we will use arrows to show substitution.

$$= 4\pi$$

The volume is $4\pi\ \text{m}^3$. Notice that we state our answer in sentence form (with units). Also notice that we do not round our answer unless a rounded answer is somehow indicated in the statement of the problem, or our only alternative is to work with a calculator or rounded results. ■

The formal study of deductive reasoning is beyond the scope of this course. However, there are certain principles and terminology associated with proof and deductive reasoning that you will encounter in calculus. For example, in Chapter 1 we discussed the meaning of the words *if and only if*, and how a mathematical statement

$$p \text{ if and only if } q$$

requires two proofs:

$$(1) \text{ if } p, \text{ then } q \quad (2) \text{ if } q, \text{ then } p$$

There is some associated terminology. If p and q are any propositions, then the statement "if p, then q," written $p \to q$, is called a **conditional** and is translated several (equivalent) ways:

Conditional Translation	Example
If p, then q.	If you are 18, then you can vote.
q, if p.	You can vote, if you are 18.
p, only if q.	You are 18 only if you can vote.
All p are q.	All 18-year-olds can vote.

In addition to these translations for the conditional, there are related statements.

LOGICAL STATEMENT

Given the conditional $p \to q$, we define
the **converse** is: $q \to p$;
the **inverse** is: not $p \to$ not q;
the **contrapositive** is: not $q \to$ not p.

EXAMPLE 3 **Logical statements**

Let p be the proposition: "It is a 300ZX" and q be the proposition: "It is a car." The given statement, symbolized by $p \to q$, is "If it is a 300ZX, then it is a car." State the converse, inverse, and contrapositive.

Solution

Converse:	$q \to p$	If it is a car, then it is a 300ZX.
Inverse:	not $p \to$ not q	If it is not a 300ZX, then it is not a car.
Contrapositive:	not $q \to$ not p	If it is not a car, then it is not a 300ZX.

As you can see from Example 3, not all these statements are equivalent in meaning. The contrapositive and the original statement always have the same truth values, as do the converse and the inverse. We accept the following law, which is frequently used not only in calculus but in all of mathematics.

LAW OF CONTRAPOSITION

A conditional may always be replaced by its contrapositive without having its truth value affected.

Mathematical Modeling

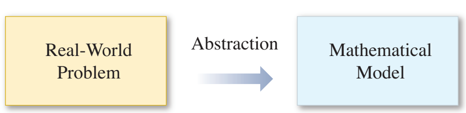
Problem solving depends not only on the substitution property but also on translating statements from English to mathematical symbols. On a much more advanced level, this process is called **mathematical modeling** and involves matching mathematical skills with knowledge about the real world. A characteristic of mathematical modeling is that it is *iterative* (requires assumptions, feedback, and revision). It often involves solving problems with either too little information (some further research is needed) or too much information (you have information that is not needed to solve the problem).*

Mathematical models are formed, modified by experimentation and the accumulation of data, and are then used to predict some future occurrence in the real world. Such mathematical models are continually being revised and modified as additional relevant information becomes known.

Some mathematical models are quite accurate—particularly those used in the physical sciences. For example, one of the first models we consider in calculus is the path of a projectile. Other models predict the time of sunrise and sunset or the distance that an object falls in a vacuum. Other modeling, however, is less accurate—in particular when examples from the life sciences and social sciences are chosen. Only recently has modeling in these disciplines become precise enough to be included in a mathematics course.

What, precisely, is a mathematical model? At the low end of the spectrum, mathematical modeling can mean nothing more than real-life word problems. At the high end of the spectrum, mathematical modeling can mean choosing appropriate mathematics to solve a problem that has previously been unsolved.

The first step of what we call mathematical modeling involves *abstraction.*



With the method of abstraction, certain assumptions about the real world are made, variables are defined, and appropriate mathematics is developed. The next step is to simplify the mathematics or derive related mathematical facts from the mathematical model.

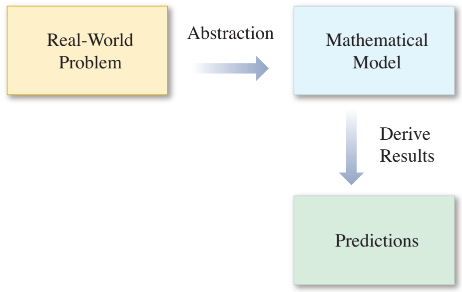
Historical Note

Mathematical modeling plays an important role in sporting events. For example, in 1992, preparing for the America's Cup competition included the following mathematical models:

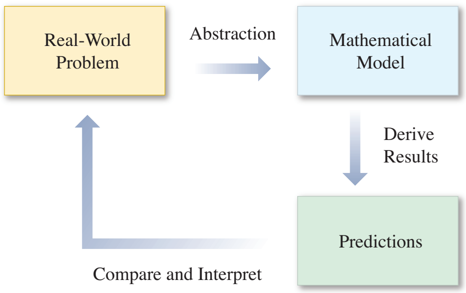
- Race Modeling Programs
- Computational Fluid
- Computer-Aided Design
- Experimental Design
- Dynamic Modeling Program

Scientific American, May 1992

*In a classroom setting, it is not practical to give too little information in a problem (unless it is designated as a research problem). However, in the real world, it is common to need additional information to find a solution. Since problem solving involves a long learning process, we will only occasionally give you "extra" information. Our goal in this book is to build your problem-solving skills to bring you to a level of competency that will allow you to know when you have too much or too little information.



The results derived from the mathematical model should lead us to some predictions about the real world. The next step is to gather data from the situation being modeled and then to compare those data with the predictions. If the two do not agree, then the gathered data are used to modify the assumptions used in the model.



Mathematical modeling is an ongoing process. As long as the predictions match the real world, the assumptions made about the real world are regarded as correct, as are the defined variables. On the other hand, as discrepancies are noticed, it is necessary to construct a closer and a more dependable mathematical model.

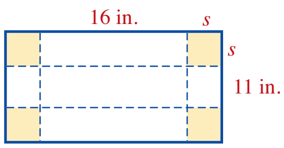
Throughout this book, we will designate certain problems MODELING APPLICATION to designate problem-solving or mathematical modeling examples. You may have noticed Example 1 of this section was so designated. Mathematical modeling necessarily involves abstraction from the real world, deriving results, making predictions, and then comparing and interpreting in the real world. As an example, consider a problem we have adapted from calculus.

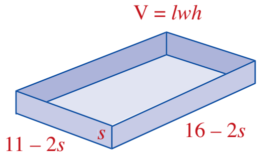
EXAMPLE 4 A volume problem from calculus **MODELING APPLICATION**

A container is to be constructed from an 11 in. by 16 in. sheet of cardboard. Squares will be cut from the corners of the sheet and discarded as waste. The domain for the variable must be restricted so that the area of the base of the container exceeds (or is equal to) the area of wasted cardboard. We will revisit this problem in Chapter 3 when we consider optimization problems to ask for the maximum volume.

Solution

Step 1: *Understand the problem.* Let s be the length of the side of a square that is cut from the cardboard.





The domain requires that all sides be nonnegative. That is,

$$s \geq 0 \qquad 16 - 2s \geq 0 \qquad 11 - 2s \geq 0$$
$$s \leq 8 \qquad s \leq 5.5$$

Step 2: *Devise a plan.* We will do a numerical, algebraic, and if we have a graphing calculator, geometrical analysis.

Step 3: *Carry out the plan.*

Numerical (tabular) approach: To get some idea about the problem, a numerical approach is often helpful. For this approach, let us assume that the dimensions must be integers. With this modeling assumption, there are few values for s, and we can easily calculate the domain conditions (as well as the volume) of the six possible boxes. This is particularly easy to do if you use a calculator or computer.

Value of s	Length $16 - 2s$	Width $11 - 2s$	Height s	Area of Base (LENGTH)(WIDTH)	Area of Waste $4s^2$	Conditions Satisfied	Volume
0	16	11	0	176	0	Yes	0
1	14	9	1	126	4	Yes	126
2	12	7	2	84	16	Yes	168
3	10	5	3	50	36	Yes	150
4	8	3	4	24	64	No	96
5	6	1	5	6	100	No	30

It looks like the domain for integer values s is $0 \leq s \leq 3$. We also note that for integer values the maximum volume is 168 in.^{3}, which occurs for a box with dimensions 12 in. $\times$ 7 in. $\times$ 2 in.

Algebraic approach: We note from the question that the area of the base of the container should exceed (or be equal to) the area of the wasted cardboard. Thus:

$$\text{AREA OF BASE} \geq \text{AREA OF WASTE}$$
$$(\text{LENGTH OF BASE})(\text{WIDTH OF BASE}) \geq 4(\text{AREA OF CORNER})$$

$$(16 - 2s)(11 - 2s) \geq 4s^2$$
$$176 - 54s + 4s^2 \geq 4s^2$$
$$-54s \geq -176$$
$$s \leq \frac{176}{54} = \frac{88}{27}$$

This means the domain for this problem is $0 \leq s \leq 88/27$. By the way, the volume of the box at the endpoint $s = 0$ is 0, and at the endpoint $s = 88/27$ it is:

$$\begin{aligned}\text{VOLUME} &= (\text{WIDTH})(\text{LENGTH})(\text{HEIGHT})\\ &= (11 - 2s)(16 - 2s)(s)\\ &\approx 138.4894579\end{aligned}$$

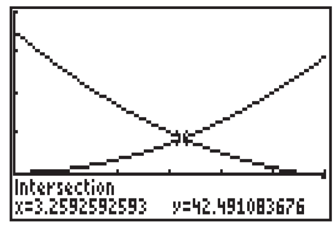
We will see in calculus that the maximum value of *this* box occurs at a value where the derivative is zero or at one of the endpoints of the domain. This means the maximum value could possibly be between $s = 0$ and $s = 88/27$. We will need to wait until we will consider optimization problems in Chapter 3 to know for sure if this is the case.

Geometrical approach: You can use a graphing calculator for the inequality:

$$\text{AREA OF BASE} \geq \text{AREA OF WASTE}$$

$$(16 - 2s)(11 - 2s) \geq 4s^2$$

Since calculators use the variables x and y, you will input Y1 $= (16 - 2X)(11 - 2X)$ and Y2 $= 4X\hat{}2$. Then, look for the intersection of these curves, which is found using the intersection capabilities of the calculator. These steps can be illustrated:




```
\Y1=(16-2X)(11-2X)
\Y2=4X^2
XMin=0    YMin=-50
XMax=6    YMax=200
XScl=1    YScl=50
```

We see that the intersection is at $s \approx 3.2592592593$.

Step 4: *Look back.* The algebraic solution yields an exact answer $(88/27)$, and the graphical method an approximate answer (3.2592592593). These values are both approximately equal to 3.26, so we say they agree. This seems to agree with the intuitively appealing numerical solution for integral values.

Word Problems

Not all word problems are modeling applications, and not all modeling applications are word problems. To do problem solving outside of a classroom setting, you need to have a great deal of practice, sometimes with rather mundane or even trivial questions. The reason for this procedure is to allow you to gain confidence and to build your problem-solving skills as you progress through the course. You might say, "I want to learn how to become a problem solver, and textbook problems are not what I have in mind; I want to do *real* problem solving." But to become a problem solver, you must first learn the basics, and word problems are part of a textbook for good reason. We start with these problems *to build a problem-solving* ***procedure*** *that can be expanded to apply to problem solving in general.* As we do this, however, we will firmly focus on our goal: to build the skills you need to succeed in calculus.

We will now rephrase Pólya's problem-solving guidelines in a setting that is appropriate to solving word problems. This procedure is summarized in the following box.

PROBLEM SOLVING WITH WORD PROBLEMS

Pólya's problem-solving guideline for problem solving can be amplified to solve word problems.

Step 1 *Understand the problem.* This means read the problem. Note what it is all about. Focus on processes rather than numbers. You cannot work a problem you do not understand. A sketch may help in understanding the problem.

Step 2 *Devise a plan.* Write down a verbal description of the problem using operation signs and an equal or inequality sign. Note the following common translations.

PROBLEM SOLVING WITH WORD PROBLEMS *(continued)*

Symbol	Verbal Description
$=$	is equal to; equals; are equal to; is the same as; is; was, becomes; will be; results in
$+$	plus; the sum of; added to; more than; greater than; increased by
$-$	minus; the difference of; the difference between; is subtracted from; less than; smaller than; decreased by; is diminished by
$\times$	times; product; is multiplied by; twice ($2\times$); triple ($3\times$)
$\div$	divided by; quotient of

Step 3 *Carry out the plan.* In the context of word problems, we need to proceed deductively by carrying out the following steps.

Choose a variable. If there is a single unknown, choose a variable. If there are several unknowns, you can use the substitution property to reduce the number of unknowns to a single variable. Later, we will consider word problems with more than one variable.

Substitute. Replace the verbal phrase for the unknown with the variable.

Solve the equation. This is generally the easiest step. Translate the symbolic statement (such as $x = 3$) into a verbal statement. Probably no variables were given as part of the word problem, so $x = 3$ is not an answer. Generally, word problems require an answer stated in words. Pay attention to units of measure and other details of the problem.

Step 4 *Look back.* Be sure your answer makes sense by checking it with the original question in the problem. *Remember to answer the question that was asked.*

EXAMPLE 5 Area problem

Two rectangles have the same width, but one is 40 ft² larger in area. The larger rectangle is 6 ft longer than it is wide. The other is only 1 ft longer than its width. What are the dimensions of the larger rectangle?

Solution

Numerical analysis: If one rectangle is 40 ft² larger in area than the other, then the difference of their areas must be 40. Also, the larger rectangle is 6 ft longer than it is wide. For example, if we start with a width (which must be the same for both rectangles) of 1, then 2, then 3, . . . , we find

First (Larger) Rectangle			Second (Smaller) Rectangle			
Length	Width	Area	Length	Width	Area	Difference of Areas
7	1	7	2	1	2	5
8	2	16	3	2	6	10
9	3	27	4	3	12	15
10	4	40	5	4	20	20
12	6	72	7	6	42	30
13	7	91	8	7	56	35
14	8	112	9	8	72	40

We have apparently found the solution numerically, but it was a great deal of work, and we would have had more difficulty with this method if the answer had not been integral. The larger rectangle is 8 ft wide by 14 ft long.

Algebraic analysis: The areas differ by 40, and each area is the product of length and width. We also note that the widths of both rectangles are the same.

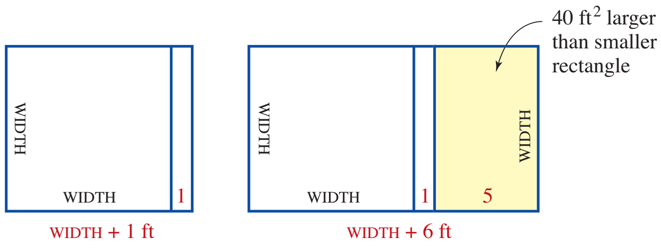
$$\begin{aligned} \text{AREA OF LARGER} - \text{AREA OF SMALLER} &= 40 \\ (\text{LENGTH OF LARGER})(\text{WIDTH}) - (\text{LENGTH OF SMALLER})(\text{WIDTH}) &= 40 \\ (\text{WIDTH} + 6)(\text{WIDTH}) - (\text{WIDTH} + 1)(\text{WIDTH}) &= 40 \end{aligned}$$

Let W be the width of each rectangle. Then,

$$\begin{aligned} (W+6)(W) - (W+1)(W) &= 40 \\ W^2 + 6W - W^2 - W &= 40 \\ 5W &= 40 \\ W &= 8, \; W + 6 = 14 \end{aligned}$$

The larger rectangle is 8 ft wide by 14 ft long.

Geometrical analysis: A sketch will frequently simplify a problem. In this case, since the widths are the same, the difference in area can be seen in the larger rectangle.



The difference is shown as the shaded region. The area of this region is stated in the problem:

$$\begin{aligned} \text{DIFFERENCE IN AREAS} &= 40 \\ (5)(\text{WIDTH}) &= 40 \end{aligned}$$

Let W be the width of each rectangle. Then

$$\begin{aligned} 5W &= 40 \\ W &= 8 \text{ and } W + 6 = 14 \end{aligned}$$

The larger rectangle is 8 ft wide by 14 ft long.

Rate Problems

There are two applications of word problems that are particularly important in calculus. One type of application involves rates, and the other type of application involves similar triangles. A dictionary definition of *rate* is "the quantity of a thing in relation to the units of something else." This definition is quite general; yet, rate is too often limited to rate–time–distance relationships. A typing rate is the number of words typed per unit of time, most prices are based on cost per unit, and interest earned is a part of the principal invested. In calculus, you will find rates to be a frequent application. In everyday usage, we might speak of miles per hour, or mph, but in scientific and mathematical applications, we write this as "mi/h." If it is included as part of the problem calculation, it is written as a fraction

$$\frac{\text{miles}}{\text{hour}}$$

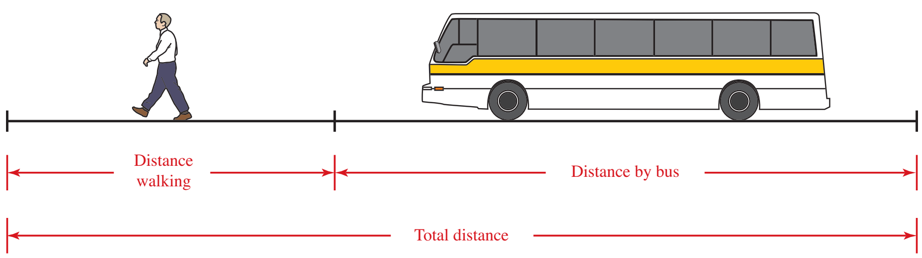
interpreted as a division of miles by hours. Similarly, feet per second in everyday usage is abbreviated as fps; in mathematics it is abbreviated as ft/s and if used as part of the problem calculation is thought of as the fraction feet divided by seconds. However, most elementary applied problems do not require that we include the units of measurement as part of the calculations, but rather are used only in the statement of the solution to the problem. This usage is illustrated in the following example.

EXAMPLE 6 **Rate–time–distance application** **MODELING APPLICATION**

An office worker takes 55 minutes to return from the job each day. This person rides a bus that averages 30 mi/hr and walks the rest of the way at 4 mi/h. If the total distance is 21 miles from office to home, what is the distance walked home each day?

Solution

Step 1: *Understand the problem.* The total trip is composed of two distinct distances. These distances are the product of rate and time. This is the familiar formula $d = rt$.



Step 2: *Devise a plan.* Don't be too eager to choose a variable. Sometimes forcing an incorrect or poorly defined variable at the beginning of a problem often results in disaster. We will use

$$\text{BUS DISTANCE} + \text{WALK DISTANCE} = \text{TOTAL DISTANCE}$$

We know that the total distance is 21 mi, but we do not know the other two distances, so we use the principle of substitution for

$$\text{BUS DISTANCE} = (\text{BUS RATE})(\text{BUS TIME})$$

$$\text{WALK DISTANCE} = (\text{WALK RATE})(\text{WALK TIME})$$

Step 3: *Carry out the plan.* By substitution, we have:

$$\underset{\substack{\uparrow\\30}}{(\text{BUS RATE})}(\text{BUS TIME}) + \underset{\substack{\uparrow\\4}}{(\text{WALK RATE})}(\text{WALK TIME}) = 21$$

We know that $\text{BUS TIME} + \text{WALK TIME} = \frac{55}{60}$ hr
We write 55 min as 55/60 hr. This means we can substitute $\text{BUS TIME} = \frac{55}{60} - \text{WALK TIME}$

$$30\left(\frac{55}{60} - \text{WALK TIME}\right) + (4)(\text{WALK TIME}) = 21$$

Let W be the time walked, in hours. Thus,

$$30\left(\frac{55}{60} - W\right) + 4W = 21$$

$$\frac{55}{2} - 30W + 4W = 21 \qquad \text{Distributive property}$$

$$27.5 - 26W = 21 \qquad \text{Similar terms}$$

$$-26W = -6.5 \qquad \text{Subtract 27.5 from both sides.}$$

$$W = 0.25 \qquad \text{Divide both sides by } -26.$$

The worker walks 0.25 hr, but the question asks for distance, so

$$\text{WALK DISTANCE} = 4(0.25) = 1$$

The worker walks 1 mile daily.

Step 4: *Look back.* If the walker walks 1 mi, then the bus ride is 20 mi and lasts (55 min − 15 min) = 40 min or 2/3 hr. (2/3)(30) = 20 mi, so the solution checks.

Similar Triangles

You will need the idea of *similar triangles* for some calculus and trigonometry problems. Two triangles are similar if they have the same shape (but not necessarily the same size, or area), as shown in Figure 2.1.

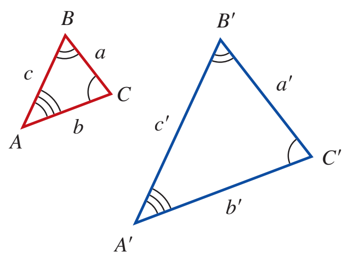


Figure 2.1 Similar triangles

SIMILAR TRIANGLES

ΔABC is similar to $\Delta A'B'C'$ if the corresponding angles are congruent (have the same measure), and we write

$$\Delta ABC \sim \Delta A'B'C'$$

IN OTHER WORDS Since the sum of the measures of any triangle is 180°, it follows that two triangles are similar if two angles of one are equal to two angles of the other.

You will also need to remember the *proportional property of similar triangles*; namely, if two triangles are similar, then the corresponding sides are proportional.

SIMILAR TRIANGLE PROPERTY

If ΔABC is similar to $\Delta A'B'C'$ with sides of lengths as shown in Figure 2.1, then

$$\frac{a}{c} = \frac{a'}{c'} \quad \frac{a}{b} = \frac{a'}{b'} \quad \frac{b}{c} = \frac{b'}{c'} \quad \frac{c}{a} = \frac{c'}{a'} \quad \frac{b}{a} = \frac{b'}{a'} \quad \frac{c}{b} = \frac{c'}{b'}$$

The similar triangle property has a wide variety of applications, some of which are provided in the problem set. We conclude this section with two applications adapted from calculus problems.

EXAMPLE 7 Streetlight problem from calculus

A person 6 ft tall is standing 7 ft from the base of a streetlight. If the light is 20 ft above ground, how long is the person's shadow due to the streetlight?

Solution Let x denote the length (in feet) of the person's shadow, as shown in Figure 2.2. (*Note*: $x > 0$.)

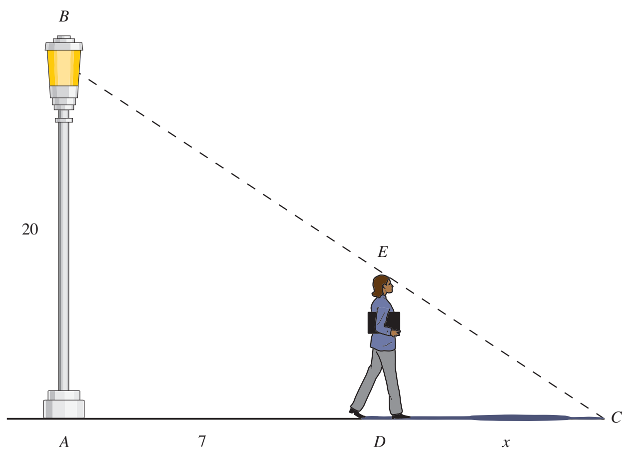


Figure 2.2 Streetlight calculus problem

Note that $\angle A$ and $\angle D$ are right angles, and $\angle C$ is common to both ΔABC and ΔDEC, so we have two angles of one triangle congruent to two angles of the other triangle, thus $\Delta ABC \sim \Delta DEC$. Since the triangles are similar, we have proportional sides, namely:

$$\begin{aligned}
\frac{6}{x} &= \frac{20}{7+x} \\
6(7+x) &= 20x \\
42+6x &= 20x \\
42 &= 14x \\
3 &= x
\end{aligned}$$

The shadow's length is 3 ft.

The last example of this section, which is the famous inverted cone problem from calculus, uses similar triangles.

EXAMPLE 8 Inverted cone problem from calculus

A water tank is in the shape of an inverted cone 20 ft high with a circular base whose radius is 5 ft. How much water is in the tank when the water is 8 ft deep?

Solution The volume of a cone is $V = \frac{1}{3}\pi r^2 h$. We are given $h = 8$ and need to find r. Once again, we use similar triangles as shown in Figure 2.3.

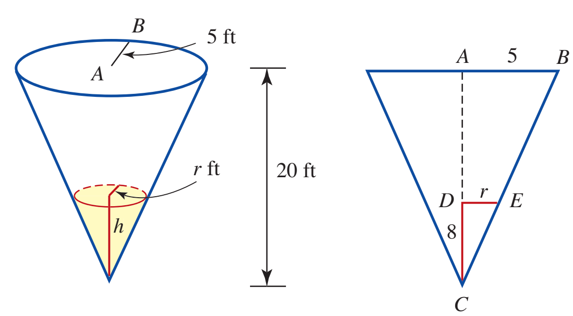


Figure 2.3 Inverted cone problem from calculus

$\Delta ABC \sim \Delta DEC$ (can you see why?); thus,

$$\begin{aligned}
\frac{5}{20} &= \frac{r}{8} \\
40 &= 20r \\
2 &= r
\end{aligned}$$

The desired volume is $V = \frac{1}{3}\pi(2)^2(8) = \frac{32}{3}\pi$. That is, the desired volume is about 33.5 ft^3.

PROBLEM SET 2.1

LEVEL 1

1. IN YOUR OWN WORDS Describe Pólya's problem-solving procedure.
2. IN YOUR OWN WORDS Compare and contrast inductive and deductive reasoning.
3. IN YOUR OWN WORDS State the substitution property.
4. IN YOUR OWN WORDS What is meant by mathematical modeling?
5. IN YOUR OWN WORDS Describe a process for solving word problems.
6. IN YOUR OWN WORDS What is the proportion property of similar triangles?

Write the converse, inverse, and contrapositive of the statements in Problems 7–12.

7. If you break the law, then you will go to jail.
8. I will go Saturday, if I get paid.
9. If a polygon has three sides, then it is a triangle.
10. If $a^2 + b^2 = c^2$, then the triangle with sides a, b, and c is a right triangle.
11. If $5 + 10 = 15$, then $15 - 10 = 2$.
12. If $8x = 16$, then $x = 2$.

Problems 13–20 give a verbal description of some of the formulas you will need to use in a calculus course. Write each formula in symbolic form.

13. The area A of a parallelogram is the product of the base b and the height h.
14. The area A of a triangle is one-half the product of the base b and the height h.
15. The area A of a rhombus is one-half the product of the diagonals p and q.
16. The area A of a trapezoid is the product of one-half the height h and the sum of the bases a and b.
17. The volume V of a cube is the cube of the length s of an edge.
18. The volume V of a rectangular solid is the product of the length l, the width w, and the height h.
19. The volume V of a cone is one-third the product of pi, the square of the radius r, and the height h.
20. The volume V of a sphere is the product of four-thirds pi and the cube of the radius r.

Solve Problems 21–26. Because you are practicing a procedure, you must show all of your work.

21. What is the sum of the first 20 positive consecutive odd numbers?
22. What is the sum of the first 1,000 positive consecutive odd numbers?
23. The sum of two consecutive odd integers is 48. What is the smaller integer?
24. The sum of two consecutive even integers is 30. What is the larger number?
25. The sum of four consecutive integers is 74. What are the integers?
26. What is the sum of the first 1,000 positive consecutive even numbers?

PROBLEMS FROM CALCULUS *Problems 27–36 use formulas you will need for calculus. State the formula you need, and then use that formula to answer the question.*

27. A rectangular field is 100 ft long and 75 ft wide. What length of fencing is necessary to enclose its perimeter?
28. How much carpeting is necessary to cover an area that is 6 yd wide and 12 yd long?
29. A square field is 540 ft on each side. What is the distance around the field, and what is its area?
30. An automotive tire has a radius of 15 in. What is the circumference of the tire?
31. If the radius of a circular region is 15 in., what is the area of the region?
32. An airplane maintains a constant speed of 670 mi/h for a 3-hr flight. How far does the plane travel?
33. Air is being pumped into a spherical balloon. What is its volume when the diameter is 50 cm?
34. A 10-ft ladder rests against a vertical wall. If the bottom of the ladder is 3 ft from the wall, how high up the building (to the nearest foot) does the ladder reach?
35. 

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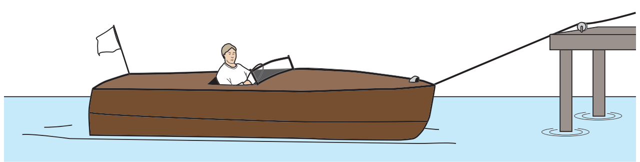
Sand is being dumped from a conveyor belt so that it forms a pile in the shape of a cone whose base diameter and height are always equal. What is the volume of the pile when it is 10 ft high?

36. A boat is pulled into a dock by a rope attached to the bow of the boat and passing through a pulley on the dock that is 1 m higher than the bow of the boat. How much rope is necessary to connect the pulley and the bow of the boat when the boat is 8 m from the dock?



LEVEL 2

Solve Problems 37–44. Because you are practicing a procedure, you must show all of your work.

37. Two rectangles have the same width, but one is 20 ft^2 larger in area. The larger rectangle is 6 ft longer than it is wide. The other is only 2 ft longer than its width. What are the dimensions of the larger rectangle?
38. Two triangles have the same height. The base of the larger one is 3 cm greater than its height. The base of the other is 1 cm greater than its height. If the areas differ by 3 cm^2, find the dimensions of the smaller figure.

39. A rectangle is 2 ft longer than it is wide. If you increase the length by a foot and reduce the width the same, the area is reduced by 3 ft^2. Find the width of the new figure.
40. The length of a rectangle is 3 m more than its width. The width is increased by 2 m, and the length is shortened by a meter. If the two figures have the same perimeter, what is it?
41. A businesswoman logs time in an airliner and a rental car to reach her destination. The total trip is 1,100 mi, the plane averaging 600 mi/h and the car 50 mi/h. How long is spent in the automobile if the trip took a total of 5 hr 30 min?
42. Barry hitchhikes back to campus from home, which is 82 mi away. He makes 4 mi/h walking, until he gets a ride. In the car, he makes 48 mi/h. If the trip took 4 hours, how far did Barry walk?
43. Two joggers set out at the same time from their homes 21 mi apart. They agree to meet at a point in between in an hour and a half. If the rate of one is 2 mi/h faster than the other, find the rate of each.
44. Two joggers set out at the same time but in opposite directions. If they were to maintain their normal rates for 4 hr, they could be 68 mi apart. If the rate of one is 1.5 mi/h faster than the other, find the rate of each.

PROBLEMS FROM CALCULUS *Problems 45–55 are modeled from problems taken from a variety of calculus textbooks.*

45. A rectangular area is to be fenced. If the space is twice as long as it is wide, for what dimensions is the area numerically greater than the perimeter?
46. A rectangular area three times as long as it is wide is to be fenced. For what dimensions is the perimeter numerically greater than the area?
47. A small manufacturer of high-gaming computers determines that the price of each item is related to the number of items, x, produced per day. The manufacturer knows that (1) the maximum number that can be produced is 10 items; (2) the price should be $400 - 25x$ dollars; (3) the overhead (the cost of producing x items) is $5x^2 + 40x + 600$ dollars; and (4) the daily profit is then found by subtracting the overhead from the revenue:

$$\begin{aligned}\text{PROFIT} &= \text{REVENUE} - \text{COST}\\ &= (\text{NUMBER OF ITEMS})(\text{PRICE/ITEM}) - \text{COST}\\ &= x(400 - 25x) - (5x^2 + 40x + 600)\\ &= 400x - 25x^2 - 5x^2 - 40x - 600\\ &= -30x^2 + 360x - 600\end{aligned}$$

What is the number of items produced if the profit is zero?
48. Current postal regulations do not permit a package to be mailed if the combined length, width, and height exceeds 72 in. What are the dimensions of the largest permissible package with length twice the length of its square end?
49. Suppose you throw a rock at 48 ft/s from the top of the Sears Tower in Chicago and the height in feet, h, from the ground after t sec is given by

$$h = -16t^2 + 48t + 1{,}454$$

   a. What is the height of the Sears Tower?
   b. How long will it take (to the nearest tenth of a second) for the rock to hit the ground?
50. If an object is shot up from the ground with an initial velocity of 256 ft/s, its distance in feet above the ground at the end of t sec is given by $d = 256t - 16t^2$ (neglecting air resistance). Find the length of time for which $d \geq 240$.
51. Find the length of time the projectile described in Problem 50 will be in the air.
52. Many materials, such as brick, steel, aluminum, and concrete, expand because of increases in temperature. This is why fillers are placed between the cement slabs in sidewalks. Suppose you have a 100-ft roof truss securely fastened at both ends, and assume that the buckle is linear. (It is not, but this assumption will serve as a worthwhile approximation.) Let the height of the buckle be x ft. If the percentage of swelling is y, then, for each half of the truss,

$$\begin{aligned}\text{NEW LENGTH} &= \text{OLD LENGTH} + \text{CHANGE IN LENGTH}\\ &= 50 + (\text{PERCENTAGE})(\text{LENGTH})\\ &= 50 + \left(\frac{y}{100}\right)(50)\\ &= 50 + \frac{y}{2}\end{aligned}$$

These relationships are shown in Figure 2.4.

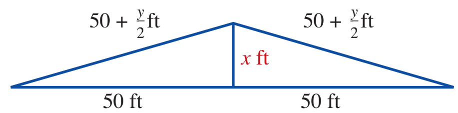


Figure 2.4 Buckling and expansion

By the Pythagorean theorem,

$$\begin{aligned}x^2 + 50^2 &= \left(50 + \frac{y}{2}\right)^2\\ x^2 + 50^2 &= \frac{(100 + y)^2}{4}\\ 4x^2 + 4 \cdot 50^2 &= 100^2 + 200y + y^2\\ 4x^2 - y^2 - 200y &= 0\end{aligned}$$

Solve this equation for x and then calculate the amount of buckling (to the nearest half-inch) for the following materials:
   a. brick; $y = 0.03$
   b. steel; $y = 0.06$
   c. aluminum; $y = 0.12$
   d. concrete; $y = 0.05$
53. A 1-mi length of pipeline connects two pumping stations.

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Special joints must be used along the line to provide for expansion and contraction due to changes in temperature. However, if the pipeline were actually one continuous length of pipe fixed at each end by the stations, then expansion would cause the pipe to bow. Approximately how high would the middle of the pipe rise if the expansion was just 1 in. over the mile? (You may make the same assumption as we did in Problem 52, namely, that the buckle is linear.)

54. Consider the following pattern:

$$9 \times 1 - 1 = 8$$
$$9 \times 21 - 1 = 188$$
$$9 \times 321 - 1 = 2{,}888$$
$$9 \times 4{,}321 - 1 = 38{,}888$$

a. Use this pattern and inductive reasoning to specify the next equation in the sequence.
b. Predict the answer to

$$9 \times 987{,}654{,}321 - 1$$

c. Predict the answer to

$$9 \times 10{,}987{,}654{,}321 - 1$$

55. Consider the following pattern:

$$123{,}456{,}789 \times 9 = 1{,}111{,}111{,}101$$
$$123{,}456{,}789 \times 18 = 2{,}222{,}222{,}202$$
$$123{,}456{,}789 \times 27 = 3{,}333{,}333{,}303$$

a. Use this pattern and inductive reasoning to specify the next equation in the sequence.
b. Predict the answer to

$$123{,}456{,}789 \times 63$$

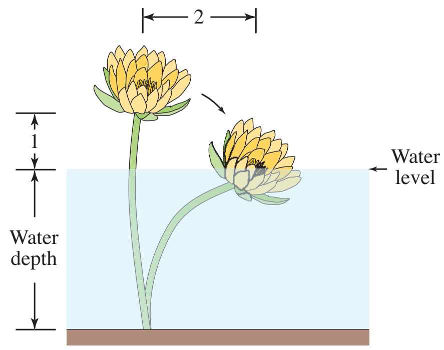
c. Predict the answer to

$$123{,}456{,}789 \times 81$$

LEVEL 3

Problems 56–60 were in the May 1989 issue of The Mathematics Teacher.

56. **Historical Quest** (From Bhaskara, ca. A.D. 1120.) "In a lake the bud of a water lily was observed, one cubit above the water, and when moved by the gentle breeze, it sunk in the water at two cubits' distance." Find the depth of the water.



57. **Historical Quest** (From Bhaskara, ca. A.D. 1120.) "One third of a collection of beautiful water lilies is offered to Mahadev, one fifth to Huri, one sixth to the Sun, one fourth to Devi, and the six which remain are presented to the spiritual teacher." Find the total number of lilies.

58. **Historical Quest** (From Bhaskara, ca. A.D. 1120.) "One fifth of a hive of bees flew to the Kadamba flower; one third flew to the Silandhara; three times the difference of these two numbers flew to an arbor, and one bee continued flying about, attracted on each side by the fragrant Keteki and the Malati." Find the number of bees.

59. **Historical Quest** (From Brahmagupta, ca. A.D. 630.) "A tree one hundred cubits high is distant from a well two hundred cubits; from this tree one monkey climbs down the tree and goes to the well, but the other leaps in the air and descends by the hypotenuse from the high point of the leap, and both pass over an equal space." Find the height of the leap.

60. **Historical Quest** "Ten times the square root of a flock of geese, seeing the clouds collect, flew to the Manus lake; one eighth of the whole flew from the edge of the water amongst a multitude of water lilies; and three couples were observed playing in the water." Find the number of geese.

Tower of Pisa

2.2 Introduction to Functions

Definitions

In the previous chapter, we considered a Cartesian coordinate system to easily see the relationship between two variables. We now introduce an algebraic characterization for certain relationships.

Suppose we drop an object from a tall structure (such as the Leaning Tower of Pisa). The distance the object falls is dependent (among other things) on the length of time it falls. If we let the variable d be the *distance* the object has fallen (in feet) and t the *time* it has fallen (in seconds), and if we disregard air resistance, the formula (from physics and calculus) is

$$d = 16t^2$$

where 16 is a constant determined by the force of gravity acting on the object. Using the formula, we can calculate the height of the tower by timing the number of seconds it takes for the object to hit the ground. If it takes 3 seconds for the object to hit the ground, then the height of the tower (in feet) is

$$
\begin{aligned}
d &= 16t^2 \\
&= 16(3)^2 \\
&= 144
\end{aligned}
$$

The formula $d = 16t^2$ gives rise to a set of data for $0 \le t \le 15$:

Time (in sec)	0	1	2	3	4	. . .	14	15
Distance (in ft)	0	16	64	144	256	. . .	3,136	3,600

For every nonnegative value of t, there is a corresponding value for d. We can represent the data in the table as a set of ordered pairs in which the first component represents a value for t and the second component represents a corresponding value for d:

First component (values for t)
$\downarrow$
(x, y)
$\uparrow$
Second component (values for d)

The determined value (distance in this example) is called the **dependent variable**, and the specified variable is the **independent variable**. For this example (see preceding table), we have a set of ordered pairs: $(0, 0)$, $(1, 16)$, $(2, 64)$, . . . , $(14, 3136)$, $(15, 3600)$. The **domain** of a function is the set of values of the independent variable for which it is defined. For this example, the domain is defined as $0 \le t \le 15$, which means that t is any value between 0 and 15 (including the endpoints). Thus, other ordered pairs (not shown in the table) are $(0.5, 4)$, $(12.75, 2601)$, ... The set of all corresponding values of the dependent variable (second components) is called the **range**. A visual representation of a function is shown in Figure 2.5.

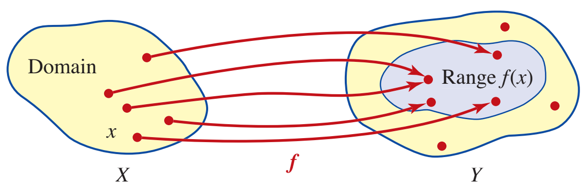


Figure 2.5 A function as a mapping

FUNCTION

A **function** is a rule that assigns to each element of the domain a single element in its range.

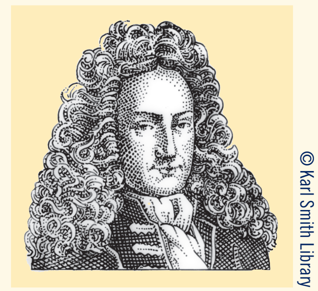
» IN OTHER WORDS To each x in the domain, there corresponds exactly one y in the range.

The pairs of corresponding values assigned in the definition of a function may be viewed as ordered pairs (x, y). This allows a rewording of the definition: *A function is a set of ordered pairs (x, y) in which no two different ordered pairs have the same first element x.*

It is customary to give functions letter names, such as f, g, f_1, or F. If y is the value of the function f corresponding to x, it is written $y = f(x)$ and is read "y is equal to the value of the function f at x," or, more briefly, "y equals f at x," or "y equals f of x."

Let $D = \{1, 2, 3, 4\}$ be the domain of a function called f. Think of the function f as a machine (*a function machine* as shown in Figure 2.6) that accepts an input x from D and produces an output $f(x)$, pronounced "f of x."

Historical Note



Gottfried Leibniz (1646–1716)

We first met Leibniz in Chapter 1 as one of the inventors of calculus. However, as you might guess, he is one of the giants in the history of mathematics. For example, the word *function* was used as early as 1694 by Leibniz to denote any quantity connected with a curve. Leibniz was one of the most universal geniuses of all time, and as a teenager, he came up with many of the great ideas in mathematics. However, his ideas were not fully accepted at the time because teenagers did not command much attention in intellectual circles. He was refused a doctorate at the University of Leipzig because of his youth, even though he had completed all the requirements. Among other things, Leibniz invented the calculus, exhibited an early calculating machine that he invented, and distinguished himself in law, philosophy, and linguistics. His ideas on functions were generalized by other mathematicians, including P. G. Lejeune-Dirichlet (1805–1859).

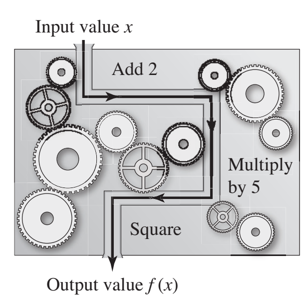


Figure 2.6 Function machine

The symbol $f(x)$ does NOT mean multiplication; it is a single symbol representing the second component of the ordered pair $(x,\ y)$.

The function machine description has the advantage of being easy to understand, but it is awkward to use. We might also describe this function in a variety of ways, as shown in Example 1.

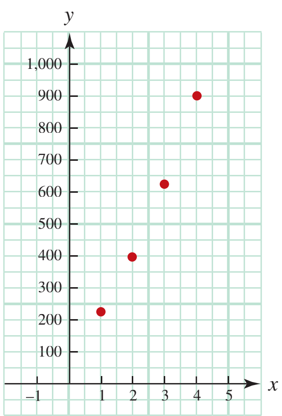
EXAMPLE 1 Alternate descriptions for a function

Describe the function machine in Figure 2.6 for the domain $D = \{1,\ 2,\ 3,\ 4\}$ using a rule, stating an equation, showing a mapping, using a set of ordered pairs, writing a table, and drawing a graph.

Solution

RULE *For each input value, add two, then multiply by five, and finally square the result to find the output.* The function in this example would probably not be defined by such a verbal rule; nevertheless, a verbal rule is often the best way we have to describe a certain function.

GRAPH

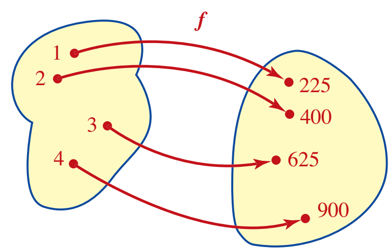


EQUATION $f(x) = [5(x + 2)]^2$

TABLE

x	$f(x)$
1	225
2	400
3	625
4	900

MAPPING

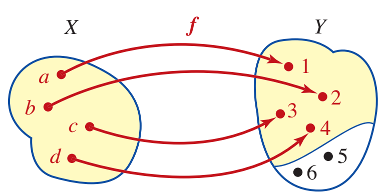


SET OF ORDERED PAIRS $\{(1,\ 225),\ (2,\ 400),\ (3,\ 625),\ (4,\ 900)\}$

EXAMPLE 2 Domain, range, and outputs

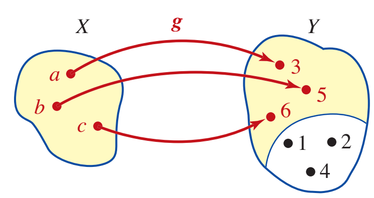
Given a mapping from X to Y, name the domain and range, and use functional notation to name the outputs for each input.

a.



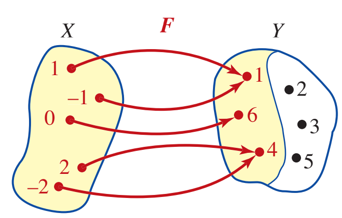
$X = \{a, b, c, d\}$; $Y = \{1, 2, 3, 4, 5, 6\}$

b.



$X = \{a, b, c\}$; $Y = \{1, 2, 3, 4, 5, 6\}$

c.



$X = \{1, -1, 0, 2, -2\}$; $Y = \{1, 2, 3, 4, 5, 6\}$

Solution

a. Domain $= \{a, b, c, d\}$; Range $= \{1, 2, 3, 4\}$;

The range consists only of those elements of Y that are actually used as outputs. However, the domain and X must be the same.

$f(a) = 1, f(b) = 2, f(c) = 3, f(d) = 4$

b. Domain $= \{a, b, c\}$; Range $= \{3, 5, 6\}$;

$g(a) = 3, g(b) = 5, g(c) = 6$

c. Domain $= \{1, -1, 0, 2, -2\}$; Range $= \{1, 4, 6\}$;

$F(0) = 6,\ F(1) = 1,\ F(-1) = 1,\ F(2) = 4,\ F(-2) = 4$

Different elements in the domain may have the same value in the range.

Notice that Example 2c showed some repeated outputs. The ultimate example of repeated outputs is a function defined by $f(x) = c$ for *all* values of x. Such a function is called a **constant function**. If the outputs are always different (that is, if there are no repeated outputs), then the function is called *one-to-one*.

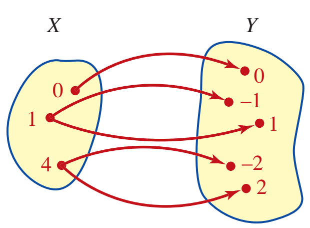
ONE-TO-ONE FUNCTIONS

If a function f maps X into Y so that for any distinct (different) elements x_1 and x_2 in X, $f(x_1) \neq f(x_2)$ then f is said to be a **one-to-one** function of X into Y.

EXAMPLE 3 A mapping that is not a function

Draw a picture of a mapping that is not one-to-one.

Solution



Do not use $f(x)$ notation unless f is a function. This is not a function because 1 and 4 are associated with more than one image; because it is not a function, it follows that the mapping is not one-to-one.

Horizontal and Vertical Line Tests

There are two tests that involve sweeping a line across a graph. The first tells us whether a graph represents a function, and the second tells us whether a graph represents a one-to-one function.

VERTICAL LINE TEST

Every vertical line passes through the graph of a function in at most one point. This means if you sweep a vertical line across a graph and it simultaneously intersects the curve at more than one point, then the curve is not the graph of a function.

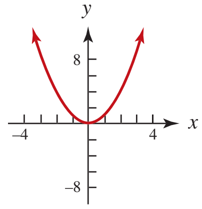
HORIZONTAL LINE TEST

Every horizontal line passes through the graph of a one-to-one function in at most one point. This means that if you sweep a horizontal line across the graph of a function and it simultaneously intersects the curve at more than one point, then the curve is not the graph of a one-to-one function.

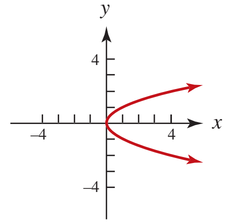
EXAMPLE 4 Horizontal and vertical line tests

Use the vertical line test to determine whether the given curve is the graph of a function, and if it is the graph of a function, use the horizontal line test to determine whether it is one-to-one. Name the probable domain and range by looking at the graphs.

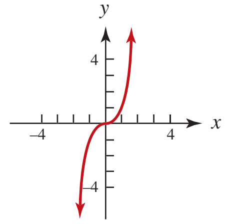
a.



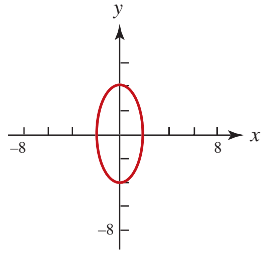
b.



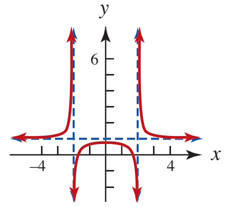
c.



d.

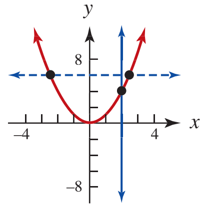


e.



Solution

a.



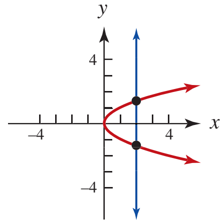
Passes the vertical line test; it is a function.

Does not pass the horizontal line test; it is not one-to-one.

Domain: $\mathbb{R}$

Range: $y \geq 0$

b.



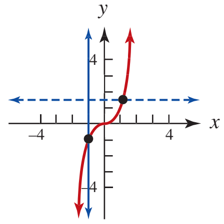
Does not pass the vertical line test; it is not a function.

If it is not a function, then the horizontal line test is not needed because if it is not a function, then it cannot be a one-to-one function.

Domain: $x \geq 0$

Range: $\mathbb{R}$

c.



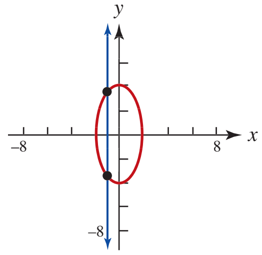
Passes the vertical line test; it is a function.

Passes the horizontal line test; it is one-to-one.

Domain: $\mathbb{R}$

Range: $\mathbb{R}$

d.

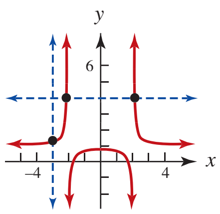


Does not pass the vertical line test; it is not a function.

If it is not a function, then the horizontal line test is not needed because if it is not a function, then it cannot be a one-to-one function.

Domain: $-1 \leq x \leq 2$

Range: $-4 \leq y \leq 4$

e. 

Passes the vertical line test; it is a function.
Does not pass the horizontal line test; it is not one-to-one.
Domain: $\mathbb{R}, x \neq \pm 2$
Range: $y \leq \frac{3}{4}$ or $y > 1$

Functional Notation

One of the most useful inventions in all the history of mathematics is the notation $f(x)$, called **functional notation**. Remember

x is a member of the domain.

$$\underbrace{f(x)}$$

f(x) is a number.

A *function* is denoted by f; $f(x)$ is a *number* associated with x.

Sometimes functions are defined by expressions such as

$$f(x) = x^2 + 1 \quad \text{or} \quad g(x) = (x+1)^2$$

To emphasize the difference between f and $f(x)$, some books use $f : x \to x^2 + 1$. In this book, however, we write "$f(x) = x^2 + 1$" to mean, "let f be the function defined by $f(x) = x^2 + 1$; this denotes the set of all ordered pairs (x, y) so that $y = x^2 + 1$."

EXAMPLE 5 Evaluating functions

Let $f(x) = x^2 + 1$ and $g(x) = (x+1)^2$.
Find **a.** $f(1)$ **b.** $g(1)$ **c.** $f(-3)$ **d.** $g(-3)$ **e.** $f(w)$ **f.** $g(w)$
g. $f(w+h)$ **h.** $g(w+h)$

Note that $f \neq g$ since $x^2 + 1 \neq (x+1)^2$.

Solution

a. $f(1) = 1^2 + 1 = 2$

b. $g(1) = (1+1)^2 = 2^2 = 4$

c. $f(-3) = (-3)^2 + 1 = 9 + 1 = 10$

d. $g(-3) = (-3+1)^2 = (-2)^2 = 4$

e. $f(w) = w^2 + 1$

f. $g(w) = (w+1)^2 = w^2 + 2w + 1$

g. $f(w+h) = (w+h)^2 + 1$
$= w^2 + 2wh + h^2 + 1$

h. $g(w+h) = (w+h+1)^2$
$= w^2 + wh + w + wh + h^2 + h + w + h + 1$
$= w^2 + 2wh + h^2 + 2w + 2h + 1$

In calculus, functional notation is used in the definition of derivative. The value

$$\frac{f(x+h) - f(x)}{h}$$

is called a **difference quotient**.

EXAMPLE 6 Difference quotient

Let $f(x) = x^3$. Find the difference quotient $\dfrac{f(x+h) - f(x)}{h}$.

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^3-x^3}{h}\\ &= \frac{x^3+3x^2h+3xh^2+h^3-x^3}{h}\\ &= \frac{3x^2h+3xh^2+h^3}{h}\\ &= 3x^2+3xh+h^2\end{aligned}$$

When finding a difference quotient, do not start with $f(x)$. Find $f(x+h)$ first, then subtract $f(x)$ and simplify; finally, divide by h.

EXAMPLE 7 Difference quotient of a polynomial function

Let $f(x) = x^2 + 2x + 3$. Find $\dfrac{f(x+h)-f(x)}{h}$.

Solution

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{[(x+h)^2+2(x+h)+3]-[x^2+2x+3]}{h}\\ &= \frac{x^2+2xh+h^2+2x+2h+3-x^2-2x-3}{h}\\ &= \frac{2xh+h^2+2h}{h}\\ &= 2x+h+2\end{aligned}$$

Using Functional Notation in Problem Solving

Functional notation can be used to work a wide variety of applied problems.

EXAMPLE 8 Falling object problem revisited

At the beginning of this section, we used the formula $d = 16t^2$ to represent the distance (in ft) that an object falls (neglecting air resistance) after t seconds. This relationship can be represented by $f(t) = 16t^2$. Use functional notation to represent each of the given ideas.

a. The distance the object will fall in one second.
b. The distance the object will fall in the next two seconds.
c. The distance the object will fall during the h seconds following the second second.
d. The average distance the object will fall in the first 5 seconds.
e. The average distance the object will fall in the next 5 seconds (after the first 5 seconds).
f. The average distance the object will fall in h seconds after the first x seconds.

Solution

a. $f(1) = 16(1)^2 = 16$; the object will fall 16 ft.
b. $f(3) = 16(3)^2 = 144$ ft is the distance the object will fall in the first 3 seconds; $f(3) - f(1)$ is the distance the object will fall in the next 2 seconds (after the first second). That is,

$$f(3) - f(1) = 144 - 16 = 128$$

In the next two seconds, the object falls 128 ft.

c. $f(2)=16(2)^2=64$ ft is the distance the object will fall in the first two seconds. $f(2+h)$ is the distance the object will fall in the next h seconds, so the distance in the h seconds following the second second is

$$\begin{aligned} f(2+h)-f(2) &= 16(2+h)^2-64 \\ &= 16(4+4h+h^2)-64 \\ &= 64+64h+16h^2-64 \\ &= 16h^2+64h \end{aligned}$$

Thus, the object will fall $(16h^2+64h)$ ft.

d. $f(5)=16(5)^2=400$ ft is the distance the object will fall in the first 5 seconds. Thus, the average distance for the first 5 seconds is

$$\frac{f(5)\text{ ft}}{5\text{ sec}}=\frac{400\text{ ft}}{5\text{ sec}}=80\text{ ft/s}$$

e. The distance the object travels in the *next* 5 seconds is

$$\begin{aligned} f(5+5)-f(5) &= 16(10)^2-16(5)^2 \\ &= 1{,}600-400 \\ &= 1{,}200 \end{aligned}$$

The average distance is

$$\frac{f(10)-f(5)}{10-5}=\frac{1{,}200}{5}=240\text{ ft/s}$$

f. In the first x seconds the object travels $f(x)$ ft; in the *next* h seconds, the object travels $f(x+h)-f(x)$. The average distance for the h seconds is

$$\begin{aligned} \frac{f(x+h)-f(x)}{(x+h)-x} &= \frac{16(x+h)^2-16x^2}{h} \qquad \text{Recognize the difference quotient.} \\ &= \frac{16(x^2+2xh+h^2)-16x^2}{h} \\ &= \frac{16x^2+32xh+16h^2-16x^2}{h} \\ &= \frac{32xh+16h^2}{h} \\ &= 32x+16h \end{aligned}$$

■

EXAMPLE 9 **Calculus example, writing a function**

Suppose you need to fence a rectangular play zone for children, to fit into a right-triangular plot with sides measuring 4 m and 12 m, as shown in Figure 2.7.

Write the area of the play zone as a function of the length of the play zone. We will continue with this example in Section 3.3.

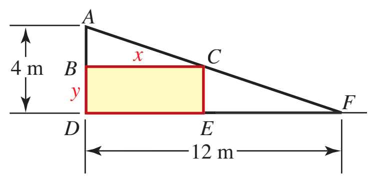


Figure 2.7 **Play Zone**

Solution Let x and y denote the length and width of the inscribed rectangle. The appropriate formula for the area is $A=lw=xy$. We wish to find a formula for this area. To write this as a single variable, x in this example, we note that $\Delta ABC \sim \Delta ADF$, which means that corresponding sides of these triangles are proportional; therefore,

$$\frac{4-y}{4} = \frac{x}{12}$$
$$4 - y = \frac{1}{3}x$$
$$y = 4 - \frac{1}{3}x$$

We now write A as a function of x alone:

$$A(x) = xy = x\left(4 - \frac{1}{3}x\right) = 4x - \frac{1}{3}x^2$$

Classification of Functions

If you have looked at the table of contents for this book, you will see that one of the unifying concepts of this book is that of a function. As a preview of what is to follow, we will define many of the functions you will encounter in this course and in your calculus course.

POLYNOMIAL FUNCTION

A **polynomial function** is a function of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where n is a nonnegative integer and $a_n, \ldots, a_2, a_1, a_0$ are constants. If $a_n \neq 0$, the integer n is called the **degree** of the polynomial. The constant a_n is called the **leading coefficient** and the constant a_0 is called the **constant term** of the polynomial function. In particular,

A **constant function** is zero degree: $f(x) = a$

A **linear function** is first degree: $f(x) = ax + b$

A **quadratic function** is second degree: $f(x) = ax^2 + bx + c$

A **cubic function** is third degree: $f(x) = ax^3 + bx^2 + cx + d$

A **quartic function** is fourth degree: $f(x) = ax^4 + bx^3 + cx^2 + dx + e$

We will consider polynomial functions in Chapter 3.

A second important algebraic function is a *rational function.*

RATIONAL FUNCTION

A **rational function** is the quotient of two polynomial functions, $p(x)$ and $d(x)$:

$$f(x) = \frac{p(x)}{d(x)}, \qquad d(x) \neq 0$$

When we write $d(x) \neq 0$ we mean that all values c for which $d(c) = 0$ are excluded from the domain of d. Here are some examples of rational functions, written in different ways.

$$f(x) = x^{-1} \qquad f(x) = \frac{x-5}{x^2+2x-3} \qquad f(x) = x^{-3} + \sqrt{2}x$$

We will consider rational functions in Chapter 4.

If r is any nonzero real number, the function $f(x) = x^r$ is called a **power function** with exponent r. You should be familiar with the following cases:

Integral powers ($r = n$, a positive integer): $f(x) = x^n = \underbrace{x \cdot x \cdot \cdots \cdot x}_{n \text{ factors}}$

Reciprocal powers (r is a negative integer): $f(x) = x^{-n} = \dfrac{1}{x^n}$ for $x \neq 0$

Roots ($r = \frac{m}{n}$ is a positive rational number): $f(x) = x^{m/n} = \sqrt[n]{x^m} = \left(\sqrt[n]{x}\right)^m$ for $x \geq 0$ if n even, $n \neq 0$ ($\frac{m}{n}$ is reduced)

Power functions can also have irrational exponents (such as $\sqrt{2}$ or π), but such functions must be defined in a special way and are not introduced until Chapter 4.

A function is called **algebraic** if it can be constructed using algebraic operations (such as adding, subtracting, multiplying, dividing, or taking roots) starting with polynomials. Any rational function is an algebraic function.

Functions that are not algebraic are called **transcendental**. The following functions are transcendental functions:

Exponential functions are functions of the form $f(x) = b^x$, where b is a positive constant. We will study these functions in Chapter 4.

Logarithmic functions are functions of the form $f(x) = \log_b x$, where b is a positive constant. We will also study these functions in Chapter 4.

Trigonometric functions are the functions sine, cosine, tangent, secant, cosecant, and cotangent. We will define these functions in Chapter 5.

PROBLEM SET 2.2

LEVEL 1

Determine whether the sets given in Problems 1–4 are functions. If it is a function, state its domain.

1. **a.** $\{(6, 3), (9, 4), (7, -1), (5, 4)\}$
b. $\{6, 9, 7, 5\}$
c. $y = 5x + 2$
d. $y = -1$ if x is a rational number, and $y = 1$ if x is an irrational number.

2. **a.** $\{(3, 6), (4, 9), (-1, 7), (4, 5)\}$
b. $\{10, 20, 30, 40\}$
c. $y \leq 5x + 2$
d. $y = -1$ if x is a positive integer, and $y = 1$ if x is a negative integer.

3. **a.** $\{(x, y) | y =$ closing price of Xerox stock on July 1 of year $x\}$
b. $\{(x, y) | x =$ closing price of Xerox stock on July 1 of year $y\}$

4. **a.** (x, y) is a point on a circle with center $(2, 3)$ and radius 4.
b. (x, y) is a point on a line passing through $(2, 3)$ and $(4, 5)$.

For each verbal description in Problems 5–8, write a rule in the form of an equation and then state the domain.

5. For each number x in the domain, the corresponding range value y is found by multiplying by three and then subtracting five.

6. For each number x in the domain, the corresponding range value y is found by squaring and then subtracting five times the domain value.

7. For each number x in the domain, the corresponding range value y is found by taking the square root of the difference of the domain value subtracted from five.

8. For each number x in the domain, the corresponding range value y is found by adding one to the domain value and then dividing that result into five added to five times the domain value.

9. Let $P(x)$ be the number of prime numbers less than x. Find
a. $P(10)$ **b.** $P(-10)$ **c.** $P(100)$

10. Let $S(x)$ be the exponent on a base 2 that gives the result x. Find
a. $S(32)$ **b.** $S\left(\frac{1}{8}\right)$
c. $S\left(\sqrt{2}\right)$

In Problems 11–18, let $f(x) = 5x - 1$ and $g(x) = 3x^2 + 1$. Find the requested values.

11. **a.** $f(0)$ **b.** $f(2)$
c. $f(-3)$ **d.** $f\left(\sqrt{5}\right)$

12. **a.** $f(w)$ **b.** $g(w)$
c. $g(t)$ **d.** $g(v)$

13. **a.** $f(t)$ **b.** $f(p)$
c. $f(t + 1)$ **d.** $g(t + 1)$

14. **a.** $f(x + 2)$ **b.** $g(x + 2)$
c. $f(t + h)$ **d.** $g(t + h)$

15. **a.** $f(t^2 - 3t) - g(t + 2)$
b. $f(t^2 + 2t + 1) - g(t + 3)$

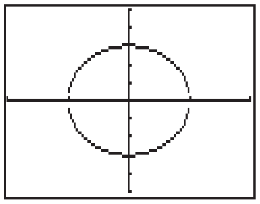
16. **a.** $\dfrac{f(t+3) - f(t)}{3}$ **b.** $\dfrac{g(t+2) - g(t)}{2}$

17. **a.** $\dfrac{f(t+h) - f(t)}{h}$ **b.** $\dfrac{g(t+h) - g(t)}{h}$

18. **a.** $\dfrac{f(x+h) - f(x)}{h}$ **b.** $\dfrac{g(x+h) - g(x)}{h}$

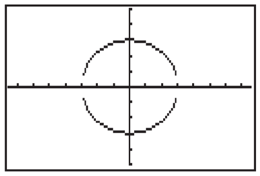
In Problems 19–24, use the vertical line test to determine whether the curve is a function and if it is the graph of a function, use the horizontal line test to determine whether it is one-to-one. Also state the probable domain and range.

19.



xMin=-2 yMin=-5
xMax=2 yMax=5
xScl=1 yScl=1

20.



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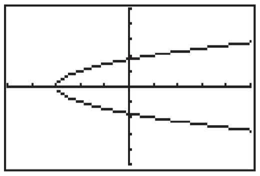
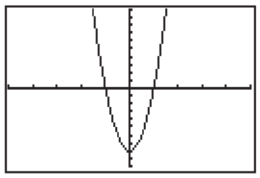
xMax=7.5806451
xScl=1
yMin=-5
yMax=5
yScl=1

21.



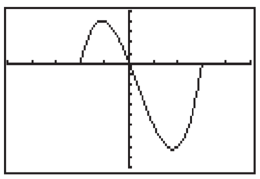
xMin=-5 yMin=-5
xMax=5 yMax=5
xScl=1 yScl=1

22.



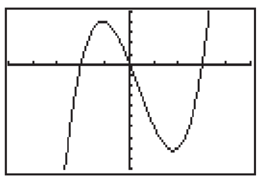
xMin=-10 yMin=-100
xMax=10 yMax=100
xScl=2 yScl=10

23.



xMin=-5 yMin=-10
xMax=5 yMax=5
xScl=1 yScl=1

24.



xMin=-5 yMin=-10
xMax=5 yMax=5
xScl=1 yScl=1

WHAT IS WRONG, *if anything, with each statement in Problems 25–34? Explain your reasoning.*

25. $f(x+2) = f(x) + f(2)$

26. $f(2x) = 2f(x)$

27. If $f(x) = 3x^2 + 5$, then $f(2)$ is a function.

28. If $f(x) = 3x^2 + 5$, then $f(x)$ is a function.

29. If $f(x) = 3x^2 + 5$, then f is a function.

30. The horizontal line test is used to determine whether a graph represents a function.

31. The horizontal line test is used to determine whether a graph represents a one-to-one function.

32. The vertical line test is used to determine whether a graph represents a function.

33. The vertical line test is used to determine whether a graph represents a one-to-one function.

34. If $f(x) = 3x^2$, then $\dfrac{f(x+h) - f(x)}{h} = \dfrac{3x^2 + h - 3x^2}{h}$

LEVEL 2

In Problems 35–40, find the difference quotient for the given function f.

35. $f(x) = 4x^2$

36. $f(x) = 6x^2$

37. $f(x) = x^2 + 3$

38. $f(x) = 6x^2 + 2x$

39. $f(x) = x^2 - x + 3$

40. $f(x) = x^2 + 2x - 3$

41. If $S(x) = \dfrac{3x+2}{4x-1}$, find $S\left(\frac{1}{x}\right)$.

42. Let $R(x) = 3x^2 + 3x^{-2} - x - x^{-1}$. Show $R\left(\frac{1}{x}\right) = R(x)$.

43. Let $f(x) = ax + b, a \neq 0$. Find a and b so that

$$f(x+y) = f(x) + f(y)$$

44. Let $f(x) = ax + bx + c, a \neq 0$. Find a, b, and c so that

$$f(x+y) = f(x) + f(y)$$

45. If $f(x) = ax + b$, $a \neq 0$, evaluate $f\left(-\frac{b}{a}\right)$.

46. Let $g(x) = ax^2 + bx + c$, $a \neq 0$. Find

$$g\left(\frac{-b + \sqrt{b^2 - 4ac}}{2a}\right)$$

47. **a.** Let $Q(x) = \dfrac{x+a}{x-a}$. Does

$$Q(2a + 3a) = Q(2a) + Q(3a)?$$

b. Give an example of a function E for which

$$E(2a + 3a) = E(2a) + E(3a)$$

48. **a.** Let $T(x) = 2^x$. Does

$$T(a+b) = T(a) \cdot T(b)?$$

b. Give an example of a function D for which $D(a+b) \neq D(a) \cdot D(b)$.

49. Let d be a function that represents the distance an object falls (neglecting air resistance) from rest in the first t seconds. Find the distance the object falls for the given intervals of time if $d(t) = 16t^2$.

a. From $t = 2$ to $t = 6$. *Hint*: This is $d(6) - d(2)$.
b. From $t = 2$ to $t = 4$.
c. From $t = 2$ to $t = 3$.
d. From $t = 2$ to $t = 2 + h$.
e. From $t = x$ to $t = x + h$.
f. Give a physical interpretation for

$$\frac{d(t+h) - d(t)}{h}$$

50. Suppose the total cost (in dollars) of manufacturing q units of a certain item is given by

$$C(q) = q^3 - 30q + 400q + 500$$

on $[0, 30]$.

a. What is the cost of manufacturing 20 units?
b. Compute the cost of manufacturing the 21st unit.

51. An efficiency study of the morning shift at a certain factory indicates that an average worker who arrives on the job at 8:00 A.M. will have assembled

$$f(x) = -x^3 + 6x + 15x^2$$

units x hours later $(0 \leq x \leq 8)$.

a. How many units will such a worker have assembled by 10:00 A.M.?
b. How many units will such a worker assemble between 9:00 A.M. and 10:00 A.M.?

52. It is estimated that t years from now the population of a certain suburban community will be

$$P(t) = 20 - \frac{6}{t+1}$$

thousand people.

a. What will the population of the community be nine years from now?
b. By how much will the population increase during the ninth year?
c. What will happen to the size of the population in the "long run"?

53. Find the area of a square as a function of its perimeter.

54. Find the area of a circle as a function of its circumference.

PROBLEMS FROM CALCULUS *Functions are, of course, central to the study of calculus. Problems 55–60 are adapted from a leading calculus textbook.**

55. A manufacturer wants to design an open box having a square base (length x) and a surface area of 108 square inches. Write the volume as a function of the length of a side of the base.

56. Write the distance between a point (x, y) on the graph of $y = 4 - x^2$ and the point $(0, 2)$ as a function of x.

57. A rectangular page is to contain 24 square inches of print where x is the height of the printed portion. The margins at the top and bottom of the page are each $1\frac{1}{2}$ inches. The margins on each side are 1 inch. Write the area of the paper as a function of x.

58. Two posts, one 12 feet high and the other 28 feet high, stand 30 feet apart. The top of each post is fastened by wire to a single stake, running from ground level to the top of each post. Write the length of the wire as a function of the distance, x, the stake is located from the 12-ft post.

59. Four feet of wire is to be used to form a square and a circle. Write the total area (sum of the area of the square and the area of the circle) as a function of the length, x, of the side of the square.

60. A hospital patient receives an intravenous glucose solution from a cylindrical bottle of radius 8 cm with height 20 cm. Suppose the fluid level drops 0.25 cm/min. (*Note*: The volume of a right circular cylinder of radius r and height h is $\pi r^2 h$.)
a. Write a formula for the amount S of solution in cubic centimeters (cm^3) that has entered the patient's vein when the height of the removed fluid is h cm.
b. Write a formula for the height of the fluid (in cm) t minutes after the full bottle is hooked up to a patient.
c. Write a formula for S as a function of t.
d. How long does it take for all the fluid to enter the patient's vein?

2.3 Graph of a Function

Graphs have visual impact. They also reveal information that may not be evident from verbal or algebraic descriptions. To represent a function $y = f(x)$ geometrically as a graph, it is traditional to use a Cartesian coordinate system on which units for the independent variable x are marked on the horizontal axis and units for the dependent variable y are marked on the vertical axis.

GRAPH OF A FUNCTION

The **graph** of a function f consists of all points whose coordinates (x, y) satisfy $y = f(x)$, for all x in the domain of f.

One of the principal tasks of this book is to discuss efficient techniques involving calculus that you can use to draw accurate graphs of functions. In beginning algebra, you began sketching lines by plotting points, but you quickly found out that this is not a very efficient way to draw more complicated graphs, especially without the aid of a graphing calculator or computer. Table 2.1 includes a few common graphs you have probably encountered in previous courses. We will assume that you are familiar with their general shape and know how to sketch each of them.

We will use the functions in Table 2.1 as a basis for discussion in this chapter as we look at *properties* of functions, and then in subsequent chapters of the book, we will use the properties of this chapter to help us graph functions in general.

Even if you do not now have access to a graphing calculator or computer software that graphs, you will no doubt be using this technology in the future. Many have a misconception that if they only had this technology they would not need to study graphing in a mathematics course. Quite the contrary is true. Even the best software will often come up with a blank screen when an equation or a curve is input. Graphing calculators require input in the form Y=, which means that you are expected to input equations that are functions.

Domain and Range

In this book, unless otherwise specified, the domain of a function is the set of real numbers for which the function is defined. We call this the **domain convention.**

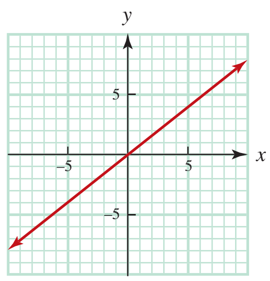
If a function f is **undefined** at x, it means that x is not in the domain of f. The most frequent exclusions from the domain are those values that cause division by 0 and negative values under a

Notice this agreement about the domain of a function that will be used throughout the book.

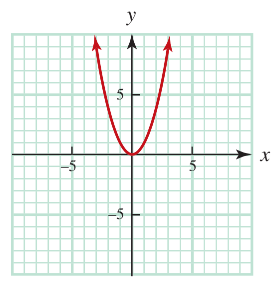
*From Section 3.7, pp 213–216, of *CALCULUS*, Fifth Edition, by Larson, Hostetler, and Edwards.

TABLE 2.1 Directory of Curves

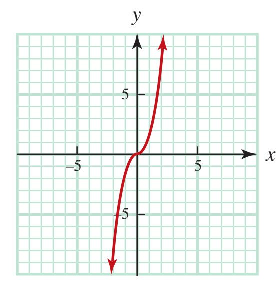
Identity Function
$y = x$



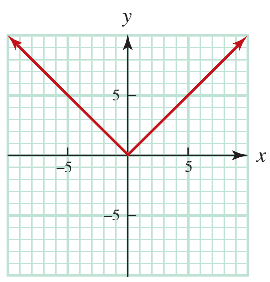
Standard Quadratic Function
$y = x^2$



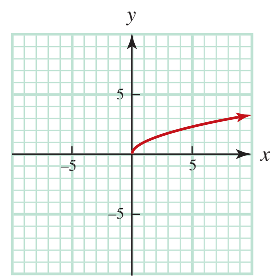
Standard Cubic Function
$y = x^3$



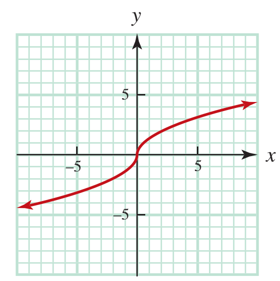
Absolute Value Function
$y = |x| = \sqrt{x^2}$



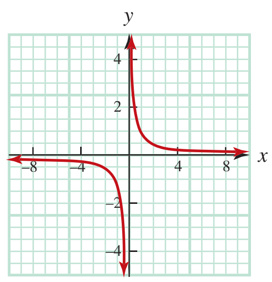
Square Root Function
$y = \sqrt{x}$



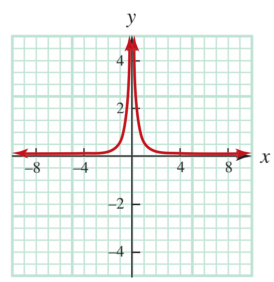
Cube Root Function
$y = \sqrt[3]{x}$



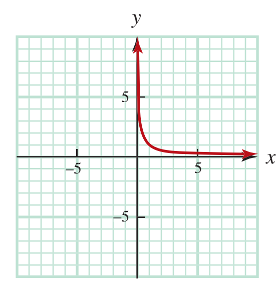
Standard Reciprocal
$y = \dfrac{1}{x}$



Standard Reciprocal Squared
$y = \dfrac{1}{x^2}$



Standard Square Root Reciprocal
$y = \dfrac{1}{\sqrt{x}}$



square root. In applications, the domain is often specified by the context. For example, if x is the number of people on an elevator, the context requires that negative numbers and nonintegers be excluded from the domain; therefore, x must be an integer such that $0 \le x \le c$, where c is the maximum capacity of the elevator.

EXAMPLE 1 Domain of a function

Find the domain for the given functions.

a. $f(x) = 2x - 1$

b. $g(x) = 2x - 1,\ x \neq -3$

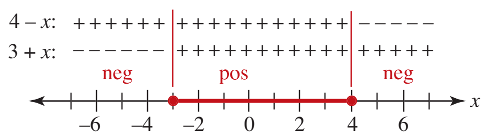
c. $h(x) = \dfrac{(2x-1)(x+3)}{x+3}$

d. $F(x) = \sqrt{12 + x - x^2}$

Solution

a. All real numbers; $D = (-\infty, \infty)$.
b. All real numbers except -3.
c. Because the expression is meaningful for all $x \neq -3$, the domain is all real numbers except -3.
d. F is defined whenever $12 + x - x^2$ is nonnegative:

$$12 + x - x^2 \geq 0$$
$$(4 - x)(3 + x) \geq 0$$



We see that x is nonnegative when $-3 \leq x \leq 4$, so $D = [-3, 4]$.

EQUAL FUNCTIONS

Two functions and f and g are **equal** if and only if

1. f and g have the same domain.
2. $f(x) = g(x)$ for all x in the domain.

In Example 1, the functions g and h are equal, but the functions f and h are not. A common mistake is to "reduce" the function h to the function f:

WRONG: $h(x) = \dfrac{(2x-1)(x+3)}{x+3} = 2x - 1 = f(x)$

RIGHT: $h(x) = \dfrac{(2x-1)(x+3)}{x+3} = 2x - 1,\ x \neq -3$; therefore, $h(x) = g(x)$.

Even though the usual graphing procedure is to find the domain, draw the graph, and then use the graph to determine the range, it is sometimes necessary to find both the domain and the range. We summarize these procedures:

FINDING THE DOMAIN

To find the domain, **solve for y** and look for exclusions for x.

FINDING THE RANGE

To find the range, **solve for x** and look for exclusions for y.

With this procedure, it is not necessary that the given relation be a function.

EXAMPLE 2 Finding the domain and range of a relation

Find the domain and range for:

a. $x^2 + y^2 = 3$
b. $y = \sqrt{3 - x^2}$
c. $xy^2 - y^2 - 1 = 0$

Solution

a. *Domain:* Solve for y and look for exclusions for x.

$$\begin{aligned} x^2 + y^2 &= 3 \\ y^2 &= 3 - x^2 \\ y &= \pm\sqrt{3 - x^2} \end{aligned}$$

Solve the inequality $3 - x^2 \geq 0$ to find critical values $\pm\sqrt{3}$. From a number line, we find

$$D = \left(-\sqrt{3}, \sqrt{3}\right)$$

Range: Solve for x and look for exclusions for y. The range is the same as the domain (the steps are identical), namely,

$$D = \left(-\sqrt{3}, \sqrt{3}\right)$$

b. *Domain:* Solve for y and look for exclusions for x. The radicand must be nonnegative, so

$$3 - x^2 \geq 0$$

Critical values are $x = \pm\sqrt{3}$, and from a number line we find $D = \left[-\sqrt{3}, \sqrt{3}\right]$.
Range: Solve for x and look for exclusions for y.

$$\begin{aligned} y &= \sqrt{3 - x^2} \\ y^2 &= 3 - x^2 \\ x^2 &= 3 - y^2 \\ x &= \sqrt{3 - y^2} \qquad \text{Positive only because } y \geq 0. \end{aligned}$$

Compare this step in part **b** with the similar step in part **a**. These steps are often confused. Do you see why the "$\pm$" is needed in part **a** but is not needed here in part **b**?

Critical values are $y = \pm\sqrt{3}$, and from a number line, we find $R = \left[0, \sqrt{3}\right]$.

c. *Domain:* Solve for y and look for exclusions for x.

$$\begin{aligned} xy^2 - y^2 - 1 &= 0 \\ y^2(x - 1) &= 1 \\ y^2 &= \frac{1}{x - 1} \\ y &= \frac{\pm 1}{\sqrt{x - 1}} \end{aligned}$$

By inspection, $D = (1, \infty)$.

Range: Solve for x and look for exclusions for y.

$$\begin{aligned} xy^2 - y^2 - 1 &= 0 \\ xy^2 &= y^2 + 1 \\ x &= \frac{y^2 + 1}{y^2} \end{aligned}$$

For this equation, we see that x is real for all y except $y = 0$. Thus, $R = (-\infty, 0) \cup (0, \infty)$.

Intercepts

As we discussed in Section 1.3, the points where a graph intersects the coordinate axes are called *intercepts*. We restate the definition here in functional notation.

INTERCEPTS

If the number zero is in the domain of f and $f(0)=b$, then the point $(0,\ b)$ is called the ***y*-intercept** of the graph of f. If a is a real number in the domain of f such that $f(a)=0$, then $(a,\ 0)$ is an ***x*-intercept** of f. Any number x such that $f(x)=0$ is called a **zero of the function**.

Functions can have several x-intercepts (or no x-intercepts) but can have at most one y-intercept. (Do you see why?)

EXAMPLE 3 Finding the intercepts

Find the intercepts and determine whether each is a function.

a. $yx^2+y-1=0$ **b.** $x^2-xy^2+4y^2-1=0$

c. $|x|+|y|=4$ **d.** $x^{2/3}+y^{2/3}=16$

e. $f(x)=-x^2+x+2$

Solution

a. If we solve for y, we see that this is a function because for each x there is exactly one y-value.

y-intercept: Let $x=0$:
$$\begin{aligned} yx^2+y-1&=0\\ 0+y-1&=0\\ y&=1\end{aligned}$$

The y-intercept is $(0,\ 1)$. STOP *Functions can have at most one y-intercept.*

x-intercept(s): Let $y=0$:
$$\begin{aligned} yx^2+y-1&=0\\ 0+0-1&=0 \qquad \text{False}\end{aligned}$$

A false equation means that there is no point; in this example, there are no x-intercepts.

b. If we solve for y, we see that for each x there are two possible values of y. Thus, this is not a function.

y-intercept(s): Let $x=0$:
$$\begin{aligned} x^2-xy^2+4y^2-1&=0\\ 0^2-0+4y^2-1&=0\\ y^2&=\frac{1}{4}\\ y&=\pm\frac{1}{2}\end{aligned}$$

The y-intercepts are $\left(0,\ \frac{1}{2}\right)$, $\left(0,\ -\frac{1}{2}\right)$

x-intercept(s): Let $y=0$:
$$\begin{aligned} x^2-xy^2+4y^2-1&=0\\ x^2-0+0-1&=0\\ x&=\pm1\end{aligned}$$

The x-intercepts are $(1,\ 0)$, $(-1,\ 0)$.

c. This is not a function.

y-intercept(s): Let $x=0$:
$$\begin{aligned} |x|+|y|&=4\\ |0|+|y|&=4\\ y&=\pm4\end{aligned}$$

The y-intercepts are $(0,\ 4)$, $(0,\ -4)$.

x-intercept(s): Let $y=0$; the x-intercepts are found similarly to be $(4,\ 0)$, $(-4,\ 0)$.

d. This is not a function.

y-intercept(s): Let $x = 0$:
$$\begin{aligned} x^{2/3} + y^{2/3} &= 16 \\ 0^{2/3} + y^{2/3} &= 16 \\ (y^{2/3})^{3/2} &= (4^2)^{3/2} \\ |y| &= 4^3 \\ y &= \pm 64 \end{aligned}$$

The y-intercepts are $(0, 64)$, $(0, -64)$.

x-intercept(s): Let $y = 0$; by symmetry, the x-intercepts are $(64, 0)$, $(-64, 0)$.

e. This is given in function notation, so it is obviously a function.

y-intercept: Let $x = 0$:
$$\begin{aligned} f(x) &= -x^2 + x + 2 \\ f(0) &= -0^2 + 0 + 2 \\ &= 2 \end{aligned}$$

The y-intercept is $(0, f(0)) = (0, 2)$.

x-intercept(s): Let $y = f(x) = 0$; factoring, we find that
$$\begin{aligned} -x^2 + x + 2 &= 0 \\ x^2 - x - 2 &= 0 \\ (x+1)(x-2) &= 0 \\ x &= -1 \text{ or } x = 2 \end{aligned}$$

The x-intercepts are $(-1, 0)$ and $(2, 0)$.

Sometimes when you are graphing a curve, you want to find a point other than an intercept so that the point is in a certain region or with certain properties. For example, if you want to know one point on the line defined by the equation $2x + 3y - 4 = 0$, where $x > 5$, then you can choose any x-value satisfying $x > 5$ and find the corresponding y-value. Consider the following example.

EXAMPLE 4 Finding points satisfying specified conditions

Find a point on the curve defined by the equation $y = \dfrac{2x^2 - 3x + 5}{x - 3}$ that also satisfies the specified conditions, if it exists.

a. $x > 5$

b. passes through the line $y = -4$

c. passes through the line $y = 2x + 1$

Solution

a. Choose any value of $x > 5$, say, $x = 10$:
$$y = \frac{2(10)^2 - 3(10) + 5}{10 - 3} = \frac{175}{7} = 25$$

One possible point is $(10, 25)$.

b. Solve
$$\begin{aligned} -4 &= \frac{2x^2 - 3x + 5}{x - 3} \\ -4(x-3) &= 2x^2 - 3x + 5 \qquad x \neq 3 \\ 2x^2 + x - 7 &= 0 \\ x &= \frac{-1 \pm \sqrt{1 - 4(2)(-7)}}{2(2)} \\ &= \frac{-1 \pm \sqrt{57}}{4} \\ &\approx 1.6, -2.1 \end{aligned}$$

The given curve passes through the line $y = -4$ at approximately the points $(1.6, -4)$ and $(-2.1, -4)$.

c. Solve

$$\begin{aligned} 2x + 1 &= \frac{2x^2 - 3x + 5}{x - 3} \\ (2x+1)(x-3) &= 2x^2 - 3x + 5 \qquad x \neq 3 \\ 2x^2 - 5x - 3 &= 2x^2 - 3x + 5 \\ -2x &= 8 \\ x &= -4 \end{aligned}$$

The curve intersects the line $y = 2x + 1$ when $x = -4$ and when $y = 2(-4) + 1 = -7$. That is at the point $(-4, -7)$.

Properties of Functions

Several different properties of functions are useful in a variety of ways.

PROPERTIES OF FUNCTIONS

Let S be a subset of the domain of a function f. Then:

f is **increasing** on S if $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ in S;
f is **decreasing** on S if $f(x_1) > f(x_2)$ whenever $x_1 < x_2$ in S;
f is **constant** on S if $f(x_1) = f(x_2)$ for every x_1 and x_2 in S.

If the value a separates an interval over which f is increasing from an interval over which f is decreasing, then $(a, f(a))$ is a **turning point**.

Note the terminology; we say that the function is increasing and the graph is rising. We say that the function is decreasing and the graph is falling.

These properties are illustrated with the following example.

EXAMPLE 5 Properties of a function

Let $y = (x-5)^2 - 4 = x^2 - 10x + 21$ with the graph as shown in Figure 2.8.

a. Where are the intercepts?
b. Where is the turning point?
c. Where is the function increasing?
d. Where is the function decreasing?

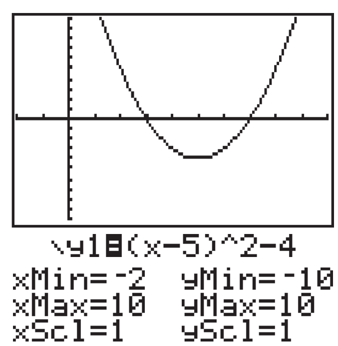


Figure 2.8 **Graph of $y = (x-5)^2 - 4$**

Solution *Note*: We state the intervals over which f is increasing, decreasing, or constant in terms of x, that is, the S that is a subset of the domain of f.

a. y-intercepts: If $x = 0$, then $y = 0^2 - 10(0) + 21 = 21$
x-intercepts: If $y = 0$, then

$$\begin{aligned} x^2 - 10x + 21 &= 0 \\ (x-7)(x-3) &= 0 \\ x &= 3, 7 \end{aligned}$$

The intercepts are $(0, 21)$, $(3, 0)$, and $(7, 0)$.

b. By inspection from the equation $y + 4 = (x-5)^2$, we note the turning point is the point $(5, -4)$.

c. We see the graph is rising to the right of the turning point, so we say the function is increasing on $(5, \infty)$.

d. We see the graph is falling to the left of the turning point, so we say the function is decreasing on $(-\infty, 5)$.

The information in Example 5 is frequently used in calculus, so we have superimposed the correct terminology on the graph in Figure 2.9.

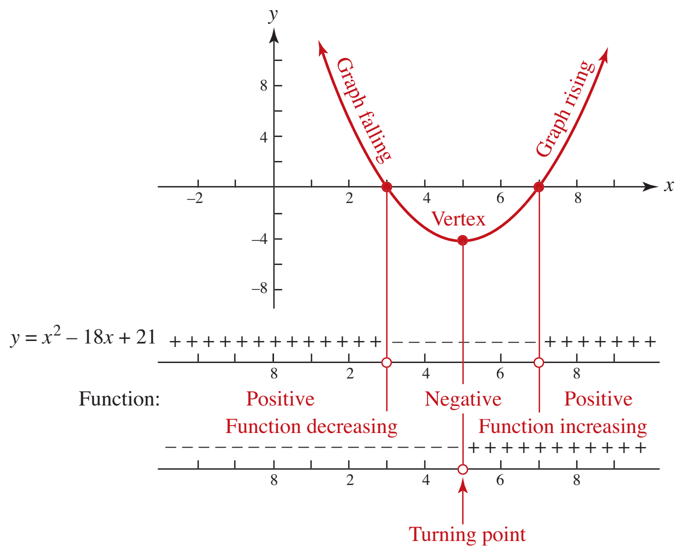


Figure 2.9 Terminology associated with the properties of functions

In the last section, we defined a *difference quotient*. There are several applications of difference quotient, and you might recall that this was the average distance we found in Example 8, Section 2.2. Another application involves the notion of an *average rate of change*.

AVERAGE RATE OF CHANGE

Let f be a function defined on some interval $[a, b]$. Then the **average rate** of change from $(a, f(a))$ to $(b, f(b))$ is the difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

You might note that the average rate of change of a function f between two points is the slope of the line (called the *secant line*) connecting these points. That is, in calculus this is often stated in terms of a starting point x, and an incremental change h so that $a = x$ and $b = x + h$, so that

$$\frac{\Delta y}{\Delta x} = \frac{f(x + h) - f(x)}{h}$$

EXAMPLE 6 Average rate of change of a function

Consider the function (from Example 5) defined by $f(x) = (x - 5)^2 - 4$

a. Find the average rate of change over the interval $[2, 6]$.

b. What is the equation of the secant line passing through $(2, f(2))$ and $(6, f(6))$?

Solution

a. $f(2) = (2-5)^2 - 4 = 5$
$f(6) = (6-5)^2 - 4 = -3$

The average rate of change is: $\dfrac{\Delta y}{\Delta x} = \dfrac{f(6) - f(2)}{6-2}$
$= \dfrac{-3-5}{4}$
$= -2$

b. The slope of the secant line is $m = -2$ (part **a**). We can use either one of the given points. We choose $(2, 5)$:

$$y - k = m(x - h) \quad \text{Point-slope form}$$
$$y - 5 = -2(x - 2) \quad \text{Substitute known values.}$$
$$y - 5 = -2x + 4 \quad \text{Simplify.}$$
$$2x + y - 9 = 0 \quad \text{Standard form}$$

Another classification of functions is related to the symmetry of its graph. A function whose graph is symmetric with respect to the y-axis is called **even**. A function whose graph is symmetric with respect to the origin is called **odd**. If the function is found to be even or odd, then the symmetry of its graph helps in the graphing of the function. This concept can be used to reduce (by half) the amount of work necessary on many problems.

EVEN AND ODD FUNCTIONS

A function f is called

even if $f(-x) = f(x)$ and

odd if $f(-x) = -f(x)$

for all x in the domain of f.

Just as not every real number is even or odd (2 is even, 3 is odd, but 2.5 is neither), not every function is even or odd.

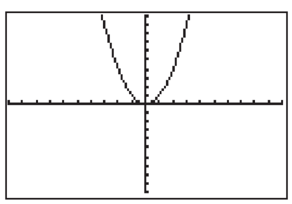
EXAMPLE 7 Even and odd functions

Classify the given functions as even, odd, or neither.

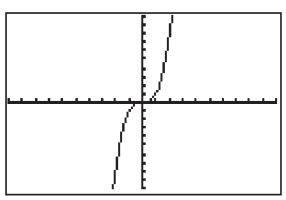
a. $f(x) = x^2$
b. $g(x) = x^3$
c. $h(x) = x^2 + 5x$

Solution

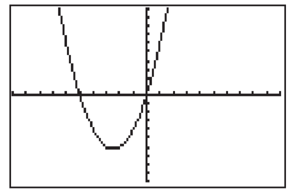
a. $f(x) = x^2$ is *even* because
$f(-x) = (-x)^2 = x^2 = f(x)$
The graph at the right shows that the graph of the even function $f(x) = x^2$ is symmetric with respect to the y-axis.



b. $g(x) = x^3$ is *odd* because
$g(-x) = (-x)^3 = -x^3 = -g(x)$
The graph at the right shows that the graph of the odd function $g(x) = x^3$ is symmetric with respect to the origin.



c. $h(x) = x^2 + 5x$ is *neither* because
$h(-x) = (-x)^2 + 5(-x) = x^2 - 5x$
Note that $h(-x) \neq h(x)$ and $h(-x) \neq -h(x)$.



xMin=-10 yMin=-10
xMax=10 yMax=10
xScl=1 yScl=1

\y1=x^2+5x

PROBLEM SET 2.3

LEVEL 1

Sketch each of the functions in Problems 1–2, and classify each as odd, even, or neither.

1. a. identity function
b. absolute value function
c. standard reciprocal function
d. standard square root reciprocal function

2. a. standard quadratic function
b. square root function
c. standard reciprocal squared function
d. standard cubic function

3. IN YOUR OWN WORDS What is the graph of a function?

4. IN YOUR OWN WORDS How do you find the domain and range of a function?

5. IN YOUR OWN WORDS Distinguish between an x-intercept and a zero of a function.

6. IN YOUR OWN WORDS In the book we state that "functions can have several x-intercepts (or no x-intercepts) but can have at most one y-intercept. Explain why this is true.

State whether the functions f and g defined in Problems 7–10 are equal.

7. a. $f(x) = \dfrac{3x^2 + x}{x}$; $g(x) = 3x + 1$
b. $f(x) = \dfrac{2x^2 - 7x - 4}{x - 4}$; $g(x) = 2x + 1,\ x \neq 4$

8. a. $f(x) = \dfrac{(3x+1)(x-4)}{x-4}$; $g(x) = 3x + 1$
b. $f(x) = \dfrac{3x^2 - 5x - 2}{3x + 1}$; $g(x) = x - 2$

9. a. $f(x) = \dfrac{2x^2 - x - 6}{x - 2}$; $g(x) = 2x + 3,\ x \neq 2$
b. $f(x) = \dfrac{3x^2 - 5x - 2}{x - 2},\ x \neq 2$; $g(x) = 3x + 1$

10. a. $f(x) = \dfrac{(3x+1)(x-2)}{x-2},\ x \neq 6$
$g(x) = \dfrac{(3x+1)(x-6)}{x-6},\ x \neq 2$
b. $f(x) = \dfrac{(5x-1)(x+4)}{x+4},\ x \neq -4$
$g(x) = \dfrac{(5x-1)(x-2)}{x-2},\ x \neq 2$

Find the domain for the functions defined by the equations in Problems 11–16.

11. a. $f(x) = 2x - 3$
b. $g(x) = 2x - 3,\ x \neq 1$

12. a. $f(x) = \dfrac{(2x+1)(x-1)}{x-1}$
b. $g(x) = 2x + 1$

13. a. $f(x) = \dfrac{(3x+1)(x-3)}{x^2+2}$
b. $g(x) = x^2 + 3x - 5$

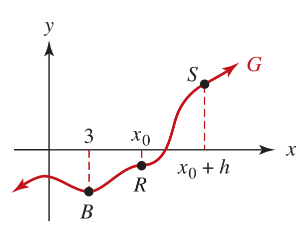
14. a. $f(x) = \sqrt{3x + 9}$
b. $g(x) = \sqrt{x^2 - 4}$

15. $f(x) = \sqrt{2 - x - x^2}$

16. $g(x) = \sqrt{2 + x - x^2}$

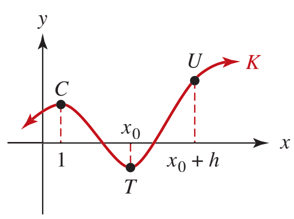
PROBLEMS FROM CALCULUS *Graphs similar to those shown in Problems 17 and 18 are common in calculus. Specify the coordinates of the requested points by looking at the given graphs.*

17. a. Point R
b. Point S



Graph of G

18. **a.** Point T
b. Point U

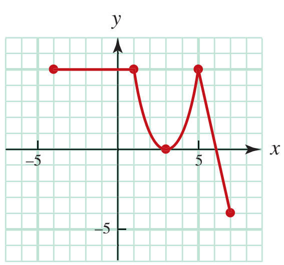


Graph of K

LEVEL 2

Find the domain, intercepts, and turning points of the functions defined by the graphs indicated in Problems 19–30. Also give the intervals for which the function is constant, where it is increasing, and where it is decreasing.

19.



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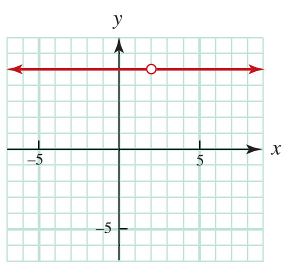

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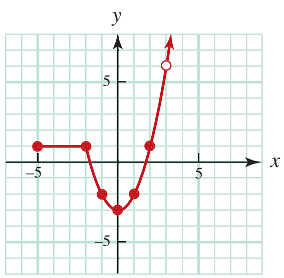

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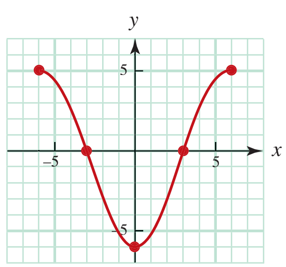

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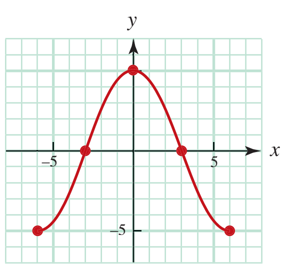

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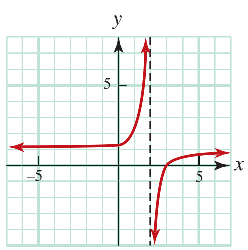

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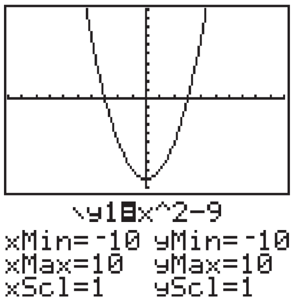

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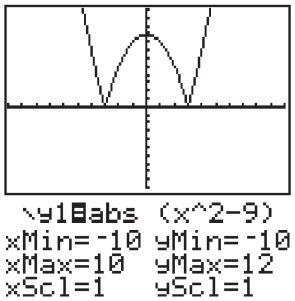

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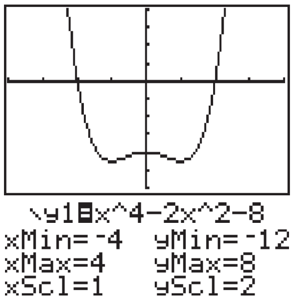

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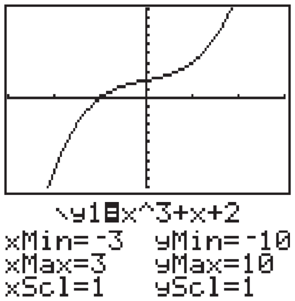

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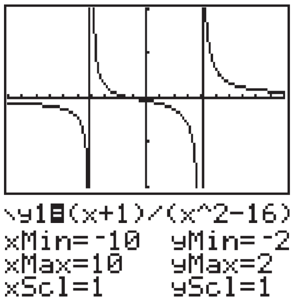

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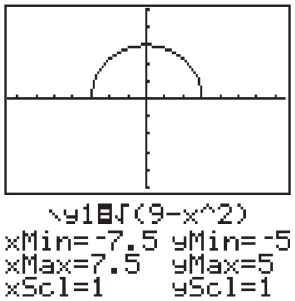


PROBLEMS FROM CALCULUS *In calculus, the average rate of change of a function f between x and $x + h$ is defined to be the quantity*

$$\frac{f(x+h) - f(x)}{h}$$

Find the average rate of change from 2 *to* $2 + h$ *for the functions in Problems 31–34.*

31. Identity function, $f(x) = x$.
32. Standard quadratic function, $f(x) = x^2$.
33. Standard reciprocal function, $f(x) = 1/x$.
34. Consider the average rate of change for the standard quadratic function, $f(x) = x^2$.
- **a.** Which is larger, the average rate of change from 2 to 3 or from 10 to 11?
- **b.** What is the average rate of change from 2 to 2.1?
- **c.** What is the average rate of change from 2 to 2.01?
- **d.** What is the average rate of change from 2 to 2.001?
- **e.** What value does the sequence of calculation seem to be approaching?

In calculus, this value is called the *instantaneous rate of change.*

Find the domain and range for the graphs defined by the equations in Problems 35–52. If the equation does not represent a function, so state, and if it does, classify it as even, odd, or neither.

35. $y = x + 4$
36. $y = \sqrt{x-4}$
37. $y = x^3$
38. $y = \sqrt{x}$
39. $y = 8x^3$
40. $y = \sqrt[3]{x}$
41. $xy = 1$
42. $y = x^2 + 4$
43. $y = x^2 - 8$
44. $y = \sqrt{x^2 - 9}$
45. $y = \sqrt{x^2 - 4}$
46. $y = \sqrt{x^3 - 9x}$
47. $y = \sqrt{x^2 + x - 12}$
48. $y = \dfrac{3}{x^2 + 1}$
49. $xy + 6 = 0$
50. $y = \dfrac{x^2 - 4}{x + 1}$
51. $2|x| - |y| = 5$
52. $|y| + 3|x| = 5$

53. Find the points (if any) where the curve defined by the equation
$$y = \frac{5x^2 - 8x}{2x + 1}$$
crosses the horizontal line $y = 3$.

54. Find the points (if any) where the curve defined by the equation
$$y = \frac{5x^2 - 8x}{2x + 1}$$
crosses the vertical line $x = -4$.

55. Find the points (if any) where the curve defined by the equation
$$y = \frac{2x^3 + 2x}{x^2 - 2}$$
crosses the vertical line $x = -1$.

56. Find the points (if any) where the curve defined by the equation
$$y = \frac{5x^2 - 8x}{2x - 1}$$
crosses the horizontal line $y = -4$.

57. Find the points (if any) where the curve defined by the equation
$$y = \frac{x^3 + 2x^2 - 2x}{x^2 - 2}$$
crosses the line $y = x + 1$.

58. Find the points (if any) where the curve defined by the equation
$$y = \frac{3x^3 + 4x^2 + 3}{3x^2 + 1}$$
crosses the horizontal line $y = x + 2$.

LEVEL 3

59. If f is increasing throughout its domain, prove that f is one-to-one.

60. If a function f is decreasing throughout its domain, prove that f is one-to-one.

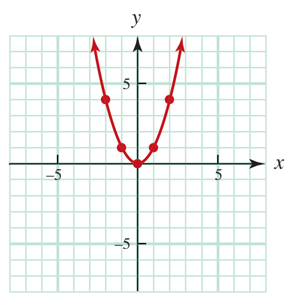
2.4 Transformations of Functions

Sometimes the graph of a function can be sketched by translating or reflecting the graph of a related function. We call these translations and reflections *transformations* of a function.

Translations

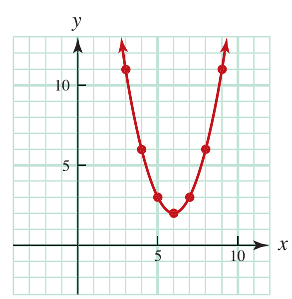
We begin by considering two examples. Graph the functions $y = x^2$ and $y - 2 = (x - 6)^2$ by plotting points, as shown in Figure 2.10.

x	$y = x^2$
0	0
1	1
−1	1
2	4
−2	4
3	9
−3	9



a. Graph of $y = x^2$

x	$y = (x - 6)^2 + 2$
0	38
1	27
2	10
3	11
4	6
5	3
6	2
7	3
8	6
9	11



b. Graph of $y - 2 = (x - 6)^2$

Figure 2.10 **Graphing by plotting points**

Notice that the graphs in Figure 2.10 are identical, except they are in different locations. You also should have noticed (if you did the arithmetic) that the first table of values was much easier to calculate than the second. When two curves are congruent (have the same size and shape) and have the same orientation, we say that one can be found from the other by a **shift** or **translation**.

TRANSLATION

The graph defined by the equation

$$y - k = f(x - h)$$

is said to be a **translation** of the graph defined by $y = f(x)$. The translation (shift, as shown in Figure 2.10) is

to the right if $h > 0$

to the left if $h < 0$

up if $k > 0$

down if $k < 0$

» IN OTHER WORDS The procedure for graphing a translated graph is a two-step process:

(1) Plot (h, k). The numbers h and k are directed distances.
Horizontal translation $|h|$ units; to the right if h is positive and to the left if h is negative.
Similarly, the vertical translation is $|k|$ units; up if k is positive and down if k is negative.

(2) Graph the curve $y = f(x)$ using (h, k) as the starting point.

EXAMPLE 1 Translations of a standard curve

Given the standard quadratic function $y = x^2$ (see Table 2.1). Graph the given curves by translation.

a. $y - 4 = x^2$ **b.** $y + 6 = x^2$ **c.** $y = (x - 5)^2$ **d.** $y = (x + 5)^2$

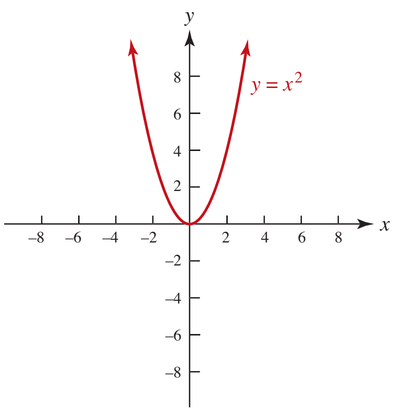
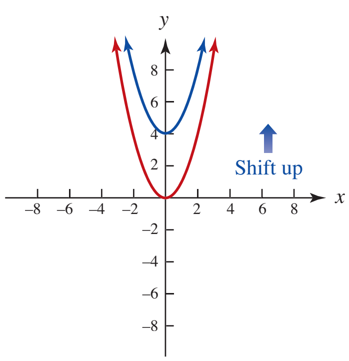


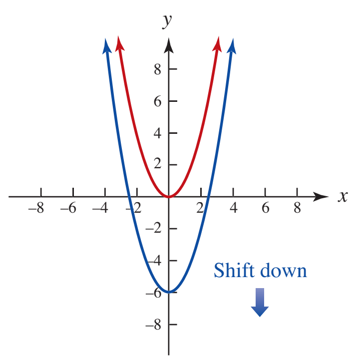
Figure 2.11 **Graph of $y = x^2$**

Solution Begin with the graph of $y = x^2$, as shown in Figure 2.11. The vertex of this standard quadratic function is $(0, 0)$. We call this the starting point.

a. Write $y - 4 = x^2$ as $y - k = (x - h)^2$, and compare it to $y = x^2$ to see that $(h, k) = (0, 4)$. Draw the curve shown in Figure 2.11 with the starting point shifted up 4 units.

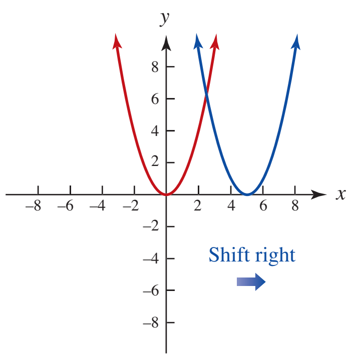


b. Write $y + 6 = x^2$ as $y - k = (x - h)^2$, and compare it to $y = x^2$ to see that $(h, k) = (0, -6)$ Draw the curve shown in Figure 2.11 with the starting point shifted down 6 units.

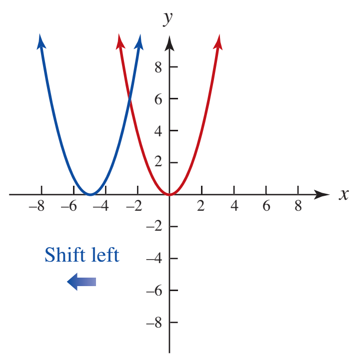


Compare the equations in parts **a** and **b** with the shift up and shift down directions of the graph.

c. Write $y = (x-5)^2$ as $y - k = (x-h)^2$, and compare it to $y = x^2$ to see that $(h, k) = (5, 0)$. Draw the curve shown in Figure 2.11 with the starting point shifted to the right 5 units.



d. Write $y = (x+5)^2$ as $y - k = (x-h)^2$, and compare it to $y = x^2$ to see that $(h, k) = (-5, 0)$. Draw the curve shown in Figure 2.11 with the starting point shifted to the left 5 units.

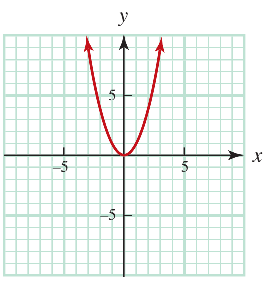


Compare the equations in parts **c** and **d** with the shift right and shift left directions of the graph. ■

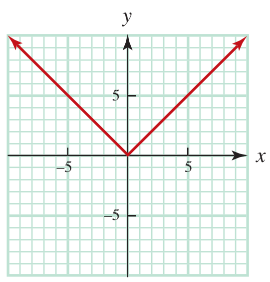
EXAMPLE 2 Translations of different standard curves

Graph: **a.** $y - \frac{1}{2} = \left(x - \frac{3}{2}\right)^2$ **b.** $f(x) = |x-3| + 2$ **c.** $y = \sqrt{x-2} - 3$

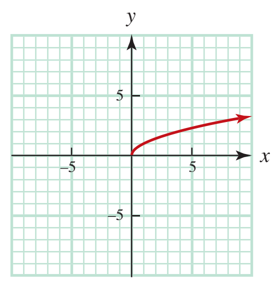
Solution We begin by looking at Table 2.1.



a. $y = x^2$



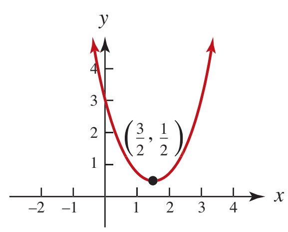
b. $y = |x|$



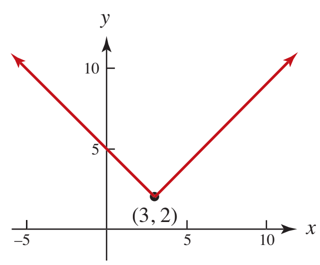
c. $y = \sqrt{x}$

a. First, plot $\left(\frac{3}{2}, \frac{1}{2}\right)$. Translate the standard quadratic function to this point, as shown at the right.

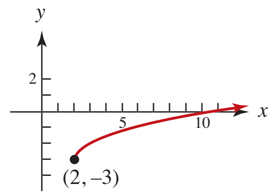
⚠ Translate the graph, do NOT calculate values.



b. This equation can be written as $y - 2 = |x - 3|$, which is identical to the graph of the function $y = |x|$ from Table 2.1. Identify and plot the point $(h, k) = (3, 2)$. The graph is shown at the right.



c. Rewrite as $y + 3 = \sqrt{x - 2}$. This graph is the same as the square root function $y = \sqrt{x}$ translated to the point $(h, k) = (2, -3)$. The graph is shown at the right.



Reflections

In Chapter 1, we introduced the notion of symmetry. In terms of functions and functional notation, we restate those notions of symmetry with respect to the coordinate axes as **reflections.**

REFLECTION

A **reflection in the x-axis** of the graph of $y = f(x)$ is the graph of

$$y = -f(x)$$

A **reflection in the y-axis** of the graph of $y = f(x)$ is the graph of

$$y = f(-x)$$

» IN OTHER WORDS The graph is reflected in the x-axis if we replace y by $-y$ in its equation; it is reflected in the y-axis if we replace x by $-x$.

This procedure is illustrated in Figure 2.12, in which we have sketched the graph of $y = x^2$ and then reflected that graph.

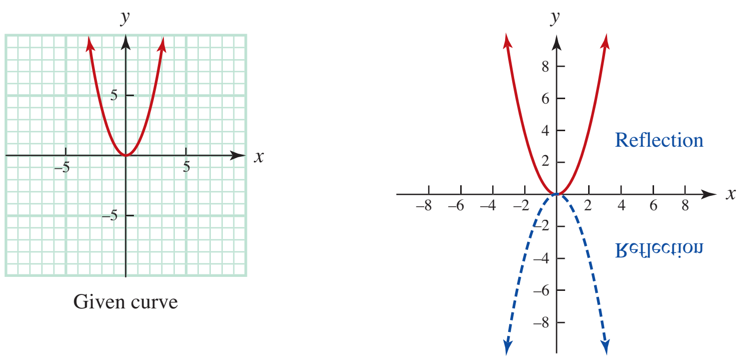


Figure 2.12 Reflection of $y = x^2$

EXAMPLE 3 Graphing with a reflection

Graph $y = -\sqrt{x}$.

Solution The graph of this function is a reflection in the x-axis of the graph of $y = \sqrt{x}$, as shown in Figure 2.13.

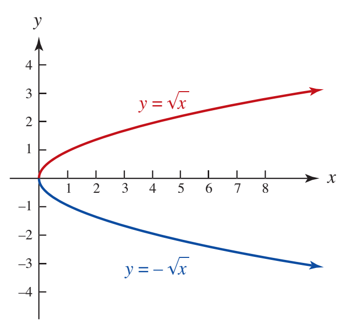


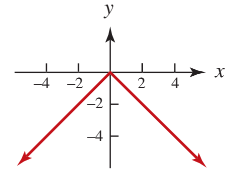
Figure 2.13 Reflection in the x-axis of $y = \sqrt{x}$

EXAMPLE 4 Reflections of the absolute value function

Graph **a.** $y = -|x|$ and **b.** $y = |-x|$.

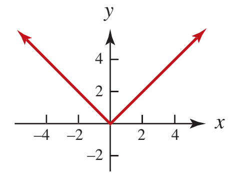
Solution

a. Recall the graph of $y = |x|$ from Table 2.1. Replacing y by $-y$ reflects the graph of $y = |x|$ in the x-axis, as shown at the right.



Graph of $y = -|x|$

b. Because x has been replaced by $-x$, the graph of $y = |-x|$ will be a reflection of $y = |x|$ in the y-axis. Note, too, that the graphs of these functions are identical. This is because (see Table 1.2, property 2) $|-a| = |a|$ for any real number a.



Graph of $y = |-x|$

An application of this reflection property helps us to graph curves such as

$$y = |(x-3)^2 - 2|$$

If $y = f(x)$, where

$$f(x) = (x-3)^2 - 2$$

the graph we seek is of the form $y = |f(x)|$. We begin by graphing

$$f(x) = (x-3)^2 - 2$$

which is the standard quadratic function shown in Table 2.1 translated to the point $(3, -2)$ as shown in Figure 2.14.

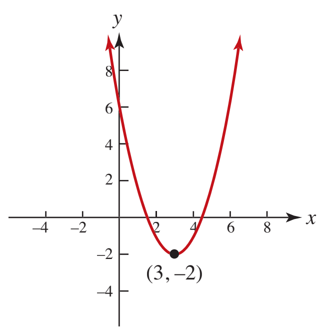


Figure 2.14 Graph of $y = x^2$ translated to $(3, -2)$

Notice that part of the graph is above the x-axis and part is below the x-axis. Since the absolute value of a positive value leaves the positive value unchanged, the absolute value on $y = |f(x)|$ will leave the graph unchanged above the x-axis. However, the portion below the x-axis will be reflected above the x-axis because of the definition of absolute value. The desired graph is shown in Figure 2.15.

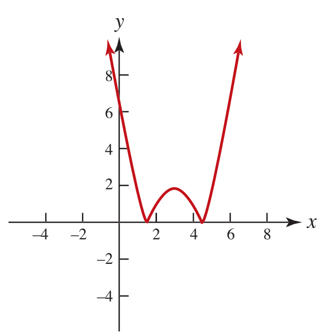


Figure 2.15 Graph of $y = |(x-3)^2 - 2|$

Contrast the graph shown in Figure 2.15 with the graph of the function

$$y = (|x| - 3)^2 - 2$$

For this curve, we also begin with the standard quadratic function shown in Table 2.1 and translate to the point $(3, -2)$. In this case, we note that for each value of x there is a corresponding value $-x$, which also satisfies the equation. Thus, we see that this curve is simply a reflection about the y-axis as shown in Figure 2.16.

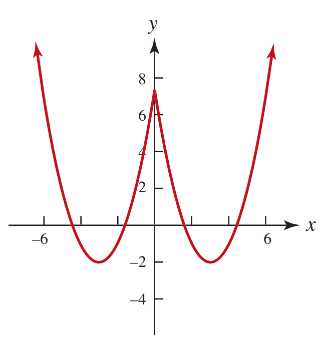


Figure 2.16 Graph of $y = (|x| - 3)^2 - 2$

We summarize our findings involving the graph of an absolute value of a function.

ABSOLUTE VALUE GRAPHS

The graph of $y = |f(x)|$ is found by graphing $y = f(x)$ and then reflecting all points of the graph that are below the x-axis through the x-axis.

The graph of $y = |f(x)|$ is an even function, so the graph is found by graphing $y = f(x)$ and then *in addition to those points* drawing all points reflected through the y-axis.

Dilations and Compressions

You may be familiar with the dilation of an eye or a compression fracture in your spine. To understand these concepts, we begin by observing that a curve can be compressed or dilated in either the x-direction, the y-direction, or both, as shown in Figure 2.17.

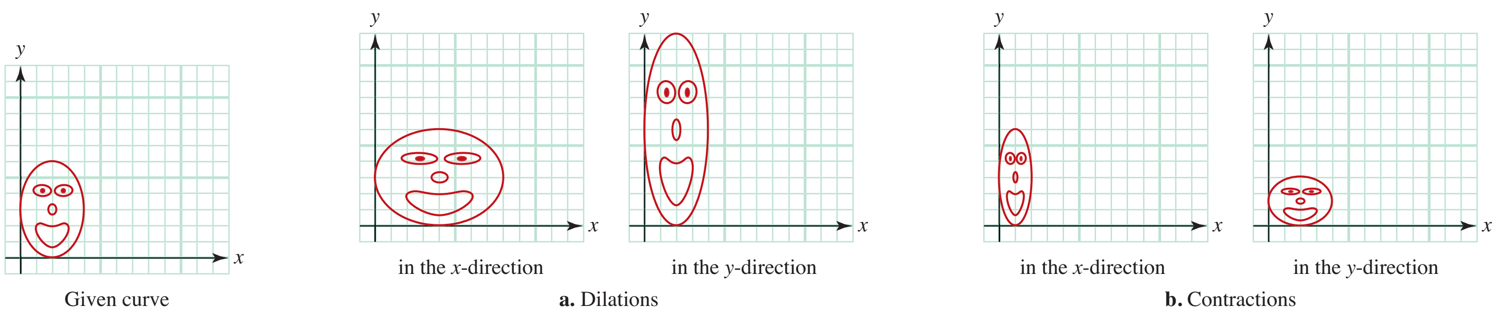


Figure 2.17 Dilations and contractions of a given curve

DILATIONS AND CONTRACTIONS

To sketch the graph of $y = af(x)$, replace each point (x, y) with (x, ay)

If $a > 1$, then we call the transformation a **y-dilation.**

If $0 < a < 1$, then we call the transformation a **y-compression.**

To sketch the graph of $y = f(bx)$, replace each point (x, y) with $\left(\frac{1}{b}x, y\right)$

If $0 < b < 1$, then we call the transformation an **x-dilation.**

If $b > 1$, then we call the transformation an **x-compression.**

Dilations and Compressions in the y-direction

We are interested in modifications of a known function, which we call f. Consider the graph of $y = f(x)$ as shown in Figure 2.18**a**. The graph of $y = 2f(x)$ has the same shape except that each y-value is double the corresponding y-value of f. On the other hand, the graph of $y = \frac{1}{2}f(x)$ has the same shape except that each y-value is one-half the corresponding y-value of f. These graphs are shown in Figures 2.18**b** and **c**.

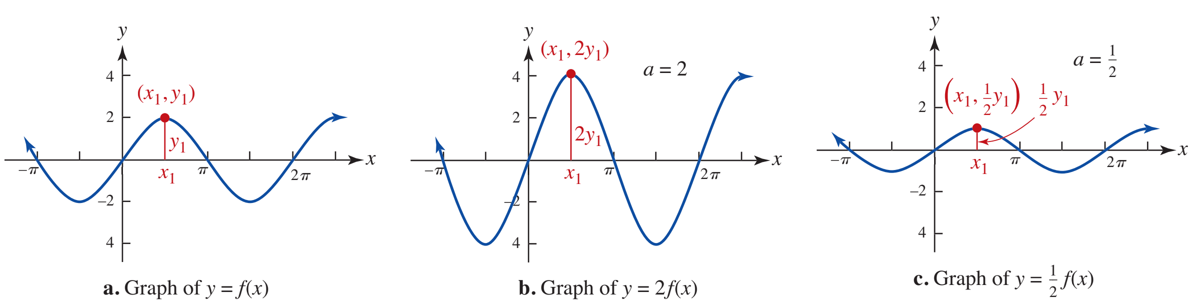
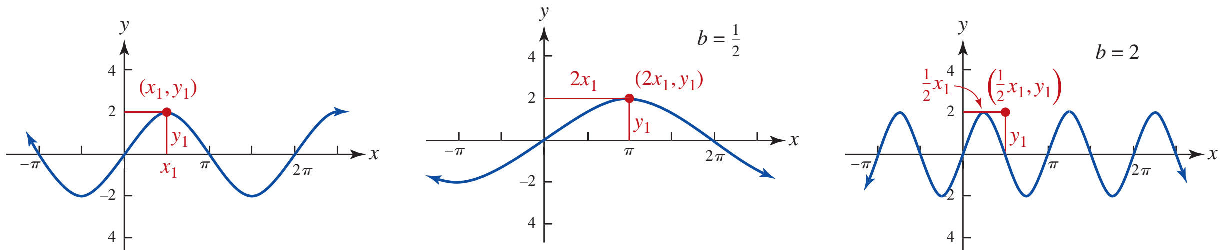


Figure 2.18 Dilations and compressions in the y-direction for a given function

Dilations and Compressions in the *x*-direction

To describe a dilation and compression in the x-direction, we will consider the function $y = f(x)$ and examine the effect of $y = f(bx)$. If we take a particular value of y, then it follows that the value bx is plotted in the same y-value as the x-value in the original curve. This means that to graph $y = f(bx)$, replace each point (x, y) on the graph of $y = f(x)$ with the point $\left(\frac{1}{b}x, y\right)$ See Figure 2.19.



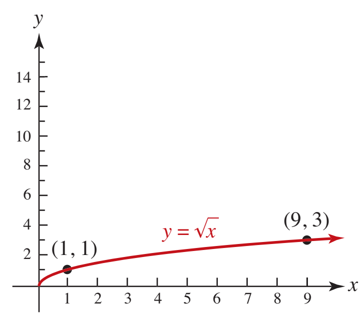
a. Graph of $y = f(x)$ **b.** Graph of $y = f\left(\frac{1}{2}x\right)$ **c.** Graph of $y = f(2x)$

Figure 2.19 Dilations and compressions in the x-direction for a given function

EXAMPLE 5 Compressions and dilations

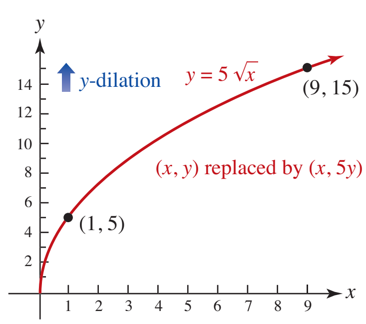
Graph each function and describe each as a dilation or compression.

a. $y = 5\sqrt{x}$ **b.** $y = \sqrt{5x}$ **c.** $y = \left|\frac{1}{5}x\right|$ **d.** $y = \frac{1}{10}x^2$

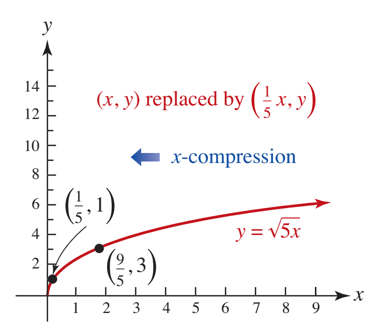


Solution Compare parts **a** and **b** with the standard square root curve from Table 2.1, which we repeat here in the margin. Note that we have plotted two points in particular, namely, $(1, 1)$ and $(9, 3)$.

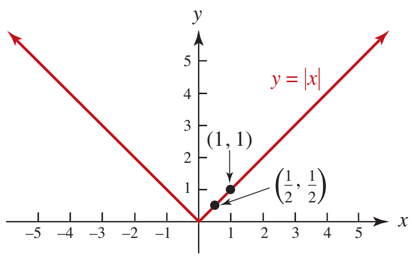
a. $y = 5\sqrt{x}$ is a y-dilation when compared with $f(x) = \sqrt{x}$. We see $a = 5 > 1$; (x, y) is replaced by $(x, 5y)$.



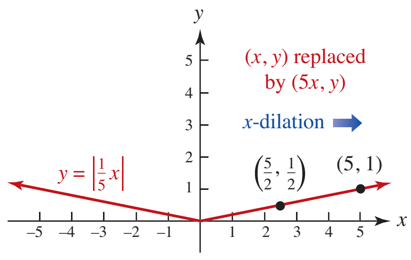
b. $y = \sqrt{5x}$ is an x-compression when compared with $f(x) = \sqrt{x}$. We see $b = 5$, so $\frac{1}{b} = \frac{1}{5}$; (x, y) is replaced by $\left(\frac{1}{5}x, y\right)$.



c. $y = \left|\frac{1}{5}x\right|$ is an x-dilation when compared with $f(x) = |x|$. We see $b = \frac{1}{5}$, so $0 < b < 1$; (x, y) is replaced by $(5x, y)$ because $\frac{1}{b} = 5$.

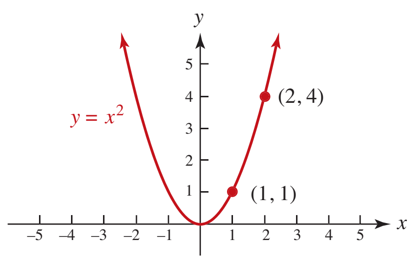


Graph of $y = |x|$ from Table 2.1

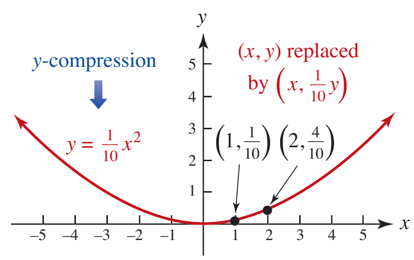


Graph of $y = \left|\frac{1}{5}x\right|$ as a dilation

d. $y = \frac{1}{10}x^2$ is a y-compression when compared with $f(x) = x^2$. We see $a = \frac{1}{10}$, so $0 < a < 1$; (x, y) is replaced by $\left(x, \frac{1}{10}y\right)$.



Graph of $y = x^2$ from Table 2.1



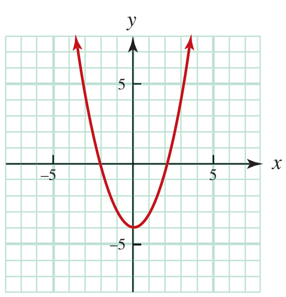
Graph of $y = \frac{1}{10}x^2$ as a compression

PROBLEM SET 2.4

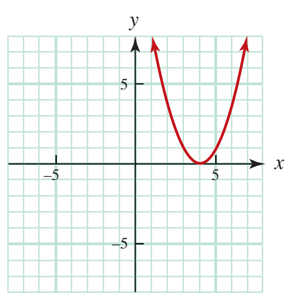
LEVEL 1

Each of the graphs in Problems 1–10 is a translation of one of the curves from Table 2.1. Write the equation of the curves illustrated in Problems 1–10.

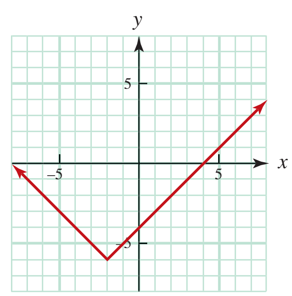
1.



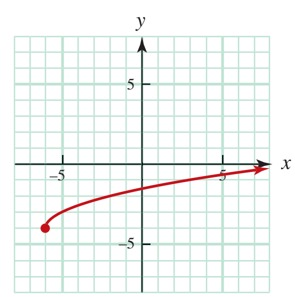
2.



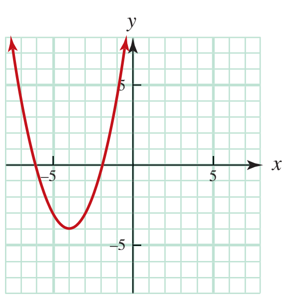
5.



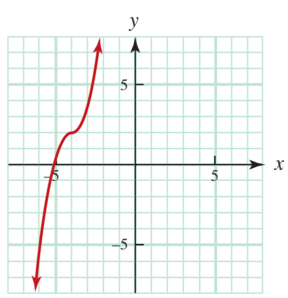
6.



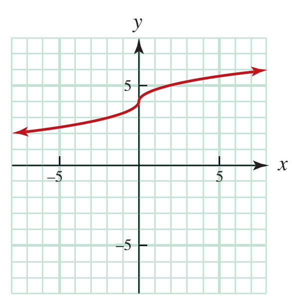
3.



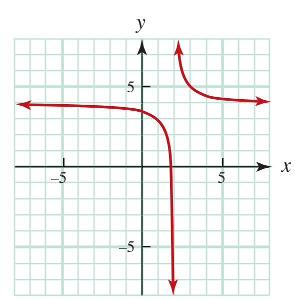
4.



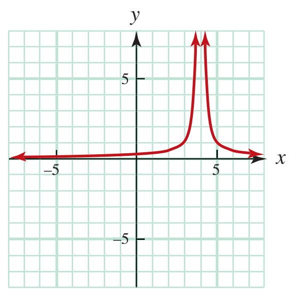
7.



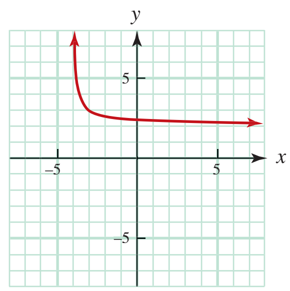
8.



9.



10.



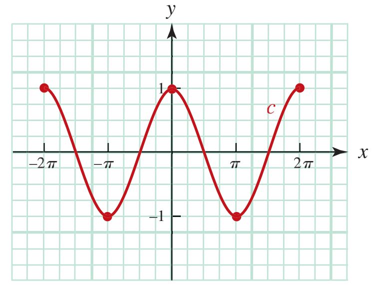
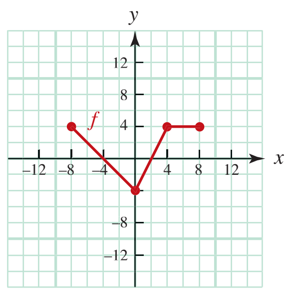
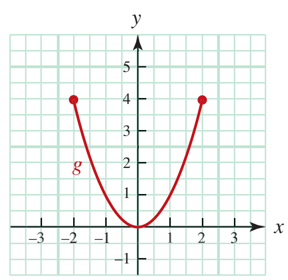


Figure 2.21 Graph of c

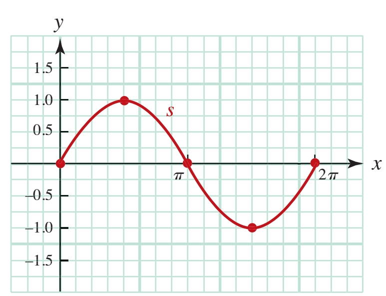
Let f, g, and s be the functions whose graphs are shown in Figure 2.20. Graph the functions indicated by the equations in Problems 11–20.



Graph of f



Graph of g



Graph of s

Figure 2.20 Graphs of curves for Problems 11–20

11. $y + 4 = f(x)$
12. $y - 4 = g(x)$
13. $y + \frac{1}{2} = s(x)$
14. $y - \pi = f(x)$
15. $y = g(x - 3)$
16. $y = s\left(x + \frac{\pi}{2}\right)$
17. $y = f(x + 4)$
18. $y - 4 = g(x - 3)$
19. $y - 1 = s\left(x + \frac{\pi}{2}\right)$
20. $y - \pi = f(x + 4)$

Let $y = c(x)$ be the function whose graph is given in Figure 2.21. Graph the curves indicated by the equations in Problems 21–30.

21. $y = 3c(x)$
22. $y = \frac{1}{2}c(x)$
23. $y = -c(x)$
24. $y = -2c(x)$
25. $y = c(2x)$
26. $y = c\left(\frac{1}{2}x\right)$
27. $y = c\left(x - \frac{\pi}{2}\right)$
28. $y + 1 = c\left(x - \frac{\pi}{2}\right)$
29. $y = 2c(x + 1)$
30. $y = \frac{3}{2}c(x - 2) + \sqrt{2}$

LEVEL 2

Graph the curves defined by the equations given in Problems 31–48. Do not plot points to graph these equations, but treat them as transformations of graphs from Table 2.1.

31. $y - 3 = (x - 2)^2$
32. $y = (x + 3)^2$
33. $y = x^2 - 1$
34. $y - 1 = |x - 7|$
35. $y = \sqrt{x} + 4$
36. $y = \sqrt{x + 4}$
37. $y = |x + \pi|$
38. $y = |x| - 6$
39. $y - \frac{\pi}{4} = (x + \pi)^2$
40. $y + 2 = \left(x + \sqrt{3}\right)^2$
41. $y + \pi = \sqrt{x - 3}$
42. $y = -3|x + 5|$
43. $y = 4\sqrt{x - 2}$
44. $y = -2(x + 4)^2$
45. $y + \sqrt{3} = \left|x - \sqrt{2}\right|$
46. $y - \sqrt{2} = \left(x + \sqrt{5}\right)^2$
47. $y - \sqrt{2} = \sqrt{x + 5}$
48. $y - 3 = -\frac{1}{2}(x + 2)^2$

LEVEL 3

Graph the equations in Problems 49–60. Do not graph these by plotting points.

49. $y = |(x - 4)^2 - 9|$
50. $y = |(x - 2)^2 - 4|$
51. $y = (|x| - 2)^2 - 4$
52. $y = (|x| - 4)^2 - 9$
53. $y + 9 = (x + 8)^2$, such that $-14 \le x < -8$
54. $y - 2 = (x + 3)^2$, such that $-7 < x \le -2$
55. $y + 12 = \left(x + \frac{25}{2}\right)^2$, such that $y > -10$
56. $y + 3 = (x + 3)^2$, such that $y < 6$
57. $y + 12 = (x - 8)^2$, such that $y < 4$
58. $y - 5 = 2|x - 1|$, such that $-4 \le x \le 2$
59. $y - 2 = \dfrac{1}{(x - 3)^2}$, such that $y < 5$
60. $y - 4 = \frac{1}{10}x^2$, such that $-2 \le x \le 2$

2.5 Piecewise Functions

Definition

There are many everyday examples of functions that cannot be defined in terms of a single equation. For example, suppose electricity is \$0.325/kwh for the first 1,000 kwh and then drops to \$0.088/kwh for usage between 1,000 and 3,000 kwh (including 3,000 kwh). A graph of this function is shown in Figure 2.22.

We see that the graph passes the vertical line test, so it is a function. To write an equation for the electric charges, C, we need to break up the domain into "pieces." If x is the kilowatt hours (kwh) and y is the price charged for the electric usage, then we find the domain: We see that $x \geq 0$. Two equations must be written, one for $0 \leq x \leq 1{,}000$ and another for $1{,}000 < x \leq 3{,}000$:

$$\begin{aligned} y &= 0.325x && \text{for } 0 \leq x \leq 1{,}000 \\ y &= 0.088x + 325 && \text{for } 1{,}000 < x \leq 3{,}000 \end{aligned}$$

Such a function is called a *piecewise function* and is usually defined using a brace:

$$C(x) = \begin{cases} 0.325x & \text{if} \quad 0 \leq x \leq 1{,}000 \\ 0.088x + 325 & \text{if} \quad 1{,}000 < x \leq 3{,}000 \end{cases}$$

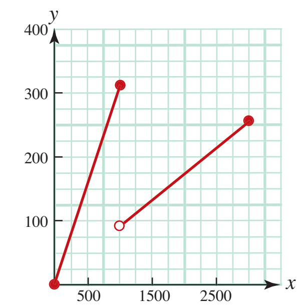


Figure 2.22 Electric charges function

PIECEWISE FUNCTION

A function whose domain D can be separated into a finite number of pieces such that the function has a different definition for each piece of the domain is called a **piecewise function.**

EXAMPLE 1 Graphing a piecewise function

Graph: $f(x) = \begin{cases} 3 - x & \text{if } -5 \leq x < 2 \\ x - 2 & \text{if } 2 \leq x \leq 5 \end{cases}$

Solution The domain is $[-5, 5]$, and we graph the line segments for each of the separate parts:

Graph $y = 3 - x$ on $[-5, 2]$.
Graph $y = x - 2$ on $[2, 5]$.

The graph is shown in Figure 2.23.

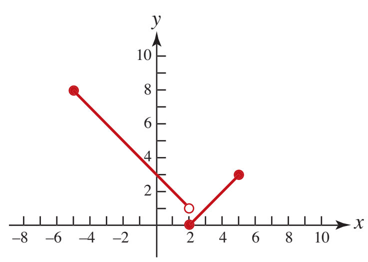


Figure 2.23 Graph of f

EXAMPLE 2 Example from calculus

Graph: $g(x) = \begin{cases} \dfrac{x^2 + 3x - 10}{x - 2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$

Solution The first piece is a rational function with a deleted point. We see (if $x \neq 2$)

$$g(x) = \frac{x^2 + 3x - 10}{x - 2} = \frac{(x - 2)(x + 5)}{x - 2} = x + 5$$

The second piece is a single point, namely, $(2, 4)$. The graph is shown in Figure 2.24.

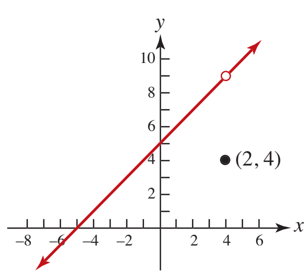


Figure 2.24 Graph of g

Consider an example very much like Example 2.

$$G(x) = \begin{cases} \dfrac{x^2 + 3x - 10}{x - 2} & \text{if } x \neq 2 \\ 7 & \text{if } x = 2 \end{cases}$$

By looking at Figure 2.24, we see that the point $(2, 7)$ for G "plugs the hole" in the deleted point from the first piece of the curve. This concept will be used in calculus when you study the topic of *continuity*.

Absolute Value Function

In Section 1.1, we defined absolute value as

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The **absolute value function** is defined by $f(x) = |x|$. In Section 1.3, Example 1, we graphed $y = \left|\frac{1}{2}x - 2\right|$ by comparing it with a standard absolute value function. We now can treat the absolute value function as a piecewise function. That is, for $y = \left|\frac{1}{2}x - 2\right|$, we note that if $\frac{1}{2}x - 2 \geq 0$, then $x \geq 4$ graph

$$f(x)\begin{cases} \frac{1}{2}x - 2 & \text{if } x \geq 4 \\ -\left(\frac{1}{2}x - 2\right) & \text{if } x < 4 \end{cases}$$

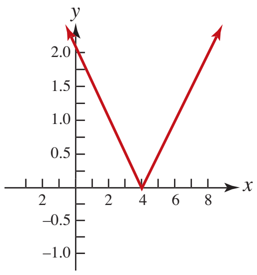


Figure 2.25 Graph of $y = \left|\frac{1}{2}x - 2\right|$

The graph is shown in Figure 2.25.

You might also notice that if we write

$$= \left|\frac{1}{2}x - 2\right| = \left|\frac{1}{2}(x - 4)\right| = \frac{1}{2}|x - 4|$$

we can consider the graph of y as a translation of $f(x) = \frac{1}{2}|x|$ where $(h, k) = (4, 0)$.

Greatest Integer Function


© CBS/Landov

The Price Is Right (1972–present) is one of several games dubbed *pricing games* where the contestants guess the price of an item, and the contestant coming the *closest without exceeding the true value* wins the item. This, basically, is the idea of another function, the *greatest integer function,* or *step function*. This rule provides a means by which to assign an integral value to the function.

Consider the following:

$$G(x) = \begin{cases} 0 & \text{if } 0 \leq x < 1 \\ 1 & \text{if } 1 \leq x < 2 \\ 2 & \text{if } 2 \leq x < 3 \\ 3 & \text{if } 3 \leq x < 4 \\ 4 & \text{if } 4 \leq x < 5 \end{cases}$$

Note that the domain for this function is $[0, 5)$ and the range is $\{0, 1, 2, 3, 4\}$. Here is the evaluation of some specific values:

$$G(3) = 3, \quad G\left(\frac{10}{3}\right) = 3, \quad G\left(\sqrt{10}\right) = 3, \quad G(\pi) = 3, \quad G(3{,}9999) = 3, \quad G(4) = 4$$

The graph is shown in Figure 2.26.

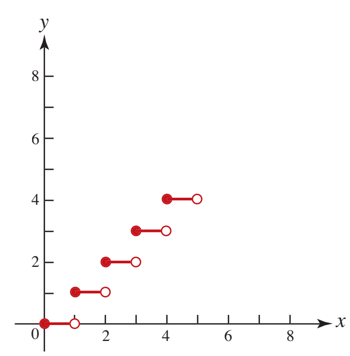


Figure 2.26 Graph of G

Notice that each part of the graph is simply a constant function and that the result looks somewhat like the steps in a stairway. As can be seen, the function is easy enough to understand, but if the domain were very large, it would be cumbersome to write, so the following notation is used.

GREATEST INTEGER FUNCTION

The **greatest integer function**, denoted by $f(x) = [\![x]\!]$, is defined by

$$[\![x]\!] = n \quad \text{if } n \le x < n+1$$

where n is an integer.

EXAMPLE 3 Evaluating a greatest integer function

Let $f(x) = [\![x]\!]$, $F(x) = [\![2x]\!]$, $g(x) = 2[\![x]\!]$, $G(x) = [\![x]\!]^2$, and $h(x) = [\![x^2]\!]$.
Evaluate these functions for $x = 5,\ 5.6,\ -5.6,\ \pi$, and $\frac{3}{4}$.

Solution We arrange the answer in tabular form; make sure you see how to find each value listed.

x	$[\![x]\!]$	$[\![2x]\!]$	$2[\![x]\!]$	$[\![x]\!]^2$	$[\![x^2]\!]$
5	5	10	10	25	25
5.6	5	11	10	25	31
-5.6	-6	-12	-12	36	31
π	3	6	6	9	9
$\frac{3}{4}$	0	1	0	0	0

■

EXAMPLE 4 Graphing a greatest integer function

Graph: $f(x) = x + [\![x]\!]$ on $[-3, 3)$

Solution Use the definition of the greatest integer function:

$$f(x) = \begin{cases} x-3 & \text{if } -3 \le x < -2; \text{ that is } [\![x]\!] = -3 \text{ on } [-3, 2) \\ x-2 & \text{if } -2 \le x < -1; \text{ that is } [\![x]\!] = -2 \text{ on } [-2, -1) \\ x-1 & \text{if } -1 \le x < -0; \text{ that is } [\![x]\!] = -1 \text{ on } [-1, 0) \\ x & \text{if } 0 \le x < 1; \text{ that is } [\![x]\!] = 0 \text{ on } [0, 1) \\ x+1 & \text{if } 1 \le x < 2; \text{ that is } [\![x]\!] = 1 \text{ on } [1, 2) \\ x+2 & \text{if } 2 \le x < 3; \text{that is } [\![x]\!] = 2 \text{ on } [2, 3) \end{cases}$$

Each of these linear functions is graphed for its respective domain, as shown by the solid line segments in Figure 2.27. ■

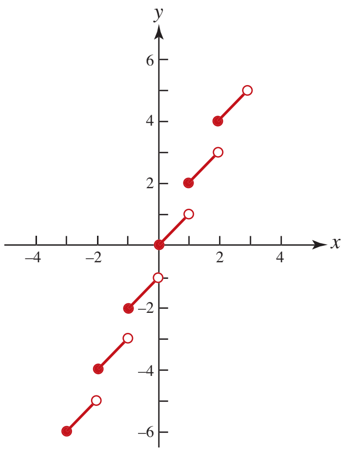


Figure 2.27 Graph of f

EXAMPLE 5 Rounding up instead of rounding down

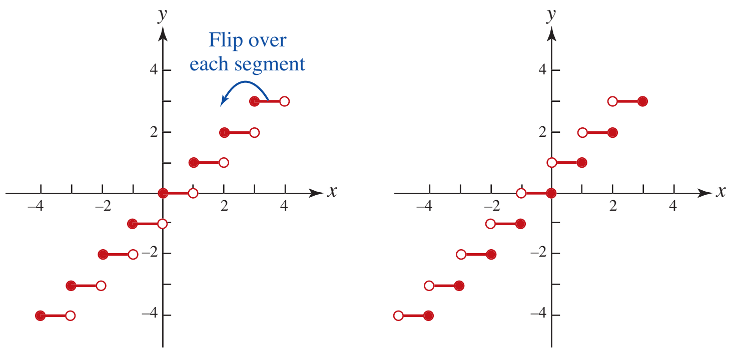
The greatest integer function $y = [\![x]\!]$ "rounds down" because

$$\begin{aligned} y &= 0 \quad \text{on } [0, 1) \\ y &= 1 \quad \text{on } [1, 2) \\ y &= 2 \quad \text{on } [2, 3) \\ &\vdots \end{aligned}$$

Write an expression using the greatest integer function that "rounds up":

$$\begin{aligned} y &= 1 \quad \text{on } (0, 1] \\ y &= 2 \quad \text{on } (1, 2] \\ y &= 3 \quad \text{on } (2, 3] \\ &\vdots \end{aligned}$$

Solution We begin by comparing the graphs of the "rounding down—greatest integer function"—and the "rounding up" function as shown in Figure 2.28.



a. Greatest integer function
Rounding down function

b. Rounding up function

Figure 2.28 Comparison of rounding down and rounding up functions

We find that if we replace y by $-y$ and x by $-x$ in the equation $y = [\![x]\!]$, we obtain

$$-y = [\![-x]\!]$$

$$y = -[\![-x]\!]$$

In graphing this function, we find that it matches the graph shown in Figure 2.28b.

The "rounding down" function is called the greatest integer function, and we give the name **ceiling function** to the "rounding up" function. Why would we want to consider a "rounding up" function? Most real-life situations require rounding up. Also, most calculations in business today are done by using computer programs, and the means of translating rounding problems for computer use is to use "rounding up" or "rounding down" functions.

EXAMPLE 6 Rental charges MODELING APPLICATION

The Rental Store charge for a special drill is \$10 for 4 hours or less usage. Additional charges are \$3.00 for each additional hour or fraction thereof. Write a function that expresses the rental charge, and graph the function for a person who rents the drill for 10 hours or less.

Solution

Step 1: *Understand the problem.* "Let's see; if I rent the drill for 2 hours, my charge is \$10. If I rent it for 5 hours, my charge is $\$10 + \$3 = \$13$; if I rent the drill for 10 hours my charge is $\$10 + \$3(6) = \$28$ because I have used 6 hours additional to my flat fee for 4 hours."

Step 2: *Devise a plan.* We will write a function to represent the charges, and we expect this function to be a step function. Then we will complete the problem by graphing the function.

Step 3: *Carry out the plan.* Let x be the number of hours the drill is rented, and let C be a function representing the cost (in dollars).

On $[0, 4]$, $C(x) = 10$

On $(4, \infty)$, the cost is dependent on the time.

If x is an integer greater than 4, then the cost for these hours is $\$3$ times $(x - 4)$. The reason we subtract 4 is that the first 4 hours are covered in the flat charge.

We consider

$$C(x) = \text{FLAT FEE} + \text{HOURLY FEE (in excess of 4 hours)}$$
$$= 10 + 3(x - 4)$$

For example, for $x = 5$, the charge is

$$C(5) = 10 + 3(5 - 4) = 13$$

If x is not an integer, then we must use a greatest integer function, and this is a rounding up function, so the desired function is

$$\begin{aligned} C(x) &= 10 - 3[\![-(x - 4)]\!] \quad \text{From Example 5} \\ &= 10 - 3[\![4 - x]\!] \end{aligned}$$

For example, if $x = 4.5$ (4 hours 30 minutes) we have

$$\begin{aligned} C(4.5) &= 10 - 3[\![4 - 4.5]\!] \\ &= 10 - 3[\![-0.5]\!] \\ &= 10 - 3(-1) \\ &= 13 \end{aligned}$$

The desired function is a piecewise function

$$C(x) = \begin{cases} 10 & \text{if } 0 < x \le 4 \\ 10 - 3[\![4 - x]\!] & \text{if } 4 < x \le 10 \end{cases}$$

The graph of $y = C(x)$ is shown in Figure 2.29.

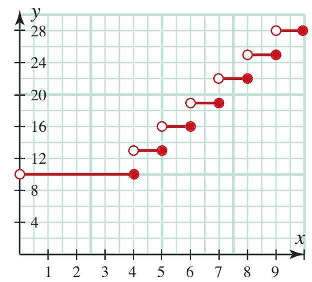


Figure 2.29 **Rental charges**

Step 4: *Look back.* We can try several examples to see whether the charges seem to be correct.

If I use the drill 4 hours or less, the cost is \$10.
If I use the drill 8 hours 15 minutes, the cost is

$$\begin{aligned} C(8.25) &= 10 - 3[\![4 - 8.25]\!] \\ &= 10 - 3[\![-4.25]\!] \\ &= 10 - 3(-5) \\ &= 25 \end{aligned}$$

These amounts seem correct, and the function makes sense.

PROBLEM SET 2.5

LEVEL 1

In Problems 1–16, find:

a. $f(1)$ **b.** $f(5.3)$ **c.** $f(\pi)$
d. $f\left(-\frac{1}{2}\right)$ **e.** $f(-5.3)$

1. $f(x) = |x|$
2. $f(x) = -|x|$
3. $f(x) = |x + 2|$
4. $f(x) = |x| - 2$
5. $f(x) = |x| + x$
6. $f(x) = |x| - x$
7. $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$
8. $f(x) = \begin{cases} 1 & \text{if } x < \pi \\ 2 & \text{if } x = \pi x > 0 \\ 3 & \text{if } x > \pi \end{cases}$
9. $f(x) = \begin{cases} -x & \text{if } x < -3 \\ 2x & \text{if } -3 \le x \le 3 \\ x^2 & \text{if } x \ge 3 \end{cases}$
10. $f(x) = \begin{cases} 10x & \text{if } x \le 0 \\ x^3 & \text{if } 0 < x < 3 \\ 0 & \text{if } x \ge 3 \end{cases}$

11. $f(x) = \begin{cases} x-1 & \text{if } x \geq 2 \\ -x+3 & \text{if } x < 2 \end{cases}$

12. $f(x) = \begin{cases} x-1 & \text{if } x \geq -1 \\ -x-3 & \text{if } x < -1 \end{cases}$

13. $f(x) = [\![x]\!]$

14. $f(x) = [\![x]\!] + 1$

15. $f(x) = [\![x]\!] + x$

16. $f(x) = 2[\![x]\!]$

WHAT IS WRONG, *if anything, with each statement in Problems 17–24? Explain your reasoning.*

17. If $f(x) = |x|$, then f is a positive function.
18. If $f(x) = [\![x]\!]$, then f is a positive function.
19. If $f(x) = [\![x]\!]$, then x is an integer.
20. $[\![8]\!] = 8$ and $[\![8.1]\!] = 8$
21. $[\![-8]\!] = -8$ and $[\![-8.1]\!] = -8$
22. $|[\![x]\!]| = [\![|x|]\!]$
23. $|x|^2 = |x^2|$
24. $[\![x]\!]^2 = [\![x^2]\!]$

LEVEL 2

Graph the functions given in Problems 25–40.

25. $f(x) = \begin{cases} x-2 & \text{if } x \geq 2 \\ 2-x & \text{if } x < 2 \end{cases}$

26. $f(x) = \begin{cases} x+1 & \text{if } x \geq -3 \\ -x-3 & \text{if } x < -3 \end{cases}$

27. $g(x) = \begin{cases} -2 & \text{if } x \leq -2 \\ 0 & \text{if } -2 < x < 2 \\ 2 & \text{if } x \geq 2 \end{cases}$

28. $r(x) = \begin{cases} -4 & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ -2 & \text{if } x > 1 \end{cases}$

29. $f(x) = \begin{cases} x-1 & \text{if } x \geq 2 \\ -x+3 & \text{if } x < 2 \end{cases}$

30. $g(x) = \begin{cases} x-1 & \text{if } x \geq -1 \\ -x-3 & \text{if } x < -1 \end{cases}$

31. $f(x) = |x| + 2$
32. $g(x) = 3 - |x|$
33. $h(x) = |x + 3|$
34. $k(x) = |x - 2|$
35. $f(x) = |2x| - 3$
36. $G(x) = |3x| + 2$
37. $m(x) = |x|^2$
38. $n(x) = |x^2|$
39. $f(x) = |x - 2| + 1$
40. $g(x) = |x + 1| - 2$

PROBLEMS FROM CALCULUS *Problems 41–50 are found in calculus.*

41. If $f(x) = |x|$, find $\dfrac{f(x+h)-f(x)}{h}$, where $x \geq 0$, $h \geq 0$.

42. If $f(x) = |x|$, find $\dfrac{f(x+h)-f(x)}{h}$, where $x < 0$, $h < 0$.

43. If $f(x) = |x + 2|$, find $\dfrac{f(x+h)-f(x)}{h}$, where $x < -2$, $h < 0$.

44. If $f(x) = |x + 2|$, find $\dfrac{f(x+h)-f(x)}{h}$, where $x \geq -2$, $h \geq 0$.

45. A national fraternity allows one delegate for each 500 state members (or fraction thereof) of the fraternity. Suppose a state has n members. How many state representatives are allowed?

46. A salesperson receives a \$500 bonus for each \$10,000 worth of sales over an established base. What is the bonus for d dollars in sales over the base?

47. If the charges for a taxi are \$2.50 plus 25¢ per each $\frac{1}{2}$ mile or fraction thereof, write a function that gives the cost of a taxi ride of x miles.

48. The charge for a certain telephone call is 75¢ for the first 3 minutes and 25¢ for each additional minute or fraction thereof. Write a function that gives the cost of a call lasting x minutes.

49. The telephone company charges \$1.00 for the first 3 minutes for a certain call and 35¢ for each additional minute or fraction thereof. Write a function that gives the cost of a call lasting x minutes.

50. A measure of the disorder or randomness in a physical system is called *entropy* and is measured in calories per Kelvin. (Note: To convert from degree Celsius to Kelvin add 273.15.) An approximate model for the entropy of one mole of water under one atmosphere pressure is given by

$$S = \begin{cases} 10 + 0.04T & \text{if } T \leq 0 \\ 10 + 0.05T & \text{if } 0 < T < 100 \\ 12 + 0.03T & \text{if } T \geq 100 \end{cases}$$

where T is the temperature in degrees Celsius and S is in calories per Kelvin. Graph S.

LEVEL 3

Graph the functions given in Problems 51–60.

51. $b(x) = [\![x]\!] + 2$
52. $d(x) = [\![x + 2]\!]$
53. $f(x) = 2[\![x]\!] + 1$
54. $g(x) = [\![x]\!] - 2$
55. $h(x) = [\![x]\!] - x$
56. $j(x) = [\![x]\!] + |x|$
57. $r(x) = |x - 2| + |x|$
58. $s(x) = |x + 1| - |x|$
59. $F(x) = |x| - |x - 1|$
60. $G(x) = |x| - |x + 2|$

2.6 Composition and Operations of Functions

There are many situations in which a quantity is given as a function of one variable that, in turn, can be written as a function of a second variable. This is known as *functional composition.*

Composition

Suppose, for example, that your job is to ship x packages of a product via Federal Express to a variety of addresses. Let x be the number of packages to ship, let f be the weight of the x objects, and let g be the cost of shipping. Then

The weight is a function of the number of objects: $f(x)$.

The cost is a function of the weight: $g[f(x)]$.

This process of evaluating a function of a function illustrates the idea of *composition of functions.*

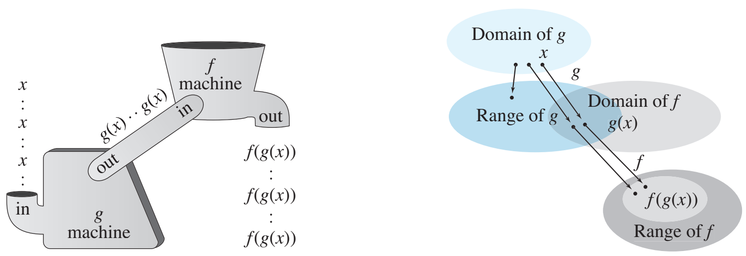
COMPOSITION

The **composite function** $f \circ g$ is defined by

$$(f \circ g)(x) = f[g(x)]$$

for each x in the domain of g for which $g(x)$ is in the domain of f.

» IN OTHER WORDS To visualize how functional composition works, think of $f \circ g$ in terms of an "assembly line" in which g and f are arranged in series, with output $g(x)$ becoming the input of f, as illustrated in Figure 2.30.



Assembly line interpretation of composite functions

Mapping diagram of composition showing domains

Figure 2.30 Composition of functions

EXAMPLE 1 Finding the composition of functions

If $f(x) = 3x + 5$ and $g(x) = \sqrt{x}$, find the composite functions $f \circ g$ and $g \circ f$.

Solution The function $f \circ g$ is defined by $f[g(x)]$:

$$(f \circ g)(x) = f[g(x)] = f\left(\sqrt{x}\right) = 3\sqrt{x} + 5$$

The function $g \circ f$ is defined by $g[f(x)]$:

$$(g \circ f)(x) = g[f(x)] = g(3x + 5) = \sqrt{3x + 5}$$

Example 1 illustrates that *functional composition is not commutative*. That is, $f \circ g$ is not, in general, the same as $g \circ f$.

Some attention must be paid to the domain of $g \circ f$. Let X be the domain of a function f, and let Y be the range of f. The situation can be viewed as shown in Figure 2.31.

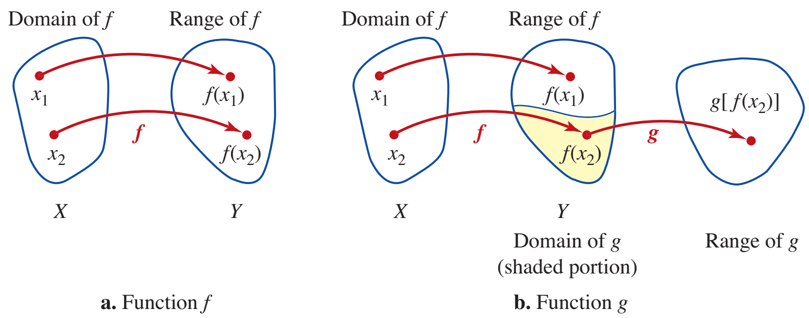


Figure 2.31 **Two functions f and g viewed as mappings**

If part of f maps into the shaded portion of Y and part of f maps into the portion that is not shaded, as indicated in Figure 2.31**b**, then, as shown in Figure 2.32, the domain of $g \circ f$ is just the part of X that maps into the shaded portion of Y.

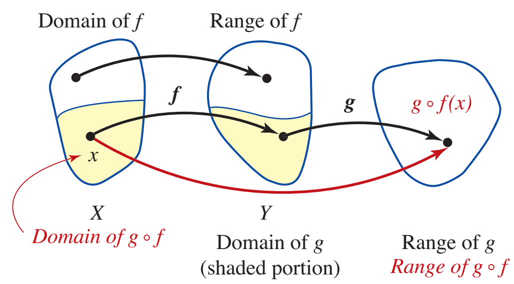


Figure 2.32 **Composition of two functions $g \circ f$. Notice that the domain of $g \circ f$ is the subset of X for which $g \circ f$ is defined (shaded portion of Y).**

EXAMPLE 2 Composition with focus on domains

Let $f = \{(0, 0), (-1, 1), (-2, 4), (-3, 9), (5, 25)\}$;
$g = \{(0, -5), (-1, 0), (2, -3), (4, -2), (5, -1)\}$

a. What are the domains of f and g?

b. Find $g \circ f$ and state its domain.

c. Find $f \circ g$ and state its domain.

Solution

a. $D_f = \{0, -1, -2, -3, 5\}$; $D_g = \{0, -1, 2, 4, 5\}$

b.

f		g	$g \circ f$
$0 \to 0$	$\cdots$	$0 \to -5$	$0 \to -5$
$-1 \to 1$	$\cdots$	not defined, exclude -1 from the domain of $g \circ f$.	
$-2 \to 4$	$\cdots$	$4 \to -2$	$-2 \to -2$
$-3 \to 9$	$\cdots$	not defined, exclude -3 from the domain of $g \circ f$.	
$5 \to 25$	$\cdots$	not defined, exclude 5 from the domain of $g \circ f$.	

Thus, $g \circ f = \{(0, -5), (-2, -2)\}$. The domain of the composite function $g \circ f$ is $\{0, -2\}$. Notice that -1, -3, and 5 are excluded from the domain of $g \circ f$ even though they are in the domain of f.

c.

g		f	$f \circ g$
$0 \to -5$	$\cdots$	not defined	
$-1 \to 0$	$\cdots$	$0 \to 0$	$-1 \to 0$
$2 \to -3$	$\cdots$	$-3 \to 9$	$2 \to 9$
$4 \to -2$	$\cdots$	$-2 \to 4$	$4 \to 4$
$5 \to -1$	$\cdots$	$-1 \to 1$	$5 \to 1$

Thus, $f \circ g = \{(-1, 0), (2, 9), (4, 4), (5, 1)\}$. The domain of the composite function $f \circ g$ is $\{-1, 2, 4, 5\}$. ■

EXAMPLE 3 An application of composite functions

Air pollution is a problem for many metropolitan areas.

Suppose that carbon monoxide is measured as a function of the number of people according to the information shown in Table 2.2.


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TABLE 2.2 Carbon Monoxide Pollution

Number of People	Daily Level (ppm)
100,000	1.41
200,000	1.83
300,000	2.43
400,000	3.05
500,000	3.72

Studies show that a refined formula for the average daily level of carbon monoxide in the air is

$$L(p) = 0.7\sqrt{p^2 + 3}$$

Further assume that the population of a given metropolitan area is growing according to the formula $p(t) = 1 + 0.02t^3$, where t is the time from now (in years) and p is the population (in hundred thousands). Based on these assumptions, what level of air pollution should be expected in four years?

Solution The level of pollution is $L(p) = 0.7\sqrt{p^2 + 3}$, where $p(t) = 1 + 0.02t^3$. Thus, the pollution level at time t is given by the composite function

$$(L \circ p)(t) = L[p(t)] = L(1 + 0.02t^3) = 0.7\sqrt{(1 + 0.02t^3)^2 + 3}$$

In particular, when $t = 4$, we have

$$(L \circ p)(4) = 0.7\sqrt{[1 + 0.02(4)^3]^2 + 3} \approx 2.00 \text{ ppm}$$

■

In calculus, it is frequently necessary to express a function as the composite of two simpler functions.

EXAMPLE 4 Separating a function into two composite functions

Express each of the following functions as the composite of two functions u and g so that $f(x) = g[u(x)]$.

a. $f(x) = (x^2 + 2x)^2$

b. $f(x) = \sqrt{x^2 + 1}$

c. $f(x) = (x^2 + 5x + 1)^5$

d. $f(x) = \sqrt{5x^2 - x}$

☠ Consider Example 4 and the following paragraph carefully. ☠

Historical Note

Nicole Oresme (1323–1382) was a Parisian scholar who was also the Bishop of Lisieux. He was the first to hint at the possibility of irrational powers (the historian Carl Boyer refers to this as his "most brilliant idea"). He expressed the exponential rules of

$$x^m x^n = x^{m+n} \quad \text{and} \quad (x^m)^n = x^{mn}$$

He was also one of the first mathematicians to comprehend the notion of a function. He interested himself in some of the great problems in mathematics, including the foundational idea from calculus, namely, the notion of the area under a curve, and the idea of using two independent variables to picture a volume. The historian Boyer traces interest in the graphical representation of a function (known as the latitude of forms) from Oresme to Galileo (1564–1643).

Solution We call f the *given function*, u the *inner function*, and g the *outer function*.

	Given Function $f(x) = g[u(x)]$	Inner Function $u(x)$	Outer Function $g[u(x)]$
a.	$f(x) = (x^2 + 2x)^2$	$u(x) = x^2 + 2x$	$g[u(x)] = [u(x)]^2$
b.	$f(x) = \sqrt{x^2 + 1}$	$u(x) = x^2 + 1$	$g[u(x)] = \sqrt{u(x)}$
c.	$f(x) = (x^2 + 5x + 1)^5$	$u(x) = x^2 + 5x + 1$	$g[u(x)] = [u(x)]^5$
d.	$f(x) = \sqrt{5x^2 - x}$	$u(x) = 5x^2 - x$	$g[u(x)] = \sqrt{u(x)}$

There are often other ways to express a composite function, but the most common procedure is to choose the function u to be the "inside" portion of the given function f. Notice that in parts **a** and **c** the "inside" portion is the portion inside the parentheses.

Operations with Functions

In algebra, you spend a great deal of time learning the algebra of real numbers. Operations with functions follows similar straightforward definitions.

FUNCTIONAL OPERATIONS

Let f and g be functions with domains D_f and D_g, respectively. Then, $f + g$, $f - g$, fg, and f/g are defined for the domain $D_f \cap D_g$:

$$(f + g)(x) = f(x) + g(x)$$

$$(f - g)(x) = f(x) - g(x)$$

$$(fg)(x) = f(x)g(x)$$

$$(f/g)(x) = \frac{f(x)}{g(x)} \qquad \text{provided } g(x) \neq 0$$

» IN OTHER WORDS Remember the operation symbols on the left of the equations, namely, $f + g$, $f - g$, fg, and f/g are operations on *functions* that are being defined; the operation on the right is defined for operations on *numbers.*

EXAMPLE 5 Operations with functions

If f and g are defined by $f(x) = x^2$ and $g(x) = x + 3$, find $f + g$, $f - g$, fg, and f/g. Evaluate these functions for $x = -1$ and $x = 5$, and state the domain for each.

Solution The domain of both f and g is the set of real numbers, so the domains of $f + g$, $f - g$, and fg are also the set of real numbers. The domain of f/g is the intersection of the domains of f and g for which values causing $g(x) = 0$ are excluded. These are the usual domains for functions we consider in precalculus and calculus.

$$(f + g)(x) = f(x) + g(x) = x^2 + (x + 3) = x^2 + x + 3 \qquad D\colon(-\infty, \infty)$$
$$(f + g)(-1) = (-1)^2 + (-1) + 3 = 3$$
$$(f + g)(5) = 5^2 + 5 + 3 = 33$$

$$\begin{aligned}(f-g)(x) &= f(x)-g(x) = x^2-(x+3) = x^2-x-3 \qquad D{:}(-\infty,\ \infty)\\ (f-g)(-1) &= (-1)^2-(-1)-3 = -1\\ (f-g)(5) &= 5^2-5-3 = 17\end{aligned}$$

$$\begin{aligned}(fg)(x) &= f(x)\cdot g(x) = x^2(x+3) = x^3+3x^2 \qquad D{:}(-\infty,\ \infty)\\ (fg)(-1) &= (-1)^3+3(-1)^2 = 2\\ (fg)(5) &= 5^3+3(5)^2 = 200\end{aligned}$$

$$\begin{aligned}(f/g)(x) &= \frac{f(x)}{g(x)} = \frac{x^3}{x+3} \qquad D{:}(-\infty,\ -3)\cup(-3,\ \infty)\\ (f/g)(-1) &= \frac{(-1)^2}{(-1)+3} = \frac{1}{2}\\ (f/g)(5) &= \frac{5^2}{5+3} = 3.125\end{aligned}$$

Functional Iteration

In calculus, you will use a process in which the result of one step is used in the following step. This process, called **iteration**, is easy to describe in terms of composition. For example, if $f(x)=\sqrt{x}$, then $f(4)=\sqrt{4}=2$. If we now take *this* answer and again evaluate f to find $f(2)=\sqrt{2}$, we have carried out an *iterative step*, which we could describe as $f(f(x))$ or as $f \circ f$.

Suppose we consider a problem from *The American Mathematical Monthly* (Vol. 92, Jan. 1985, pp. 3–23).* Define a function f, with domain the positive integers as follows:

$$f(x) = \begin{cases} 3x+1 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even}\end{cases}$$

Let x_0 be some number in the domain of a function f. The **iterates** of x_0 are the numbers $f(x_0)$, $f(f(x_0))$, $f(f(f(x_0)))$, ... We can write this using composition: f, $f \circ f$, $f \circ f \circ f$, ... The article asserts that for every positive integer, the iterates eventually return to 1.

EXAMPLE 6 **Iteration conjecture** **MODELING APPLICATION**

Show that the iterates return to 1 for the case $x = 3$.

Solution

Step 1: *Understand the problem.* Let's see if we can understand what this problem is about. We calculate the first few values of f: (Remember the domain is the set of positive integers.)

$$\begin{aligned} f(1) &= 3(1)+1 = 4 & f(2) &= \frac{2}{2} = 1\\ f(3) &= 3(3)+1 = 10 & f(4) &= \frac{4}{2} = 2\\ &\vdots & &\vdots\\ f(99) &= 3(99)+1 = 298 & f(100) &= \frac{100}{2} = 50\end{aligned}$$

*"The $3x+1$ Problem and Its Generalizations," by Jeffrey C. Lagarias.

Step 2: *Devise a plan.* Calculate the iterates for the smallest member of the domain.

The first iterate of $x_0 = 1$: $f(1) = 3(1) + 1 = 4$

the second iterate of $x_0 = 1$: $f(f(1)) = f(4) = \frac{4}{2} = 2$

the third iterate of $x_0 = 1$: $f(f(f(1))) = f(f(4)) = f(2) = 1$

We see for the first member, the iterates return to 1. The second number, $x = 2$, iterates to 1 in the first step (since 2 is even).

Step 3: *Carry out the plan.* Calculate the iterates for the number $x = 3$.

$f(3) = 10$; $f(10) = \frac{10}{2} = 5$; $f(5) = 3(5) + 1 = 16$; $f(16) = \frac{16}{2} = 8$; $f(8) = \frac{8}{2} = 4$; $f(4) = \frac{4}{2} = 2$; $f(2) = \frac{2}{2} = 1$. Ah ha, we ended back at 1! Thus, $f(f(f(f(f(f(f(3))))))) = 1$.

Step 4: *Look back.* The hypothesis that all values return to 1 is hardly proved. We showed it true for 1, 2, 3; you are asked to show it true for 4 and 5 in the problem set (Problem 55) and then asked to present an argument that it is always true.

PROBLEM SET 2.6

LEVEL 1

In Problems 1–4 find the indicated values where $f(x) = 3x - 2$ and $g(x) = 2x^2 + 1$.

1. a. $(f + g)(4)$ **b.** $(fg)(2)$
2. a. $(f - g)(3)$ **b.** $(f/g)(1)$
3. a. $(f \circ g)(2)$ **b.** $(g \circ f)(2)$
4. What is the domain of $(f + g)$, $(f - g)$, (fg), and (f/g)?

In Problems 5–8, find the indicated values, where $f(x) = \dfrac{x-2}{x+1}$ and $g(x) = x^2 - x - 2$.

5. a. $(f + g)(2)$ **b.** $(fg)(102)$
6. a. $(f - g)(5)$ **b.** $(f/g)(99)$
7. a. $(f \circ g)(1)$ **b.** $(g \circ f)(1)$
8. What is the domain of $(f + g)$, $(f - g)$, (fg), and (f/g)?

In Problems 9–12, find the indicated values, where $f(x) = \dfrac{2x^2 - x - 3}{x - 2}$ and $g(x) = x^2 - x - 2$.

9. a. $(f + g)(-2)$ **b.** $(f/g)(2)$
10. a. $(f - g)(2)$ **b.** $(f/g)(102)$
11. a. $(f \circ g)(0)$ **b.** $(g \circ f)(0)$
12. What is the domain of $(f + g)$, $(f - g)$, (fg), and (f/g)?

In Problems 13–16, find the indicated values, where

$$f = \{(0, 1), (1, 4), (2, 7), (3, 10)\}$$

and

$$g = \{(0, 3), (1, -1), (2, 1)(3, 3)\}$$

13. a. $(f + g)(1)$ **b.** $(fg)(2)$
14. a. $(f - g)(3)$ **b.** $(f/g)(0)$
15. a. $(f \circ g)(2)$ **b.** $(g \circ f)(2)$
16. What is the domain of $(f + g)$, $(f - g)$, (fg), and (f/g)?

LEVEL 2

WHAT IS WRONG, *if anything, with each statement in Problems 17–20? Explain your reasoning.*

17. If $f(x) = \sqrt[3]{x}$ and $u(x) = 2x^2 + 1$, then $f[u(x)] = 2\sqrt[3]{x} + 1$.
18. If $f(x) = \dfrac{1}{2x^2 + 1}$ and $u(x) = \sqrt{x - 1}$, then $f[u(x)] = \dfrac{1}{2x - 1}$.
19. If $f(x) = 3x^2 + 5x + 1$ and $u(x) = x^3$, then $f[u(x)] = (3x^2 + 5x + 1)^3$.
20. If $f(x) = \dfrac{x-1}{x+2}$ and $g(x) = \dfrac{2x+1}{x-3}$, then $(f \circ g)$ is the function defined by $\left(\dfrac{x-1}{x+2}\right)\left(\dfrac{2x+1}{x-3}\right)$.
21. Given $f(x) = x^2 - 1$ and $(f - g)(x) = 2x + 1$. Find $g(x)$.
22. Given $f(x) = x^3 + 2$ and $(f + g)(x) = 4x - 6$. Find $g(x)$.
23. Given $f(x) = x^{-1}$ and $(fg)(x) = x$ $(x \neq 0)$. Find $g(x)$.
24. Given $f(x) = \dfrac{x-2}{x+3}$ and $(f/g)(x) = \dfrac{x+1}{x+2}$. Find $g(x)$.
25. Given $f(x) = \sqrt{x}$ and $(f \circ g)(x) = \sqrt{x^3 + 1}$. Find $g(x)$.
26. Given $f(x) = x^4$ and $(f \circ g)(x) = (2x + \sqrt{5})^4$. Find $g(x)$.

PROBLEMS FROM CALCULUS *In Problems 27–38, express f as a composition of two functions u and g so that $f(x) = g[u(x)]$.*

27. $f(x) = (x^2 + 1)^2$ **28.** $f(x) = (x^2 - 1)^3$
29. $f(x) = (2x^2 - 1)^4$ **30.** $f(x) = (x^2 + 4)^{3/2}$
31. $f(x) = (3x^2 + 4x - 5)^3$ **32.** $f(x) = (2x^2 - x + 1)^2$
33. $f(x) = \sqrt{5x - 1}$ **34.** $f(x) = \sqrt{x^2 - 1}$
35. $f(x) = \sqrt[3]{x^2 - 4}$ **36.** $f(x) = \sqrt[4]{x^3 - x + 1}$
37. $f(x) = (x^2 - 1)^3 + \sqrt{x^2 - 1} + 5$
38. $f(x) = |x + 1|^2 + 6$

In Problems 39–42, find the sum, difference, product, and quotient of the given functions. Also state the domain for each.

39. $f(x) = 2x - 3$ and $g(x) = x^2 + 1$

40. $f(x) = \dfrac{x-2}{x+1}$ and $g(x) = x^2 - x - 2$

41. $f(x) = \dfrac{2x^2 - x - 3}{x-2}$ and $g(x) = x^2 - x - 2$

42. $f(x) = 4x + 2$ and $g(x) = x^3 + 3$

In Problems 43–46, find $f \circ g$ and $g \circ f$ for the given functions.

43. $f(x) = 2x - 3$ and $g(x) = x^2 + 1$

44. $f(x) = \dfrac{x-2}{x+1}$ and $g(x) = x^2 - x - 2$

45. $f(x) = \dfrac{2x^2 - x - 3}{x+1}$ and $g(x) = x^2 - x - 2$

46. $f(x) = 4x + 2$ and $g(x) = x^3 + 3$

47. If $f(x) = x^2$, $g(x) = 2x - 1$, and $h(x) = 3x + 2$, find:
a. $(f \circ g) \circ h$ **b.** $f \circ (g \circ h)$

48. If $f(x) = x^2$, $g(x) = 3x - 2$, and $h(x) = x^2 + 1$, find:
a. $(f \circ g) \circ h$ **b.** $f \circ (g \circ h)$

49. If $f(x) = \sqrt{x}$, $g(x) = x^2$, and $h(x) = x + 2$ all within domain $(0, \infty)$, find
a. $(f \circ g) \circ h$ **b.** $f \circ (g \circ h)$

50. If $f(x) = \sqrt{-x}$, $g(x) = x^2$, and $h(x) = x$ all with domain $(-\infty, 0)$, find:
a. $(f \circ g) \circ h$ **b.** $f \circ (g \circ h)$

51. Consider the volume of a particular cone as a function of its height by the formula

$$V(h) = \frac{\pi h^3}{12}$$

Suppose the height is expressed as a function of time by letting $h(t) = 2t$.
a. Find the volume for $t = 2$.
b. Express the volume as a function of time by finding $V \circ h$.
c. If the domain of V is $(0, 6]$, find the domain of h; that is, what are the permissible values for t?

52. The surface area of a spherical balloon is given by $S(r) = 4\pi r^2$. Suppose the radius is expressed as a function of time by $r(t) = 3t$.
a. Find the surface area for $t = 2$.
b. Express the surface area as a function of time by finding $S \circ r$.
c. If the domain of S is $(0, 8)$, find the domain of r; that is, what are the permissible values for t?

53. If $f(x) = x^2$, then

$$f\left(\frac{1}{x}\right) = \left(\frac{1}{x}\right)^2 = \frac{1}{x^2} = \frac{1}{f(x)}$$

Give an example of a function for which

$$f\left(\frac{1}{x}\right) \neq \frac{1}{f(x)}$$

54. If $f(x) = x$, then $f(x^2) = [f(x)]^2$. Give an example of a function for which

$$f(x^2) \neq [f(x)]^2$$

55. In Example 6, an unproved conjecture was checked for $x = 1$, 2, and 3. Check this conjecture for $x = 4$ and $x = 5$. Make a statement about values for which x is even.

56. PROBLEM FROM CALCULUS* A buffalo herd at Yellowstone has population P, which is a function of the amount of grass cover available for the herd.

© Tony Campbell/ShutterStock, Inc.

Grass yield is estimated in lb/ft^2 by weighing the harvest from a 20 ft^2 test plot. The number of grasshoppers affects this yield, Y. An approximation for Y is given by

$$g(N) = 1 - \frac{1}{20}N\left(\frac{1}{20}N - 1\right)$$

where N is the number of grasshoppers contained on the test plot on July 1. An approximation for the population of the buffalo herd on October 1 is

$$P = f(Y) = 1{,}000\sqrt{Y^2 + 3Y + 1}$$

Find a formula for P as a function of N.

LEVEL 3

57. Let $f(x) = 1 + \dfrac{1}{x}$, find:
a. $(f \circ f)(x)$ **b.** $(f \circ f \circ f)(x)$
c. $(f \circ f \circ f \circ f)(x)$
d. Can you predict the output for further iterations of compositions?

58. Choose *any* positive x. Find a numerical value for $(f \circ f)(x)$, $(f \circ f \circ f)(x)$, and $(f \circ f \circ f \circ f)(x)$. If you continue this iterative procedure, predict the outcome for any x for each of the given functions.
a. $f(x) = \sqrt{x}$ **b.** $f(x) = 2\sqrt{x}$
c. $f(x) = 3\sqrt{x}$ **d.** $f(x) = k\sqrt{x}$

59. PROBLEMS FROM CALCULUS A process in calculus called the *Newton-Raphson method* uses an iterative process for approximating roots of a given equation. A leading calculus book shows that the solution to the equation $x^2 = 5$ (which we know has a solution $x = \sqrt{5}$) can be found iteratively by evaluating the function

$$f(x) = \frac{x^2 + 5}{2x}$$

Compute the iterates of this function for $x_0 = 2$ and compare with a calculator approximation of $\sqrt{5}$.

60. Journal Problem From *School Science and Mathematics,* Vol. 83, No. 1, Jan. 1983. Let $f(x) = \dfrac{(x^2+1)^2}{2x^2}$ and $g(t) = \sqrt{t} \pm \sqrt{t-1}$, where t is a positive integer. Find $f[g(t)]$.

*From *Calculus,* James F. Hurley. Belmont, CA: Wadsworth, Inc., 1987, p. 61

2.7 Inverse Functions

The Idea of an Inverse

In mathematics, the ideas of "opposite operations" and "inverse properties" are very important. The basic notion of an opposite operation or an inverse property is to "undo" a previously performed operation. For example, pick a number, and call it x; then:

	x	*Think:* I pick 8.
Add 5:	$x + 5$	*Think:* Now, I have $8 + 5 = 13$.
The next operation returns you to x:		*Think:* I want to find an operation to get back to my original number
Subtract 5:	$x + 5 - 5 = x$	*Think:* $13 - 5 = 8$, my original number.

We now want to apply this idea to functions. Pick a number in the domain of a function f; call this number a:

	a	*Think:* I'll pick 4 this time.
Now, evaluate f for the number you picked; suppose we let f be defined by $f(x) = 2x + 7$:		
Evaluate f:	$f(a) = 2a + 7$	*Think:* $f(4) + 7 = 15$.
The next operation returns us back to x:		*Think:* I want to find a function, called the *inverse function*, denoted by f^{-1} if it exists, so that $f^{-1}(a) = a$.
Let $f^{-1}(x) = \frac{1}{2}(x - 7)$		*Think:* Where did this come from? This is the topic of this section!
Evaluate f^{-1}:	$f^{-1}(2a + 7) = a$	*Think:* $f^{-1}(2a+7) = \frac{1}{2}(2a + 7 - 7)$ $= \frac{1}{2}(2a)$ $= a$

The symbol f^{-1} means the inverse of f and does not mean $\frac{1}{f}$.

Of course, for f^{-1} to be an inverse function, it must "undo" the effect of f for *each and every member* of the domain. This may be impossible if f is a function such that two x values give the same y value. For example, if $g(x) = x^2$, then $g(2) = 4$ and $g(-2) = 4$, so we cannot find a function g^{-1} such that $g^{-1}(4)$ equals *both* 2 and -2 because that would violate the very definition of a function. We see it is necessary to limit the given function so that it is *one-to-one*. Recall the *horizontal line test* from Section 2.2 to determine whether a function is one-to-one.

Inverse Functions

For a given function f, we write $b = f(a)$ to indicate that f maps the number a in its domain into the corresponding number b in the range. If f has an inverse f^{-1}, it is the function that reverses the *inverse* of f and does not affect f in the sense that

$$f^{-1}(b) = a$$

This means that

$$(f^{-1} \circ f)(a) = a$$

Furthermore, for every b in the domain of f^{-1},

$$(f \circ f^{-1})(b) = b$$

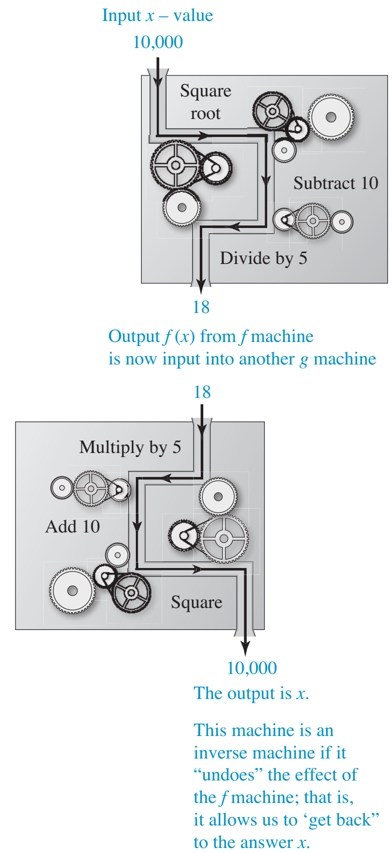
INVERSE FUNCTION

Let f be a function with domain D and range R. Then the function f^{-1} with domain R and range D is the **inverse of f** if

$$f^{-1}[f(a)] = a \quad \text{for all } a \text{ in } D$$

$$f[f^{-1}(b)] = b \quad \text{for all } b \text{ in } R$$

» IN OTHER WORDS Start with some x value in the domain of a function f. You can think of this in terms of a function machine.



EXAMPLE 1 Showing that two given functions are inverses

Show that f and g defined by $f(x) = 5x + 4$ and $g(x) = \dfrac{x-4}{5}$ are inverse functions.

Solution We must show that f and g are inverse functions in two parts:

$$\begin{aligned}(g \circ f)(a) &= g(5a+4) \\ &= \frac{(5a+4)-4}{5} \\ &= \frac{5a}{5} \\ &= a\end{aligned} \qquad \text{and} \qquad \begin{aligned}(f \circ g)(b) &= f\left(\frac{b-4}{5}\right) \\ &= 5\left(\frac{b-4}{5}\right)+4 \\ &= (b-4)+4 \\ &= b\end{aligned}$$

Thus, $(g \circ f)(a) = a$ and $(f \circ g)(b) = b$, so f and g are inverse functions. ■

Once you are certain that a function g is the inverse of a function f, you can denote it by f^{-1}. This relationship is shown in Figure 2.33. Note that the range of f is the domain of f^{-1}.

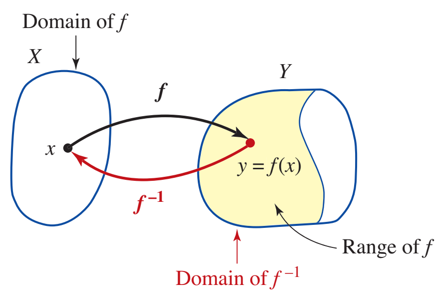


Figure 2.33 Inverse functions

Even though the definition of inverse functions shows us how to check to see whether two functions are inverses, it does not tell us how to *find* the inverse of a given function. To find the inverse, it is helpful to visualize a function as a set of ordered pairs. Suppose we pick a number, say 4, and evaluate a function f at 4 to find $f(4) = 15$. Then $(4, 15)$ is an element of f. Now the inverse function f^{-1} requires 15 to be changed back into 4; that is, $f^{-1}(15) = 4$ so that $(15, 4)$ is an element of f^{-1}. Thus, *if a function f has an element (a, b), then the inverse function f^{-1} must have element (b, a).*

EXAMPLE 2 Inverse of a given function defined as a set of ordered pairs

Let $f = \{(0, 3), (1, 5), (3, 9), (5, 13)\}$; find f^{-1}, if it exists.

Solution The inverse simply reverses the ordered pairs:

$$f^{-1} = \{(3, 0), (5, 1), (9, 3), (13, 5)\}$$

The inverse of a function may not exist. For example,

$$f = \{(0, 0), (1, 1), (-1, 1), (2, 4), (-2, 4)\}$$

and

$$g(x) = x^2$$

do not have inverses because if we attempt to find the inverses, we obtain relations that are not functions. In the first case, we find

$$\text{Possible inverse of } f\text{: } \{(0, 0), (1, 1), (1, -1), (4, 2), (4, -2)\}$$

This is not a function because not every member of the domain is associated with a single member in the range: $(1, 1)$ and $(1, -1)$, for example.

In the second case, if we interchange the x and y in the equation for the function g where $y = x^2$ and then solve for y, we find:

$$x = y^2 \quad \text{or} \quad y = \pm\sqrt{x} \text{ for } x \geq 0$$

But this is not a function of x, because for any positive value of x, there are two corresponding values of y, namely, $\sqrt{x}$ and $-\sqrt{x}$. These examples show why we impose the one-to-one condition.

Let us summarize the procedure for finding an inverse as illustrated in Example 3.

FINDING AN INVERSE

The procedure for finding an equation for a given inverse function, f.

Step 1 Let $y = f(x)$ be a given *one-to-one* function.

Step 2 Replace all x's and all y's (that is, interchange x and y) in the given equation.

Step 3 Solve for y. The resulting function defined by the equation $y = f^{-1}(x)$ is the inverse of f.

The domain of f and the range of f^{-1} must be equal as well as the domain of f^{-1} and the range of f. This property is evident in the following example (part **c**).

EXAMPLE 3 Finding the inverse using functional notation

a. If $s(x) = x^3 + 3$, find s^{-1}, if it exists.
b. If $u(x) = x^2$, find u^{-1}, if it exists.
c. If $t(x) = x^2$ on $(-\infty, 0]$ find t^{-1}, if it exists.

Solution

a. Note that s is a one-to-one function and thus will have an inverse.

$y = x^3 + 3$	Step 1: Given function.
$x = y^3 + 3$	Step 2: Interchange x and y.
$y^3 = x - 3$	Step 3: Solve for y.
$y = (x - 3)^{1/3}$	Thus, $s^{-1}(x) = (x - 3)^{1/3}$

b. Since u is not a one-to-one function, we say it has no inverse function.
c. With the given restriction on the domain, t is a one-to-one function, so the inverse exists.

$y = x^2$ for $-\infty < x \le 0$	Step 1: Given function
$x = y^2$ for $-\infty < y \le 0$	Step 2: Interchange x and y.
$y = -\sqrt{x}$	Step 3: Solve for y.
Thus, $t^{-1}(x) = -\sqrt{x}$	Note that y is negative, and $\sqrt{x}$ is positive, so the opposite is necessary. The implied domain here is $(0, \infty]$.

Graph of f^{-1}

The graphs of f and its inverse f^{-1} are closely related. In particular, if (a, b) is a point on the graph of f, then $b = f(a)$ and $a = f^{-1}(b)$, so (b, a) is on the graph of f^{-1}. It can be shown that (a, b) and (b, a) are reflections of one another in the line $y = x$. (See Figure 2.34.)

These observations yield the following procedure for sketching the graph of an inverse function.

INVERSE GRAPHS

If f^{-1} exists, its graph may be obtained by reflecting the graph of f in the line $y = x$.

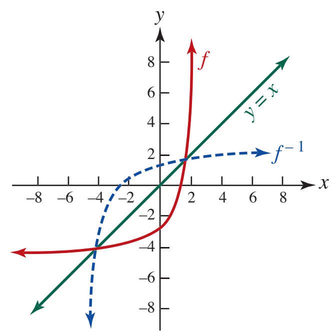


Figure 2.34 The graphs of f and f^{-1} are reflections in the line $y = x$

EXAMPLE 4 Graphing an inverse

Show that $t(x) = x^2$ on $(-\infty, 0]$ and $t^{-1}(x) = -\sqrt{x}$ (inverse functions from Example 3c) are symmetric with respect to the line $y = x$ by graphing each.

Solution The graphs are shown in Figure 2.35.

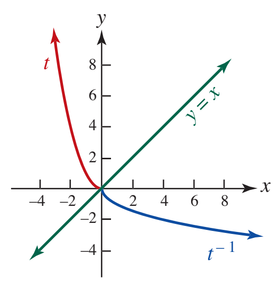


Figure 2.35 Graphs of t, t^{-1}, and $y = x$

EXAMPLE 5 Function and inverses from graph

Consider the function f defined by the graph in Figure 2.36. **a.** Find $f(5)$ **b.** Find $f^{-1}(6)$.

Solution Use the graph as shown in Figure 2.36.

This is a member of the domain of f; locate this on the x-axis.

a. $f(5) = 3$

This is found by following the dots in Figure 2.36.

This is a member of the domain of f^{-1}; locate this on the y-axis.

b. $f^{-1}(6) = 9$

This is found by following the dashes in Figure 2.36.

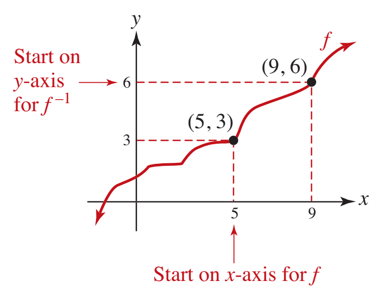


Figure 2.36 Graph of f

PROBLEM SET 2.7

LEVEL 1

WHAT IS WRONG, *if anything, with each statement in Problems 1–4? Explain your reasoning.*

1. If $f(x)=x+1$, then $f^{-1}(x)=\dfrac{1}{x+1}$.

2. a. If $f(3)=5$, then $f^{-1}(5)=3$.

b. If $f(5)=3$, then $f^{-1}(5)=\frac{1}{3}$.

3. Every function has an inverse because $f[f^{-1}(x)]=x$ for all x.

4. If $f^{-1}[f(a)]=a$, then f and f^{-1} are inverse functions.

Determine which pairs of functions defined by the equations in Problems 5–6 are inverses.

5. a. $f(x)=3x;\ g(x)=\frac{1}{3}x$

b. $f(x)=5x+3;\ g(x)=\dfrac{x-3}{5}$

c. $f(x)=\dfrac{1}{x};\ g(x)=\dfrac{1}{x}$

d. $f(x)=x^2, x<0;\ g(x)=\sqrt{x}, x>0$

6. a. $f(x)=-5x;\ g(x)=\frac{1}{5}x$

b. $f(x)=\frac{2}{3}x+2;\ g(x)=\frac{3}{2}x+3$

c. $f(x)=\dfrac{1}{x+1};\ g(x)=\dfrac{1}{x-1}$

d. $f(x)=x^2, x\geq 0;\ g(x)=\sqrt{x}, x\geq 0$

Find the inverse function, if it exists, of each function given in Problems 7–12.

7. a. $f=\{(4,5),(6,3),(7,1),(2,4)\}$

b. $f(x)=x+3$

c. $f(x)=5x$

d. $f(x)=5$

8. a. $g=\{(1,4),(6,1),(4,5),(3,4)\}$

b. $g(x)=2x+3$

c. $g(x)=\frac{1}{5}x$ **d.** $g(x)=-3$

9. a. $f(x)=x^2-4$ **b.** $f(x)=\dfrac{1}{x-3}$

10. a. $g(x)=\sqrt{x}+4$ **b.** $g(x)=\dfrac{2x+1}{x}$

11. $f(x)=\dfrac{2x-6}{3x+3}$ **12.** $g(x)=\dfrac{3x+1}{2x-3}$

LEVEL 2

If s and c are defined by the graphs in Figure 2.37, find the values requested in Problems 13–16.

13. a. $s(2)$ **b.** $s(-2)$

c. $s(0)$ **d.** $s(6)$

14. a. $s^{-1}(2)$ **b.** $s^{-1}(-2)$

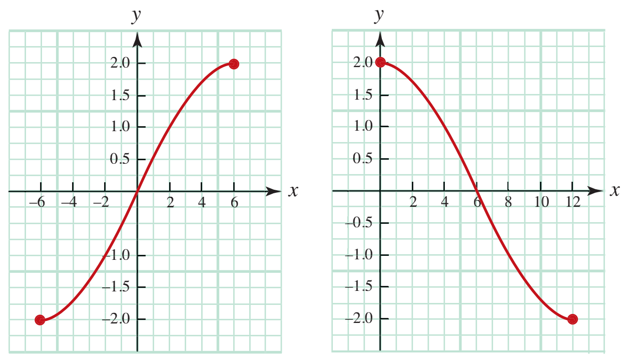
c. $s^{-1}(0)$ **d.** $s^{-1}(1)$

15. a. $c(0)$ **b.** $c^{-1}(1)$

c. $c^{-1}(0)$ **d.** $c(8)$

16. a. $c^{-1}(-1)$ **b.** $c(4)$

c. $c(12)$ **d.** $c^{-1}(-2)$



a. Graph of s **b.** Graph of c

Figure 2.37 Graphs for Problems 13–16

The functions s and c in Problems 17–24 are defined by the graphs in Figure 2.37.

17. What are the domain and range of the function s?

18. What are the domain and range of the function c?

19. Graph $y=s(x+2)$. **20.** Graph $y=c(x-2)$.

21. Graph $y=s(x)+2$. **22.** Graph $y=c(x)-2$.

23. Graph $y=s^{-1}(x)$. **24.** Graph $y=c^{-1}(x)$.

In problems 25–26, use the function defined in Figure 2.38 to graph the requested functions.

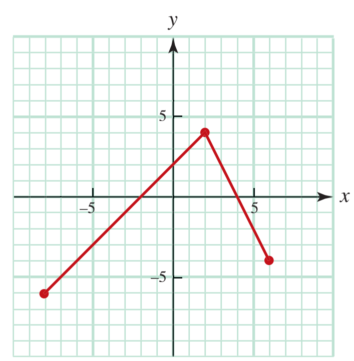


Figure 2.38 Graph of f

25. a. $y=f^{-1}(x)$ **b.** $y=f^{-1}(-x)$

26. a. $y=f^{-1}(x-2)$ **b.** $y=f^{-1}(x)-2$

For each of the Problems 27–38, graph the defined function and its inverse on the same coordinate axes, and use the graphs to decide whether the functions are inverses.

27. $f(x)=x+2;\ f^{-1}(x)=x-2$

28. $f(x)=5x-2;\ f^{-1}(x)=\dfrac{x+2}{5}$

29. $f(x)=x^2, x\geq 0;\ f^{-1}(x)=\sqrt{x}$

30. $f(x)=x^2, x\leq 0;\ f^{-1}(x)=-\sqrt{x}$

31. $f(x)=x^2-2, x\geq 0;\ f^{-1}(x)=\sqrt{x+2}$

32. $f(x)=|x-1|, x\geq 1;\ f^{-1}(x)=|x+1|, x\geq 0$

33. $f(x)=\frac{1}{4}x-2$

34. $f(x)=\frac{1}{3}x+1$

35. $f(x) = \frac{1}{3}x - \frac{5}{3}$

36. $f(x) = -\frac{1}{4}x + \frac{3}{4}$

37. $f(x) = 4x - 2$

38. $f(x) = 3x + 6$

39. If $f(x) = x^3 + 5x + 3$, find:

a. $f[f^{-1}(5)]$ **b.** $f^{-1}[f(-2)]$

c. $(f \circ f^{-1})(\pi)$ **d.** $(f^{-1} \circ f)\left(\sqrt{3}\right)$

40. If $f(x) = 5x + 1$

a. Find $f^{-1}(x)$. **b.** Find $f^{-1}(3)$.

c. Find $\dfrac{1}{f(3)}$. **d.** Does $f^{-1}(3) = \dfrac{1}{f(3)}$?

41. If $f(x) = x^4 - 3x^2 + 6$, find $f^{-1}(6)$.

42. If $F(x) = \dfrac{ax + b}{cx + d}$, find:

a. $F^{-1}(x)$ **b.** $F[F^{-1}(x)]$

c. $F[F(0)]$ **d.** $F^{-1}[F(0)]$

43. Show that $f(x) = \dfrac{5x + 3}{8x - 5}$ is its own inverse.

44. The function

$$C(F) = \frac{5}{9}(F - 32)$$

gives the temperature in Celsius when the Fahrenheit temperature (F) is known. Find the inverse function and give a verbal description.

45. The function

$$C(x) = 2.54x$$

gives the approximate number of centimeters when the length is x inches. Find the inverse function and give a verbal description.

46. Let $f(x) = \begin{cases} 2x - 3, x \le 1 \\ x - 2, x > 1 \end{cases}$

Sketch the graphs of f and f^{-1}.

47. Let $f(x) = \begin{cases} \frac{1}{2}x - 4, x \le 1 \\ x - 2, x > 1 \end{cases}$

Sketch the graphs of f and f^{-1}.

48. Let $f(x) = 10^x$.

a. Graph f, f^{-1}, and the line $y = x$ on the same coordinate axes.

b. Show that $f(x + y) = f(x)f(y)$.

49. Let $f(x) = 2^x$.

a. Graph f, f^{-1}, and the line $y = x$ on the same coordinate axes.

b. Show that $f(x - y) = \dfrac{f(x)}{f(y)}$.

50. Find the coordinates of A, B, C, and D in Figure 2.39.

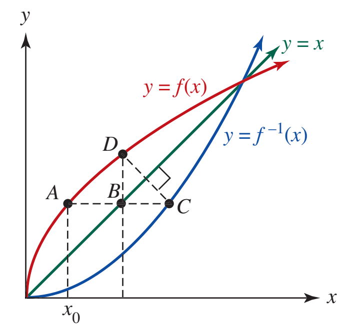


Figure 2.39 Graphs of f and f^{-1}

LEVEL 3

Determine which pairs of functions defined by the equations in Problems 51–56 are inverses.

51. $f(x) = 2x^2 + 1, x \ge 0;\ g(x) = \frac{1}{2}\sqrt{2x - 2}, x \ge 1$

52. $f(x) = 2x^2 + 1, x \le 0;\ g(x) = -\frac{1}{2}\sqrt{2x - 2}, x \ge 1$

53. $f(x) = (x + 1)^2, x \ge 1;\ g(x) = -1 - \sqrt{x}, x \ge 0$

54. $f(x) = (x + 1)^2, x \ge -1;\ g(x) = -1 + \sqrt{x}, x \ge 0$

55. $f(x) = |x + 1|, x \ge -1;\ g(x) = |x - 1|, x \ge 1$

56. $f(x) = |x + 1|, x \ge -1;\ g(x) = x - 1, x \ge 0$

Find the inverse for each function defined by the equations in Problems 57–59.

57. **a.** $f(x) = x^2$ on $[0, \infty)$ **b.** $f(x) = x^2$ on $(-\infty, 0]$

58. **a.** $f(x) = x^2 + 1$ on $(-\infty, 0]$ **b.** $f(x) = x^2 + 1$ on $[0, \infty)$

59. **a.** $f(x) = 2x^2$ on $[2, 10]$ **b.** $f(x) = 2x^2$ on $[-10, -1]$

60. If $f(x) = \dfrac{x + 1}{x - 3}$, find:

a. $f^{-1}(x)$ **b.** $f(x^{-1})$ **c.** $[f(x)]^{-1}$

2.8 Limits and Continuity

Two essential ideas in understanding the nature of functions are limits and continuity. You will investigate these ideas at length in calculus, but because of their importance, we introduce them in this section as a foretaste of things to come.

Intuitive Notion of a Limit

The limit of a function f is a tool for investigating the behavior of $f(x)$ as x gets closer and closer to a particular number c. To visualize this concept, we return to the falling object example of Section 2.2.

EXAMPLE 1 **Velocity as a limit** **MODELING APPLICATION**

A freely falling body experiencing no air resistance falls $s(t) = 16t^2$ feet in t seconds. Express the body's velocity at time $t = 2$ as a limit.

Solution

Step 1: *Understand the problem.* We know how to find the average velocity over a period of time, but here we need to define some sort of "mathematical speedometer" for measuring the *instantaneous velocity* of the body at time $t = 2$.

Step 2: *Devise a plan.* We first compute the *average velocity* $\overline{v}(t)$ of the body between time $t = 2$ and any other time t by the formula $\text{AVERAGE VELOCITY} = \dfrac{\text{DISTANCE TRAVELED}}{\text{ELAPSED TIME}}$.

Step 3: *Carry out the plan.*

$$\overline{v}(t) = \frac{\text{DISTANCE TRAVELED}}{\text{ELAPSED TIME}} = \frac{s(t) - s(2)}{t - 2} = \frac{16t^2 - 16(2)^2}{t - 2} = \frac{16t^2 - 64}{t - 2}$$

As t gets closer and closer to 2, it is reasonable to expect the average velocity $\overline{v}(t)$ to approach the value of the required instantaneous velocity at time $t = 2$.

$$\lim_{t \to 2} \overline{v}(t) = \underbrace{\lim_{t \to 2} \frac{16t^2 - 64}{t - 2}}_{\text{This is the instantaneous velocity at } t = 2.}$$

Step 4: *Look back.* Notice that we cannot find the instantaneous velocity at time $t = 2$ by simply substituting $t = 2$ into the average velocity formula because this would yield the meaningless form $0/0$.

Historical Note*

The Diné (also known as the Navajo) are a Native American people who, despite considerable interchange and assimilation with the surrounding dominant culture, maintain a worldview that remains vital and distinctive. The Navajo believe in a dynamic universe. Rather than consisting of objects and situations, the universe is made up of processes. Central to our Western mode of thought is the idea that things are separable entities that can be subdivided into smaller, discrete units. For us, things that change through time do so by going from one specific state to another specific state. Although we believe time to be continuous, we often even break it into discrete units or freeze it and talk about an instant or point in time. Among the Navajo, where the focus is on process, change is ever present; interrelationship and motion are of primary significance. These incorporate and subsume space and time.

*From *Ethnomathematics* by Marcia Ascher, pp. 128–129.

We now devote the remainder of this section to an intuitive introduction of how we can find the value of limits such as the one that appears in Example 1.

LIMIT OF A FUNCTION (informal definition)

The notation

$$\lim_{x \to c} f(x) = L$$

is read "the **limit** of $f(x)$ as x approaches c is L" and means that the functional value $f(x)$ can be made arbitrarily close to L by choosing x sufficiently close to c (but not equal to c).

» IN OTHER WORDS If $f(x)$ becomes arbitrarily close to a single number L as x approaches c from either side, then we say that L is the limit of $f(x)$ as x approaches c.

Limits by Graphing or by Table

Figure 2.40 shows the graph of a function f and the number $c = 3$.

The arrowheads are used to illustrate possible sequences of numbers along the x-axis, approaching from both the left and the right. As x approaches $c = 3$, $f(x)$ gets closer and closer to 5. We write this as

$$\lim_{x \to 3} f(x) = 5$$

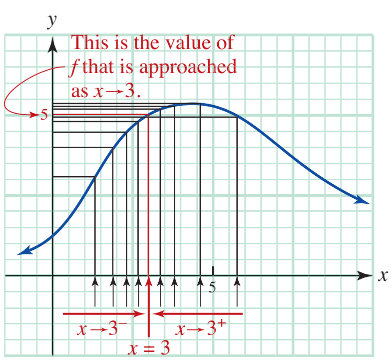


Figure 2.40 Limit as $x \to c$

As x approaches 3 from the left, we write $x \to 3^-$, and as x approaches 3 from the right, we write $x \to 3^+$. We say that the limit at $x = 3$ exists only if the value approached from the left is the same as the value approached from the right.

STOP Pay attention to this notation.

EXAMPLE 2 Estimating limits by graphing

Given the functions f, g, and h defined by the graphs in Figure 2.41, find the following limits by inspection, if they exist: **a.** $\lim\limits_{x\to 0} f(x)$ **b.** $\lim\limits_{x\to 1} g(x)$ **c.** $\lim\limits_{x\to 1} h(x)$

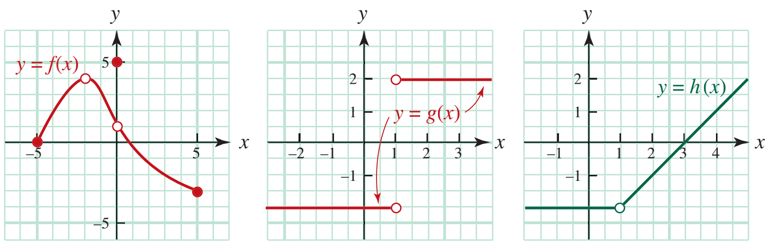


Figure 2.41 Limits from a graph

Solution

a. Take a good look at the given graph; notice the open circles on the graph at $x = 0$ and $x = -2$; also notice that $f(0) = 5$. To find $\lim\limits_{x\to 0} f(x)$, we need to look at both the left-hand and right-hand limits. Look at Figure 2.41 (left graph) to find

$$\lim_{x\to 0^-} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 0^+} f(x) = 1$$

so $\lim\limits_{x\to 0} f(x)$ exists and $\lim\limits_{x\to 0} f(x) = 1$. Notice here that the value of the limit as x approaches 0 is not the same as the value of the function at $x = 0$.

b. Look at the center graph in Figure 2.41 to find

$$\lim_{x\to 1^-} g(x) = -2 \qquad \text{and} \qquad \lim_{x\to 1^+} g(x) = 2$$

so the limit as x approaches 1 does not exist.

c. Look at the graph at the right to find

$$\lim_{x\to 1^-} h(x) = -2 \qquad \text{and} \qquad \lim_{x\to 1^+} h(x) = -2$$

so $\lim\limits_{x\to 1} h(x) = -2$.

EXAMPLE 3 Velocity limit

In Example 1, we found the velocity of a falling object at time $t = 2$ as a limit:

$$\lim_{t\to 2} \frac{16t^2 - 64}{t - 2}$$

Find this limit using the graphical approach and the numerical approach (tabular).

Solution *Graphical approach*

$$\overline{v}(t) = \frac{16t^2 - 64}{t - 2} = \frac{16(t^2 - 4)}{t - 2} = \frac{16(t-2)(t+2)}{t - 2} = 16(t + 2) \qquad t \neq 2$$

The graph of $\overline{v}(t)$ is a line with a deleted point, as shown in Figure 2.42.

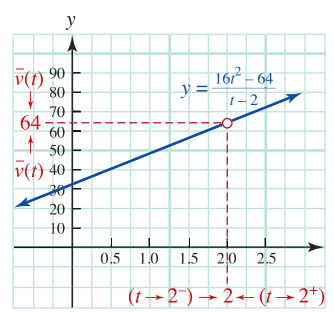


Figure 2.42 Graph of $\overline{v}(t)$

The limit can now be seen:

$$\lim_{t\to 2} \overline{v}(t) = 64$$

That is, the instantaneous velocity of the falling body at time $t = 2$ is 64 ft/s.

Numerical (tabular) approach

We begin by selecting sequences of numbers for $t \to 2^-$ and $t \to 2^+$:

t approaches from the left $t \to 2^-$		t approaches from the right $t \to 2^+$	
t	$\overline{v}(t)$	t	$\overline{v}(t)$
1.950	63.200	2.100	65.600
1.995	63.920	2.015	64.240
1.999	63.984	2.001	64.016
2	undefined	2	undefined

That is, the pattern of numbers suggests

$$\lim_{t\to 2} \frac{16t^2 - 64}{t - 2} = 64$$

This tabular approach agrees with what we found with the graphical approach. ■

Example 3 clearly shows that we cannot evaluate limits by substitution because if we attempt to substitute $t = 2$ into the velocity function $\frac{16t^2 - 64}{t - 2}$ we find that it does not exist (because we cannot divide by zero). However, the limit *does* exist and is 64. However, in calculus, you will prove that you may find the limit of a polynomial by direct substitution.

LIMIT OF A POLYNOMIAL

If P is a polynomial function, then

$$\lim_{x\to c} P(x) = P(c)$$

EXAMPLE 4 Limit of a polynomial function

Evaluate **a.** $\lim_{x\to 2}(2x^5 - 9x^3 + 3x^2 - 11)$ **b.** $\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3}$

Solution

a. It would be difficult to graph this function, and a table of values would be lengthy. However, because this is a polynomial function, we can find the limit by direct substitution.

$$\lim_{x\to 2}(2x^5 - 9x^3 + 3x^2 - 11) = 2(2)^5 - 9(2)^3 + 3(2)^2 - 11 = -7$$

b. $\lim_{x\to -3} \frac{x^2 + x - 6}{x + 3} = \lim_{x\to -3} \frac{(x + 3)(x - 2)}{x + 3}$ Note that direct substitution gives an undefined expression (can't divide by 0).

$= \lim_{x\to -3}(x - 2)$ Because $x - 2$ is a polynomial, evaluate by substitution.

$= 3 - 2$

$= 1$ ■

Notice from the preceding examples that when we write

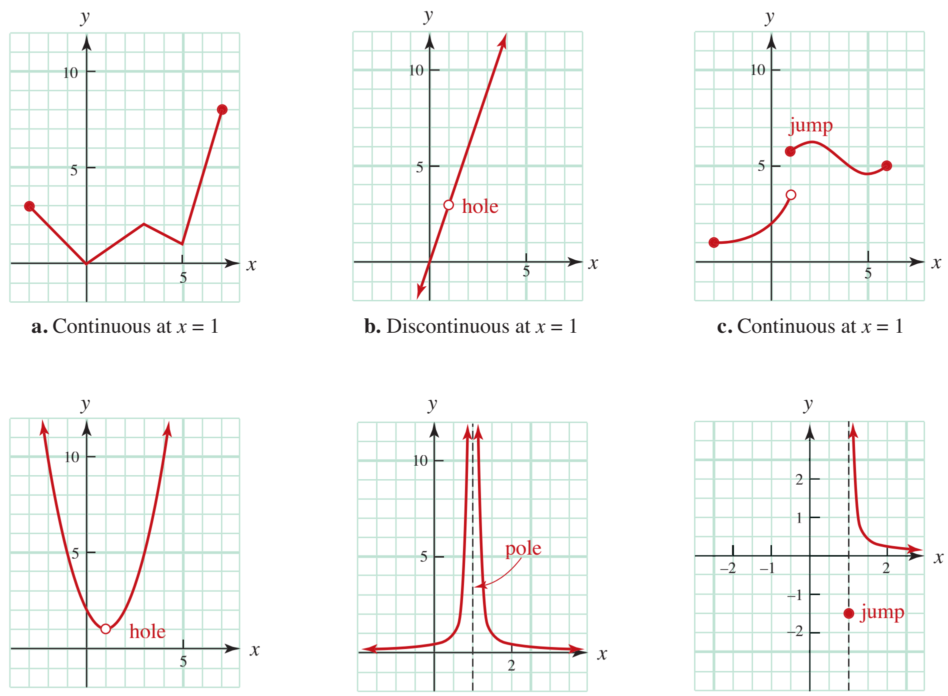
$$\lim_{x\to c} f(x) = L$$

we do not require c itself to be in the domain of f, nor do we require $f(c)$, if it is defined, to be equal to the limit. Functions with the special property that $\lim\limits_{x\to c} f(x) = f(c)$ are said to be *continuous* at $x = c$.

Intuitive Notion of Continuity

Continuity may be thought of informally as the quality of having parts that are in immediate connection with one another. This idea evolved from the vague or intuitive notion of a curve "without breaks or jumps" to a rigorous definition first given toward the end of the 19th century.

We begin with a discussion of *continuity at a point*. It may seem strange to talk about continuity *at a point*, but it should seem natural to talk about a curve being "discontinuous at a point," as illustrated by Figure 2.43.



a. Continuous at $x = 1$

b. Discontinuous at $x = 1$

c. Continuous at $x = 1$

d. Discontinuous at $x = 1$

e. Discontinuous at $x = 1$ (Sometimes a pole is called a vertical asymptote.)

f. Discontinuous at $x = 1$

Figure 2.43 Holes, poles, and jumps

Definition of Continuity

Let us consider the conditions that must be satisfied for a function f to be continuous at a point c. First, $f(c)$ must be defined. For example, the curves in Figures 2.43**b** and **e** are not continuous at $x = 1$ because they are not defined for $x = 1$. (An open dot indicates an excluded point.) A second condition for continuity at a point $x = c$ is that the function makes no jumps there. This means that if "x is close to c," then "$f(x)$ must be close to $f(c)$." This condition is satisfied if $\lim\limits_{x\to c} f(x)$ exists. Looking at Figure 2.43, we see that the graphs in parts **c**, **d**, and **f** have a jump at the point $x = 1$. A third condition for continuity at point $x = c$ is that $\lim\limits_{x\to c} f(x) = f(c)$. Note that in the curve in Figure 2.43**d**, $\lim\limits_{x\to 1} f(x)$ exists but is not equal to $f(1)$. These considerations lead us to a formal definition of continuity of a function at a point.

Historical Note*

The idea of continuity evolved from the notion of a curve "without breaks or jumps" to a rigorous definition given by Karl Weierstraß. Our definition of continuity is a refinement of a definition first given by Bernhard Bolzano (1781–1848). Galileo and Leibniz had thought of continuity in terms of the density of points on a curve, but using today's standards, we would say they were in error because the rational numbers have this property, yet do not form a continuous curve. However, this was a difficult concept, which evolved over a period of time. Another mathematician, J. W. R. Dedekind (1831–1916), took an entirely different approach to conclude that continuity is due to the division of a segment into two parts by a point on the segment. As Dedekind wrote, "By this commonplace remark, the secret of continuity is to be revealed."

*From Carl Boyer, *A History of Mathematics* (New York, John Wiley & Sons, Inc., 1968), p. 607.

CONTINUITY AT A POINT

A function f is **continuous at a point $x = c$** if

1. $f(c)$ is defined;
2. $\lim_{x \to c} f(x)$ exists;
3. $\lim_{x \to c} f(x) = f(c)$.

A function that is not continuous at c is said to have a **discontinuity** at that point.

» IN OTHER WORDS Step 1 refers to the domain of the function and ignores what happens at points $x \neq c$, whereas step 2 refers to points close to c but ignores the point $x = c$. Continuity looks at the whole picture: at $x = c$ and at points close to $x = c$ and checks to see whether they are somehow "alike."

If f is continuous at $x = c$, the difference between $f(x)$ and $f(c)$ is small whenever x is close to c because $\lim_{x \to c} f(x) = f(c)$ Geometrically, this means that the points $(x, f(x))$ on the graph of f converge to the point $(c, f(c))$ as $x \to c$ and this is what guarantees that the graph is unbroken at $(c, f(c))$ with no "gap" or "hole."

EXAMPLE 5 Testing the definition of continuity with a given function

Test the continuity of each of the following functions at $x = 1$. If it is not continuous at $x = 1$, explain.

a. $f(x) = \dfrac{x^2 + 2x - 3}{x - 1}$

b. $g(x) = \dfrac{x^2 + 2x - 3}{x - 1}$ if $x \neq 1$, and $g(x) = 6$ if $x = 1$

c. $h(x) = \dfrac{x^2 + 2x - 3}{x - 1}$ if $x \neq 1$, and $h(x) = 4$ if $x = 1$

d. $F(x) = \dfrac{x + 3}{x - 1}$ if $x \neq 1$, and $F(x) = 4$ if $x = 1$

e. $G(x) = 7x^3 + 3x^2 - 2$

Solution We verify that the three criteria for continuity are satisfied for $c = 1$.

a. The function f is not continuous at $x = 1$ (hole; $f(c)$ not defined) because it is not defined at this point.

b. 1. $g(1)$ is defined; $g(1) = 6$.

2. $$\begin{aligned}\lim_{x \to 1} g(x) &= \lim_{x \to 1} \frac{x^2 + 2x - 3}{x - 1}\\ &= \lim_{x \to 1} \frac{(x - 1)(x + 3)}{x - 1}\\ &= \lim_{x \to 1} (x + 3)\\ &= 4\end{aligned}$$

3. $\lim_{x \to 1} g(x) \neq g(1)$, so g is not continuous at $= 1$ (hole; $g(c)$ defined).

c. Compare h with g of part **b**. We see that all three conditions of continuity are satisfied, so h is continuous at $x = 1$.

d. 1. $F(1)$ is defined; $F(1) = 4$.

2. $\lim_{x\to 1} F(x) = \lim_{x\to 1} \dfrac{x+3}{x-1}$; the limit does not exist.

The function F is not continuous at $x = 1$ (pole; $F(c)$ defined).

e. 1. $G(1)$ is defined; $G(1) = 8$.

2. $\lim_{x\to 1} G(x) = 7(1)^3 + 3(1)^2 - 2$ G is a polynomial function

$= 8$

3. $\lim_{x\to 1} G(x) = G(1)$.

Because the three conditions of continuity are satisfied, G is continuous at $x = 1$.

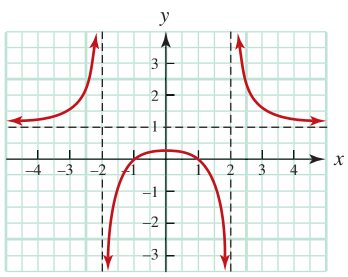
Continuity on an Interval

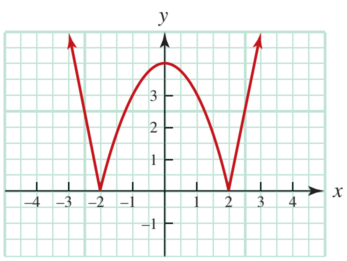
The function f is said to be **continuous on the open interval** (a, b) if it is continuous at each number in this interval. If f is also continuous from the right at a, we say it is **continuous on the half-open interval** $[a, b)$. Similarly, f is **continuous on the half-open interval** $(a, b]$ if it is continuous at each number between a and b and is continuous from the left at the endpoint b. Finally, f is **continuous on the closed interval** $[a, b]$ if it is continuous at each number between a and b and is both continuous from the right at a and continuous from the left at b.

EXAMPLE 6 Testing for continuity on an interval

Find the intervals on which each of the given functions is continuous.

a. $f_1(x) = \dfrac{x^2 - 1}{x^2 - 4}$ **b.** $f_2(x) = |x^2 - 4|$





Solution

a. Function f_1 is not defined when $x^2 - 4 = 0$ or when $x = 2$ or $x = -2$. The curve is continuous on $(-\infty, -2) \cup (-2, 2) \cup (2, \infty)$.

b. Function f_2 is continuous on $(-\infty, \infty)$.

Because we do not always have the graph of a function readily available, as we did in Example 6, and because the task of checking for continuity will focus on certain values, we consider a procedure involving identifying and then checking those values of concern. To help us describe the situation, we define a **suspicious point** as a point having an x value for which the definition of the function changes, or a value that causes division by zero for the given function.

For Example 6, the suspicious points can be listed:

a. $\dfrac{x^2 - 1}{x^2 - 4}$ has suspicious points for division by zero when and $x = 2$ and $x = -2$.

b. $|x^2 - 4| = x^2 - 4$ when $x^2 - 4 \geq 0$, and $|x^2 - 4| = 4 - x^2$ when $x^2 - 4 < 0$. This means the definition of the function changes when $x^2 - 4 = 0$, namely, $x = 2$ and $x = -2$ are suspicious points.

EXAMPLE 7 Checking continuity at suspicious points

Let $f(x) = \begin{cases} 3 - x & \text{if } -5 \le x < 2 \\ x - 2 & \text{if } 2 \le x < 5 \end{cases}$ and $g(x) = \begin{cases} 2 - x & \text{if } -5 \le x < 2 \\ x - 2 & \text{if } 2 \le x < 5 \end{cases}$

Find the intervals on which f and g are continuous.

Solution The domain for both functions is $[-5, 5)$; the functions are continuous everywhere on that interval except possibly at the suspicious points.

Examining f, we see $f(x) = 3 - x$ on $[-5, 2)$, which is a polynomial function and thus is continuous; $f(x) = x - 2$ on $[2, 5)$, which is a polynomial function and thus is continuous. The suspicious point on the real number line is the x value for which the definition of f changes—in this case, $x = 2$.

For g, we likewise see that the function is continuous except possibly at the suspicious point $x = 2$, the value where the definition of the function changes.

	Function f	Function g
Suspicious point(s):	$x = 2$	$x = 2$
1.	$f(2) = 2 - 2 = 0$ f is defined at $x = 2$.	$g(2) = 2 - 2 = 0$ g is defined at $x = 2$.
2.	$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2^-}(3 - x)$ $= 1$ $\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2^+}(x - 2)$ $= 0$ Thus, $\lim\limits_{x\to 2^-} f(x)$ does not exist because the left- and right-hand limits are not equal.	$\lim\limits_{x\to 2^-} g(x) = \lim\limits_{x\to 2^-}(2 - x)$ $= 0$ $\lim\limits_{x\to 2^+} g(x) = \lim\limits_{x\to 2^+}(x - 2)$ $= 0$ Thus, $\lim\limits_{x\to 2} g(x) = 0$
3.	The third condition of continuity cannot hold at $x = 2$ because the limit does not exist.	$\lim\limits_{x\to 2^-} g(x) = g(2)$
Conclusion:	Continuous on $[-5, 2)$ and on $[2, 5)$ but not on $[-5, 5)$.	Continuous on $[-5, 5)$.

Root Location and Limits That Do Not Exist

In calculus, you will use continuity to prove an important result that we will need to approximate the roots of equations in algebra. This result asserts that if a function is continuous on a closed interval and has opposite signs someplace on that interval, then there is a root somewhere between its positive and negative values.

ROOT LOCATION THEOREM

If f is continuous on the closed interval $[a, b]$ and if $f(a)$ and $f(b)$ have opposite algebraic signs (one positive and the other negative), then $f(c) = 0$ for at least one number c on the open interval (a, b).

EXAMPLE 8 Using the root location theorem

Show that $x^4 = 5\sqrt{x} - 1 = 0$ for at least one number c on $[0, 2]$.

Solution Notice that the function $f(x) = x^4 - 5\sqrt{x} - 1$ is continuous on $[0, 2]$. Also notice that $f(0) = -1$ (it is negative) and $f(2) = 15 - 5\sqrt{2}$ (it is positive), so the conditions of the root location theorem apply.

Therefore, there is at least one number c on $(0, 2)$ for which $f(c) = 0$. This means that on $(0, 2)$, $c^4 - 5\sqrt{c} - 1 = 0$, as required.

It may happen that a function f does not have a (finite) limit as $x \to c$. When $\lim\limits_{x \to c} f(x)$ fails to exist, we say that $f(x)$ **diverges** as x approaches c. The following examples illustrate how divergence may occur.

EXAMPLE 9 A function that diverges

Evaluate $\lim\limits_{x \to 0} \dfrac{1}{x^2}$.

Solution As $x \to 0$, the corresponding functional values of $f(x) = \frac{1}{x^2}$ grow arbitrarily large, as indicated in the table below:

x approaches from the left $x \to 0^-$		x approaches from the right $x \to 0^+$	
x	$f(x)$	x	$f(x)$
-0.1	$100 = 1 \times 10^2$	0.01	1×10^2
-0.05	$400 = 4 \times 10^2$	0.05	4×10^2
-0.001	1×10^6	0.001	1×10^6
0	undefined	0	undefined

The graph of f from Table 2.1 is shown in Figure 2.44.

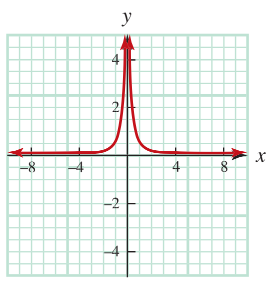


Figure 2.44 Graph of $y = \dfrac{1}{x^2}$

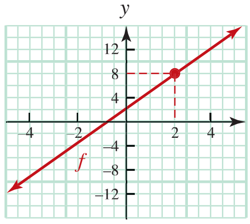
Geometrically, the graph of $y = f(x)$ rises without bound as $x \to 0$. Thus, $\lim\limits_{x \to 0} \dfrac{1}{x^2}$ does not exist, so we say f diverges as $x \to 0$.

PROBLEM SET 2.8

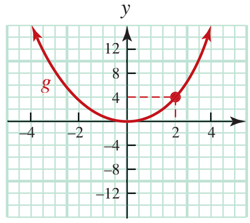
LEVEL 1

Describe each illustration in Problems 1–4 using a limit statement.

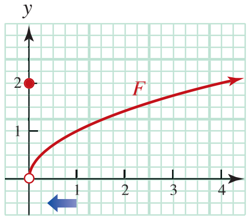
1.



2.



3.



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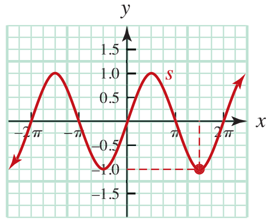


5. State the domain of each function, if possible, and determine whether it represents a continuous function.

a. The temperature on a specific day at a given location considered as a function of time

b. The humidity on a specific day at a given location considered as a function of time

c. The selling price of AT&T stock on a specific day considered as a function of time

d. The number of unemployed people in the United States during January 2007 considered as a function of time

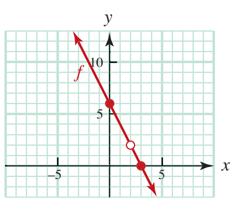
e. The charges for a taxi ride across town considered as a function of mileage

f. The charges to mail a package as a function of its weight

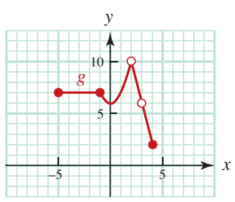
6. Let $f(x) = \begin{cases} x^2 & \text{if } x > 2 \\ x+1 & \text{if } x \le 2 \end{cases}$

Show that f is continuous from the left at 2 but not from the right.

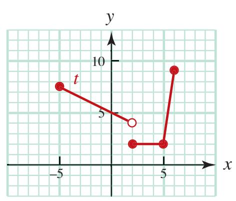
Given the functions defined by the graphs in Problems 7–8, find the limits.



Graph of f



Graph of g



Graph of t

7. a. $\lim_{x\to 3} f(x)$ **b.** $\lim_{x\to 2} f(x)$ **c.** $\lim_{x\to -3} g(x)$ **d.** $\lim_{x\to -1} g(x)$ **e.** $\lim_{x\to 2^-} t(x)$ **f.** $\lim_{x\to 2^+} t(x)$

8. a. $\lim_{x\to 0} f(x)$ **b.** $\lim_{x\to -1} f(x)$ **c.** $\lim_{x\to 2} g(x)$ **d.** $\lim_{x\to 3^+} g(x)$ **e.** $\lim_{x\to 4} t(x)$ **f.** $\lim_{x\to -4} t(x)$

PROBLEMS FROM CALCULUS *Evaluate the limits in Problems 9–18. You may use the numerical, graphical, or algebraic method of solution. If the limit does not exist, explain why.*

9. $\lim_{x\to 2^-}(x^2-4)$

10. $\lim_{x\to 3^+}(x^2-4)$

11. $\lim_{x\to -2}(x^2+3x-7)$

12. $\lim_{x\to 0}(x^3-5x^2+4)$

13. $\lim_{x\to 3}\dfrac{x^2+3x-10}{x-2}$

14. $\lim_{x\to 2}\dfrac{x^2+3x-10}{x-2}$

15. $\lim_{x\to -3^+}\dfrac{1}{x-3}$

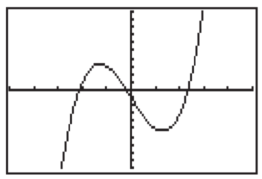
16. $\lim_{x\to 3^-}\dfrac{1}{x-3}$

17. $\lim_{x\to 0}\dfrac{\frac{1}{x+3}-\frac{1}{3}}{x}$

18. $\lim_{x\to 0}\dfrac{\frac{1}{x}-\frac{1}{3}}{x-3}$

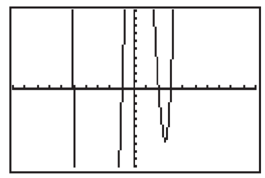
Which of the functions defined in Problems 19–24 are continuous? If it is not continuous, explain why.

19.



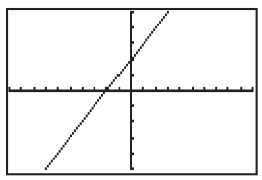
\y1=x^3-5x-1
xMin=-5 yMin=-10
xMax=5 yMax=10
xScl=1 yScl=1

20.



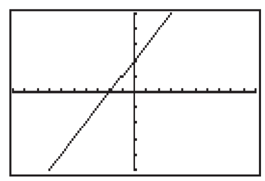
\y1=(x-2)(x-3)(x+1)(x+5)
xMin=-10 yMin=-10
xMax=10 yMax=10
xScl=1 yScl=1

21.



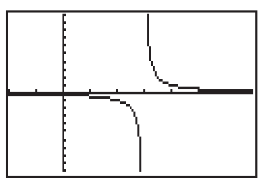
\y1=(x^2-4)/(x-2)
xMin=-10 yMin=-5
xMax=10 yMax=5
xScl=1 yScl=1

22.



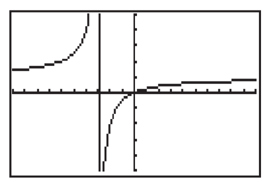
\y1=(x^2-x-6)/(x-3)
xMin=-10 yMin=-5
xMax=10 yMax=5
xScl=1 yScl=1

23.



\y1=1/(x-3)
xMin=-2 yMin=-10
xMax=7 yMax=10
xScl=1 yScl=1

24.



\y1=x/(x+3)
xMin=-10 yMin=-5
xMax=10 yMax=5
xScl=1 yScl=1

PROBLEMS FROM CALCULUS *In Problems 25–28, compute the one-sided limit.*

25. $\lim_{x\to 2^-}(x^2-2x)$

26. $\lim_{x\to 2^-} f(x)$, where $f(x)=\begin{cases}3-2x & \text{if } x\le 2\\ x^2-5 & \text{if } x>2\end{cases}$

27. $\lim_{s\to 1^-} g(x)$, where $g(x)=\begin{cases}\dfrac{s^2-s}{s-1} & \text{if } s<1\\ \sqrt{1-s} & \text{if } s\ge 1\end{cases}$

28. $\lim_{s\to 1^+} g(x)$, where $g(x)=\begin{cases}\dfrac{s^2-s}{s-1} & \text{if } s<1\\ \sqrt{1-s} & \text{if } s\ge 1\end{cases}$

PROBLEMS FROM CALCULUS *Identify all suspicious points and determine all points of discontinuity in Problems 29–38.*

29. $f(x)=x^3-3x+5$

30. $f(x)=\dfrac{2x+4}{x-6}$

31. $f(x)=\dfrac{x}{x^2-x}$

32. $f(x)=3-(5+2x)^3$

33. $f(x)=\sqrt{x}+\dfrac{5}{x}$

34. $f(x)=\sqrt[3]{x^2-1}$

35. $f(x)=\dfrac{1}{x}-\dfrac{3}{x+1}$

36. $f(x)=\dfrac{x^2-1}{x^2+x-2}$

37. $f(x)=\begin{cases}x^2-2 & \text{if } x>1\\ 2x-3 & \text{if } x\le 1\end{cases}$

38. $f(x)=\begin{cases}3x^2-2 & \text{if } x>3\\ 5 & \text{if } 1<x\le 3\\ 3x+2 & \text{if } x\le 1\end{cases}$

PROBLEMS FROM CALCULUS *In Problems 39–42, determine whether or not the given function is continuous on the prescribed interval.*

39. $f(x)=\dfrac{1}{x}$

a. on $[1, 2]$ **b.** on $[0, 1]$ **c.** on $[-3, 0)$

40. $f(x)=\dfrac{1}{5-x}$

a. on $[0, 7)$ **b.** on $[0, 5]$ **c.** on $[-1, 1]$

41. $f(x)=\begin{cases}x^2 & \text{if } 0\le x<2\\ 3x+1 & \text{if } 2\le x<5\end{cases}$

42. $g(t)=\begin{cases}2t & \text{if } 0<t\le 3\\ 15-t^2 & \text{if } -3\le t\le 0\end{cases}$

LEVEL 2

PROBLEMS FROM CALCULUS *In Problems 43–50, either evaluate the limit or explain why it does not exist.*

43. $\lim_{x\to 1}\dfrac{1}{x-1}$

44. $\lim_{x\to 2}\dfrac{x^2-4}{x^2-4x+4}$

45. $\lim_{x\to 1} f(x)$, where $f(x)=\begin{cases}2 & \text{if } x\ge 1\\ -5 & \text{if } x<1\end{cases}$

46. $\lim_{t\to -1} g(t)$, where $g(t)=\begin{cases}2t+1 & \text{if } t\ge -1\\ 5t^2 & \text{if } t<-1\end{cases}$

47. $\lim_{t\to 5} f(t)$, where $f(t)=\begin{cases}t+3 & \text{if } t\ne 5\\ 4 & \text{if } t=5\end{cases}$

48. $\lim_{t\to 2} g(t)$, where $g(t)=\begin{cases}t^2 & \text{if } t\ne 2\\ 5 & \text{if } t=2\end{cases}$

49. $\lim_{x\to 2} F(x)$, where $F(x)=\begin{cases}2(x+1) & \text{if } x<2\\ 4 & \text{if } x=2\\ x^2-1 & \text{if } x>2\end{cases}$

50. $\lim_{x\to 3} G(x)$, where $G(x)=\begin{cases}2(x+1) & \text{if } x<3\\ 4 & \text{if } x=3\\ x^2-1 & \text{if } x>3\end{cases}$

In Problems 51–56, show that the given equation has at least one solution on the indicated interval.

51. $x^3-4x-5=0$ on $[-5, 5]$

52. $x^5-4x+6=0$ on $[-5, 5]$

53. $x^4-4x^2=0$ on $[-5, 5]$

54. $x^3-4x^2=0$ on $[-5, 5]$

55. $x^3-24x^2+188x-465=0$ on $[-10, 10]$

56. $x^3+36x^2+424x+1{,}657=0$ on $[-20, 20]$

57. A ball is thrown directly upward from the edge of a cliff and travels in such a way that t seconds later, its height above the ground at the base of the cliff (in feet) is

$$s(t)=-16t^2+40t+24$$

a. Compute the limit

$$v(t)=\lim_{x\to t}\frac{s(x)-s(t)}{x-t}$$

to find the instantaneous velocity of the ball at time t.

b. What is the ball's initial velocity?

c. When does the ball hit the ground, and what is its impact velocity?

d. When does the ball have velocity 0? What physical interpretation should be given to this time?

58. Tom and Sue are driving along a straight, level road in a car whose speedometer needle is broken but which has a trip odometer that can measure the distance traveled from an arbitrary starting point in tenths of a mile. At 2:50 P.M., Tom says he would like to know how fast they are traveling at 3:00 P.M., so Sue takes down the odometer readings listed in the table below, makes a few calculations, and announces the desired velocity. What is her result?

Time, t	2:50	2:55	2:59	
Odometer reading	33.9	38.2	41.5	
Time, t	**3:00**	**3:01**	**3:03**	**3:06**
Odometer reading	42.4	43.2	44.9	47.4

LEVEL 3

In Problems 59–60, find constants a *and* b *so that the given function will be continuous for all* x *throughout its domain.*

59. $f(x) = \begin{cases} ax + 12 & \text{if } x > 2 \\ 20 & \text{if } x = 2 \\ x^2 + bx + 5 & \text{if } x < 2 \end{cases}$

60. $g(x) = \begin{cases} ax + 3 & \text{if } x > 5 \\ 8 & \text{if } x = 5 \\ x^2 + bx + 1 & \text{if } x < 5 \end{cases}$

CHAPTER 2 SUMMARY AND REVIEW

The business of concrete mathematics is to discover the equations which express the mathematical laws of the phenomenon under consideration; and these equations are the starting-point of the calculus.

Auguste Compte

Take some time getting ready to work the review problems in this section. First, look back at the definition and property boxes. You will maximize your understanding of this chapter by working the problems in this section only after you have studied the material.

STOP

SELF TEST *All of the answers for this self test are given in the back of the book.*

1. If $f(x) = \dfrac{6x^2 + x - 2}{2x - 1}$, find:

a. $f(2x)$ **b.** $2f(x)$

c. $f(-\frac{2}{3})$ **d.** $|f(3) - f(2)|$

e. $f^{-1}(x)$

2. Let $f(x) = \sqrt{34 - 2x^2}$.

a. What is the domain? **b.** What is the range?

c. If $f(x) = g[u(x)]$, find g and u.

d. What is $f(-3)$? **e.** Find $\lim\limits_{x \to -3} f(x)$.

3. If $16x^2 - x^2y^2 + y^2 - 1 = 0$

a. What is the domain? **b.** What is the range?

c. What are the x-intercepts? **d.** What are the y-intercepts?

e. Does this equation represent a function? What or why not?

4. If $f(x) = \frac{1}{2}(x - 2)^2$

a. Describe f by comparing it to $y = x^2$; classify it and then specify the shift, compression, or dilation, as appropriate.

b. Graph f. **c.** Find $\lim\limits_{x \to 0} f(x)$.

d. Find $\dfrac{f(x + h) - f(x)}{h}$. **e.** Find $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$.

5. If $f(x) = \begin{cases} x^2 + 1 & \text{if } x > 1 \\ |x + 1| & \text{if } x \le 1 \end{cases}$

a. What is the domain of this function?

b. What are the intercepts? **c.** Graph f.

d. Is f continuous at $x = 1$?

e. If $y = f(x)$, graph $y - 1 = f(x + 2)$.

6. Suppose f is defined by the graph shown in Figure 2.45.

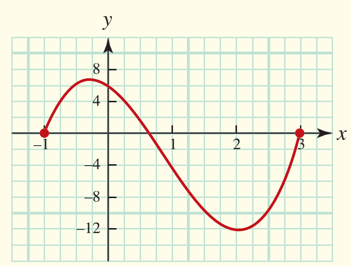


Figure 2.45 Graph of *f*

a. Is f a function?

b. If f one-to-one? Does it have an inverse?

c. What are the domain and range of f?

d. If $y = f(x)$, draw the graph of $y = \frac{1}{2}f(x)$.

e. Graph $y + 2 = f(x - 1)$.

7. Consider the function defined by the graph in Figure 2.46.

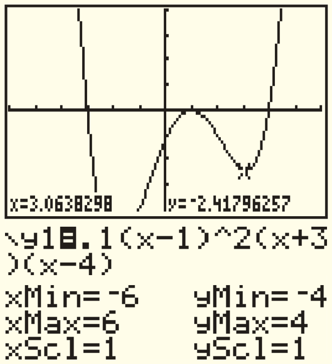


Figure 2.46 Function of *f* defined by a graph

a. Is f continuous on $[-6, 6]$?
b. Is f one-to-one?
c. What are the intercepts?
d. What are the approximate coordinates of the turning points for $x \geq 0$?
e. What is $\lim_{x \to 0} f(x)$?

8. Let $f(x) = 5 - x^2$, and $g(x) = \dfrac{5 - x}{x + 2}$.

a. Find g^{-1}, if it exists. **b.** Find $f \circ f$.
c. Find $g \circ f$.
d. Classify f and g as even, odd, or neither.
e. For which values is $f(x) \geq -20$?

9. a. An efficiency expert found that at a particular company that employs x workers $(x \geq 3)$, it takes

$$H = \frac{3x + 4}{2x - 5}$$

hours to complete a certain task. How many hours will the task take for 3 workers? How many hours will the task take for 10 workers? For 20 workers? If the task must be completed in 2 hours, how many workers are required? If each worker earns \$25/hr, express the total labor cost, C, as a function of x.

b. If f and g are defined by

$$f(x) = \frac{6x^2 - x - 2}{3x - 2} \quad \text{and} \quad g(x) = 2x + 1$$

is it true that $f = g$? Why or why not?

10. An open box with a square base is to be built for \$96. The sides of the box will cost \$3/ft² and the base will cost \$8/ft². Express the volume of the box as a function of the length of its base.

STOP STUDY HINTS *Compare your solutions and answers to the self test.*

Practice for Calculus—*Supplementary Problems: Cumulative Review Chapters 1–2*

***Simplify** each expression in Problems 1–4.*

1. Find $|f(x) - L|$, where $f(x) = x^2 - 2$ and $L = 2$.
2. Find $|f(x) - L|$, where $f(x) = x^2 + 2$ and $L = 6$.
3. Find $|f(x) - L|$, where $f(x) = \dfrac{2x^2 - 3x - 2}{x - 2}$ and $L = 6$.
4. Find $|f(x) - L|$, where $f(x) = \dfrac{x^2 - 2x + 2}{x - 4}$ and $L = -1$.

***Solve** each inequality in Problems 5–12.*

5. $-0.001 \leq x + 2 \leq 0.001$ **6.** $-0.001 \leq 2x + 5 \leq 0.001$
7. $|3x + 1| < 0.25$ **8.** $|x - 5| < 0.01$
9. $|5 - 3x| < 0.001$ **10.** $|1 - 8x| < 0.001$
11. $|2x - 1| < 0.0001$ **12.** $|5x + 3| < 0.0001$

***Solve** each equation for x in Problems 13–24.*

13. $x^2 - 5x + 3 = 0$ **14.** $2x^2 - 5x - 3 = 0$
15. $x^2 + 9x + 20 = 0$ **16.** $x^2 - 3x + 1 = 0$
17. $|2x + 3| = 8$ **18.** $|5x + 1| = 10$
19. $|2 - 3x| = 11$ **20.** $|5 - 4x| = 20$
21. $x^2 = 4cy$ **22.** $x^2 + y^2 = 9$
23. $4x^2 + 3y^2 = 1$ **24.** $3x^2 - 4y^2 = 12$

***Factor** the expressions in Problems 25–32.*

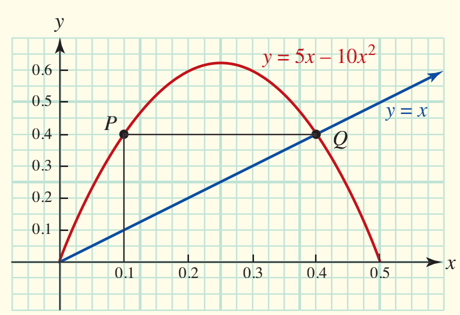
25. $1 - x^6$ **26.** $x^6 - 64$
27. $\left(x^2 - \frac{1}{9}\right)\left(x^2 - \frac{1}{25}\right)$ **28.** $x^4 - 41x^2 + 400$
29. $-4(3x^2 - 2x)^{-5}(6x - 4)$ **30.** $-2(2x^3 - 8x)^{-3}(6x - 8)$
31. $2(2x^2 + 3)(4x)(x^3 - 1)^3 + 3(2x^2 + 3)^2(x^3 - 1)^2(3x^2)$
32. $4(x^2 - 8)^3(2x)(5x^2 - 1)^3 + 3(x^2 - 8)^4(5x^2 - 1)(10x)$

In Problems 33–38, find $\dfrac{f(x + h) - f(x)}{h}$.

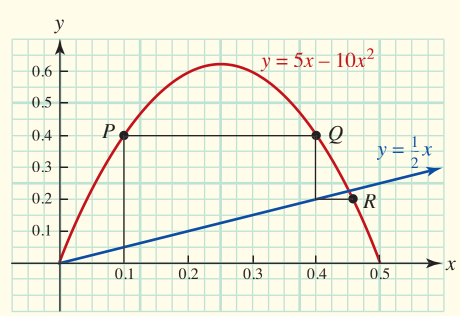
33. $f(x) = x + 5$ **34.** $f(x) = 6 - 3x$
35. $f(x) = 5x^2$ **36.** $f(x) = 6$
37. $f(x) = \frac{1}{x}$ **38.** $f(x) = 2x^2 + x$

*In the **graphs** in Problems 39–44, assume the dashed lines are parallel to the coordinate axes.*

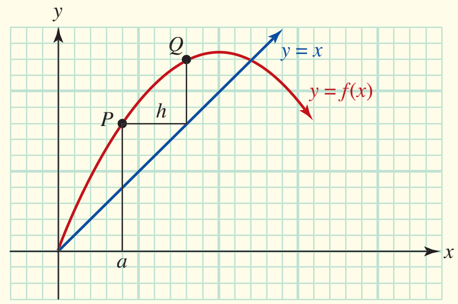
39. What are the coordinates of P and Q?



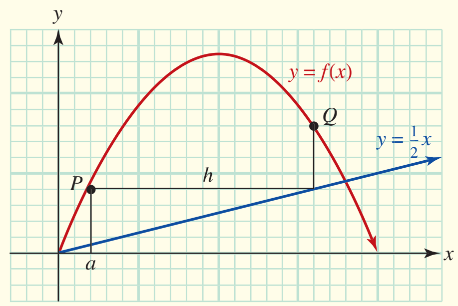
40. What are the coordinates of P, Q, and R?



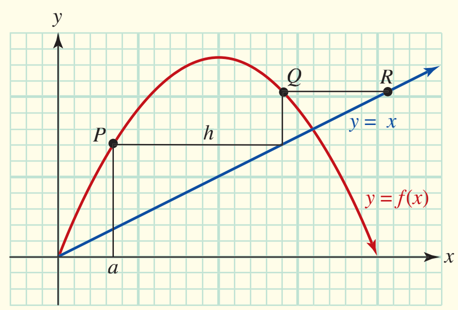
41. What are the coordinates of P and Q?



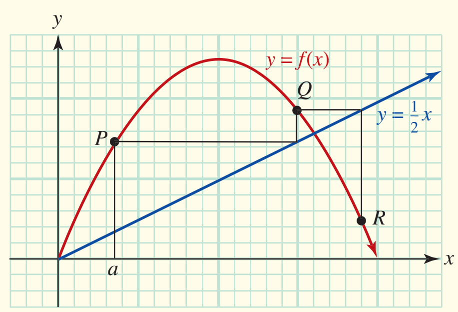
42. What are the coordinates of P and Q?



43. What are the coordinates of R?



44. What is the second component of R?



Evaluate the limits in Problems 45–52.

45. $\lim\limits_{x\to 3}(2x^2 - x - 10)$

46. $\lim\limits_{x\to 3}\dfrac{2x^2 - x - 10}{x+2}$

47. $\lim\limits_{x\to 2}(2x^2 - x - 10)$

48. $\lim\limits_{x\to 2}\dfrac{2x^2 - x - 10}{x+2}$

49. $\lim\limits_{h\to 0}\dfrac{[(x+h)+5]-(x+5)}{h}$

50. $\lim\limits_{h\to 0}\dfrac{[6-3(x+h)]-(6-3x)}{h}$

51. $\lim\limits_{h\to 0}\dfrac{(x+h)^2 - x^2}{h}$

52. $\lim\limits_{h\to 0^-}\dfrac{|2+h|-|2|}{h}$

In Problems 53–58, express each function as the composite of two functions. That is, express the given function f as $f(x) = g[u(x)]$, and then state the inner function u and the outer function g.

53. $f(x) = (3x^2 - 5x)^2$

54. $f(x) = \sqrt{x^2+9}$

55. $f(x) = (3x^4 - 1)^{3/2}$

56. $f(x) = |x^2 + 8|$

57. $f(x) = (x^2+3)^4 - (x^2+3)^{5/2}$

58. $f(x) = \sqrt[5]{x^2 + 3x + 1}$

59. To study the rate at which animals learn, a psychology student performed an experiment in which a rat was sent repeatedly through a laboratory maze. Suppose that the time (in minutes) required for the rat to traverse the maze on the nth trial was approximately

$$f(n) = 3 + \frac{12}{n}$$

a. What is the domain of the function f?

b. For what values of n does $f(n)$ have meaning in the context of the psychology experiment?

c. How long did it take the rat to traverse the maze on the third trial?

d. On which trial did the rat first traverse the maze in 4 minutes or less?

e. According to the function f, what will happen to the time required for the rat to traverse the maze as the number of trials increases?

60. Biologists have found that the speed of blood in an artery is a function of the distance of the blood from the artery's central axis. According to *Poiseuille's law*, the speed (cm/sec) of blood that is r cm from the central axis of an artery is given by the function

$$S(r) = C(R^2 - r^2)$$

where C is a constant and R is the radius of the artery.* Suppose that for a certain artery, $C = 1.76 \times 10^5$ cm/sec² and $R = 1.2 \times 10^{-2}$ cm.

a. Compute the speed of the blood at the central axis of this artery.

b. Compute the speed of the blood midway between the artery's wall and central axis.

*The law and the unit poise, a unit of viscosity, are both named for the French physician Jean Louis Poiseuille (1799–1869).

CHAPTER 2 Group Research Projects

Working in small groups is typical of most work environments, and this book seeks to develop skills with group activities. At the end of each chapter, we present a list of suggested projects, and even though they could be done as individual projects, we suggest that these projects be done in groups of three or four students.

G2.1 Let $S(x) = \dfrac{4^x - 4^{-x}}{2}$ and $C(x) = \dfrac{4^x + 4^{-x}}{2}$. Show that

$$[C(x)]^2 - [S(x)]^2 = 1$$

G2.2 Suppose $f(x) = \frac{1}{1-x}$. Define $f_1(x) = f \circ f$; $f_2(x) = f \circ f \circ f$; $f_3(x) = f \circ f \circ f \circ f$; . . . . Find $f_{100}(x)$.

G2.3 Use functional iteration to find $(f \circ f \circ f \circ f \ldots)(x)$ when:

a. $f(x) = \sqrt{1+x}$ **b.** $f(x) = \dfrac{1}{1+x}$ **c.** $f(x) = |x| - 1$

G2.4 *Historical Quest* Write a paper on George Pólya. Include, as part of this paper, a report on his book *How to Solve It* (Princeton University Press, 1971).

G2.5 Journal Problem (From Fourth U.S.A. Olympiad, May 6, 1975.) Prove that

$$[\![5x]\!] + [\![5y]\!] \geq [\![3x + y]\!] + [\![3y + x]\!]$$

G2.6 Journal Problem (From *The Mathematics Teacher*, December 1995, pp. 734–737, "Precalculus Explorations of Function Composition with a Graphing Calculator," by Lewis Lum.) This article describes a process called *carom paths*, as follows. Let

$$y_1 = f(x) = -2\sqrt{x-3}, \qquad y_2 = x, \qquad y_3 = g(x) = 3\sqrt{-2-x}$$

We describe the "carom path" as follows. Start at some value of x in the domain of $g \circ f$, say $x = 6$. Move vertically to the point on y_1 (point A), and bounce off horizontally to point B on y_2. Carom vertically to the point on y_3 (point C), and finally rebound horizontally to the point D on the graph of y_2. The graphs are shown in Figure 2.47.

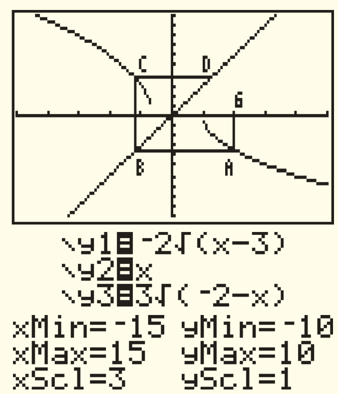


Figure 2.47 Carom path for f and g

We can calculate the coordinates of the indicated points.

A: $\left(6, -2\sqrt{3}\right)$ since $f(6) = -2\sqrt{6-3} = -2\sqrt{3}$

B: $\left(-2\sqrt{3}, -2\sqrt{3}\right)$ since $y = x$

C: $\left(-2\sqrt{3}, 3\sqrt{2\sqrt{3}-2}\right)$ since $g\left(-2\sqrt{3}\right) = 3\sqrt{-2-\left(-2\sqrt{3}\right)}$

D: $\left(3\sqrt{2\sqrt{3}-2}, 3\sqrt{2\sqrt{3}-2}\right)$ since $y = x$

a. What are the points A, B, C, and D for the starting point $x = 12$?

b. Attempt to plot the carom path of $x = 2$. Explain why the path cannot be completed. Find other points with the same problem as $x = 2$.

c. Attempt to plot the carom path of $x = 3.5$. Explain why the path cannot be completed. Find other points with the same problem as $x = 3.5$.

d. What is the domain for x that assures that the carom path exists?

G2.7 *Historical Quest* The notion of a function is not only central for this book but also fundamental to the study of calculus. In the article "The Mathematical Way of Thinking," Hermann Weyl begins by saying, "By the mathematical way of thinking I mean first that form of reasoning through which mathematics penetrates into the sciences of the external world." He goes on to say that the average education of every person should include mathematics to teach everyone to think in terms of variables and functions. Do some research into the history of the idea of a function. Write a report of your research, and include a statement about whether you agree or disagree with Weyl's theses.

G2.8 "There are, in every culture, groups or individuals who think more about some ideas than do others. For other cultures, we know about the ideas of some professional groups or some ideas of the culture at large. We know little, however, about the mathematical thoughts of individuals in those cultures who are specially inclined toward mathematical ideas. In Western culture, on the other hand, we focus on, and record much about, those special individuals while including little about everyone else. Realization of this difference should make us particularly wary of any comparisons across cultures. Even more important, it should encourage finding out more about the ideas of mathematically oriented innovators in other cultures and, simultaneously, encourage expanding the scope of Western history to recognize and include mathematical ideas held by different groups within our culture or by our culture as a whole."*

Write a paper discussing this quotation.

G2.9 Journal Problem (From *The American Mathematical Monthly*, Vol. 92, January 1985, pp. 3–23.) Write a paper on the $3x + 1$ conjecture. See Example 6 on page 80 of "The $3x + 1$ Problem and Its Generalizations," by Jeffrey C. Lagarias.

G2.10 We introduced problem solving in this chapter. Here is an excerpt of a poem followed by three questions.

Yes, weekly from Southampton,
Great steamers, white and gold,
Go rolling down to Rio
(Roll down—roll down to Rio!),
And I'd like to roll to Rio
Some day before I'm old!

Rudyard Kipling
Just So Stories

a. How many steamers wending their way home will I see on this ocean?
b. On what days of the week will I see them?
c. How far from Southampton will I meet them?

Build a mathematical model to answer these questions. Not enough information is given, so here is a start. "Well, so weekly (say each Thursday) from Southampton great steamers go rolling down to Rio . . . It takes 14 days for a great white and gold steamer to cover the entire distance of 9,800 km (700 km per day) and arrive at Rio de Janeiro exactly at noon on Thursday. After a four-day stopover, the ship sets off on the return trip, and in a fortnight, at noon on Monday, it arrives at Southampton. Three days later—again on a Thursday—it leaves on its next voyage to Brazil . . . I wanted to roll to Rio, too, so I stepped onto an ocean liner at Southampton on Thursday and my voyage began."†

* From *Ethnomathematics* by Marcia Ascher (Pacific Grove: Brooks/Cole, 1991), pp. 188–189.
† From "Atlantic Crossings," by A. Rozental, *Quantum*, July/Aug 1993, pp. 46–47.