3 Unit Circle Trigonometry

In This Chapter

The shape of a plucked guitar string, fixed at both ends, can be described by trigonometric functions of a real variable.

A Bit of History The discussion in Section 2.4 leads directly to a more analytical approach to trigonometry where the cosine and sine are defined as the x- and y-coordinates, respectively, of a point (x, y) on a unit circle. It is this interpretation of the sine and cosine that enables us to define the trigonometric functions of a real number instead of an angle. It is this last approach to trigonometry that is used in calculus and in advanced applications of trigonometry. Moreover, a trigonometric function of a real number can then be graphed as we would an ordinary function $y = f(x)$, where the variable x represents a real number in the domain of f.

From a historical viewpoint, it is not known who made this important leap from sines and cosines of angles to sines and cosines of real numbers.

3.1 The Circular Functions

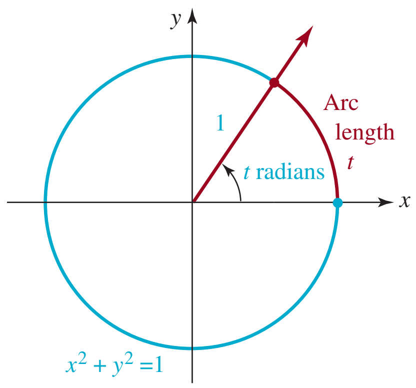


FIGURE 3.1.1 Unit circle

Introduction In Chapter 2 we considered trigonometric functions of *angles* measured either in degrees or in radians. For calculus and the sciences it is necessary to consider trigonometric functions whose domains consist of *real numbers* rather than angles. The transition from angles to real numbers is made by recognizing that to each real number t, there corresponds an angle of measure t radians. As we see next, this correspondence can be visualized using a circle with radius 1 centered at the origin in a rectangular coordinate system. This circle is called the **unit circle**. From Section 1.3 it follows that the equation of the unit circle is $x^2 + y^2 = 1$. In this section the focus will be on the sine and cosine functions. The other four trigonometric functions will be considered in detail in Section 3.3.

We now consider a **central angle** t in standard position; that is, an angle with its vertex at the center of a circle and initial side coinciding with the positive x-axis. From the definition of radian measure, (3) of Section 2.1, the angle t is defined to be $t = s/r$, the ratio of the subtended arc of length s to the radius r of the circle. For the unit circle shown in **FIGURE 3.1.1**, $r = 1$, and so $t = s/1$ or $t = s$. In other words:

- *On a unit circle, the radian measure of an angle of t radians is equal to the measure t of the subtended arc.*

It follows that for each real number t, the terminal side of an angle of t radians in standard position has traversed a distance of $|t|$ units along the circumference of the unit circle—counterclockwise if $t > 0$, clockwise if $t < 0$. This association of each real number t with an angle of t radians is illustrated in **FIGURE 3.1.2**.

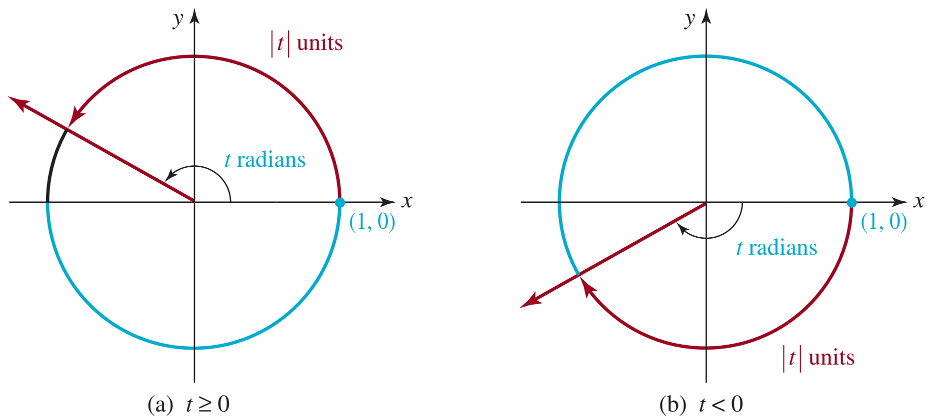


FIGURE 3.1.2 Angle of t radians subtends an arc of length $|t|$ units

Trigonometric Functions of Real Numbers We are now in a position to define **trigonometric functions** of a real number. Before proceeding we need the following important definition.

DEFINITION 3.1.1 Values of the Trigonometric Functions

The value of a trigonometric function at a real number t is defined to be its value at an angle of t radians, provided that value exists.

For example, the sine of the real number $\pi/6 = 0.62359\ldots$ is simply the sine of the angle $\pi/6$ radian that, as we know, is $\frac{1}{2}$. Thus there is really nothing new in evaluating the trigonometric functions of a real number.

The unit circle is very helpful in describing the trigonometric functions of real numbers. If $P(t)$ denotes the point of intersection of the terminal side of the angle t with the unit circle $x^2 + y^2 = 1$ and $P(x, y)$ are the rectangular coordinates of this point, then from (2) of Section 2.4 we have

$$\sin t = \frac{y}{r} = \frac{y}{1} = y \quad \text{and} \quad \cos t = \frac{x}{r} = \frac{x}{1} = x.$$

These definitions, along with the definitions of the remaining four trigonometric functions, are summarized next.

DEFINITION 3.1.2 Trigonometric Functions

Let t be any real number and $P(t) = P(x, y)$ be the point of intersection on the unit circle with the terminal side of the angle of t radians in standard position. Then the six trigonometric functions of the real number t are

$$\begin{aligned} \sin t &= y & \cos t &= x \\ \tan t &= \frac{y}{x} & \cot t &= \frac{x}{y} \\ \sec t &= \frac{1}{x} & \csc t &= \frac{1}{y}. \end{aligned} \tag{1}$$

From the first line in (1) of Definition 3.1.2 we see immediately that

- *For any real number t, the* ***cosine*** *and* ***sine*** *of t are the x- and y-coordinates, respectively, of the point P of intersection of the terminal side of the angle of t radians (in standard position) with the unit circle.*

See **FIGURE 3.1.3**.

As we will soon see, a number of important properties of the sine and cosine functions can be obtained from this result. Because of the role played by the unit circle in this discussion, the trigonometric functions (1) are often referred to as the **circular functions**.

A number of properties of the sine and cosine functions follow from the fact that $P(t) = (\cos t, \sin t)$ lies on the unit circle. For instance, the coordinates of $P(t)$ must satisfy the equation of the circle:

$$x^2 + y^2 = 1.$$

Substituting $x = \cos t$ and $y = \sin t$ into the foregoing equation gives the familiar result $\cos^2 t + \sin^2 t = 1$. This relationship between the sine and cosine functions is the most fundamental of trigonometric identities, the **Pythagorean identity**. Bear in mind this identity is not just valid for angles as discussed in Sections 2.2 and 2.4; we see now that it is valid for all real numbers t.

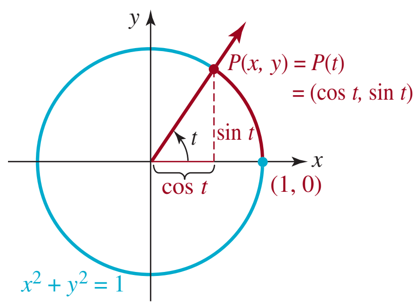


FIGURE 3.1.3 Coordinates of $P(t)$ are $(\cos t, \sin t)$

THEOREM 3.1.1 Pythagorean Identity

For all real numbers t,

$$\sin^2 t + \cos^2 t = 1. \tag{2}$$

Bounds on the Values of Sine and Cosine A number of properties of the sine and cosine functions follow from the fact that $P(t) = P(x, y)$ lies on the unit circle. For instance, it follows that

$$-1 \leq x \leq 1 \qquad \text{and} \qquad -1 \leq y \leq 1.$$

Since $x = \cos t$ and $y = \sin t$, the foregoing inequalities are equivalent to

$$-1 \leq \cos t \leq 1 \qquad \text{and} \qquad -1 \leq \sin t \leq 1. \tag{3}$$

The inequalities in (3) can also be expressed using absolute values as $|\cos t| \leq 1$ and $|\sin t| \leq 1$. Thus, for example, there is no real number t for which $\sin t = \frac{3}{2}$.

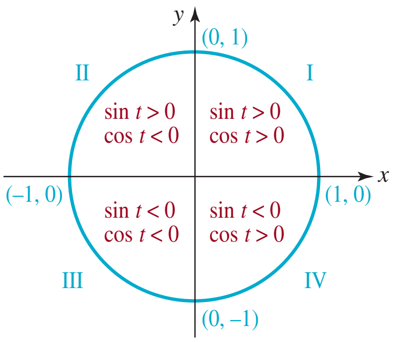


FIGURE 3.1.4 Algebraic signs of $\sin t$ and $\cos t$ in the four quadrants

Domain and Range The observations in (3) indicate that both $\cos t$ and $\sin t$ can be any number in the interval $[-1, 1]$. Thus we have the sine and cosine functions,

$$f(t) = \sin t \qquad \text{and} \qquad g(t) = \cos t,$$

respectively, each with domain the set R of all real numbers and range the interval $[-1, 1]$. The domains and ranges of the other four trigonometric functions will be discussed in Section 3.3.

Signs of the Circular Functions The signs of the function values $\sin t$ and $\cos t$ are determined by the quadrant in which the point $P(t)$ lies, and conversely. For example, if $\sin t$ and $\cos t$ are both negative, then the point $P(t)$ and terminal side of the corresponding angle of t radians must lie in quadrant III. **FIGURE 3.1.4** displays the signs of the cosine and sine functions in each of the four quadrants.

EXAMPLE 1 Sine and Cosine of a Real Number

Use a calculator to approximate $\sin 3$ and $\cos 3$ and give a geometric interpretation of these values.

Solution From a calculator set in *radian mode*, we obtain $\cos 3 \approx -0.9899925$ and $\sin 3 \approx 0.1411200$. These values represent the x and y coordinates, respectively, of the point of intersection $P(3)$ of the terminal side of the angle of 3 radians in standard position, with the unit circle. As shown in **FIGURE 3.1.5**, this point lies in the second quadrant because $\pi/2 < 3 < \pi$. This would also be expected in view of Figure 3.1.4 since $\cos 3$, the x-coordinate, is *negative* and $\sin 3$, the y-coordinate, is *positive*. ≡

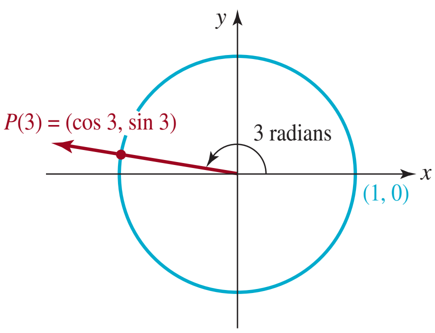


FIGURE 3.1.5 The point $P(3)$ in Example 1

Periodicity In Section 2.1 we saw that the angles of t radians and $t \pm 2\pi$ radians are coterminal. Thus they determine the same point $P(x, y)$ on the unit circle. Therefore

$$\sin t = \sin(t \pm 2\pi) \qquad \text{and} \qquad \cos t = \cos(t \pm 2\pi). \tag{4}$$

In other words, the sine and cosine functions repeat their values every 2π units. It also follows that for any integer n:

$$\begin{aligned} \sin(t + 2n\pi) &= \sin t \\ \cos(t + 2n\pi) &= \cos t. \end{aligned} \tag{5}$$

DEFINITION 3.1.3 Periodic Functions

A nonconstant function f is said to be **periodic** if there is a positive number p such that

$$f(t) = f(t + p) \tag{6}$$

for every t in the domain of f. If p is the smallest positive number for which (6) is true, then p is called the **period** of the function f.

The equations in (4) imply that the sine and cosine functions are periodic with period $p \leq 2\pi$. To see that the period of $\sin t$ is actually 2π, we observe that there is only one point on the unit circle with y-coordinate 1, namely, $P(\pi/2) = (\cos(\pi/2), \sin(\pi/2)) = (0, 1)$. Therefore,

$$\sin t = 1 \qquad \text{only for} \qquad t = \frac{\pi}{2}, \frac{\pi}{2} \pm 2\pi, \frac{\pi}{2} \pm 4\pi,$$

and so on. Thus the smallest possible positive value of p is 2π. In summary, the sine function $f(t) = \sin t$ and cosine function $g(t) = \cos t$ are periodic with **period 2π;** that is, $f(t) = f(t + 2\pi)$ and $g(t) = g(t + 2\pi)$, respectively. For future reference, we have

$$\sin(t + 2\pi) = \sin t \qquad \text{and} \qquad \cos(t + 2\pi) = \cos t \tag{7}$$

for every real number t.

EXAMPLE 2 Using Periodicity

Evaluate **(a)** $\sin(7\pi/3)$ **(b)** $\cos(13\pi/3)$.

Solution (a) Because $7\pi/3$ is greater than 2π and can be written

$$\frac{7\pi}{3} = 2\pi + \frac{\pi}{3},$$

it follows from $\sin(t + 2\pi) = \sin t$ with $t = \pi/3$, that

$$\sin\frac{7\pi}{3} = \sin\left(\frac{\pi}{3} + 2\pi\right) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}. \quad \leftarrow \text{See Table 2.3.1}$$

◀ See the first equation in (7).

(b) Because

$$\frac{19\pi}{3} = 6\pi + \frac{\pi}{3},$$

it follows from $\cos(t + 2n\pi) = \cos t$ with $n = 3$ and $t = \pi/3$, that

$$\cos\frac{19\pi}{3} = \cos\left(\frac{\pi}{3} + 6\pi\right) = \cos\frac{\pi}{3} = \frac{1}{2}.$$

◀ See the second equation in (7).

≡

☐ Odd–Even Properties

The symmetry of the unit circle endows the circular functions with several additional properties. For any real number t, the points $P(t)$ and $P(-t)$ on the unit circle are located on the terminal side of an angle of t and $-t$ radians, respectively. These two points will always be symmetric with respect to the x-axis. **FIGURE 3.1.6** illustrates the situation for a point $P(t)$ lying in the first quadrant: the x-coordinates of the two points are identical, but the y-coordinates have equal magnitudes but opposite signs. The same symmetries will hold regardless of which quadrant contains $P(t)$. Thus, for $f(t) = \sin t$ and $g(t) = \cos t$ and any real number t, $f(-t) = -f(t)$ and $g(-t) = g(t)$, respectively. Applying the definitions of **odd** and **even functions** from Section 1.6 we have the following result.

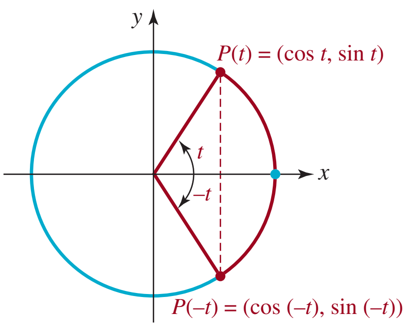


FIGURE 3.1.6 Coordinates of $P(t)$ and $P(-t)$

THEOREM 3.1.2 Odd and Even Functions

The sine function $f(t) = \sin t$ is **odd** and the cosine function $g(t) = \cos t$ is **even**; that is, for every real number t,

$$\sin(-t) = -\sin t \qquad \text{and} \qquad \cos(-t) = \cos t. \tag{8}$$

EXAMPLE 3 Using the Odd–Even Properties

Find exact values of $\sin t$ and $\cos t$ for the real number $t = -\pi/6$.

Solution From (8) we have

$$\overbrace{\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6}}^{\text{sine is an odd function}} = -\frac{1}{2}, \qquad \leftarrow \text{See Table 2.3.1}$$

and

$$\overbrace{\cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6}}^{\text{cosine is an even function}} = \frac{\sqrt{3}}{2}.$$

Note that the signs of the answers are consistent with the fact that the terminal side of the angle $-\pi/6$ radian lies in quadrant IV. ≡

The following additional properties of the sine and cosine functions can be verified by considering the symmetries of appropriately chosen points on the unit circle. We first saw the results in (*i*) and (*ii*) in the next theorem stated for acute angles in (5) of Section 2.2.

THEOREM 3.1.3 Additional Properties

For all real numbers t,

$$(i)\ \cos\left(\frac{\pi}{2} - t\right) = \sin t \qquad (ii)\ \sin\left(\frac{\pi}{2} - t\right) = \cos t$$
$$(iii)\ \cos(t + \pi) = -\cos t \qquad (iv)\ \sin(t + \pi) = -\sin t$$
$$(v)\ \cos(\pi - t) = -\cos t \qquad (vi)\ \sin(\pi - t) = \sin t$$

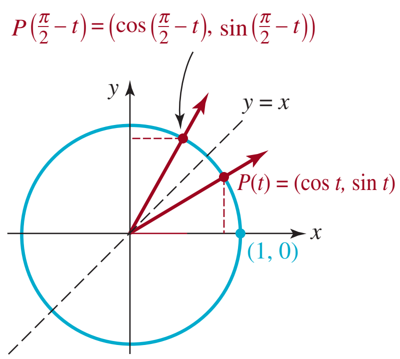


FIGURE 3.1.7 Geometric justification of (*i*) and (*ii*) of Theorem 3.1.3

For example, to justify properties (*i*) and (*ii*) of Theorem 3.1.3 for $0 < t < \pi/2$, consider **FIGURE 3.1.7**. Since the points $P(t)$ and $P(\pi/2 - t)$ are symmetric with respect to the line $y = x$, we can obtain the coordinates of $P(\pi/2 - t)$ by interchanging the coordinates of $P(t)$. Thus,

$$\cos t = x = \sin\left(\frac{\pi}{2} - t\right) \qquad \text{and} \qquad \sin t = y = \cos\left(\frac{\pi}{2} - t\right).$$

In Section 3.4 we will use properties (*i*) and (*ii*) to justify two important formulas for the sine function.

EXAMPLE 4 Using Theorem 3.1.3

In Table 2.3.1 in Section 2.3 we saw that $\cos(\pi/3) = \sin(\pi/6)$. This result is a special case of property (*i*) of Theorem 3.1.3; with $t = \pi/3$ we see that

$$\sin\frac{\pi}{6} = \overbrace{\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right) = \cos\frac{\pi}{3}}^{\text{using property } (i) \text{ of Theorem 3.1.3}}.$$

≡

□ **Reference Angle—Revisited** As we noted at the beginning of this section, for each real number t there is a unique angle of t radians in standard position that determines the point $P(t)$, with coordinates $(\cos t, \sin t)$, on the unit circle. As shown in **FIGURE 3.1.8**,

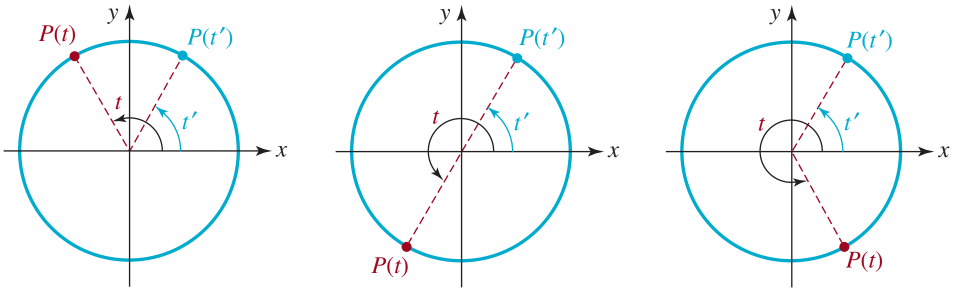


FIGURE 3.1.8 Reference angle t' is an acute angle

the terminal side of any angle of t radians (with $P(t)$ not on an axis) will form an acute angle with the x-axis. We can then locate an angle of t' radians in the first quadrant that is congruent to this acute angle. The angle of t' radians is called the **reference angle** for the real number t. Because of the symmetry of the unit circle, the coordinates of $P(t')$ will be equal *in absolute value* to the respective coordinates of $P(t)$. Hence

$$\sin t = \pm\sin t' \qquad \text{and} \qquad \cos t = \pm\cos t'.$$

As the following examples will show, reference angles can be used to find the trigonometric function values of any integer multiples of $\pi/6$, $\pi/4$, and $\pi/3$.

EXAMPLE 5 Using a Reference Angle

Find the exact values of $\sin t$ and $\cos t$ for the given real number:
(a) $t = 5\pi/3$ **(b)** $t = -3\pi/4$.

Solution In each part we begin by finding the reference angle corresponding to the given real number t.

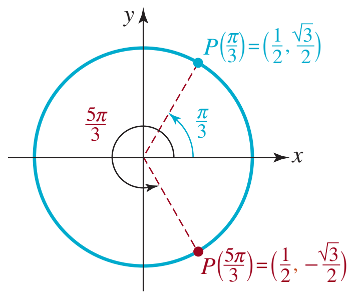


FIGURE 3.1.9 Reference angle in part (a) of Example 5

(a) From **FIGURE 3.1.9** we find that an angle of $t = 5\pi/3$ radians determines a point $P(5\pi/3)$ in the fourth quadrant and has the reference angle $t' = \pi/3$ radians. After adjusting the signs of the coordinates of $P(\pi/3) = \left(1/2, \sqrt{3}/2\right)$ to obtain the fourth-quadrant point $P(5\pi/3) = \left(1/2, -\sqrt{3}/2\right)$, we find that

reference angle

$$\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} \qquad \text{and} \qquad \cos\frac{5\pi}{3} = \cos\frac{\pi}{3} = \frac{1}{2}.$$

(b) The point $P(-3\pi/4)$ lies in the third quadrant and has reference angle $\pi/4$ as shown in **FIGURE 3.1.10**. Therefore

$$\sin\left(-\frac{3\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \qquad \text{and} \qquad \cos\left(-\frac{3\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}. \equiv$$

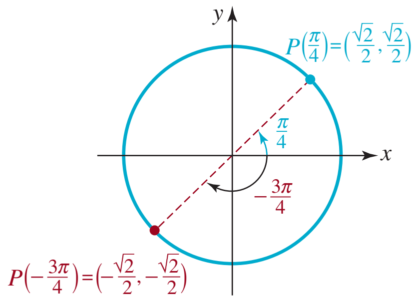


FIGURE 3.1.10 Reference angle in part (b) of Example 5

Sometimes, in order to find the trigonometric values of multiples of our basic fractions of π we must use periodicity or the even–odd function properties in addition to reference angles.

EXAMPLE 6 Using Periodicity and a Reference Angle

Find the exact values of the coordinates of $P(29\pi/6)$ on the unit circle.

Solution The point $P(29\pi/6)$ has coordinates $(\cos(29\pi/6), \sin(29\pi/6))$. We begin by observing that $29\pi/6$ is greater than 2π, and so we must rewrite $29\pi/6$ as an integer multiple of 2π plus a number less than 2π. By division we have

$$\frac{29\pi}{6} = 4\pi + \frac{5\pi}{6} = 2(2\pi) + \frac{5\pi}{6}.$$

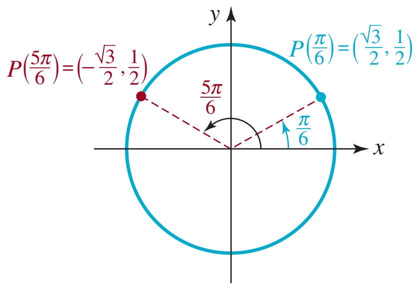


FIGURE 3.1.11 Reference angle in Example 6

Next, from the periodicity equations in (5) with $n = 2$ we know that

$$\sin\left(\frac{29\pi}{6}\right) = \sin\left(\frac{5\pi}{6}\right) \quad \text{and} \quad \cos\left(\frac{29\pi}{6}\right) = \cos\left(\frac{5\pi}{6}\right).$$

Next we see from **FIGURE 3.1.11** that the reference angle for $5\pi/6$ is $\pi/6$. Since $P(5\pi/6)$ is a second-quadrant point its x-coordinate $\cos(5\pi/6)$ is negative and its y-coordinate $\sin(5\pi/6)$ is positive. Finally, using the reference angle as shown in Figure 3.1.11 we simply adjust the algebraic signs of the coordinates of the $P(\pi/6) = (\cos(\pi/6), \sin(\pi/6))$:

$$\cos\frac{29\pi}{6} = \cos\frac{5\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}$$

and

$$\sin\frac{29\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}.$$

Thus, $P(29\pi/6) = \left(-\sqrt{3}/2, 1/2\right)$. ≡

3.1 Exercises

Answers to selected odd-numbered problems begin on page ANS-8.

In Problems 1–8, for the given real number t, **(a)** locate the point $P(t) = (\cos t, \sin t)$ on the unit circle and **(b)** find the exact values of the coordinates of $P(t)$. Do not use a calculator.

1. $\frac{7\pi}{6}$	**2.** $-\frac{4\pi}{3}$	**3.** $-\frac{\pi}{2}$	**4.** 2π
5. $\frac{5\pi}{3}$	**6.** $-\frac{3\pi}{2}$	**7.** $-\frac{11\pi}{6}$	**8.** $\frac{5\pi}{4}$

In Problems 9–16, for the given real number t, **(a)** locate the point $P(t) = (\cos t, \sin t)$ on the unit circle and **(b)** use a calculator to approximate the coordinates of $P(t)$.

9. 1.3	**10.** -4.4	**11.** -7.2	**12.** 0.5
13. 6.1	**14.** 3.2	**15.** -2.6	**16.** 15.3

In Problems 17–24, use periodicity of $\sin t$ and $\cos t$ to find the exact value of the given trigonometric function. Do not use a calculator.

17. $\sin\frac{13\pi}{6}$	**18.** $\cos\frac{61\pi}{3}$	**19.** $\cos\frac{9\pi}{4}$	**20.** $\sin\left(-\frac{5\pi}{3}\right)$
21. $\cos 9\pi$	**22.** $\sin 20\pi$	**23.** $\sin\frac{7\pi}{2}$	**24.** $\cos\frac{27\pi}{4}$

In Problems 25–30, justify the given statement by one of the properties of $\sin t$ and $\cos t$ given in this section.

25. $\sin\pi = \sin 3\pi$	**26.** $\cos(\pi/4) = \sin(\pi/4)$
27. $\sin(-3 - \pi) = -\sin(3 + \pi)$	**28.** $\cos 16.8\pi = \cos 14.8\pi$
29. $\cos 0.43 = \cos(-0.43)$	**30.** $\cos(2.5 + \pi) = -\cos 2.5$

31. Given that $\cos t = -\frac{2}{5}$ and that $P(t)$ is a point on the unit circle in the second quadrant, find $\sin t$.
32. Given that $\sin t = \frac{1}{4}$ and that $P(t)$ is a point on the unit circle in the second quadrant, find $\cos t$.
33. Given that $\sin t = -\frac{2}{3}$ and that $P(t)$ is a point on the unit circle in the third quadrant, find $\cos t$.
34. Given that $\cos t = \frac{3}{4}$ and that $P(t)$ is a point on the unit circle in the fourth quadrant, find $\sin t$.

In Problems 35–38, the y-coordinate of the point $P(5\pi/8)$ on the unit circle is $\frac{1}{2}\sqrt{2+\sqrt{2}}$. Find the exact value of the given trigonometric function. Do not use a calculator.

35. $\cos\dfrac{5\pi}{8}$
36. $\sin\left(\dfrac{5\pi}{8} - 2\pi\right)$
37. $\sin\left(-\dfrac{5\pi}{8}\right)$
38. $\cos\left(-\dfrac{5\pi}{8}\right)$

In Problems 39–42, use the unit circle to determine all real numbers t for which the given equality is true.

39. $\sin t = \sqrt{2}/2$
40. $\cos t = -\frac{1}{2}$
41. $\cos t = -1$
42. $\sin t = -1$

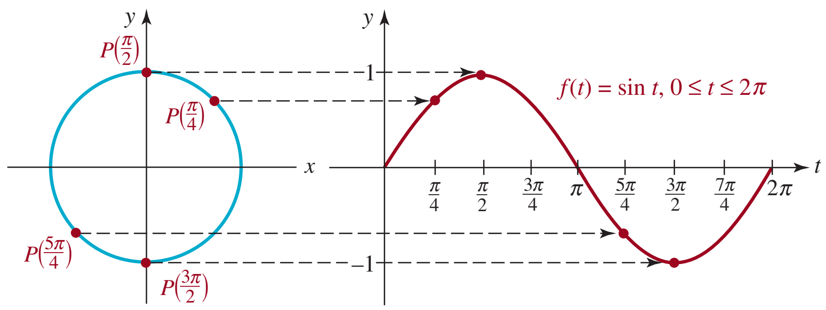
For Discussion

43. Suppose f is a periodic function with period p. Show that $F(x) = f(ax)$, $a > 0$, is periodic with period p/a.

3.2 Graphs of Sine and Cosine Functions

Introduction One way to further your understanding of the trigonometric functions is to examine their graphs. In this section we consider the graphs of the sine and cosine functions.

Graphs of Sine and Cosine In Section 3.1 we saw that the domain of the sine function $f(t) = \sin t$ is the set of real numbers $(-\infty, \infty)$ and the interval $[-1, 1]$ is its range. Since the sine function has period 2π, we begin by sketching its graph on the interval $[0, 2\pi]$. We obtain a rough sketch of the graph given in FIGURE 3.2.1(b) by considering various positions of the point $P(t)$ on the unit circle, as shown in Figure 3.2.1(a). As t varies from 0 to $\pi/2$, the value $\sin t$ increases from 0 to its maximum value 1. But as t varies from $\pi/2$ to $3\pi/2$, the value $\sin t$ decreases from 1 to its minimum value -1. We note that $\sin t$ changes from positive to negative at $t = \pi$. For t between $3\pi/2$ and 2π, we see that the corresponding values of $\sin t$ increase from –1 to 0. The graph of *any* periodic function over an interval of length equal to its period is said to be one **cycle** of its graph. In the case of the sine function, the graph over the interval $[0, 2\pi]$ in Figure 3.2.1(b) is one cycle of the graph of $f(t) = \sin t$.



(a) Unit circle (b) One cycle of sine graph

FIGURE 3.2.1 Points $P(t)$ on the unit circle corresponding to points on the graph

Note: Change of symbols ▶

From this point on we will switch to the traditional symbols x and y when graphing trigonometric functions. Thus, $f(t) = \sin t$ will either be written $f(x) = \sin x$ or simply $y = \sin x$.

The graph of a periodic function is easily obtained by repeatedly drawing one cycle of its graph. In other words, the graph of $y = \sin x$ on, say, the intervals $[-2\pi, 0]$ and $[2\pi, 4\pi]$ is the same as that given in Figure 3.2.1(b). Recall from Section 3.1 that the sine function is an odd function since $f(-x) = \sin(-x) = -\sin x = -f(x)$. Thus, as can be seen in **FIGURE 3.2.2**, the graph of $y = \sin x$ is symmetric with respect to the origin.

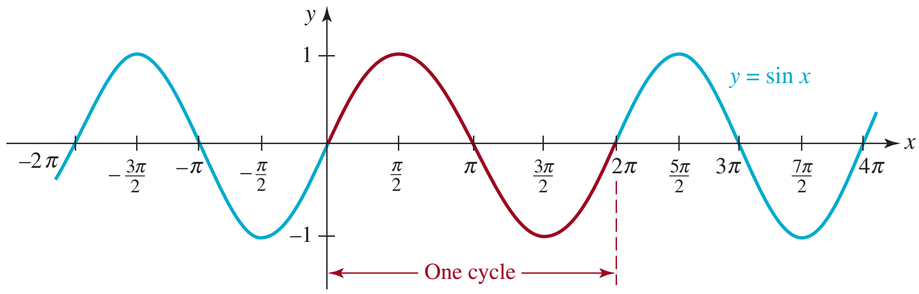


FIGURE 3.2.2 Graph of $y = \sin x$

By working again with the unit circle we can obtain one cycle of the graph of the cosine function $g(x) = \cos x$ on the interval $[0, 2\pi]$. In contrast to the graph of $f(x) = \sin x$ where $f(0) = f(2\pi) = 0$, for the cosine function we have $g(0) = g(2\pi) = 1$. **FIGURE 3.2.3** shows one cycle (in red) of $y = \cos x$ on $[0, 2\pi]$ along with the extension of that cycle (in blue) to the adjacent intervals $[-2\pi, 0]$ and $[2\pi, 4\pi]$. We see from this figure that the graph of the cosine function is symmetric with respect to the y-axis. This is a consequence of the fact that g is an even function: $g(-x) = \cos(-x) = \cos x = g(x)$.

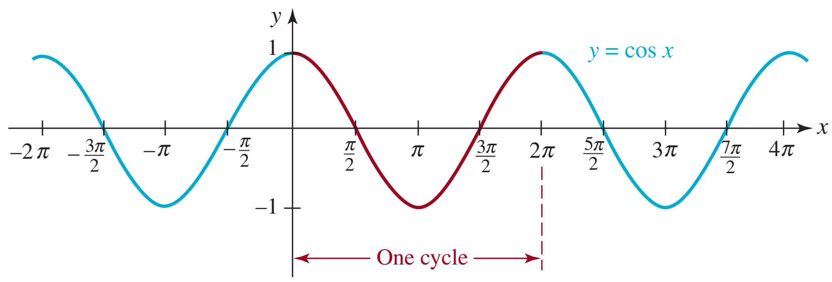


FIGURE 3.2.3 Graph of $y = \cos x$

Properties of the Sine and Cosine Functions In this and subsequent courses in mathematics it is important that you know the x-coordinates of the x-intercepts of the sine and cosine graphs, in other words, the zeros of $f(x) = \sin x$ and $g(x) = \cos x$. From the sine graph in Figure 3.2.2 we see that the zeros of the sine function, or the numbers for which $\sin x = 0$, are $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. These numbers are integer multiples of π. From the cosine graph in Figure 3.2.3 we see that $\cos x = 0$ when $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$. These numbers are odd-integer multiples of $\pi/2$, that is, $x = (2n + 1)\pi/2$, where n is an integer. Using the distributive law, the zeros of $g(x) = \cos x$ are often written as $x = \pi/2 + n\pi$. The following list summarizes some of the important properties of the sine and cosine functions that are apparent from their graphs.

◀ An odd integer can be written as $2n + 1$, where n is an integer.

PROPERTIES OF THE SINE AND COSINE FUNCTIONS

- The domain of $f(x) = \sin x$ and the domain of $g(x) = \cos x$ is the set of real numbers, that is, $(-\infty, \infty)$.
- The range of $f(x) = \sin x$ and the range of $g(x) = \cos x$ is the interval $[-1, 1]$ on the y-axis.
- The zeros of $f(x) = \sin x$ are $x = n\pi$, n an integer. The zeros of $g(x) = \cos x$ are $x = (2n + 1)\pi/2$, n an integer.
- The graph of $f(x) = \sin x$ is symmetric with respect to the origin. The graph of $g(x) = \cos x$ is symmetric with respect to the y-axis.
- The functions $f(x) = \sin x$ and $g(x) = \cos x$ are continuous on the interval $(-\infty, \infty)$.

As we did in Chapter 3 we can obtain variations of the basic sine and cosine graphs through rigid and nonrigid transformations. For the remainder of the discussion we will consider graphs of functions of the form

$$y = A\sin(Bx + C) + D \qquad \text{or} \qquad y = A\cos(Bx + C) + D, \tag{1}$$

where A, B, C, and D are real constants.

Graphs of $y = A\sin x + D$ and $y = A\cos x + D$ We begin by considering the special cases of (1):

$$y = A\sin x \qquad \text{and} \qquad y = A\cos x.$$

For $A > 0$ graphs of these functions are either a vertical stretch or a vertical compression of the graphs of $y = \sin x$ or $y = \cos x$. For $A < 0$ the graphs are also reflected in the x-axis. For example, as **FIGURE 3.2.4** shows we obtain the graph of $y = 2\sin x$ by stretching the graph of $y = \sin x$ vertically by a factor of 2. Note that the maximum and minimum values of $y = 2\sin x$ occur at the same x-values as the maximum and minimum values of $y = \sin x$. In general, the maximum distance from any point on the graph of $y = A\sin x$ or $y = A\cos x$ to the x-axis is $|A|$. The number $|A|$ is called the **amplitude** of the functions or of their graphs. The amplitude of the basic functions $y = \sin x$ and $y = \cos x$ is $|A| = 1$. In general, if a periodic function f is continuous, then over a closed interval of length equal to its period, f has both a maximum value M and a minimum value m. The amplitude is defined by

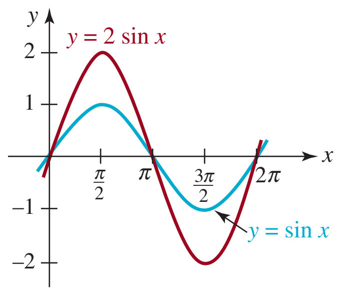


FIGURE 3.2.4 Vertical stretch of $y = \sin x$

$$\text{amplitude} = \frac{1}{2}[M - m]. \tag{2}$$

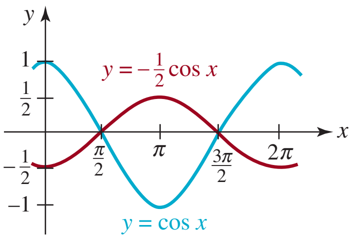


FIGURE 3.2.5 Graph of function in Example 1

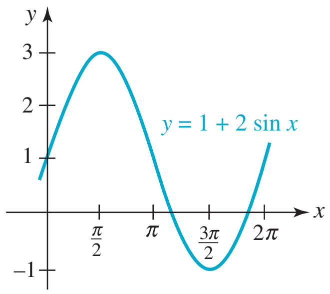


FIGURE 3.2.6 Graph of $y = 2\sin x$ shifted up 1 unit

EXAMPLE 1 Vertically Compressed Cosine Graph

Graph $y = -\frac{1}{2}\cos x$.

Solution The graph of $y = -\frac{1}{2}\cos x$ is the graph of $y = \cos x$ compressed vertically by a factor of $\frac{1}{2}$ and then reflected in the x-axis. With the identification $A = -\frac{1}{2}$ we see that the amplitude of the function is $|A| = |-\frac{1}{2}| = \frac{1}{2}$. The graph of $y = -\frac{1}{2}\cos x$ on the interval $[0, 2\pi]$ is shown in red in **FIGURE 3.2.5**. ≡

The graphs of

$$y = A\sin x + D \quad \text{and} \quad y = A\cos x + D$$

are the respective graphs of $y = A\sin x$ and $y = A\cos x$ shifted vertically, up for $D > 0$ and down for $D < 0$. For example, the graph of $y = 1 + 2\sin x$ is the graph of $y = 2\sin x$ (Figure 3.2.4) shifted up 1 unit. The amplitude of the graph of either $y = A\sin x + D$ or $y = A\cos x + D$ is still $|A|$. Observe in **FIGURE 3.2.6**, the maximum of $y = 1 + 2\sin x$ is $y = 3$ at $x = \pi/2$ and the minimum is $y = -1$ at $x = 3\pi/2$. From (2), the amplitude of $y = 1 + 2\sin x$ is then $\frac{1}{2}[3 - (-1)] = 2$.

By interpreting x as a placeholder we can find the x-coordinates of the x-intercepts of the graphs of sine and cosine functions of the form $y = A\sin Bx$ and $y = A\cos Bx$ (considered next). For example, to solve $\sin 2x = 0$, we use the fact that the zeros of $f(x) = \sin x$ are $x = n\pi$, where n is an integer. We simply replace x by $2x$ to obtain

$$2x = n\pi \quad \text{so that} \quad x = \tfrac{1}{2}n\pi,\ n = 0, \pm 1, \pm 2, \ldots;$$

that is, $\sin 2x = 0$ for $x = 0, \pm\frac{1}{2}\pi, \pm\frac{2}{2}\pi = \pi, \pm\frac{3}{2}\pi, \pm\frac{4}{2}\pi = 2\pi$, and so on. See **FIGURE 3.2.7**.

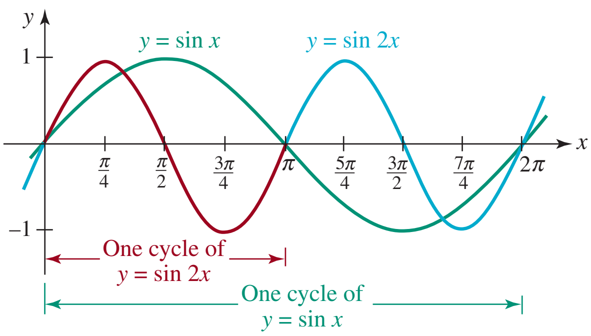


FIGURE 3.2.7 Comparison of the graphs of $y = \sin x$ and $y = \sin 2x$

□ Graphs of $y = A\sin Bx$ and $y = A\cos Bx$

We now consider the graph of $y = \sin Bx$, for $B > 0$. The function has amplitude 1 since $A = 1$. Since the period of $y = \sin x$ is 2π, a cycle of the graph of $y = \sin Bx$ begins at $x = 0$ and will start to repeat its values when $Bx = 2\pi$. In other words, a cycle of the function $y = \sin Bx$ is completed on the interval defined by $0 \le Bx \le 2\pi$. Dividing the last inequality by B shows that the **period** of the function $y = \sin Bx$ is $2\pi/B$ and that the graph over the interval $[0, 2\pi/B]$ is one **cycle** of its graph. For example, the period of $y = \sin 2x$ is $2\pi/2 = \pi$, and therefore one cycle of the graph is completed on the interval $[0, \pi]$. Figure 3.2.7 shows that two cycles of the graph of $y = \sin 2x$ (in red and blue) are completed on the interval $[0, 2\pi]$, whereas the graph of $y = \sin x$ (in green) has completed only one cycle. In terms of transformations, we can characterize the cycle of $y = \sin 2x$ on $[0, \pi]$ as a **horizontal compression** of the cycle of $y = \sin x$ on $[0, 2\pi]$.

Careful here; $\sin 2x \neq 2\sin x$ ▶

In summary, the graphs of

$$y = A\sin Bx \quad \text{and} \quad y = A\cos Bx$$

for $B > 0$, each have amplitude $|A|$ and period $2\pi/B$.

EXAMPLE 2 Horizontally Compressed Cosine Graph

Find the period of $y = \cos 4x$ and graph the function.

Solution Since $B = 4$, we see that the period of $y = \cos 4x$ is $2\pi/4 = \pi/2$. We conclude that the graph of $y = \cos 4x$ is the graph of $y = \cos x$ compressed horizontally. To graph the function, we draw one cycle of the cosine graph with amplitude 1 on the interval $[0, \pi/2]$ and then use periodicity to extend the graph. FIGURE 3.2.8 shows four complete cycles of $y = \cos 4x$ (the basic cycle in red and the extended graph in blue) and one cycle of $y = \cos x$ (in green) on $[0, 2\pi]$. Notice that $y = \cos 4x$ attains its minimum at $x = \pi/4$ since $\cos 4(\pi/4) = \cos \pi = -1$ and its maximum at $x = \pi/2$ since $\cos 4(\pi/2) = \cos 2\pi = 1$. ≡

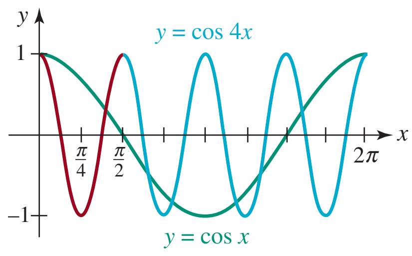


FIGURE 3.2.8 Graph of function in Example 2

If $B < 0$ in either $y = A\sin Bx$ or $y = A\cos Bx$, we can use the odd/even properties, (8) of Section 3.1, to rewrite the function with positive B. This is illustrated in the next example.

EXAMPLE 3 Horizontally Stretched Sine Graph

Find the amplitude and period of $y = \sin\left(-\frac{1}{2}x\right)$. Graph the function.

Solution Since we require $B > 0$, we use $\sin(-x) = -\sin x$ to rewrite the function as

$$y = \sin\left(-\tfrac{1}{2}x\right) = -\sin\tfrac{1}{2}x.$$

With the identification $A = -1$, the amplitude is seen to be $|A| = |-1| = 1$. Now with $B = \frac{1}{2}$ we find that the period is $2\pi/\frac{1}{2} = 4\pi$. Hence we can interpret the cycle of $y = -\sin\frac{1}{2}x$ on $[0, 4\pi]$ as a horizontal stretch and a reflection (in the x-axis because $A < 0$) of the cycle of $y = \sin x$ on $[0, 2\pi]$. FIGURE 3.2.9 shows that on the interval $[0, 4\pi]$ the graph of $y = -\sin\frac{1}{2}x$ (in blue) completes one cycle, whereas the graph of $y = \sin x$ (in green) completes two cycles. ≡

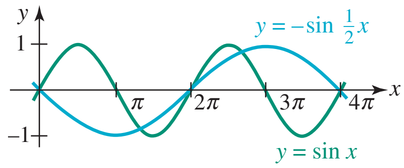


FIGURE 3.2.9 Graph of function in Example 3

□ Graphs of $y = A\sin(Bx + C)$ and $y = A\cos(Bx + C)$ We have seen that the basic graphs of $y = \sin x$ and $y = \cos x$ can, in turn, be stretched or compressed vertically,

$$y = A\sin x \quad \text{and} \quad y = A\cos x,$$

shifted vertically,

$$y = A\sin x + D \quad \text{and} \quad y = A\cos x + D,$$

and stretched or compressed horizontally,

$$y = A\sin Bx + D \quad \text{and} \quad y = A\cos Bx + D.$$

The graphs of

$$y = A\sin(Bx + C) + D \quad \text{and} \quad y = A\cos(Bx + C) + D,$$

are the graphs of $y = A\sin Bx + D$ and $y = A\cos Bx + D$ shifted horizontally.

In the remaining discussion we are going to focus on the graphs of $y = A\sin(Bx + C)$ and $y = A\cos(Bx + C)$. For example, we know from Section 3.2 that the graph of $y = \cos(x - \pi/2)$ is the basic cosine graph shifted to the right. In

FIGURE 3.2.10 the graph of $y = \cos(x - \pi/2)$ (in red) on the interval $[0, 2\pi]$ is one cycle of $y = \cos x$ on the interval $[-\pi/2, 3\pi/2]$ (in blue) shifted horizontally $\pi/2$ units to the right. Similarly, the graphs of $y = \sin(x + \pi/2)$ and $y = \sin(x - \pi/2)$ are the basic sine graph shifted $\pi/2$ units to the left and to the right, respectively. See FIGURES 3.2.11 and 3.2.12.

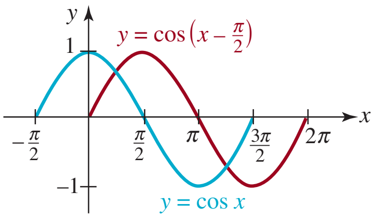


FIGURE 3.2.10 Horizontally shifted cosine graph

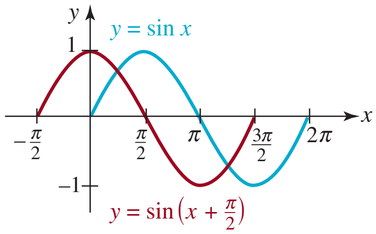


FIGURE 3.2.11 Horizontally shifted sine graph

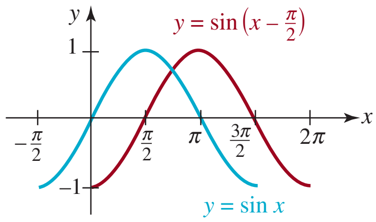


FIGURE 3.2.12 Horizontally shifted sine graph

By comparing the red graphs in Figures 3.2.10–3.2.12 with the graphs in Figures 3.2.2 and 3.2.3 we see that

- the cosine graph shifted $\pi/2$ units to the right is the sine graph,
- the sine graph shifted $\pi/2$ units to the left is the cosine graph, and
- the sine graph shifted $\pi/2$ units to the right is the cosine graph reflected in the x-axis.

In other words, we have graphically verified the identities

$$\cos\left(x - \frac{\pi}{2}\right) = \sin x, \quad \sin\left(x + \frac{\pi}{2}\right) = \cos x, \quad \text{and} \quad \sin\left(x - \frac{\pi}{2}\right) = -\cos x. \qquad (3)$$

We now consider the graph of $y = A\sin(Bx + C)$, for $B > 0$. Since the values of $\sin(Bx + C)$ range from -1 to 1, it follows that $A\sin(Bx + C)$ varies between $-A$ and A. That is, the **amplitude** of $y = A\sin(Bx + C)$ is $|A|$. Also, as $Bx + C$ varies from 0 to 2π, the graph will complete one cycle. By solving $Bx + C = 0$ and $Bx + C = 2\pi$, we find that one cycle is completed as x varies from $-C/B$ to $(2\pi - C)/B$. Therefore, the function $y = A\sin(Bx + C)$ has the **period**

$$\frac{2\pi - C}{B} - \left(-\frac{C}{B}\right) = \frac{2\pi}{B}.$$

Moreover, if $f(x) = A\sin Bx$, then

$$f\left(x + \frac{C}{B}\right) = A\sin B\left(x + \frac{C}{B}\right) = A\sin(Bx + C). \qquad (4)$$

The result in (4) shows that the graph of $y = A\sin(Bx + C)$ can be obtained by shifting the graph of $f(x) = A\sin Bx$ horizontally a distance $|C|/B$. If $C < 0$ the shift is to the right, whereas if $C > 0$ the shift is to the left. The number $|C|/B$ is called the **phase shift** of the graph of $y = A\sin(Bx + C)$.

EXAMPLE 4 Equation of a Shifted Cosine Graph

The graph of $y = 10\cos 4x$ is shifted $\pi/12$ units to the right. Find its equation.

Solution By writing $f(x) = 10\cos 4x$ and using (4), we find

$$f\left(x - \frac{\pi}{12}\right) = 10\cos 4\left(x - \frac{\pi}{12}\right) \quad \text{or} \quad y = 10\cos\left(4x - \frac{\pi}{3}\right).$$

In the last equation we would identify $C = -\pi/3$. The phase shift is $\pi/12$. ≡

As a practical matter the phase shift of $y = A\sin(Bx + C)$ can be obtained by factoring the number B from $Bx + C$: ◀ Note

$$y = A\sin(Bx + C) = A\sin B\left(x + \frac{C}{B}\right).$$

For convenience the foregoing information is summarized next.

SHIFTED SINE AND COSINE GRAPHS

The graphs of

$$y = A\sin(Bx + C) \quad \text{and} \quad y = A\cos(Bx + C), \quad B > 0,$$

are, respectively, the graphs of $y = A\sin Bx$ and $y = A\cos Bx$ shifted horizontally by $|C|/B$. The shift is to the right if $C < 0$ and to the left if $C > 0$. The number $|C|/B$ is called the **phase shift**. The **amplitude** of each graph is $|A|$ and the **period** of each graph is $2\pi/B$.

EXAMPLE 5 Horizontally Shifted Sine Graph

Graph $y = 3\sin(2x - \pi/3)$.

Solution For purposes of comparison we will first graph $y = 3\sin 2x$. The amplitude of $y = 3\sin 2x$ is $|A| = 3$ and its period is $2\pi/2 = \pi$. Thus one cycle of $y = 3\sin 2x$ is completed on the interval $[0, \pi]$. Then we extend this graph to the adjacent interval $[\pi, 2\pi]$ as shown in blue in FIGURE 3.2.13. Next, we rewrite $y = 3\sin(2x - \pi/3)$ by factoring 2 from $2x - \pi/3$:

$$y = 3\sin\left(2x - \frac{\pi}{3}\right) = 3\sin 2\left(x - \frac{\pi}{6}\right).$$

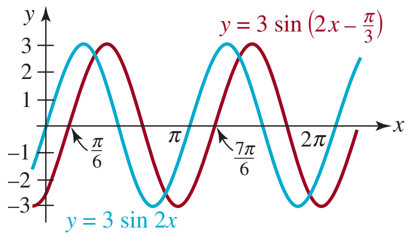


FIGURE 3.2.13 Graph of function in Example 5

From the last form we see that the phase shift is $\pi/6$. The graph of the given function, shown in red in Figure 3.2.13, is obtained by shifting the graph of $y = 3\sin 2x$ to the right $\pi/6$ units. Remember, this means that if (x, y) is a point on the blue graph, then $(x + \pi/6, y)$ is the corresponding point on the red graph. For example, $x = 0$ and $x = \pi$ are the x-coordinates of two x-intercepts of the blue graph. Thus $x = 0 + \pi/6 = \pi/6$ and $x = \pi + \pi/6 = 7\pi/6$ are x-coordinates of the x-intercepts of the red or shifted graph. These numbers are indicated by the black arrows in Figure 3.2.13. ≡

EXAMPLE 6 Horizontally Shifted Graphs

Determine the amplitude, the period, the phase shift, and the direction of horizontal shift for each of the following functions.

(a) $y = 15\cos\left(5x - \frac{3\pi}{2}\right)$ **(b)** $y = -8\sin\left(2x + \frac{\pi}{4}\right)$

Solution (a) We first make the identifications $A = 15$, $B = 5$, and $C = -3\pi/2$. Thus the amplitude is $|A| = 15$ and the period is $2\pi/B = 2\pi/5$. The phase shift can be computed either by $(|-3\pi|/2)/5 = 3\pi/10$ or by rewriting the function as

$$y = 15\cos 5\left(x - \frac{3\pi}{10}\right).$$

The last form indicates that the graph of $y = 15\cos(5x - 3\pi/2)$ is the graph of $y = 15\cos 5x$ shifted $3\pi/10$ units to the right.

(**b**) Since $A = -8$ the amplitude is $|A| = |-8| = 8$. With $B = 2$ the period is $2\pi/2 = \pi$. By factoring 2 from $2x + \pi/4$, we see from

$$y = -8\sin\left(2x + \frac{\pi}{4}\right) = -8\sin 2\left(x + \frac{\pi}{8}\right)$$

that the phase shift is $\pi/8$. The graph of $y = -8\sin(2x + \pi/4)$ is the graph of $y = -8\sin 2x$ shifted $\pi/8$ units to the left. ≡

EXAMPLE 7 Horizontally Shifted Cosine Graph

Graph $y = 2\cos(\pi x + \pi)$.

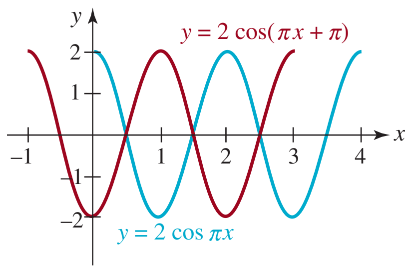


FIGURE 3.2.14 Graph of function in Example 7

Solution The amplitude of $y = 2\cos\pi x$ is $|A| = 2$ and the period is $2\pi/\pi = 2$. Thus one cycle of $y = 2\cos\pi x$ is completed on the interval $[0, 2]$. In FIGURE 3.2.14 two cycles of the graph of $y = 2\cos\pi x$ (in blue) are shown. The x-coordinates of the x-intercepts of this graph are the values of x for which $\cos\pi x = 0$. The last equation implies $\pi x = (2n + 1)\pi/2$ or $x = (2n + 1)/2$, n an integer. In other words, for $n = 0, -1, 1, -2, 2, -3, \ldots$ we get $x = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$, and so on. Now by rewriting the given function $y = 2\cos(\pi x + \pi)$ as

$$y = 2\cos\pi(x + 1)$$

we see the phase shift is 1. The graph of $y = 2\cos(\pi x + \pi)$ (in red) in Figure 3.2.14, is obtained by shifting the graph of $y = 2\cos\pi x$ to the left 1 unit. This means that the x-intercepts are the same for both graphs. ≡

EXAMPLE 8 Alternating Current

The current I (in amperes) in a wire of an alternating-current circuit is given by $I(t) = 30\sin 120\pi t$, where t is time measured in seconds. Sketch one cycle of the graph. What is the maximum value of the current?

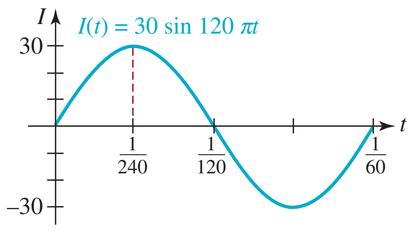


FIGURE 3.2.15 Graph of current in Example 8

Solution The graph has amplitude 30 and period $2\pi/120\pi = \frac{1}{60}$. Therefore, we sketch one cycle of the basic sine curve on the interval $[0, \frac{1}{60}]$, as shown in FIGURE 3.2.15. From the figure it is evident that the maximum value of the current is $I = 30$ amperes and occurs at $t = \frac{1}{240}$ second since

$$I\left(\tfrac{1}{240}\right) = 30\sin\left(120\pi \cdot \tfrac{1}{240}\right) = 30\sin\frac{\pi}{2} = 30.$$ ≡

3.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-8.

In Problems 1–6, use the techniques of shifting, stretching, compressing, and reflecting to sketch at least one cycle of the graph of the given function.

1. $y = \frac{1}{2} + \cos x$
2. $y = -1 + \cos x$
3. $y = 2 - \sin x$
4. $y = 3 + 3\sin x$
5. $y = -2 + 4\cos x$
6. $y = 1 - 2\sin x$

In Problems 7–10, the given figure shows one cycle of a sine or cosine graph. From the figure determine A and D and write an equation of the form $y = A\sin x + D$ or $y = A\cos x + D$ for the graph.

7.

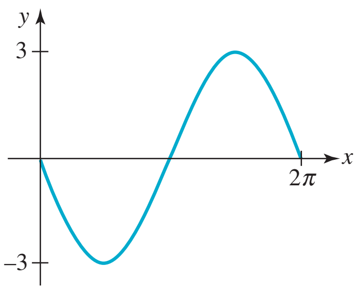


FIGURE 3.2.16 Graph for Problem 7

8.

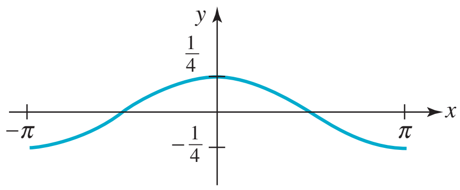


FIGURE 3.2.17 Graph for Problem 8

9.

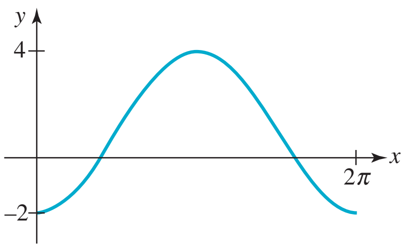


FIGURE 3.2.18 Graph for Problem 9

10.

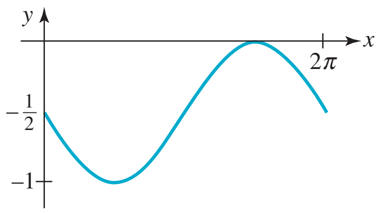


FIGURE 3.2.19 Graph for Problem 10

In Problems 11–16, find the x-intercepts for the graph of the given function. Do not graph.

11. $y = \sin \pi x$

12. $y = -\cos 2x$

13. $y = 10\cos\dfrac{x}{2}$

14. $y = 3\sin(-5x)$

15. $y = \sin\left(x - \dfrac{\pi}{4}\right)$

16. $y = \cos(2x - \pi)$

In Problems 17 and 18, find the x-intercepts of the graph of the given function on the interval $[0, 2\pi]$. Then find all intercepts using periodicity.

17. $y = -1 + \sin x$

18. $y = 1 - 2\cos x$

In Problems 19–24, the given figure shows one cycle of a sine or cosine graph. From the figure determine A and B and write an equation of the form $y = A\sin Bx$ or $y = A\cos Bx$ for the graph.

19.

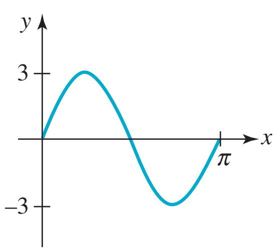


FIGURE 3.2.20 Graph for Problem 19

20.

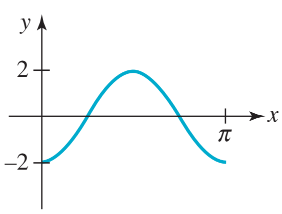


FIGURE 3.2.21 Graph for Problem 20

21.

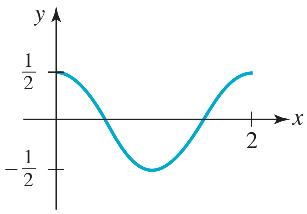


FIGURE 3.2.22 Graph for Problem 21

22.

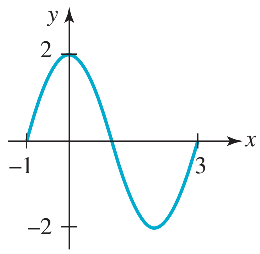


FIGURE 3.2.23 Graph for Problem 22

23.

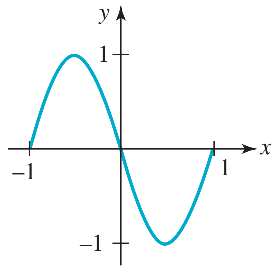


FIGURE 3.2.24 Graph for Problem 23

24.

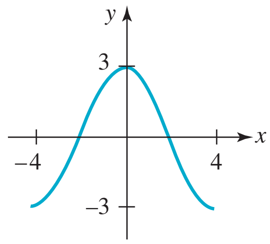


FIGURE 3.2.25 Graph for Problem 24

In Problems 25–32, find the amplitude and period of the given function. Sketch at least one cycle of the graph.

25. $y = 4\sin \pi x$

26. $y = -5\sin \dfrac{x}{2}$

27. $y = -3\cos 2\pi x$

28. $y = \dfrac{5}{2}\cos 4x$

29. $y = 2 - 4\sin x$

30. $y = 2 - 2\sin \pi x$

31. $y = 1 + \cos \dfrac{2x}{3}$

32. $y = -1 + \sin \dfrac{\pi x}{2}$

In Problems 33–42, find the amplitude, period, and phase shift of the given function. Sketch at least one cycle of the graph.

33. $y = \sin\left(x - \dfrac{\pi}{6}\right)$

34. $y = \sin\left(3x - \dfrac{\pi}{4}\right)$

35. $y = \cos\left(x + \dfrac{\pi}{4}\right)$

36. $y = -2\cos\left(2x - \dfrac{\pi}{6}\right)$

37. $y = 4\cos\left(2x - \dfrac{3\pi}{2}\right)$

38. $y = 3\sin\left(2x + \dfrac{\pi}{4}\right)$

39. $y = 3\sin\left(\dfrac{x}{2} - \dfrac{\pi}{3}\right)$

40. $y = -\cos\left(\dfrac{x}{2} - \pi\right)$

41. $y = -4\sin\left(\dfrac{\pi}{3}x - \dfrac{\pi}{3}\right)$

42. $y = 2\cos\left(-2\pi x - \dfrac{4\pi}{3}\right)$

In Problems 43 and 44, write an equation of the function whose graph is described in words.

43. The graph of $y = \cos x$ is vertically stretched up by a factor of 3 and shifted down by 5 units. One cycle of $y = \cos x$ on $[0, 2\pi]$ is compressed to $[0, \pi/3]$ and then the compressed cycle is shifted horizontally $\pi/4$ units to the left.

44. One cycle of $y = \sin x$ on $[0, 2\pi]$ is stretched to $[0, 8\pi]$ and then the stretched cycle is shifted horizontally $\pi/12$ units to the right. The graph is also compressed vertically by a factor of $\frac{3}{4}$ and then reflected in the x-axis.

In Problems 45–48, find horizontally shifted sine and cosine functions so that each function satisfies the given conditions. Graph the functions.

45. Amplitude 3, period $2\pi/3$, shifted by $\pi/3$ units to the right
46. Amplitude $\frac{2}{3}$, period π, shifted by $\pi/4$ units to the left
47. Amplitude 0.7, period 0.5, shifted by 4 units to the right
48. Amplitude $\frac{5}{4}$, period 4, shifted by $1/2\pi$ units to the left

In Problems 49 and 50, graphically verify the given identity.

49. $\cos(x + \pi) = -\cos x$

50. $\sin(x + \pi) = -\sin x$

Miscellaneous Applications

51. Pendulum The angular displacement θ of a pendulum from the vertical at time t seconds is given by $\theta(t) = \theta_0 \cos \omega t$, where θ_0 is the initial displacement at $t = 0$ seconds. See FIGURE 3.2.26. For $\omega = 2$ rad/s and $\theta_0 = \pi/10$, sketch two cycles of the resulting function.

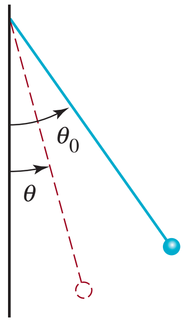


FIGURE 3.2.26 Pendulum in Problem 51

52. Current In a certain kind of electrical circuit, the current I measured in amperes at time t seconds is given by

$$I(t) = 10\cos\left(120\pi t + \frac{\pi}{3}\right).$$

Sketch two cycles of the graph of I as a function of time t.

53. Depth of Water The depth d of water at the entrance to a small harbor at time t is modeled by a function of the form

$$d(t) = A \sin B\left(t - \frac{\pi}{2}\right) + C,$$

where A is one-half the difference between the high- and low-tide depths, $2\pi/B$, $B > 0$, is the tidal period, and C is the average depth. Assume that the tidal period is 12 hours, the depth at high tide is 18 feet, and the depth at low tide is 6 feet. Sketch two cycles of the graph of d.

54. Fahrenheit Temperature Suppose that

$$T(t) = 50 + 10 \sin \frac{\pi}{12}(t - 8),$$

$0 \le t \le 24$, is a mathematical model of the Fahrenheit temperature at t hours after midnight on a certain day of the week.

(a) What is the temperature at 8 AM?
(b) At what time(s) does $T(t) = 60$?
(c) Sketch the graph of T.
(d) Find the maximum and minimum temperatures and the times at which they occur.

Calculator Problems

In Problems 55–58, use a calculator to investigate whether the given function is periodic.

55. $f(x) = \sin\left(\frac{1}{x}\right)$

56. $f(x) = \dfrac{1}{\sin 2x}$

57. $f(x) = 1 + (\cos x)^2$

58. $f(x) = x \sin x$

For Discussion

59. The function $f(x) = \sin \frac{1}{2}x + \sin 2x$ is periodic. What is the period of f?
60. Discuss and then sketch the graphs of $y = |\sin x|$ and $y = |\cos x|$.

3.3 Graphs of Other Trigonometric Functions

Introduction Four additional trigonometric functions are defined in terms of quotients and reciprocals of the sine and cosine functions. In this section we will consider the properties and graphs of these new functions.

We begin with a definition that follows directly from (1) of Section 3.1.

DEFINITION 3.3.1 Four More Trigonometric Functions

The **tangent**, **cotangent**, **secant**, and **cosecant** functions are denoted by $\tan x$, $\cot x$, $\sec x$, and $\csc x$, respectively, and are defined as follows:

$$\tan x = \frac{\sin x}{\cos x}, \qquad \cot x = \frac{\cos x}{\sin x}, \tag{1}$$

$$\sec x = \frac{1}{\cos x}, \qquad \csc x = \frac{1}{\sin x}. \tag{2}$$

Note that the tangent and cotangent functions are related by

$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\dfrac{\sin x}{\cos x}} = \frac{1}{\tan x}.$$

In view of the definitions in (2) and the foregoing result, $\cot x$, $\sec x$, and $\csc x$ are referred to as the **reciprocal functions**.

Domain and Range Because the functions in (1) and (2) are quotients, the **domain** of each function consists of the set of real numbers except those numbers for which the denominator is zero. We have seen in Section 3.2 that $\cos x = 0$ for $x = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$, and so

- the domain of $\tan x$ and of $\sec x$ is $\{x \mid x \neq (2n + 1)\pi/2, n = 0, \pm 1, \pm 2, \ldots\}$.

Similarly, since $\sin x = 0$ for $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, it follows that

- the domain of $\cot x$ and of $\csc x$ is $\{x \mid x \neq n\pi, n = 0, \pm 1, \pm 2, \ldots\}$.

We know that the values of the sine and cosine are bounded, that is, $|\sin x| \leq 1$ and $|\cos x| \leq 1$. From these last inequalities we have

$$|\sec x| = \left|\frac{1}{\cos x}\right| = \frac{1}{|\cos x|} \geq 1 \tag{3}$$

and

$$|\csc x| = \left|\frac{1}{\sin x}\right| = \frac{1}{|\sin x|} \geq 1. \tag{4}$$

Recall, an inequality such as (3) means that $\sec x \geq 1$ *or* $\sec x \leq -1$. Hence the range of the secant function is $(-\infty, -1] \cup [1, \infty)$. The inequality in (4) implies that the cosecant function has the same range $(-\infty, -1] \cup [1, \infty)$. When we consider the graphs of the tangent and cotangent functions we will see that they have the same range: $(-\infty, \infty)$.

If we interpret x as an angle, then FIGURE 3.3.1 illustrates the algebraic signs of the tangent, cotangent, secant, and cosecant functions in each of the four quadrants. This is easily verified using the signs of the sine and cosine functions displayed in Figure 3.1.4.

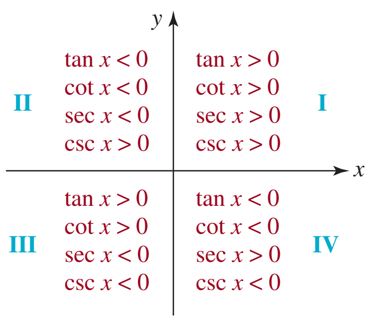


FIGURE 3.3.1 Signs of $\tan x$, $\cot x$, $\sec x$, and $\csc x$ in the four quadrants

EXAMPLE 1 Example 3 of Section 3.1 Revisited

Find $\tan x$, $\cot x$, $\sec x$, and $\csc x$ for $x = -\pi/6$.

Solution In Example 3 of Section 3.1 we saw that

$$\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2} \quad \text{and} \quad \cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Therefore, by the definitions in (1) and (2):

$$\tan\left(-\frac{\pi}{6}\right) = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}, \quad \cot\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3} \quad \leftarrow \text{we could also use } \cot x = 1/\tan x$$

$$\sec\left(-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}, \quad \csc\left(-\frac{\pi}{6}\right) = \frac{1}{-1/2} = -2.$$

≡

Table 3.3.1 summarizes some important values of the tangent, cotangent, secant, and cosecant and was constructed using values of the sine and cosine from Section 2.3. A dash in the table indicates the trigonometric function is not defined at that particular value of x.

TABLE 3.3.1

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–
$\cot x$	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–
$\csc x$	–	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Periodicity

Because the cosine and sine functions are 2π periodic necessarily the secant and cosecant function have the same period. But from Theorem 3.1.3 of Section 3.1 we have

$$\tan(x + \pi) = \frac{\overbrace{\sin(x+\pi)}^{(iv)\text{ of Theorem 3.1.3}}}{\underbrace{\cos(x+\pi)}_{(iii)\text{ of Theorem 3.1.3}}} = \frac{-\sin x}{-\cos x} = \tan x. \tag{5}$$

◀ Also, see Problems 49 and 50 in Exercises 3.2.

Thus (5) implies that $\tan x$ and $\cot x$ are periodic with a period $p \le \pi$. In the case of the tangent function, $\tan x = 0$ only if $\sin x = 0$, that is, only if $x = 0, \pm\pi, \pm 2\pi$, and so on. Therefore, the smallest positive number p for which $\tan(x + p) = \tan x$ is $p = \pi$. The cotangent function, since it is the reciprocal of the tangent function, has the same period.

In summary, the secant and cosecant functions are periodic with **period 2π**:

$$\sec(x + 2\pi) = \sec x \quad \text{and} \quad \csc(x + 2\pi) = \csc x. \tag{6}$$

The tangent and cotangent function are periodic with **period π**:

$$\tan(x + \pi) = \tan x \quad \text{and} \quad \cot(x + \pi) = \cot x. \tag{7}$$

Of course it is understood that (6) and (7) hold for every real number x for which the functions are defined.

Graphs of $y = \tan x$ and $y = \cot x$

The numbers that make the denominators of $\tan x$, $\cot x$, $\sec x$, and $\csc x$ equal to zero correspond to **vertical asymptotes** of their graphs. For example, we encourage you to verify using a calculator that

$$\tan x \to -\infty \text{ as } x \to -\frac{\pi}{2}^{+} \quad \text{and} \quad \tan x \to \infty \text{ as } x \to \frac{\pi}{2}^{-}.$$

In other words, $x = -\pi/2$ and $x = \pi/2$ are vertical asymptotes. The graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$ given in **FIGURE 3.3.2** is one **cycle** of the graph of $y = \tan x$. Using periodicity we extend the cycle in Figure 3.3.2 to adjacent intervals of length π as shown in **FIGURE 3.3.3**. The x-intercepts of the graph of the tangent function are $(0, 0), (\pm\pi, 0), (\pm 2\pi, 0), \ldots$ and the vertical asymptotes of the graph are $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$.

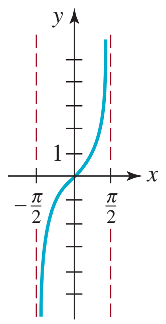


FIGURE 3.3.2 One cycle of the graph of $y = \tan x$

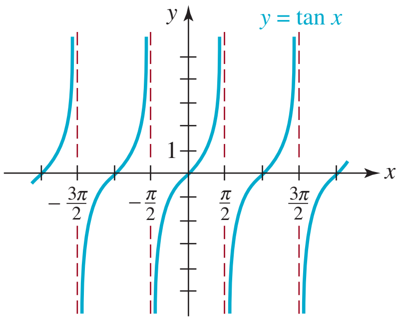


FIGURE 3.3.3 Graph of of $y = \tan x$

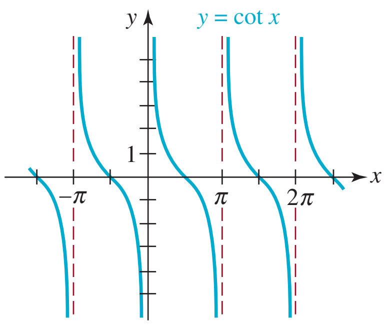


FIGURE 3.3.4 Graph of $y = \cot x$

The graph of $y = \cot x$ is similar to the graph of the tangent function and is given in FIGURE 3.3.4. In this case, the graph of $y = \cot x$ on the interval $(0, \pi)$ is one **cycle** of the graph of $y = \cot x$. The x-intercepts of the graph of the cotangent function are $(\pm\pi/2, 0)$, $(\pm 3\pi/2, 0)$, $(\pm 5\pi/2, 0)$, . . . and the vertical asymptotes of the graph are the vertical lines $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$.

Note that the graphs of $y = \tan x$ and $y = \cot x$ are symmetric with respect to the origin.

THEOREM 3.3.1 ODD FUNCTIONS

The tangent function $f(x) = \tan x$ and the cotangent function $g(x) = \cot x$ are **odd** functions, that

$$\tan(-x) = -\tan x \quad \text{and} \quad \cot(-x) = -\cot x \tag{8}$$

for every real number x for which the functions are defined.

Graphs of sec *x* and csc *x*

For both $y = \sec x$ and $y = \csc x$ we know that $|y| \geq 1$ and so no portion of their graphs can appear in the horizontal strip $-1 < y < 1$ of the Cartesian plane. Hence the graphs of $y = \sec x$ and $y = \csc x$ have no x-intercepts. As we have already seen, $y = \sec x$ and $y = \csc x$ have period 2π. The vertical asymptotes for the graph of $y = \sec x$ are the same as $y = \tan x$, namely, $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$. Because $y = \cos x$ is an even function so is $y = \sec x = 1/\cos x$. The graph of $y = \sec x$ is symmetric with respect to the y-axis. On the other hand, the vertical asymptotes for the graph of $y = \csc x$ are the same as $y = \cot x$, namely, $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Because $y = \sin x$ is an odd function so is $y = \csc x = 1/\sin x$. The graph of $y = \csc x$ is symmetric with respect to the origin. One cycle of the graph of $y = \sec x$ on $[0, 2\pi]$ is extended to the interval $[-2\pi, 0]$ by periodicity (or y-axis symmetry) in FIGURE 3.3.5. Similarly, in FIGURE 3.3.6 we extend one cycle of $y = \csc x$ on $(0, 2\pi)$ to the interval $(-2\pi, 0)$ by periodicity (or origin symmetry).

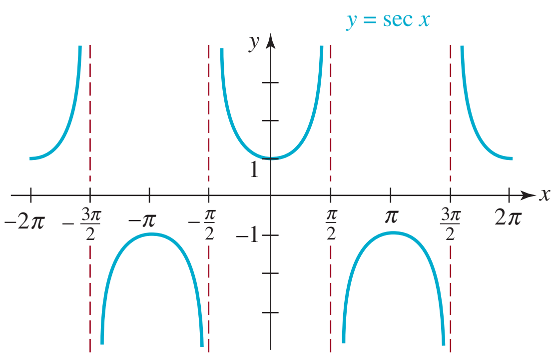


FIGURE 3.3.5 Graph of $y = \sec x$

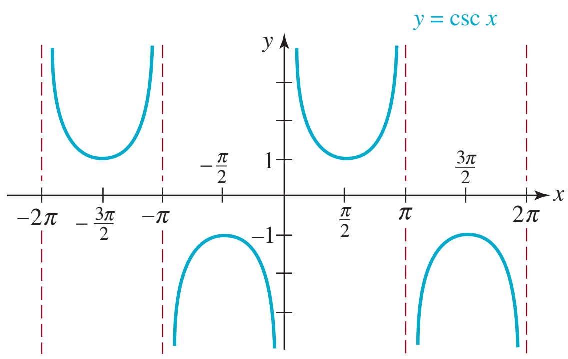


FIGURE 3.3.6 Graph of $y = \csc x$

Transformations and Graphs

Like the sine and cosine graphs, rigid and nonrigid transformations can be applied to the graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$. For example, a function such as $y = A\tan(Bx + C) + D$ can be analyzed in the following manner:

$$y = \overset{\substack{\text{vertical stretch/compression/reflection}\\ \downarrow}}{A}\tan(\underset{\substack{\uparrow\\ \text{horizontal stretch/compression}\\ \text{by changing period}}}{B}x + \underset{\substack{\uparrow\\ \text{horizontal shift}}}{C}) + \overset{\substack{\text{vertical shift}\\ \downarrow}}{D}. \tag{9}$$

If $B > 0$, then the period of

$$y = A\tan(Bx + C) \qquad \text{and} \qquad y = A\cot(Bx + C) \text{ is } \pi/B, \tag{10}$$

whereas the period of

$$y = A\sec(Bx + C) \qquad \text{and} \qquad y = A\csc(Bx + C) \text{ is } 2\pi/B. \tag{11}$$

As we see in (9) the number A in each case can be interpreted as either a vertical stretch or compression of a graph. However, you should be aware of the fact that the functions in (10) and (11) have no amplitude, because none of the functions have a maximum *and* a minimum value.

◀ Of the six trigonometric functions, only the sine and cosine functions have an amplitude.

EXAMPLE 2 Comparison of Graphs

Find the period, x-intercepts, and vertical asymptotes for the graph of $y = \tan 2x$. Graph the function on $[0, \pi]$.

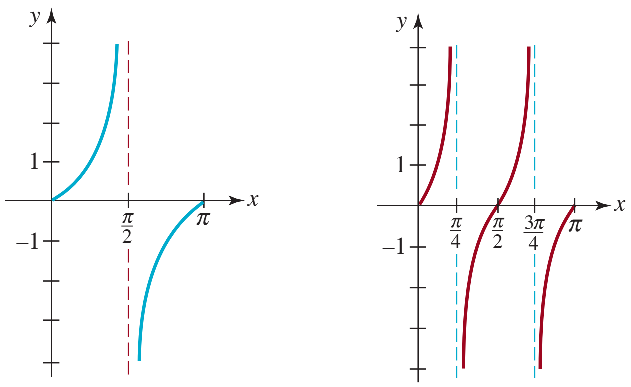
Solution With the identification $B = 2$, we see from (10) that the period is $\pi/2$. Since $\tan 2x = \sin 2x/\cos 2x$, the x-intercepts of the graph occur at the zeros of $\sin 2x$. From the properties of the sine function given in Section 3.2, we know that $\sin 2x = 0$ for

$$2x = n\pi \quad \text{so that} \quad x = \tfrac{1}{2}n\pi, n = 0, \pm 1, \pm 2, \ldots.$$

That is, $x = 0, \pm\pi/2, \pm 2\pi/2 = \pi, \pm 3\pi/2, \pm 4\pi/2 = 2\pi$, and so on. The x-intercepts are $(0, 0), (\pm\pi/2, 0), (\pm\pi, 0), (\pm 3\pi/2, 0), \ldots$. The vertical asymptotes of the graph occur at zeros of $\cos 2x$. Moreover, the numbers for which $\cos 2x = 0$ are found in the following manner:

$$2x = (2n + 1)\frac{\pi}{2} \qquad \text{so that} \qquad x = (2n + 1)\frac{\pi}{4}, n = 0, \pm 1, \pm 2, \ldots.$$

That is, the vertical asymptotes are $x = \pm\pi/4, \pm 3\pi/4, \pm 5\pi/4, \ldots$. On the interval $[0, \pi]$ the graph of $y = \tan 2x$ has three intercepts, $(0, 0), (\pi/2, 0), (\pi, 0)$, and two vertical asymptotes, $x = \pi/4$ and $x = 3\pi/4$. In **FIGURE 3.3.7** we have compared the graphs of $y = \tan x$ and $y = \tan 2x$ on the interval. The graph of $y = \tan 2x$ is a horizontal compression of the graph of $y = \tan x$.



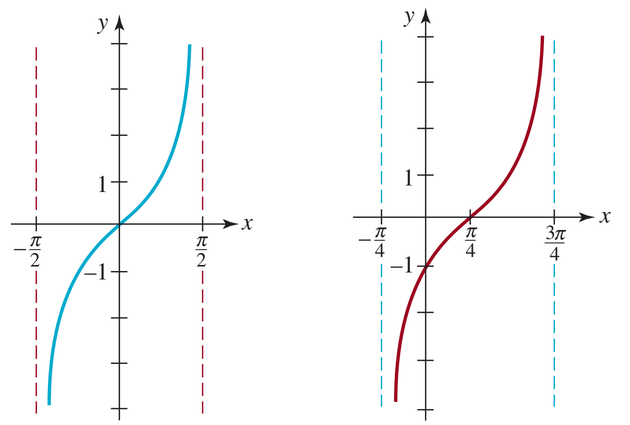
(a) $y = \tan x$ on $[0, \pi]$ (b) $y = \tan 2x$ on $[0, \pi]$

FIGURE 3.3.7 Graphs of functions in Example 2

EXAMPLE 3 Comparisons of Graphs

Compare one cycle of the graphs of $y = \tan x$ and $y = \tan(x - \pi/4)$.

Solution The graph of $y = \tan(x - \pi/4)$ is the graph of $y = \tan x$ shifted horizontally $\pi/4$ units to the right. The intercept $(0, 0)$ for the graph of $y = \tan x$ is shifted to $(\pi/4, 0)$ on the graph of $y = \tan(x - \pi/4)$. The vertical asymptotes $x = -\pi/2$ and $x = \pi/2$ for the graph of $y = \tan x$ are shifted to $x = -\pi/4$ and $x = 3\pi/4$ for the graph of $y = \tan(x - \pi/4)$. In FIGURES 3.3.8(a) and 3.3.8(b) we see, respectively, a cycle of the graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$ is shifted to the right to yield a cycle of the graph of $y = \tan(x - \pi/4)$ on the interval $(-\pi/4, 3\pi/4)$.



(a) Cycle of $y = \tan x$ on $(-\pi/2, \pi/2)$

(b) Cycle of $y = \tan(x - \pi/4)$ on $(-\pi/4, 3\pi/4)$

FIGURE 3.3.8 Graph of functions in Example 3

As we did in the analysis of the graphs of $y = A\sin(Bx + C)$ and $y = A\cos(Bx + C)$ we can determine the amount of horizontal shift for graphs of functions such as $y = A\tan(Bx + C)$ and $y = A\sec(Bx + C)$ by factoring the number $B > 0$ from $Bx + C$.

EXAMPLE 4 Two Shifts and Two Compressions

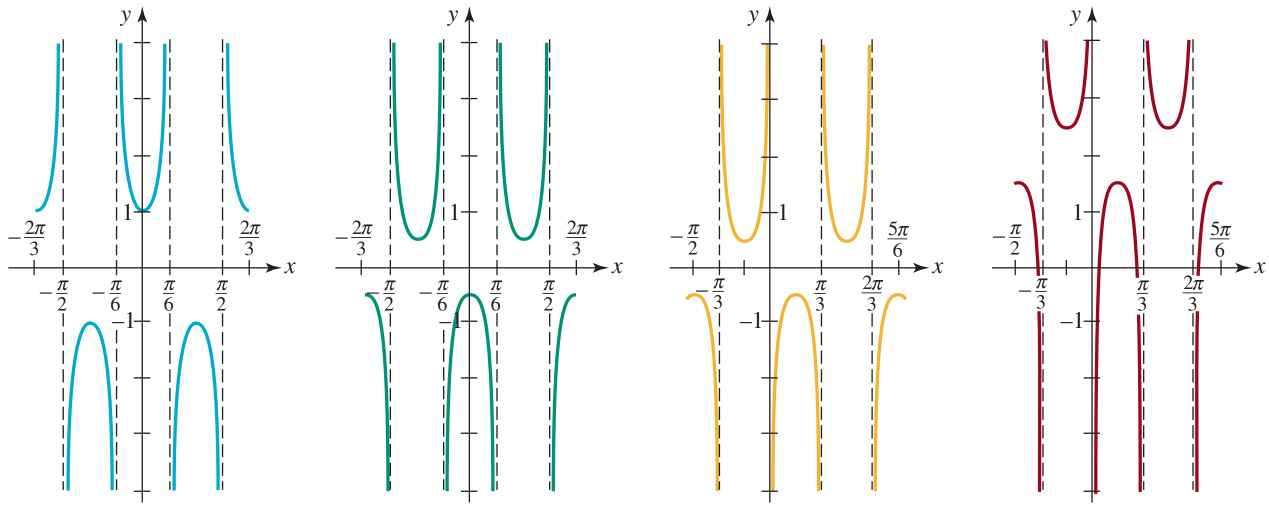
Graph $y = 2 - \frac{1}{2}\sec(3x - \pi/2)$.

Solution Let's break down the analysis of the graph into four parts, namely, by transformations.

(*i*) One cycle of the graph of $y = \sec x$ occurs on $[0, 2\pi]$. Since the period of $y = \sec 3x$ is $2\pi/3$, one cycle of its graph occurs on the interval $[0, 2\pi/3]$. In other words, the graph of $y = \sec 3x$ is a horizontal compression of the graph of $y = \sec x$. Since $\sec 3x = 1/\cos 3x$ the vertical asymptotes occur at the zeros of $\cos 3x$. Using the zeros of the cosine function given in Section 3.2 we find

$$3x = (2n + 1)\frac{\pi}{2} \qquad \text{or} \qquad x = (2n + 1)\frac{\pi}{6},\ n = 0, \pm 1, \pm 2, \ldots .$$

FIGURE 3.3.9(a) shows two cycles of the graph $y = \sec 3x$; one cycle on $[-2\pi/3, 0]$ and another on $[0, 2\pi/3]$. Within those intervals the vertical asymptotes are $x = -\pi/2$, $x = -\pi/6$, $x = \pi/6$, and $x = \pi/2$.



(a) Horizontal compression (b) Vertical compression and reflection in x-axis (c) Horizontal shift (d) Vertical shift

FIGURE 3.3.9 Graph of function in Example 4

(*ii*) The graph of $y = -\frac{1}{2}\sec 3x$ is the graph of $y = \sec 3x$ compressed vertically by a factor of $\frac{1}{2}$ and then reflected in the x-axis. See Figure 3.3.9(b).
(*iii*) By factoring 3 from $3x - \pi/2$, we see from

$$y = -\frac{1}{2}\sec\left(3x - \frac{\pi}{2}\right) = -\frac{1}{2}\sec 3\left(x - \frac{\pi}{6}\right)$$

that the graph of $y = -\frac{1}{2}\sec(3x - \pi/2)$ is the graph of $y = -\frac{1}{2}\sec 3x$ shifted $\pi/6$ units to the right. By shifting the two intervals $[-2\pi/3, 0]$ and $[0, 2\pi/3]$ in Figure 3.3.9(b) to the right $\pi/6$ units, we see in Figure 3.3.9(c) two cycles of $y = -\frac{1}{2}\sec(3x - \pi/2)$ on the intervals $[-\pi/2, \pi/6]$ and $[\pi/6, 5\pi/6]$. The vertical asymptotes $x = -\pi/2$, $x = -\pi/6$, $x = \pi/6$, and $x = \pi/2$ shown in Figure 3.3.9(b) are shifted to $x = -\pi/3$, $x = 0$, $x = \pi/3$, and $x = 2\pi/3$. Observe that the y-intercept $\left(0, -\frac{1}{2}\right)$ in Figure 3.3.9(b) is now moved to $\left(\pi/6, -\frac{1}{2}\right)$ in Figure 3.3.9(c).
(*iv*) Finally, we obtain the graph $y = 2 - \frac{1}{2}\sec(3x - \pi/2)$ in Figure 3.3.9(d) by shifting the graph of $y = -\frac{1}{2}\sec(3x - \pi/2)$ in Figure 3.3.9(c) 2 units upward. ≡

3.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-9.

In Problems 1 and 2, complete the given table.

1.

x	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\tan x$												
$\cot x$												

2.

x	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sec x$												
$\csc x$												

In Problems 3–18, find the indicated value without the use of a calculator.

3. $\cot \frac{13\pi}{6}$ **4.** $\csc\left(-\frac{3\pi}{2}\right)$ **5.** $\tan \frac{9\pi}{2}$ **6.** $\sec 7\pi$

7. $\csc\left(-\frac{\pi}{3}\right)$ **8.** $\cot\left(-\frac{13\pi}{3}\right)$ **9.** $\tan \frac{23\pi}{4}$ **10.** $\tan\left(-\frac{5\pi}{6}\right)$

11. $\sec \frac{10\pi}{3}$ **12.** $\cot \frac{17\pi}{6}$ **13.** $\csc 5\pi$ **14.** $\sec \frac{29\pi}{4}$

15. $\sec(-120°)$ **16.** $\tan 405°$ **17.** $\csc 495°$ **18.** $\cot(-720°)$

In Problems 19–22, use the given information to find the values of the remaining five trigonometric functions.

19. $\tan x = -2, \quad \pi/2 < x < \pi$ **20.** $\cot x = \frac{1}{2}, \quad \pi < x < 3\pi/2$

21. $\csc x = \frac{4}{3}, \quad 0 < x < \pi/2$ **22.** $\sec x = -5, \quad \pi/2 < x < \pi$

23. If $3\cos x = \sin x$, find all values of $\tan x$, $\cot x$, $\sec x$, and $\csc x$.

24. If $\csc x = \sec x$, find all values of $\tan x$, $\cot x$, $\sin x$, and $\cos x$.

In Problems 25–32, find the period, x-intercepts, and the vertical asymptotes of the given function. Sketch at least one cycle of the graph.

25. $y = \tan \pi x$ **26.** $y = \tan \frac{x}{2}$

27. $y = \cot 2x$ **28.** $y = -\cot \frac{\pi x}{3}$

29. $y = \tan\left(\frac{x}{2} - \frac{\pi}{4}\right)$ **30.** $y = \frac{1}{4}\cot\left(x - \frac{\pi}{2}\right)$

31. $y = -1 + \cot \pi x$ **32.** $y = \tan\left(x + \frac{5\pi}{6}\right)$

In Problems 33–40, find the period and the vertical asymptotes of the given function. Sketch at least one cycle of the graph.

33. $y = -\sec x$ **34.** $y = 2\sec \frac{\pi x}{2}$

35. $y = 3\csc \pi x$ **36.** $y = -2\csc \frac{x}{3}$

37. $y = \sec\left(3x - \frac{\pi}{2}\right)$ **38.** $y = \csc x(4x + \pi)$

39. $y = 3 + \csc\left(2x + \frac{\pi}{2}\right)$ **40.** $y = -1 + \sec(x - 2\pi)$

In Problems 41 and 42, use the graphs of $y = \tan x$ and $y = \sec x$ to find numbers A and C for which the given equality is true.

41. $\cot x = A\tan(x + C)$ **42.** $\csc x = A\sec(x + C)$

For Discussion

43. Using a calculator in radian mode, compare the values of tan 1.57 and tan 1.58. Explain the difference in these values.

44. Using a calculator in radian mode, compare the values of cot 3.14 and cot 3.15.

45. Can $9\csc x = 1$ for any real number x?

46. Can $7 + 10\sec x = 0$ for any real number x?

47. For which real numbers x is **(a)** $\sin x \le \csc x$? **(b)** $\sin x < \csc x$?

48. For which real numbers x is **(a)** $\sec x \le \cos x$? **(b)** $\sec x < \cos x$?

49. Discuss and then sketch the graphs of $y = |\sec x|$ and $y = |\csc x|$.

50. Use Definition 1.6.1 to prove Theorem 3.3.1, that is, $f(x) = \tan x$ and $g(x) = \cot x$ are odd functions.

3.4 Special Identities

Introduction In this section we will examine identities for trigonometric functions. We have already seen some of these identities, such as the Pythagorean identities, in earlier sections. A **trigonometric identity** is an equation or formula involving trigonometric functions that is valid for all angles or real numbers for which both sides of the equality are defined. There are *numerous* trigonometric identities, but we are going to develop only those of special importance in courses in mathematics and science.

Pythagorean Identities Revisited In Sections 2.2 and 2.4 we saw that the sine and cosine of an angle θ are related by the fundamental identity $\sin^2\theta + \cos^2\theta = 1$. We saw that by dividing this identity, in turn, by $\cos^2\theta$ and then by $\sin^2\theta$ we obtain two more identities, one relating $\tan^2\theta$ to $\sec^2\theta$ and the other relating $\cot^2\theta$ to $\csc^2\theta$. These so-called **Pythagorean identities** are also valid for a real number x as well as to an angle θ measured in degrees or in radians. Also, see (2) in Section 3.1.

The Pythagorean identities are so basic to trigonometry that we give them again for future reference.

THEOREM 3.4.1 Pythagorean Identities

For x a real number for which the functions are defined,

$$\sin^2 x + \cos^2 x = 1 \tag{1}$$

$$1 + \tan^2 x = \sec^2 x \tag{2}$$

$$1 + \cot^2 x = \csc^2 x. \tag{3}$$

Trigonometric Substitutions In calculus it is often useful to make use of trigonometric substitution to change the form of certain algebraic expressions involving radicals. Generally, this is done using the Pythagorean identities. The following example illustrates the technique.

EXAMPLE 1 Rewriting a Radical

Rewrite $\sqrt{a^2 - x^2}$ as a trigonometric expression without radicals by means of the substitution $x = a\sin\theta$, $a > 0$ and $-\pi/2 \le \theta \le \pi/2$.

Solution If $x = a\sin\theta$, then

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - (a\sin\theta)^2}\\ &= \sqrt{a^2 - a^2\sin^2\theta}\\ &= \sqrt{a^2(1 - \sin^2\theta)} \qquad \leftarrow \text{now use (1) of Theorem 3.4.1}\\ &= \sqrt{a^2\cos^2\theta}.\end{aligned}$$

Since $a > 0$ and $\cos\theta \ge 0$ for $-\pi/2 \le \theta \le \pi/2$, the original radical is the same as

$$\sqrt{a^2 - x^2} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$

Sum and Difference Formulas The **sum** and **difference formulas** for the cosine and sine functions are identities that reduce $\cos(x_1 + x_2)$, $\cos(x_1 - x_2)$, $\sin(x_1 + x_2)$, and $\sin(x_1 - x_2)$ to expressions that involve $\cos x_1$, $\cos x_2$, $\sin x_1$, and $\sin x_2$. We will derive the formula for $\cos(x_1 - x_2)$ first, and then we will use that result to obtain the others.

For convenience, let us suppose that x_1 and x_2 represent angles measured in radians. As shown in FIGURE 3.4.1(a), let d denote the distance between $P(x_1)$ and $P(x_2)$. If we place the angle $x_1 - x_2$ in standard position as shown in Figure 3.4.1(b), then d is also the distance between $P(x_1 - x_2)$ and $P(0)$. Equating the squares of these distances gives

$$(\cos x_1 - \cos x_2)^2 + (\sin x_1 - \sin x_2)^2 = (\cos(x_1 - x_2) - 1)^2 + \sin^2(x_1 - x_2)$$

or

$$\begin{aligned}\cos^2 x_1 - 2\cos x_1 \cos x_2 + \cos^2 x_2 + \sin^2 x_1 - 2\sin x_1 \sin x_2 + \sin^2 x_2 \\ = \cos^2(x_1 - x_2) - 2\cos(x_1 - x_2) + 1 + \sin^2(x_1 - x_2).\end{aligned}$$

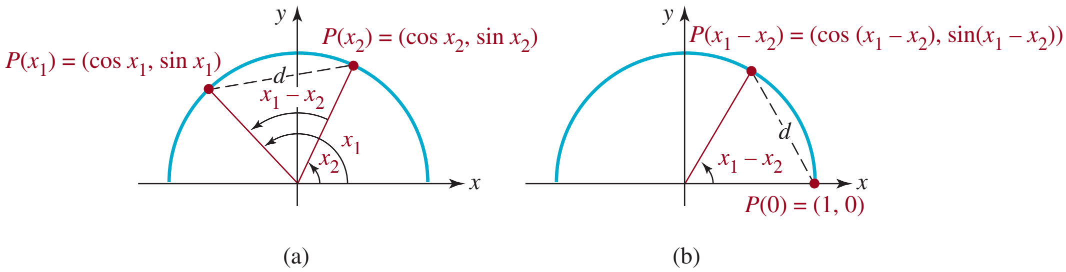


FIGURE 3.4.1 The difference of two angles

In view of (1),

$$\cos^2 x_1 + \sin^2 x_1 = 1, \quad \cos^2 x_2 + \sin^2 x_2 = 1, \quad \cos^2(x_1 - x_2) + \sin^2(x_1 - x_2) = 1$$

and so the preceding equation simplifies to

$$\cos(x_1 - x_2) = \cos x_1 \cos x_2 + \sin x_1 \sin x_2.$$

This last result can be put to work immediately to find the cosine of the sum of two angles. Since $x_1 + x_2$ can be rewritten as the difference $x_1 - (-x_2)$,

$$\begin{aligned}\cos(x_1 + x_2) &= \cos(x_1 - (-x_2)) \\ &= \cos x_1 \cos(-x_2) + \sin x_1 \sin(-x_2).\end{aligned}$$

By the even–odd identities, $\cos(-x_2) = \cos x_2$ and $\sin(-x_2) = -\sin x_2$, it follows that the last line is the same as

$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2.$$

The two results just obtained are summarized next.

THEOREM 3.4.2 Sum and Difference Formulas for the Cosine

For all real numbers x_1 and x_2,

$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2 \quad (4)$$

$$\cos(x_1 - x_2) = \cos x_1 \cos x_2 + \sin x_1 \sin x_2. \quad (5)$$

EXAMPLE 2 Cosine of a Sum

Evaluate $\cos(7\pi/12)$.

Solution We have no way of evaluating $\cos(7\pi/12)$ directly. However, observe that

$$\frac{7\pi}{12}\text{ radians} = 105° = 60° + 45° = \frac{\pi}{3} + \frac{\pi}{4}.$$

Because $7\pi/12$ radians is a second-quadrant angle we know that the value of $\cos(7\pi/12)$ is negative. Proceeding, the sum formula (4) gives

$$\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \overbrace{\cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4}}^{\text{this is (4) of Theorem 3.4.2}}$$
$$= \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(1 - \sqrt{3}).$$

Using $\sqrt{2}\sqrt{3} = \sqrt{6}$ this result can also be written as $\cos(7\pi/12) = (\sqrt{2} - \sqrt{6})/4$. Since $\sqrt{6} > \sqrt{2}$, we see that $\cos(7\pi/12) < 0$ as expected. ≡

To obtain the corresponding sum/difference identities for the sine function we will make use of two identities:

$$\cos\left(x - \frac{\pi}{2}\right) = \sin x \quad\text{and}\quad \sin\left(x - \frac{\pi}{2}\right) = -\cos x. \tag{6}$$

These identities were first presented in Section 2.2 as cofunction identities and then rediscovered in Section 3.2 by shifting the graphs of the cosine and sine. However, both results in (6) can now be proved using (5):

$$\cos\left(x - \frac{\pi}{2}\right) = \cos x\cos\frac{\pi}{2} + \sin x\sin\frac{\pi}{2} = \cos x \cdot 0 + \sin x \cdot 1 = \sin x$$

◀ This proves the first equation in (6).

$$\cos x = \cos\left(\underbrace{\frac{\pi}{2} - \frac{\pi}{2}}_{\text{zero}} + x\right) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right)$$
$$= \cos\frac{\pi}{2}\cos\left(\frac{\pi}{2} - x\right) + \sin\frac{\pi}{2}\sin\left(\frac{\pi}{2} - x\right)$$
$$= 0 \cdot \cos\left(\frac{\pi}{2} - x\right) + 1 \cdot \sin\left(\frac{\pi}{2} - x\right)$$
$$= -\sin\left(x - \frac{\pi}{2}\right).$$

◀ This proves the second equation in (6).

We put the first equation in (6) to work immediately by writing the sine of the sum $x_1 + x_2$ as

$$\sin(x_1 + x_2) = \cos\left((x_1 + x_2) - \frac{\pi}{2}\right)$$
$$= \cos\left(x_1 + \left(x_2 - \frac{\pi}{2}\right)\right) = \cos x_1\cos\left(x_2 - \frac{\pi}{2}\right) - \sin x_1\sin\left(x_2 - \frac{\pi}{2}\right) \quad \leftarrow \text{by (4)}$$
$$= \cos x_1\sin x_2 - \sin x_1(-\cos x_2). \quad \leftarrow \text{by (6)}$$

The preceding line is traditionally written as

$$\sin(x_1 + x_2) = \sin x_1\cos x_2 + \cos x_1\sin x_2.$$

To obtain the sine of the difference $x_1 - x_2$, we use again $\cos(-x_2) = \cos x_2$ and $\sin(-x_2) = -\sin x_2$:

$$\begin{aligned}\sin(x_1 - x_2) &= \sin(x_1 + (-x_2)) = \sin x_1 \cos(-x_2) + \cos x_1 \sin(-x_2)\\ &= \sin x_1 \cos x_2 - \cos x_1 \sin x_2.\end{aligned}$$

THEOREM 3.4.3 Sum and Difference Formulas for the Sine

For all real numbers x_1 and x_2,

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2, \tag{7}$$

$$\sin(x_1 - x_2) = \sin x_1 \cos x_2 - \cos x_1 \sin x_2. \tag{8}$$

EXAMPLE 3 Sine of a Sum

Evaluate $\sin(7\pi/12)$.

Solution We proceed as in Example 2, except we use the sum formula (7):

$$\begin{aligned}\sin\frac{7\pi}{12} &= \overbrace{\sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4}}^{\text{this is (7) of Theorem 3.4.3}}\\ &= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(1 + \sqrt{3}).\end{aligned}$$

As in Example 2, the result can be rewritten as $\sin(7\pi/12) = \left(\sqrt{2} + \sqrt{6}\right)/4$. ≡

Since we know the value of $\cos(7\pi/12)$ from Example 2 we can also compute the value of $\sin(7\pi/12)$ using the Pythagorean identity (1):

$$\sin^2\frac{7\pi}{12} + \cos^2\frac{7\pi}{12} = 1.$$

We solve for $\sin(7\pi/12)$ and take the positive square root:

$$\begin{aligned}\sin\frac{7\pi}{12} &= \sqrt{1 - \cos^2\frac{7\pi}{12}} = \sqrt{1 - \left[\frac{\sqrt{2}}{4}(1 - \sqrt{3})\right]^2}\\ &= \sqrt{\frac{4 + 2\sqrt{3}}{8}} = \frac{\sqrt{2 + \sqrt{3}}}{2}.\end{aligned} \tag{9}$$

Although the number in (9) does not look like the result obtained in Example 3, the values are the same. See Problem 62 in Exercises 3.4.

There are sum and difference formulas for the tangent function as well. We can derive the sum formula using the sum formulas for the sine and cosine as follows:

$$\tan(x_1 + x_2) = \frac{\sin(x_1 + x_2)}{\cos(x_1 + x_2)} = \frac{\sin x_1 \cos x_2 + \cos x_1 \sin x_2}{\cos x_1 \cos x_2 - \sin x_1 \sin x_2}. \tag{10}$$

We now divide the numerator and denominator of (10) by $\cos x_1 \cos x_2$ (assuming that x_1 and x_2 are such that $\cos x_1 \cos x_2 \neq 0$),

$$\tan(x_1 + x_2) = \frac{\dfrac{\sin x_1}{\cos x_1}\dfrac{\cos x_2}{\cos x_2} + \dfrac{\cos x_1}{\cos x_1}\dfrac{\sin x_2}{\cos x_2}}{\dfrac{\cos x_1}{\cos x_1}\dfrac{\cos x_2}{\cos x_2} - \dfrac{\sin x_1}{\cos x_1}\dfrac{\sin x_2}{\cos x_2}} = \frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2}. \tag{11}$$

The derivation of the difference formula for $\tan(x_1 - x_2)$ is obtained in a similar manner. We summarize the two results.

THEOREM 3.4.4 Sum and Difference Formulas for the Tangent

For all real numbers x_1 and x_2 for which the functions are defined,

$$\tan(x_1 + x_2) = \frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2} \tag{12}$$

$$\tan(x_1 - x_2) = \frac{\tan x_1 - \tan x_2}{1 + \tan x_1 \tan x_2}. \tag{13}$$

EXAMPLE 4 Tangent of a Sum

Evaluate $\tan(\pi/12)$.

Solution If we think of $\pi/12$ as an angle in radians, then

$$\frac{\pi}{12} \text{ radians } = 15^\circ = 45^\circ - 30^\circ = \frac{\pi}{4} - \frac{\pi}{6} \text{ radians.}$$

It follows from formula (13):

$$\tan\frac{\pi}{12} = \overbrace{\tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}}}^{\text{this is (13) of Theorem 3.4.4}}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \quad \leftarrow \text{this is the answer but we can simplify the expression}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \quad \leftarrow \text{rationalizing the denominator}$$

$$= \frac{(\sqrt{3} - 1)^2}{2} = \frac{4 - 2\sqrt{3}}{2} = \frac{2(2 - \sqrt{3})}{2} = 2 - \sqrt{3}.$$

≡

◀ You should rework this example using $\pi/12 = \pi/3 - \pi/4$ to see that the result is the same.

Strictly speaking, we really do not need the identities for $\tan(x_1 \pm x_2)$ since we can always compute $\sin(x_1 \pm x_2)$ and $\cos(x_1 \pm x_2)$ using (4)–(8) and then proceed as in (10), that is, from the quotient $\sin(x_1 \pm x_2)/\cos(x_1 \pm x_2)$.

□ **Double-Angle Formulas** Many useful trigonometric formulas can be derived from the sum and difference formulas. The **double-angle formulas** express the cosine and sine of $2x$ in terms of the cosine and sine of x.

If we set $x_1 = x_2 = x$ in (4) and use $\cos(x + x) = \cos 2x$, then

$$\cos 2x = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x.$$

Similarly, by setting $x_1 = x_2 = x$ in (7) and using $\sin(x + x) = \sin 2x$, then

these two terms are equal

$$\sin 2x = \underset{\downarrow}{\sin x \cos x} + \underset{\downarrow}{\cos x \sin x} = 2 \sin x \cos x.$$

We summarize the last two results.

THEOREM 3.4.5 Double-Angle Formulas for the Cosine and Sine

For any real number x,

$$\cos 2x = \cos^2 x - \sin^2 x \tag{14}$$
$$\sin 2x = 2\sin x \cos x. \tag{15}$$

EXAMPLE 5 Using the Double-Angle Formulas

If $\sin x = -\frac{1}{4}$ and $\pi < x < 3\pi/2$, find the exact values of $\cos 2x$ and $\sin 2x$.

Solution First, we compute $\cos x$ using $\sin^2 x + \cos^2 x = 1$. Since $\pi < x < 3\pi/2$, $\cos x < 0$ and so we choose the negative square root:

$$\cos x = -\sqrt{1 - \sin^2 x} = -\sqrt{1 - \left(-\frac{1}{4}\right)^2} = -\frac{\sqrt{15}}{4}.$$

From the double-angle formula (14),

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= \left(-\frac{\sqrt{15}}{4}\right)^2 - \left(-\frac{1}{4}\right)^2 \\ &= \frac{15}{16} - \frac{1}{16} = \frac{14}{16} = \frac{7}{8}. \end{aligned}$$

Finally, from the double-angle formula (15),

$$\sin 2x = 2\sin x \cos x = 2\left(-\frac{1}{4}\right)\left(-\frac{\sqrt{15}}{4}\right) = \frac{\sqrt{15}}{8}.$$

≡

The formula in (14) has two useful alternative forms. By (1), we know that $\sin^2 x = 1 - \cos^2 x$. Substituting the last expression into (14) yields $\cos 2x = \cos^2 x - (1 - \cos^2 x)$ or

$$\cos 2x = 2\cos^2 x - 1. \tag{16}$$

On the other hand, if we substitute $\cos^2 x = 1 - \sin^2 x$ into (14) we get

$$\cos 2x = 1 - 2\sin^2 x. \tag{17}$$

☐ **Half-Angle Formulas** The alternative forms of (16) and (17) of the double-angle formula (14) are the source of two **half-angle formulas**. Solving (16) and (17) for $\cos^2 x$ and $\sin^2 x$ gives, respectively,

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x). \tag{18}$$

By replacing the symbol x in (18) by $x/2$ and using $2(x/2) = x$, we obtain the following formulas.

THEOREM 3.4.6 Half-Angle Formulas for the Cosine and Sine

For any real number x,

$$\cos^2 \frac{x}{2} = \frac{1}{2}(1 + \cos x) \tag{19}$$
$$\sin^2 \frac{x}{2} = \frac{1}{2}(1 - \cos x). \tag{20}$$

EXAMPLE 6 Using the Half-Angle Formulas

Find the exact values of $\cos(5\pi/8)$ and $\sin(5\pi/8)$.

Solution If we let $x = 5\pi/4$, then $x/2 = 5\pi/8$ and formulas (19) and (20) yield, respectively

$$\cos^2\left(\frac{5\pi}{8}\right) = \frac{1}{2}\left(1 + \cos\frac{5\pi}{4}\right) = \frac{1}{2}\left[1 + \left(-\frac{\sqrt{2}}{2}\right)\right] = \frac{2 - \sqrt{2}}{4},$$

and $$\sin^2\left(\frac{5\pi}{8}\right) = \frac{1}{2}\left(1 - \cos\frac{5\pi}{4}\right) = \frac{1}{2}\left[1 - \left(-\frac{\sqrt{2}}{2}\right)\right] = \frac{2 + \sqrt{2}}{4}.$$

Because $5\pi/8$ radians is a second-quadrant angle, $\cos(5\pi/8) < 0$ and $\sin(5\pi/8) > 0$. Therefore, we take the negative square root for the value of the cosine,

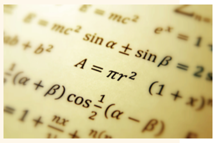
$$\cos\left(\frac{5\pi}{8}\right) = -\sqrt{\frac{2 - \sqrt{2}}{4}} = -\frac{\sqrt{2 - \sqrt{2}}}{2},$$

and the positive square root for the value of the sine,

$$\sin\left(\frac{5\pi}{8}\right) = \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{2 + \sqrt{2}}}{2}.$$

≡

NOTES FROM THE CLASSROOM



(*i*) Should you memorize all the identities presented in this section? You should consult your instructor about this, but in the opinion of the authors, you should at the very least memorize formulas (1)–(8), (14), (15), and the two formulas in (18).

(*ii*) If you eventually enroll in a calculus course, check the title of your text. If it has the words *Early Transcendentals* in its title, then your knowledge of the graphs and properties of the trigonometric functions will come into play almost immediately. The sum identities (4) and (7) are used in differential calculus to compute functions known as derivatives of $\sin x$ and $\cos x$. Identities are especially useful in integral calculus. Replacing a radical by a trigonometric function as illustrated in Example 1 in this section is a standard technique for evaluating some types of integrals. Also, to evaluate integrals of $\cos^2 x$ and $\sin^2 x$ you would use the half-angle formulas in the form given in (18):

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x) \qquad \text{and} \qquad \sin^2 x = \tfrac{1}{2}(1 - \cos 2x).$$

At some point in your study of integral calculus you may be required to evaluate integrals of products such as

$$\sin 2x \sin 5x \qquad \text{and} \qquad \sin 10x \cos 4x.$$

One way of doing this is to use the sum/difference formulas to devise an identity that converts these products into either a sum of sines or a sum of cosines. See Problems 66–70 in Exercises 3.4.

3.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-10.

In Problems 1–8, proceed as in Example 1 and rewrite the given expression as a trigonometric expression without radicals by making the indicated substitution. Assume that $a > 0$.

1. $\sqrt{a^2 - x^2}$, $x = a\cos\theta$, $0 \le \theta \le \pi$
2. $\sqrt{a^2 + x^2}$, $x = a\tan\theta$, $-\pi/2 < \theta < \pi/2$
3. $\sqrt{x^2 - a^2}$, $x = a\sec\theta$, $0 \le \theta < \pi/2$
4. $\sqrt{16 - 25x^2}$, $x = \frac{4}{5}\sin\theta$, $-\pi/2 \le \theta \le \pi/2$
5. $\dfrac{x}{\sqrt{9 - x^2}}$, $x = 3\sin\theta$, $-\pi/2 < \theta < \pi/2$
6. $\dfrac{\sqrt{x^2 - 3}}{x^2}$, $x = \sqrt{3}\sec\theta$, $0 < \theta < \pi/2$
7. $\dfrac{1}{\sqrt{7 + x^2}}$, $x = \sqrt{7}\tan\theta$, $-\pi/2 < \theta < \pi/2$
8. $\dfrac{\sqrt{5 - x^2}}{x}$, $x = \sqrt{5}\cos\theta$, $0 \le \theta \le \pi$

In Problems 9–30, use a sum or difference formula to find the exact value of the given expression.

9. $\cos\dfrac{\pi}{12}$
10. $\sin\dfrac{\pi}{12}$
11. $\sin 75°$
12. $\cos 75°$
13. $\sin\dfrac{7\pi}{12}$
14. $\cos\dfrac{11\pi}{12}$
15. $\tan\dfrac{5\pi}{12}$
16. $\cos\left(-\dfrac{5\pi}{12}\right)$
17. $\sin\left(-\dfrac{\pi}{12}\right)$
18. $\tan\dfrac{11\pi}{12}$
19. $\sin\dfrac{11\pi}{12}$
20. $\tan\dfrac{7\pi}{12}$
21. $\cos 165°$
22. $\sin 165°$
23. $\tan 165°$
24. $\cos 195°$
25. $\sin 195°$
26. $\tan 195°$
27. $\cos 345°$
28. $\sin 345°$
29. $\cos\dfrac{13\pi}{12}$
30. $\tan\dfrac{17\pi}{12}$

In Problems 31–34, use a double-angle formula to write the given expression as a single trigonometric function of twice the angle.

31. $2\cos\beta\sin\beta$
32. $\cos^2 2t - \sin^2 2t$
33. $1 - 2\sin^2\dfrac{\pi}{5}$
34. $2\cos^2\left(\dfrac{19}{2}x\right) - 1$

In Problems 35–40, use the given information to find the exact values of **(a)** $\cos 2x$, **(b)** $\sin 2x$, and **(c)** $\tan 2x$.

35. $\sin x = \sqrt{2}/3, \quad \pi/2 < x < \pi$
36. $\cos x = \sqrt{3}/5, \quad 3\pi/2 < x < 2\pi$
37. $\tan x = \frac{1}{2}, \ \pi < x < 3\pi/2$
38. $\csc x = -3, \quad \pi < x < 3\pi/2$
39. $\sec x = -\frac{13}{5}, \quad \pi/2 < x < \pi$
40. $\cot x = \frac{4}{3}, \quad 0 < x < \pi/2$

In Problems 41–48, use a half-angle formula to find the exact value of the given expression.

41. $\cos(\pi/12)$
42. $\sin(\pi/8)$
43. $\sin(3\pi/8)$
44. $\tan(\pi/12)$
45. $\cos 67.5°$
46. $\sin 15°$
47. $\csc(13\pi/12)$
48. $\sec(-3\pi/8)$

In Problems 49–54, use the given information to find the exact values of **(a)** $\cos(x/2)$, **(b)** $\sin(x/2)$, and **(c)** $\tan(x/2)$.

49. $\sin x = \frac{12}{13}, \quad \pi/2 < x < \pi$
50. $\cos x = \frac{4}{5}, \quad 3\pi/2 < x < 2\pi$
51. $\tan x = 2, \quad \pi < x < 3\pi/2$
52. $\csc x = 9, \quad 0 < x < \pi/2$
53. $\sec x = \frac{3}{2}, \quad 0 < x < 90°$
54. $\cot x = -\frac{1}{4}, \quad 90° < x < 180°$

55. If $P(x_1)$ and $P(x_2)$ are points in quadrant II on the terminal side of the angles x_1 and x_2, respectively, with $\cos x_1 = -\frac{1}{3}$ and $\sin x_2 = \frac{2}{3}$, find
(a) $\sin(x_1 + x_2)$ **(b)** $\cos(x_1 + x_2)$
(c) $\sin(x_1 - x_2)$ **(d)** $\cos(x_1 - x_2)$.

56. If x_1 is a quadrant II angle, x_2 is a quadrant III angle, $\sin x_1 = \frac{8}{17}$, and $\tan x_2 = \frac{3}{4}$, find **(a)** $\sin(x_1 + x_2)$, **(b)** $\sin(x_1 - x_2)$, **(c)** $\cos(x_1 + x_2)$, and **(d)** $\cos(x_1 - x_2)$.

Miscellaneous Applications

57. Mach Number The ratio of the speed of an airplane to the speed of sound is called the Mach number M of the plane. If $M > 1$, the plane makes sound waves that form a (moving) cone, as shown in FIGURE 3.4.2. A sonic boom is heard at the intersection of the cone with the ground. If the vertex angle of the cone is θ, then

$$\sin\frac{\theta}{2} = \frac{1}{M}.$$

If $\theta = \pi/6$, find the exact value of the Mach number.

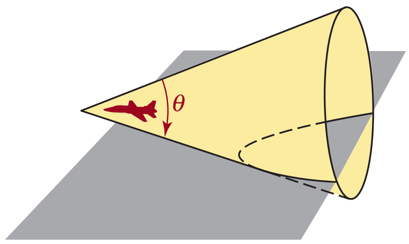


FIGURE 3.4.2 Airplane in Problem 57

58. Cardiovascular Branching A mathematical model for blood flow in a large blood vessel predicts that the optimal values of the angles θ_1 and θ_2, which represent the (positive) angles of the smaller daughter branches (vessels) with respect to the axis of the parent branch, are given by

$$\cos\theta_1 = \frac{A_0^2 + A_1^2 - A_2^2}{2A_0A_1} \quad \text{and} \quad \cos\theta_2 = \frac{A_0^2 - A_1^2 + A_2^2}{2A_0A_2},$$

where A_0 is the cross-sectional area of the parent branch and A_1 and A_2 are the cross-sectional areas of the daughter branches. See FIGURE 3.4.3. Let $\psi = \theta_1 + \theta_2$ be the junction angle, as shown in the figure.

(a) Show that

$$\cos\psi = \frac{A_0^2 - A_1^2 - A_2^2}{2A_1A_2}.$$

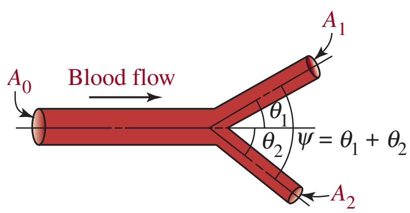


FIGURE 3.4.3 Branching of a large blood vessel in Problem 58

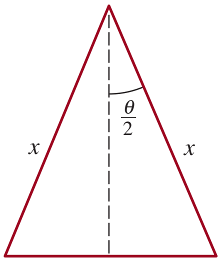


FIGURE 3.4.4 Isosceles triangle in Problem 59

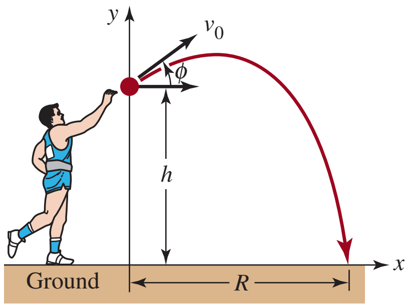


FIGURE 3.4.5 Projectile in Problem 60

(b) Show that for the optimal values of θ_1 and θ_2, the cross-sectional area of the daughter branches, $A_1 + A_2$, is greater than or equal to that of the parent branch. Therefore, the blood must slow down in the daughter branches.

59. Area of a Triangle Show that the area of an isosceles triangle with equal sides of length x is $A = \frac{1}{2}x^2 \sin\theta$, where θ is the angle formed by the two equal sides. See FIGURE 3.4.4. [*Hint*: Consider $\theta/2$ as shown in the figure.]

60. Putting the Shot We saw in Problem 66 in Exercises 2.4 that when a projectile, such as a shot put, is released from a height h, upward at an angle ϕ with velocity v_0, the range R at which it strikes the ground is given by

$$R = \frac{v_0 \cos\phi}{g}\left(v_0 \sin\phi + \sqrt{v_0^2 \sin^2\phi + 2gh}\right),$$

where g is the acceleration due to gravity. See FIGURE 3.4.5. It can be shown that the maximum range R_{max} is achieved if the angle ϕ satisfies the equation

$$\cos 2\phi = \frac{gh}{v_0^2 + gh}.$$

Show that
$$R_{max} = \frac{v_0\sqrt{v_0^2 + 2gh}}{g},$$

by using the expressions for R and $\cos 2\phi$ and the half-angle formulas for the sine and the cosine with $t = 2\phi$.

For Discussion

61. Discuss: Why would you expect to get an error message from your calculator when you try to evaluate

$$\frac{\tan 35° + \tan 55°}{1 - \tan 35° \tan 55°}?$$

62. In Example 3 we showed that $\sin\frac{7\pi}{12} = \frac{\sqrt{2} + \sqrt{6}}{4}$. Following the example, we then showed that $\sin\frac{7\pi}{12} = \frac{\sqrt{2 + \sqrt{3}}}{2}$. Demonstrate that these answers are equivalent.

63. Discuss: How would you express $\sin 3\theta$ in terms of $\sin\theta$? Carry out your ideas.

64. In Problem 55, in what quadrants do the points $P(x_1 + x_2)$ and $P(x_1 - x_2)$ lie?

65. In Problem 56, in which quadrant does the terminal side of $x_1 + x_2$ lie? The terminal side of $x_1 - x_2$?

66. Use the sum/difference formulas (4), (5), (7), and (8) to derive the **product-to-sum formulas**:

$$\sin x_1 \sin x_2 = \tfrac{1}{2}[\cos(x_1 - x_2) - \cos(x_1 + x_2)]$$
$$\cos x_1 \cos x_2 = \tfrac{1}{2}[\cos(x_1 - x_2) + \cos(x_1 + x_2)]$$
$$\sin x_1 \cos x_2 = \tfrac{1}{2}[\sin(x_1 + x_2) + \sin(x_1 - x_2)].$$

In Problems 67–70, use a product-to-sum formula in Problem 66 to write the given product as a sum of sines or a sum of cosines.

67. $\cos 4\theta \cos 3\theta$

68. $\sin\frac{3t}{2}\cos\frac{t}{2}$

69. $\sin 2x \sin 5x$

70. $\sin 10x \cos 4x$

In Problems 71 and 72, use one of the formulas in Problem 66 to find the exact value of the given expression.

71. $\sin 15° \sin 45°$ **72.** $\sin 75° \cos 15°$

73. Prove the double-angle formula for the tangent function:

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x}.$$

74. Prove the half-angle formula for the tangent function:

$$\tan^2 \frac{x}{2} = \frac{1 - \cos x}{1 + \cos x}.$$

In Problems 75 and 76, prove the alternative half-angle formulas for the tangent function. [*Hint*: In Problem 75, multiply the numerator and denominator of $\dfrac{\sin(x/2)}{\cos(x/2)}$ by $2\sin(x/2)$ and then look at (15) and (20).]

75. $\tan \dfrac{x}{2} = \dfrac{1 - \cos x}{\sin x}$ **76.** $\tan \dfrac{x}{2} = \dfrac{\sin x}{1 + \cos x}$

77. Discuss: Why are the formulas in Problems 75 and 76 more useful than the formula in Problem 74?

3.5 Inverse Trigonometric Functions

Introduction Although we can find the values of the trigonometric functions of real numbers or angles, in many applications we must do the reverse: Given the value of a trigonometric function, find a corresponding angle or number. This suggests we consider inverse trigonometric functions. Before we define the inverse trigonometric functions, let's recall from Section 1.9 some of the properties of a one-to-one function f and its inverse f^{-1}.

Recall, a function f is one-to-one if every y in its range corresponds to exactly one x in its domain.

Properties of Inverse Functions If $y = f(x)$ is a one-to-one function, then there is a unique inverse function f^{-1} with the following properties:

PROPERTIES OF INVERSE FUNCTIONS

- The domain of f^{-1} = range of f.
- The range of f^{-1} = domain of f.
- $y = f(x)$ is equivalent to $x = f^{-1}(y)$.
- The graphs of f and f^{-1} are reflections in the line $y = x$.
- $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}.
- $f^{-1}(f(x)) = x$ for x in the domain of f.

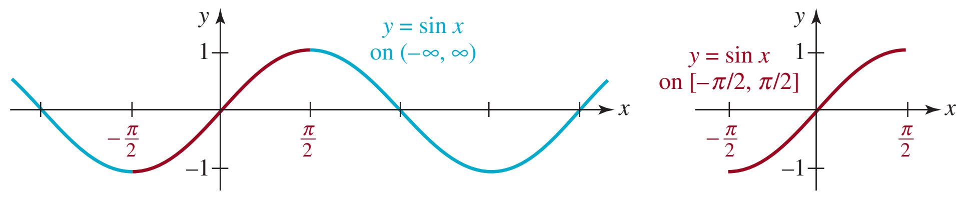
Inspection of the graphs of the various trigonometric functions clearly shows that *none* of these functions are one-to-one. In Section 1.9 we discussed the fact that if a function f is not one-to-one, it may be possible to restrict the function to a portion of its domain where it is one-to-one. Then we can define an inverse for f on that restricted domain. Normally, when we restrict the domain, we make sure to preserve the entire range of the original function.

See Example 7 in Section 1.9.

□ **Arcsine Function** From **FIGURE 3.5.1(a)** we see that the function $y = \sin x$ on the closed interval $[-\pi/2, \pi/2]$ takes on all values in its range $[-1, 1]$. Notice that any horizontal line drawn to intersect the red portion of the graph can do so at most once. Thus the sine function on this restricted domain is one-to-one and has an inverse. There are two commonly used notations to denote the inverse of the function shown in Figure 3.5.1(b):

$$\arcsin x \qquad \text{or} \qquad \sin^{-1} x,$$

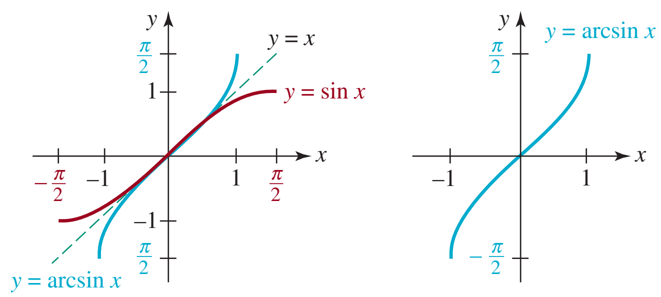
and are read **arcsine of** $\boldsymbol{x}$ and **inverse sine of** $\boldsymbol{x}$, respectively.



(a) Not a one-to-one function (b) A one-to-one function

FIGURE 3.5.1 Restricting the domain of $y = \sin x$ to produce a one-to-one function

In **FIGURE 3.5.2(a)** we have reflected the portion of the graph of $y = \sin x$ on the interval $[-\pi/2, \pi/2]$ (the red graph in Figure 3.5.1(b)) about the line $y = x$ to obtain the graph of $y = \arcsin x$ (in blue). For clarity, we have reproduced this blue graph in Figure 3.5.2(b). As this curve shows, the domain of the arcsine function is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$.



(a) (b)

FIGURE 3.5.2 Graph of $y = \arcsin x$ is the blue curve

DEFINITION 3.5.1 Arcsine Function

The **arcsine function**, or **inverse sine function**, is defined by

$$y = \arcsin x \qquad \text{if and only if} \qquad x = \sin y, \tag{1}$$

where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$.

In other words:

The arcsine of the number x is that number y (or radian-measured angle) between $-\pi/2$ and $\pi/2$ whose sine is x.

When using the notation $\sin^{-1}x$ it is important to realize that "–1" is not an exponent; rather, it denotes an inverse function. The notation $\arcsin x$ has an advantage over the notation $\sin^{-1}x$ in that there is no "-1" and hence no potential for misinterpretation; moreover, the prefix "arc" refers to an angle—*the* angle whose sine is x. But since $y = \arcsin x$ and $y = \sin^{-1}x$ are used interchangeably in calculus and in applications, we will continue to alternate their use so that you become comfortable with both notations.

◀ **Note of Caution:** $(\sin x)^{-1} = \dfrac{1}{\sin x} \neq \sin^{-1}x$

EXAMPLE 1 Evaluating the Inverse Sine Function

Find **(a)** $\arcsin\frac{1}{2}$, **(b)** $\sin^{-1}(-\frac{1}{2})$, and **(c)** $\sin^{-1}(-1)$.

Solution **(a)** If we let $y = \arcsin\frac{1}{2}$, then by (1) we must find the number y (or radian-measured angle) that satisfies $\sin y = \frac{1}{2}$ *and* $-\pi/2 \le y \le \pi/2$. Since $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ satisfies the inequality $-\pi/2 \le y \le \pi/2$ it follows that $y = \pi/6$.

(b) If we let $y = \sin^{-1}(-\frac{1}{2})$, then $\sin y = -\frac{1}{2}$. Since we must choose y such that $-\pi/2 \le y \le \pi/2$, we find that $y = -\pi/6$.

(c) Letting $y = \sin^{-1}(-1)$, we have that $\sin y = -1$ and $-\pi/2 \le y \le \pi/2$.

Hence $y = -\pi/2$. ≡

In parts (b) and (c) of Example 1 we were careful to choose y so that $-\pi/2 \le y \le \pi/2$. For example, it is a common error to think that because $\sin(3\pi/2) = -1$, then necessarily $\sin^{-1}(-1)$ can be taken to be $3\pi/2$. Remember: If $y = \sin^{-1}x$, then y is subject to the restriction $-\pi/2 \le y \le \pi/2$ and $3\pi/2$ does not satisfy this inequality.

◀ Read this paragraph several times.

EXAMPLE 2 Evaluating a Composition

Without using a calculator, find $\tan(\sin^{-1}\frac{1}{4})$.

Solution We must find the tangent of the angle of t radians with sine equal to $\frac{1}{4}$, that is, $\tan t$, where $t = \sin^{-1}\frac{1}{4}$. The angle t is shown in **FIGURE 3.5.3**. Since

$$\tan t = \frac{\sin t}{\cos\ t} = \frac{\frac{1}{4}}{\cos t},$$

we want to determine the value of $\cos t$. From Figure 3.5.3 and the Pythagorean identity $\sin^2 t + \cos^2 t = 1$, we see that

$$\left(\frac{1}{4}\right)^2 + \cos^2 t = 1 \qquad \text{or} \qquad \cos t = \frac{\sqrt{15}}{4}.$$

Hence we have

$$\tan t = \frac{1/4}{\sqrt{15}/4} = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15},$$

and so

$$\tan\left(\sin^{-1}\frac{1}{4}\right) = \tan t = \frac{\sqrt{15}}{15}.$$ ≡

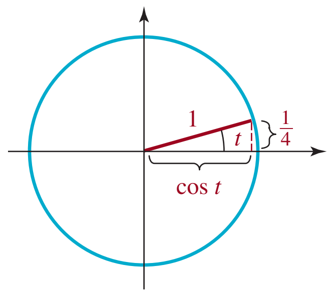


FIGURE 3.5.3 The angle $x = \sin^{-1}\frac{1}{4}$ in Example 2

□ **Arccosine Function** If we restrict the domain of the cosine function to the closed interval $[0, \pi]$, the resulting function is one-to-one and thus has an inverse. We denote this inverse by

$$\arccos x \qquad \text{or} \qquad \cos^{-1} x,$$

which gives us the following definition.

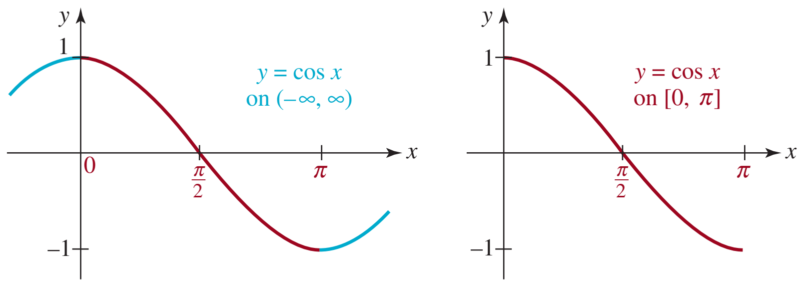
DEFINITION 3.5.2 Arccosine Function

The **arccosine function**, or **inverse cosine function**, is defined by

$$y = \arccos x \quad \text{if and only if} \quad x = \cos y, \tag{2}$$

where $-1 \leq x \leq 1$ and $0 \leq y \leq \pi$.

The graphs shown in FIGURE 3.5.4 illustrate how the function $y = \cos x$ restricted to the interval $[0, \pi]$ becomes a one-to-one function. The inverse of the function shown in Figure 3.5.4(b) is $y = \arccos x$.



(a) Not a one-to-one function (b) A one-to-one function

FIGURE 3.5.4 Restricting the domain of $y = \cos x$ to produce a one-to-one function

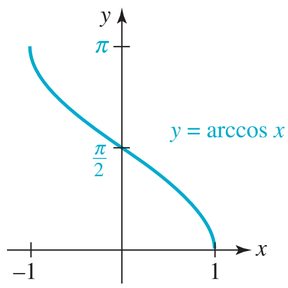


FIGURE 3.5.5 Graph of $y = \arccos x$

By reflecting the graph of the one-to-one function in Figure 3.5.4(b) in the line $y = x$ we obtain the graph of $y = \arccos x$ shown in FIGURE 3.5.5.

Note that the figure clearly shows that the domain and range of $y = \arccos x$ are $[-1, 1]$ and $[0, \pi]$, respectively.

EXAMPLE 3 Evaluating the Inverse Cosine Function

Find **(a)** $\arccos\left(\sqrt{2}/2\right)$ and **(b)** $\cos^{-1}(-\sqrt{3}/2)$.

Solution **(a)** If we let $y = \arccos\left(\sqrt{2}/2\right)$, then $\cos y = \sqrt{2}/2$ and $0 \leq y \leq \pi$. Thus $y = \pi/4$.

(b) Letting $y = \cos^{-1}(-\sqrt{3}/2)$, we have that $\cos y = -\sqrt{3}/2$, and we must find y such that $0 \leq y \leq \pi$. Therefore, $y = 5\pi/6$ since $\cos(5\pi/6) = -\sqrt{3}/2$. ≡

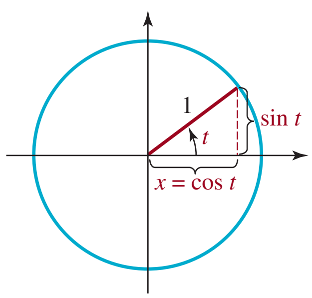


FIGURE 3.5.6 The angle $t = \cos^{-1}x$ in Example 4

EXAMPLE 4 Evaluating the Compositions of Functions

Write $\sin(\cos^{-1}x)$ as an algebraic expression in x.

Solution In FIGURE 3.5.6 we have constructed an angle of t radians with cosine equal to x. Then $t = \cos^{-1}x$, or $x = \cos t$, where $0 \leq t \leq \pi$. Now to find $\sin(\cos^{-1}x) = \sin t$, we use the identity $\sin^2 t + \cos^2 t = 1$. Thus

$$\begin{aligned} \sin^2 t + x^2 &= 1 \\ \sin^2 t &= 1 - x^2 \\ \sin t &= \sqrt{1 - x^2} \\ \sin(\cos^{-1} x) &= \sqrt{1 - x^2}. \end{aligned}$$

We use the positive square root of $1 - x^2$, since the range of $\cos^{-1} x$ is $[0, \pi]$, and the sine of an angle t in the first or second quadrant is positive. ≡

Arctangent Function If we restrict the domain of $\tan x$ to the open interval $(-\pi/2, \pi/2)$, then the resulting function is one-to-one and thus has an inverse. This inverse is denoted by

$$\arctan x \qquad \text{or} \qquad \tan^{-1} x.$$

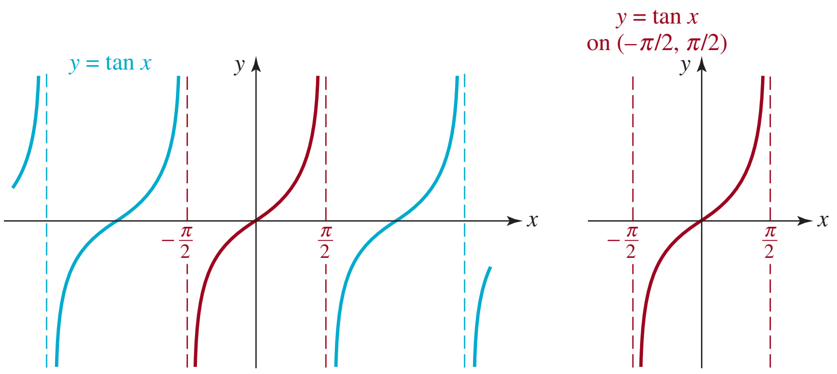
DEFINITION 3.5.3 Arctangent Function

The **arctangent**, or **inverse tangent**, function is defined by

$$y = \arctan x \qquad \text{if and only if} \qquad x = \tan y, \tag{3}$$

where $-\infty < x < \infty$ and $-\pi/2 < y < \pi/2$.

The graphs shown in FIGURE 3.5.7 illustrate how the function $y = \tan x$ restricted to the open interval $(-\pi/2, \pi/2)$ becomes a one-to-one function.



(a) Not a one-to-one function

(b) A one-to-one function

FIGURE 3.5.7 Restricting the domain of $y = \tan x$ to produce a one-to-one function

By reflecting the graph of the one-to-one function in Figure 3.5.7(b) in the line $y = x$ we obtain the graph of $y = \arctan x$ shown in FIGURE 3.5.8. We see in the figure that the domain and range of $y = \arctan x$ are, in turn, the intervals $(-\infty, \infty)$ and $(-\pi/2, \pi/2)$.

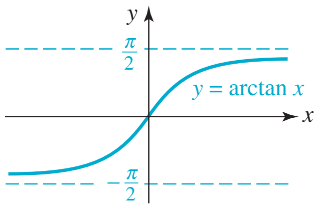


FIGURE 3.5.8 Graph of $y = \arctan x$

EXAMPLE 5 Evaluating the Inverse Tangent Function

Find $\tan^{-1}(-1)$.

Solution If $\tan^{-1}(-1) = y$, then $\tan y = -1$, where $-\pi/2 < y < \pi/2$. It follows that $\tan^{-1}(-1) = y = -\pi/4$. ≡

EXAMPLE 6 Evaluating the Compositions of Functions

Without using a calculator, find $\sin\left(\arctan\left(-\frac{5}{3}\right)\right)$.

Solution If we let $t = \arctan\left(-\frac{5}{3}\right)$, then $\tan t = -\frac{5}{3}$. The Pythagorean identity $1 + \tan^2 t = \sec^2 t$ can be used to find $\sec t$:

$$1 + \left(-\frac{5}{3}\right)^2 = \sec^2 t$$

$$\sec t = \sqrt{\frac{25}{9} + 1} = \sqrt{\frac{34}{9}} = \frac{\sqrt{34}}{3}.$$

In the preceding line we take the positive square root because $t = \arctan\left(-\frac{5}{3}\right)$ is in the interval $(-\pi/2, \pi/2)$ (the range of the arctangent function) and the secant of an angle t in the first or fourth quadrant is positive. Also, from $\sec t = \sqrt{34}/3$ we find the value of $\cos t$ from the reciprocal identity:

$$\cos t = \frac{1}{\sec t} = \frac{1}{\sqrt{34}/3} = \frac{3}{\sqrt{34}}.$$

Finally, we can use the identity $\tan t = \sin t/\cos t$ in the form $\sin t = \tan t \cos t$ to compute $\sin\left(\arctan\left(-\frac{5}{3}\right)\right)$. It follows that

$$\sin t = \tan t \cos t = \left(-\frac{5}{3}\right)\left(\frac{3}{\sqrt{34}}\right) = -\frac{5}{\sqrt{34}}.$$

□ **Properties of the Inverses** Recall from Section 1.9 that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ hold for any function f and its inverse under suitable restrictions on x. Thus for the inverse trigonometric functions, we have the following properties.

THEOREM 3.5.1 Properties of the Inverse Trigonometric Functions

(i)	$\arcsin(\sin x) = \sin^{-1}(\sin x) = x$	if	$-\pi/2 \le x \le \pi/2$
(ii)	$\sin(\arcsin x) = \sin(\sin^{-1} x) = x$	if	$-1 \le x \le 1$
(iii)	$\arccos(\cos x) = \cos^{-1}(\cos x) = x$	if	$0 \le x \le \pi$
(iv)	$\cos(\arccos x) = \cos(\cos^{-1} x) = x$	if	$-1 \le x \le 1$
(v)	$\arctan(\tan x) = \tan^{-1}(\tan x) = x$	if	$-\pi/2 < x < \pi/2$
(vi)	$\tan(\arctan x) = \tan(\tan^{-1} x) = x$	if	$-\infty < x < \infty$

EXAMPLE 7 Using the Inverse Properties

Without using a calculator, evaluate:

(a) $\sin^{-1}\left(\sin\frac{\pi}{12}\right)$ **(b)** $\cos\left(\cos^{-1}\frac{1}{3}\right)$ **(c)** $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Solution In each case we use the properties of the inverse trigonometric functions given in Theorem 3.5.1.

(a) Because $\pi/12$ satisfies $-\pi/2 \le x \le \pi/2$ it follows from property (i) that

$$\sin^{-1}\left(\sin\frac{\pi}{12}\right) = \frac{\pi}{12}.$$

(b) By property (*iv*), $\cos\left(\cos^{-1}\frac{1}{3}\right) = \frac{1}{3}$.

(c) In this case we *cannot* apply property (*v*), since $3\pi/4$ is not in the interval $(-\pi/2, \pi/2)$. If we first evaluate $\tan(3\pi/4) = -1$, then we have

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \overset{\text{see Example 5}}{\overset{\downarrow}{\tan^{-1}(-1)}} = -\frac{\pi}{4}. \qquad \equiv$$

In the next section we illustrate how inverse trigonometric functions can be used to solve trigonometric equations.

□ **Postscript—The Other Inverse Trig Functions** The functions $\cot x$, $\sec x$, and $\csc x$ also have inverses when their domains are suitably restricted. See Problems 49–51 in Exercises 3.5. Because these functions are not used as often as arctan, arccos, and arcsin, most scientific calculators do not have keys for them. However, any calculator that computes arcsin, arccos, and arctan can be used to obtain values for **arccsc**, **arcsec**, and **arccot**. Unlike the fact that $\sec x = 1/\cos x$, we note that $\sec^{-1}x \neq 1/\cos^{-1}x$; rather, $\sec^{-1}x = \cos^{-1}(1/x)$ for $|x| \geq 1$. Similar relationships hold for $\csc^{-1}x$ and $\cot^{-1}x$. See Problems 56–58 in Exercises 3.5.

3.5 Exercises

Answers to selected odd-numbered problems begin on page ANS-10.

In Problems 1–14, find the exact value of the given trigonometric expression. Do not use a calculator.

1. $\sin^{-1}0$

2. $\tan^{-1}\sqrt{3}$

3. $\arccos(-1)$

4. $\arcsin\dfrac{\sqrt{3}}{2}$

5. $\arccos\frac{1}{2}$

6. $\arctan(-\sqrt{3})$

7. $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

8. $\cos^{-1}\dfrac{\sqrt{3}}{2}$

9. $\tan^{-1}1$

10. $\sin^{-1}\dfrac{\sqrt{2}}{2}$

11. $\arctan\left(-\dfrac{\sqrt{3}}{3}\right)$

12. $\arccos\left(-\frac{1}{2}\right)$

13. $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

14. $\arctan 0$

In Problems 15–32, find the exact value of the given trigonometric expression. Do not use a calculator.

15. $\sin\left(\cos^{-1}\frac{3}{5}\right)$

16. $\cos\left(\sin^{-1}\frac{1}{3}\right)$

17. $\tan\left(\arccos\left(-\frac{2}{3}\right)\right)$

18. $\sin\left(\arctan\frac{1}{4}\right)$

19. $\cos(\arctan(-2))$

20. $\tan\left(\sin^{-1}\left(-\frac{1}{6}\right)\right)$

21. $\csc\left(\sin^{-1}\frac{3}{5}\right)$

22. $\sec(\tan^{-1}4)$

23. $\sin\left(\sin^{-1}\frac{1}{5}\right)$

24. $\cos\left(\cos^{-1}\left(-\frac{4}{5}\right)\right)$

25. $\tan(\tan^{-1}1.2)$

26. $\sin(\arcsin 0.75)$

27. $\arcsin\left(\sin\dfrac{\pi}{16}\right)$

28. $\arccos\left(\cos\dfrac{2\pi}{3}\right)$

29. $\tan^{-1}(\tan\pi)$

30. $\sin^{-1}\left(\sin\dfrac{5\pi}{6}\right)$

31. $\cos^{-1}\left(\cos\left(-\dfrac{\pi}{4}\right)\right)$

32. $\arctan\left(\tan\dfrac{\pi}{7}\right)$

In Problems 33–40, write the given expression as an algebraic expression in x.

33. $\sin(\tan^{-1}x)$

34. $\cos(\tan^{-1}x)$

35. $\tan(\arcsin x)$

36. $\sec(\arccos x)$

37. $\cot(\sin^{-1}x)$

38. $\cos(\sin^{-1}x)$

39. $\csc(\arctan x)$

40. $\tan(\arccos x)$

In Problems 41–48, sketch the graph of the given function.

41. $y = \arctan|x|$

42. $y = \dfrac{\pi}{2} - \arctan x$

43. $y = |\arcsin x|$

44. $y = \sin^{-1}(x + 1)$

45. $y = 2\cos^{-1}x$

46. $y = \cos^{-1}2x$

47. $y = \arccos(x - 1)$

48. $y = \cos(\arcsin x)$

49. The **arccotangent** function can be defined by $y = \operatorname{arccot} x$ (or $y = \cot^{-1}x$) if and only if $x = \cot y$, where $0 < y < \pi$. Graph $y = \operatorname{arccot} x$, and give the domain and the range of this function.

50. The **arccosecant** function can be defined by $y = \operatorname{arccsc} x$ (or $y = \csc^{-1}x$) if and only if $x = \csc y$, where $-\pi/2 \le y \le \pi/2$ and $y \ne 0$. Graph $y = \operatorname{arccsc} x$, and give the domain and the range of this function.

51. One definition of the **arcsecant** function is $y = \operatorname{arcsec} x$ (or $y = \sec^{-1}x$) if and only if $x = \sec y$, where $0 \le y \le \pi$ and $y \ne \pi/2$. (See Problem 52 for an alternative definition.) Graph $y = \operatorname{arcsec} x$, and give the domain and the range of this function.

52. An alternative definition of the arcsecant function can be made by restricting the domain of the secant function to $[0, \pi/2) \cup [\pi, 3\pi/2)$. Under this restriction, define the arcsecant function. Graph $y = \operatorname{arcsec} x$, and give the domain and the range of this function.

53. Using the definition of the arccotangent function from Problem 49, for what values of x is it true that **(a)** $\cot(\operatorname{arccot} x) = x$ and **(b)** $\operatorname{arccot}(\cot x) = x$?

54. Using the definition of the arccosecant function from Problem 50, for what values of x is it true that **(a)** $\csc(\operatorname{arccsc} x) = x$ and **(b)** $\operatorname{arccsc}(\csc x) = x$?

55. Using the definition of the arcsecant function from Problem 51, for what values of x is it true that **(a)** $\sec(\operatorname{arcsec} x) = x$ and **(b)** $\operatorname{arcsec}(\sec x) = x$?

56. Verify that $\operatorname{arccot} x = \dfrac{\pi}{2} - \arctan x$, for all real numbers x.

57. Verify that $\operatorname{arccsc} x = \arcsin(1/x)$ for $|x| \ge 1$.

58. Verify that $\operatorname{arcsec} x = \arccos(1/x)$ for $|x| \ge 1$.

In Problems 59–64, use the results of Problems 56–58 and a calculator to find the indicated value.

59. $\cot^{-1}0.75$

60. $\csc^{-1}(-1.3)$

61. $\operatorname{arccsc}(-1.5)$

62. $\operatorname{arccot}(-0.3)$

63. $\operatorname{arcsec}(-1.2)$

64. $\sec^{-1}2.5$

Miscellaneous Applications

65. Projectile Motion The departure angle θ for a bullet to hit a target at a distance R (assuming that the target and the gun are at the same height) satisfies

$$R = \frac{v_0^2 \sin 2\theta}{g},$$

where v_0 is the muzzle velocity and g is the acceleration due to gravity. If the target is 800 ft from the gun and the muzzle velocity is 200 ft/s, find the departure angle. Use $g = 32$ ft/s^2. [*Hint*: There are two solutions.]

66. Olympic Sports For the Olympic event, the hammer throw, it can be shown that the maximum distance is achieved for the release angle θ (measured from the horizontal) that satisfies

$$\cos 2\theta = \frac{gh}{v_0^2 + gh},$$

where h is the height of the hammer above the ground at release, v_0 is the initial velocity, and g is the acceleration due to gravity. For $v_0 = 13.7$ m/s and $h = 2.25$ m, find the optimal release angle. Use $g = 9.81$ m/s^2.

67. Highway Design In the design of highways and railroads, curves are banked to provide centripetal force for safety. The optimal banking angle θ is given by $\tan\theta = v^2/Rg$, where v is the speed of the vehicle, R is the radius of the curve, and g is the acceleration due to gravity. See FIGURE 3.5.9. As the formula indicates, for a given radius there is no one correct angle for all speeds. Consequently, curves are banked for the average speed of the traffic over them. Find the correct banking angle for a curve of radius 600 ft on a country road where speeds average 30 mi/h. Use $g = 32$ ft/s^2. [*Hint*: Use consistent units.]

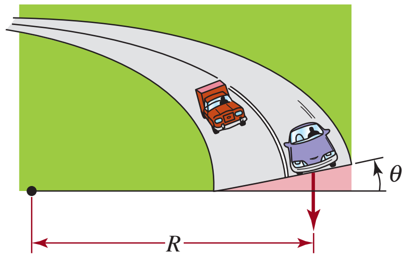


FIGURE 3.5.9 Banked curve in Problem 67

68. Highway Design—Continued If μ is the coefficient of friction between the car and the road, then the maximum velocity v_m that a car can travel around a curve without slipping is given by $v_m^2 = gR\tan(\theta + \tan^{-1}\mu)$, where θ is the banking angle of the curve. Find v_m for the country road in Problem 67 if $\mu = 0.26$.

69. Geology Viewed from the side, a volcanic cinder cone usually looks like an isosceles trapezoid. See FIGURE 3.5.10. Studies of cinder cones less than 50,000 years old indicate that cone height H_{co} and crater width W_{cr} are related to the cone width W_{co} by the equations $H_{co} = 0.18W_{co}$ and $W_{cr} = 0.40W_{co}$. If $W_{co} = 1.00$, use these equations to determine the base angle ϕ of the trapezoid in Figure 3.5.10.

Volcanic cone

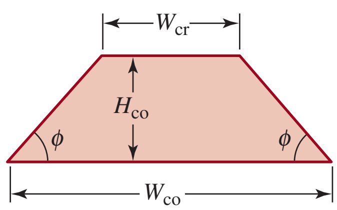


FIGURE 3.5.10 Volcanic cinder cone in Problem 69

For Discussion

70. Using a calculator set in radian mode, evaluate arctan (tan 1.8), arccos (cos 1.8), and arcsin (sin 1.8). Explain the results.

71. Using a calculator set in radian mode, evaluate $\tan^{-1}(\tan(-1))$, $\cos^{-1}(\cos(-1))$, and $\sin^{-1}(\sin(-1))$. Explain the results.

72. In Section 3.2 we saw that the graphs of $y = \sin x$ and $y = \cos x$ are related by shifting and reflecting. Justify the identity

$$\arcsin x + \arccos x = \frac{\pi}{2},$$

for all x in $[-1, 1]$, by finding a similar relationship between the graphs of $y = \arcsin x$ and $y = \arccos x$.

73. With a calculator set in radian mode determine which of the following inverse trigonometric evaluations result in an error message: **(a)** $\sin^{-1}(-2)$, **(b)** $\cos^{-1}(-2)$, **(c)** $\tan^{-1}(-2)$. Explain.

74. Discuss: Can any periodic function be one-to-one?

75. Show that $\arcsin\frac{3}{5} + \arcsin\frac{5}{13} = \arcsin\frac{56}{65}$. [*Hint*: See (7) of Section 3.4.]

3.6 Trigonometric Equations

Introduction In Section 3.4 we considered trigonometric identities, which are equations involving trigonometric functions that are satisfied by all values of the variable for which both sides of the equality are defined. In this section we examine **conditional trigonometric equations**, that is, equations that are true for only certain values of the variable. We discuss techniques for finding those values of the variable (if any) that satisfy the equation.

We begin by considering the problem of finding all real numbers x that satisfy $\sin x = \sqrt{2}/2$. Interpreted as the x-coordinates of the points of intersection of the graphs of $y = \sin x$ and $y = \sqrt{2}/2$, FIGURE 3.6.1 shows that there exists infinitely many solutions of the equation $\sin x = \sqrt{2}/2$:

$$\ldots, -\frac{7\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \ldots \tag{1}$$

$$\ldots, -\frac{5\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{4}, \frac{19\pi}{4}, \ldots. \tag{2}$$

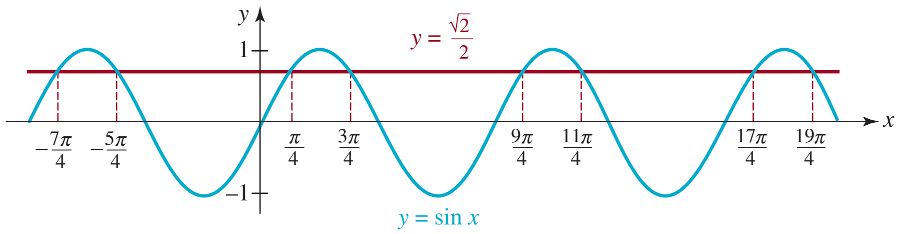


FIGURE 3.6.1 Graphs of $y = \sin x$ and $y = \frac{\sqrt{2}}{2}$

Note that in each of the lists (1) and (2), two successive solutions differ by $2\pi = 8\pi/4$. This is a consequence of the periodicity of the sine function. It is common for trigonometric equations to have an infinite number of solutions because of the periodicity of the trigonometric functions. In general, to obtain solutions of an equation such as $\sin x = \sqrt{2}/2$, it is more convenient to use a unit circle and reference angles rather than a graph of the trigonometric function. We illustrate this approach in the following example.

EXAMPLE 1 Using the Unit Circle

Find all real numbers x satisfying $\sin x = \sqrt{2}/2$.

Solution If $\sin x = \sqrt{2}/2$, the reference angle for x is $\pi/4$ radian. Since the value of $\sin x$ is positive, the terminal side of the angle x lies in either the first or second quadrant. Thus, as shown in FIGURE 3.6.2, the only solutions between 0 and 2π are

$$x = \frac{\pi}{4} \quad \text{and} \quad x = \frac{3\pi}{4}.$$

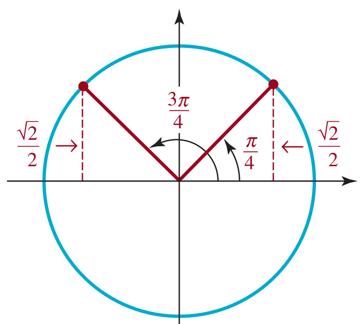


FIGURE 3.6.2 Unit circle in Example 1

Since the sine function is periodic with period 2π, all of the remaining solutions can be obtained by adding integer multiples of 2π to these solutions. The two solutions are

$$x = \frac{\pi}{4} + 2n\pi \quad \text{and} \quad x = \frac{3\pi}{4} + 2n\pi, \tag{3}$$

where n is an integer The numbers that you see in (1) and (2) correspond, respectively, to letting $n = -1$, $n = 0$, $n = 1$, and $n = 2$ in the first and second formulas in (3). ≡

When we are faced with a more complicated equation, such as

$$4\sin^2 x - 8\sin x + 3 = 0,$$

the basic approach is to solve for a single trigonometric function (in this case, it would be $\sin x$) by using methods similar to those for solving algebraic equations.

EXAMPLE 2 Solving a Trigonometric Equation by Factoring

Find all solutions of $4\sin^2 x - 8\sin x + 3 = 0$.

Solution We first observe that this is a quadratic equation in $\sin x$, and that it factors as

$$(2\sin x - 3)(2\sin x - 1) = 0.$$

This implies that either

$$\sin x = \frac{3}{2} \quad \text{or} \quad \sin x = \frac{1}{2}.$$

The first equation has no solution since $|\sin x| \leq 1$. As we see in **FIGURE 3.6.3** the two angles between 0 and 2π for which $\sin x$ equals $\frac{1}{2}$ are

$$x = \frac{\pi}{6} \quad \text{and} \quad x = \frac{5\pi}{6}.$$

Therefore, by the periodicity of the sine function, the solutions are

$$x = \frac{\pi}{6} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{6} + 2n\pi,$$

where n is an integer. ≡

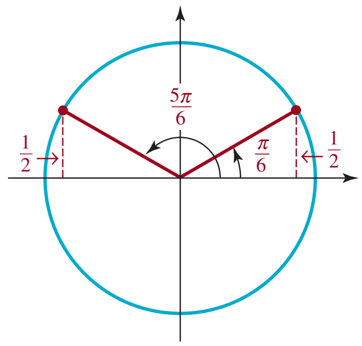


FIGURE 3.6.3 Unit circle in Example 2

EXAMPLE 3 Checking for Lost Solutions

Find all solutions of

$$\sin x = \cos x. \tag{4}$$

Solution In order to work with a single trigonometric function, we divide both sides of the equation by $\cos x$ to obtain

$$\tan x = 1. \tag{5}$$

Equation (5) is equivalent to (4) *provided* that $\cos x \neq 0$. We observe that if $\cos x = 0$, then as we have seen in Section 3.2, $x = (2n + 1)\pi/2 = \pi/2 + n\pi$, for n an integer. By the sum formula for the sine,

◀ $\cos 0 = 1$, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, and so on. In general, $\cos n\pi = (-1)^n$, where n is an integer.

$$\sin\left(\frac{\pi}{2} + n\pi\right) = \sin\frac{\pi}{2}\underbrace{\cos n\pi}_{\text{see (7) of Section 3.4}} + \cos\frac{\pi}{2}\,\sin n\pi = \underbrace{(-1)^n}_{(-1)^n} \neq \underbrace{0}_{0},$$

we see that these values of x do not satisfy the original equation. Thus we will find *all* the solutions to (4) by solving equation (5).

Now $\tan x = 1$ implies that the reference angle for x is $\pi/4$ radian. Since $\tan x = 1 > 0$, the terminal side of the angle of x radians can lie either in the first or in the third quadrant, as shown in **FIGURE 3.6.4**. Thus the solutions are

$$x = \frac{\pi}{4} + 2n\pi \quad \text{and} \quad x = \frac{5\pi}{4} + 2n\pi,$$

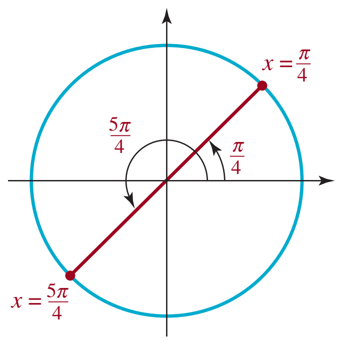


FIGURE 3.6.4 Unit circle in Example 3

where n is an integer. We can see from Figure 3.6.4 that these two sets of numbers can be written more compactly as

This also follows from the fact that $\tan x$ is π-periodic.

$$x = \frac{\pi}{4} + n\pi,$$

where n is an integer.

Losing Solutions When solving an equation, if you divide by an expression containing a variable, you may lose some solutions of the original equation. For example, in algebra a common mistake in solving equations such as $x^2 = x$ is to divide by x to obtain $x = 1$. But by writing $x^2 = x$ as $x^2 - x = 0$ or $x(x - 1) = 0$ we see that in fact $x = 0$ or $x = 1$. To prevent the loss of a solution you must determine the values that make the expression zero and check to see whether they are solutions of the original equation. Note that in Example 3, when we divided by $\cos x$, we took care to check that no solutions were lost.

Whenever possible, it is preferable to avoid dividing by a variable expression. As illustrated with the algebraic equation $x^2 = x$, this can frequently be accomplished by collecting all nonzero terms on one side of the equation and then factoring (something we could not do in Example 3). Example 4 illustrates this technique.

EXAMPLE 4 Solving a Trigonometric Equation by Factoring

Solve
$$2\sin x\cos^2 x = -\frac{\sqrt{3}}{2}\cos x. \tag{6}$$

Solution To avoid dividing by $\cos x$, we write the equation as

$$2\sin x\cos^2 x + \frac{\sqrt{3}}{2}\cos x = 0$$

and factor:
$$\cos x\left(2\sin x\cos x + \frac{\sqrt{3}}{2}\right) = 0.$$

Thus either

$$\cos x = 0 \quad \text{or} \quad 2\sin x\cos x + \frac{\sqrt{3}}{2} = 0.$$

Since the cosine is zero for all odd multiples of $\pi/2$, the solutions of $\cos x = 0$ are

$$x = (2n + 1)\frac{\pi}{2} = \frac{\pi}{2} + n\pi,$$

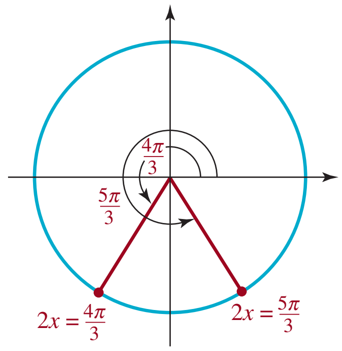


FIGURE 3.6.5 Unit circle in Example 4

where n is an integer.

In the second equation we replace $2\sin x\cos x$ by $\sin 2x$ from the double-angle formula for the sine function to obtain an equation with a single trigonometric function:

See (15) in Section 3.4.

$$\sin 2x + \frac{\sqrt{3}}{2} = 0 \quad \text{or} \quad \sin 2x = -\frac{\sqrt{3}}{2}.$$

Thus the reference angle for $2x$ is $\pi/3$. Since the sine is negative, the angle $2x$ must be in either the third quadrant or the fourth quadrant. As **FIGURE 3.6.5** illustrates, either

$$2x = \frac{4\pi}{3} + 2n\pi \quad \text{or} \quad 2x = \frac{5\pi}{3} + 2n\pi.$$

Dividing by 2 gives

$$x = \frac{2\pi}{3} + n\pi \qquad \text{or} \qquad x = \frac{5\pi}{6} + n\pi.$$

Therefore, all solutions of (6) are

$$x = \frac{\pi}{2} + n\pi, \quad x = \frac{2\pi}{3} + n\pi, \quad \text{and} \quad x = \frac{5\pi}{6} + n\pi,$$

where n is an integer. ≡

In Example 4 had we simplified the equation by dividing by $\cos x$ and not checked to see whether the values of x for which $\cos x = 0$ satisfied equation (6), we would have lost the solutions $x = \pi/2 + n\pi$, where n is an integer.

EXAMPLE 5 Using a Trigonometric Identity

Solve $3\cos^2 x - \cos 2x = 1$.

Solution We observe that the given equation involves both the cosine of x and the cosine of $2x$. Consequently, we use the double-angle formula for the cosine in the form

$$\cos 2x = 2\cos^2 x - 1$$ ← See (16) of Section 3.4

to replace the equation by an equivalent equation that involves $\cos x$ only. We find that

$$3\cos^2 x - (2\cos^2 x - 1) = 1 \qquad \text{becomes} \qquad \cos^2 x = 0.$$

Therefore, $\cos x = 0$, and the solutions are

$$x = (2n + 1)\frac{\pi}{2} = \frac{\pi}{2} + n\pi,$$

where n is an integer. ≡

So far in this section we have viewed the variable in the trigonometric equation as representing either a real number or an angle measured in radians. If the variable represents an angle measured in degrees, the technique for solving is the same.

EXAMPLE 6 Equation When the Angle Is in Degrees

Solve $\cos 2\theta = -\frac{1}{2}$, where θ is an angle measured in degrees.

Solution Since $\cos 2\theta = -\frac{1}{2}$, the reference angle for 2θ is $60°$ and the angle 2θ must be in either the second or the third quadrant. FIGURE 3.6.6 illustrates that either $2\theta = 120°$ or $2\theta = 240°$. Any angle that is coterminal with one of these angles will also satisfy $\cos 2\theta = -\frac{1}{2}$. These angles are obtained by adding any integer multiple of $360°$ to $120°$ or to $240°$:

$$2\theta = 120° + 360°n \qquad \text{or} \qquad 2\theta = 240° + 360°n,$$

where n is an integer. Dividing by 2 the last line yields the two solutions

$$\theta = 60° + 180°n \qquad \text{and} \qquad \theta = 120° + 180°n.$$ ≡

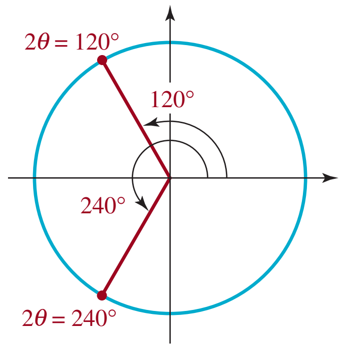


FIGURE 3.6.6 Unit circle in Example 6

□ Extraneous Solutions

The next example shows that by squaring a trigonometric equation we may introduce extraneous solutions. In other words, the resulting equation after squaring may *not* be equivalent to the original.

EXAMPLE 7 Extraneous Roots

Find all solutions of $1 + \tan\alpha = \sec\alpha$, where α is an angle measured in degrees.

Solution The equation does not factor, but we see that if we square both sides, we can use a fundamental identity to obtain an equation involving a single trigonometric function:

$$\begin{aligned}(1 + \tan\alpha)^2 &= (\sec\alpha)^2\\ 1 + 2\tan\alpha + \tan^2\alpha &= \sec^2\alpha \qquad \leftarrow \text{See (2) of Section 3.4.}\\ 1 + 2\tan\alpha + \tan^2\alpha &= 1 + \tan^2\alpha\\ 2\tan\alpha &= 0\\ \tan\alpha &= 0.\end{aligned}$$

The values of α in $[0°, 360°)$ for which $\tan\alpha = 0$ are

$$\alpha = 0° \qquad \text{and} \qquad \alpha = 180°.$$

Since we squared each side of the original equation, we may have introduced extraneous solutions. Therefore, it is important that we check all solutions in the original equation. Substituting $\alpha = 0°$ into $1 + \tan\alpha = \sec\alpha$, we obtain the *true* statement $1 + 0 = 1$. But after substituting $\alpha = 180°$, we obtain the *false* statement $1 + 0 = -1$. Therefore, $180°$ is an extraneous solution and $\alpha = 0°$ is the only solution in the interval $[0°, 360°)$. Thus, all the solutions of the equation are given by

$$\alpha = 0° + 360°n = 360°n,$$

where n is an integer. For $n \neq 0$, these are the angles that are coterminal with $0°$. ≡

Recall from Section 1.5 that to find the x-intercepts of the graph of a function $y = f(x)$ we find the zeros of f, that is, we must solve the equation $f(x) = 0$. The following example makes use of this fact.

EXAMPLE 8 Intercepts of a Graph

Find the first three x-intercepts of the graph of $f(x) = \sin 2x\cos x$ on the positive x-axis.

Solution We must solve $f(x) = 0$, that is, $\sin 2x\cos x = 0$. It follows that either $\sin 2x = 0$ or $\cos x = 0$.

From $\sin 2x = 0$, we obtain $2x = n\pi$, where n is an integer, or $x = n\pi/2$, where n is an integer. From $\cos x = 0$, we find $x = \pi/2 + n\pi$, where n is an integer. Then for $n = 2$, $x = n\pi/2$ gives $x = \pi$, whereas for $n = 0$ and $n = 1$, $x = \pi/2 + n\pi$ gives $x = \pi/2$ and $x = 3\pi/2$, respectively. Thus the first three x-intercepts on the positive x-axis are $(\pi/2, 0)$, $(\pi, 0)$, and $(3\pi/2, 0)$. ≡

☐ **Using Inverse Functions** So far all of the trigonometric equations have had solutions that were related by reference angles to the special angles 0, $\pi/6$, $\pi/4$, $\pi/3$, or $\pi/2$. If this is not the case, we will see in the next example how to use inverse trigonometric functions and a calculator to find solutions.

EXAMPLE 9 Solving Equations Using Inverse Functions

Find the solutions of $4\cos^2 x - 3\cos x - 2 = 0$ in the interval $[0, \pi]$.

Solution We recognize that this is a quadratic equation in $\cos x$. Since the left-hand side of the equation does not readily factor, we apply the quadratic formula to obtain

$$\cos x = \frac{3 \pm \sqrt{41}}{8}.$$

At this point we can discard the value $(3 + \sqrt{41})/8 \approx 1.18$, because $\cos x$ cannot be greater than 1. We then use the inverse cosine function (and the aid of a calculator) to solve the remaining equation:

$$\cos x = \frac{3 - \sqrt{41}}{8} \quad \text{which implies} \quad x = \cos^{-1}\left(\frac{3 - \sqrt{41}}{8}\right) \approx 2.01. \quad \equiv$$

Of course in Example 9, had we attempted to compute $\cos^{-1}[(3 + \sqrt{41})/8]$ with a calculator, we would have received an error message.

3.6 Exercises

Answers to selected odd-numbered problems begin on page ANS-11.

In Problems 1–6, find all solutions of the given trigonometric equation if x represents an angle measured in radians.

1. $\sin x = \sqrt{3}/2$
2. $\cos x = -\sqrt{2}/2$
3. $\sec x = \sqrt{2}$
4. $\tan x = -1$
5. $\cot x = -\sqrt{3}$
6. $\csc x = 2$

In Problems 7–12, find all solutions of the given trigonometric equation if x represents a real number.

7. $\cos x = -1$
8. $2\sin x = -1$
9. $\tan x = 0$
10. $\sqrt{3}\sec x = 2$
11. $-\csc x = 1$
12. $\sqrt{3}\cot x = 1$

In Problems 13–18, find all solutions of the given trigonometric equation if θ represents an angle measured in degrees.

13. $\csc\theta = 2\sqrt{3}/3$
14. $2\sin\theta = \sqrt{2}$
15. $1 + \cot\theta = 0$
16. $\sqrt{3}\sin\theta = \cos\theta$
17. $\sec\theta = -2$
18. $2\cos\theta + \sqrt{2} = 0$

In Problems 19–46, find all solutions of the given trigonometric equation if x is a real number and θ is an angle measured in degrees.

19. $\cos^2 x - 1 = 0$
20. $2\sin^2 x - 3\sin x + 1 = 0$
21. $3\sec^2 x = \sec x$
22. $\tan^2 x + (\sqrt{3} - 1)\tan x - \sqrt{3} = 0$
23. $2\cos^2\theta - 3\cos\theta - 2 = 0$
24. $2\sin^2\theta - \sin\theta - 1 = 0$
25. $\cot^2\theta + \cot\theta = 0$
26. $2\sin^2\theta + (2 - \sqrt{3})\sin\theta - \sqrt{3} = 0$
27. $\cos 2x = -1$
28. $\sec 2x = 2$
29. $2\sin 3\theta = 1$
30. $\tan 4\theta = -1$
31. $\cot(x/2) = 1$
32. $\csc(\theta/3) = -1$
33. $\sin 2x + \sin x = 0$
34. $\cos 2x + \sin^2 x = 1$
35. $\cos 2\theta = \sin\theta$
36. $\sin 2\theta + 2\sin\theta - 2\cos\theta = 2$
37. $\sin^4 x - 2\sin^2 x + 1 = 0$
38. $\tan^4\theta - 2\sec^2\theta + 3 = 0$
39. $\sec x \sin^2 x = \tan x$
40. $\dfrac{1 + \cos\theta}{\cos\theta} = 2$
41. $\sin\theta + \cos\theta = 1$
42. $\sin x + \cos x = 0$
43. $\sqrt{\dfrac{1 + 2\sin x}{2}} = 1$
44. $\sin x + \sqrt{\sin x} = 0$
45. $\cos\theta - \sqrt{\cos\theta} = 0$
46. $\cos\theta\sqrt{1 + \tan^2\theta} = 1$

In Problems 47–54, find the first three x-intercepts of the graph of the given function on the positive x-axis.

47. $f(x) = -5\sin(3x + \pi)$

48. $f(x) = 2\cos\left(x + \dfrac{\pi}{4}\right)$

49. $f(x) = 2 - \sec\dfrac{\pi}{2}x$

50. $f(x) = 1 + \cos \pi x$

51. $f(x) = \sin x + \tan x$

52. $f(x) = 1 - 2\cos\left(x + \dfrac{\pi}{3}\right)$

53. $f(x) = \sin x - \sin 2x$

54. $f(x) = \cos x + \cos 3x$
[*Hint*: Write $3x = x + 2x$.]

In Problems 55–58, by graphing determine whether the given equation has any solutions.

55. $\tan x = x$ [*Hint*: Graph $y = \tan x$ and $y = x$ on the same set of axes.]
56. $\sin x = x$
57. $\cot x - x = 0$
58. $\cos x + x + 1 = 0$

In Problems 59–64, using a inverse trigonometric function find the solutions of the given equation in the indicated interval. Round your answers to two decimal places.

59. $20\cos^2 x + \cos x - 1 = 0, \quad [0, \pi]$
60. $3\sin^2 x - 8\sin x + 4 = 0, \quad [-\pi/2, \pi/2]$
61. $\tan^2 x + \tan x - 1 = 0, \quad (-\pi/2, \pi/2)$
62. $3\sin 2x + \cos x = 0, \quad [-\pi/2, \pi/2]$
63. $5\cos^3 x - 3\cos^2 x - \cos x = 0, \quad [0, \pi]$
64. $\tan^4 x - 3\tan^2 x + 1 = 0, \quad (-\pi/2, \pi/2)$

Miscellaneous Applications

65. Isosceles Triangle From Problem 59 in Exercises 3.4, the area of the isosceles triangle with vertex angle θ as shown in Figure 3.4.4 is given by $A = \frac{1}{2}x^2\sin\theta$. If the length x is 4, what value of θ will give a triangle with area 4?

66. Circular Motion An object travels in a circular path centered at the origin with constant angular speed. The y-coordinate of the object at any time t seconds is given by $y = 8\cos(\pi t - \pi/12)$. At what time(s) does the object cross the x-axis?

67. Mach Number Use Problem 57 in Exercises 3.4 to find the vertex angle of the cone of sound waves made by an airplane flying at Mach 2.

68. Alternating Current An electric generator produces a 60-cycle alternating current given by $I(t) = 30\sin 120\pi\left(t - \frac{7}{36}\right)$, where $I(t)$ is the current in amperes at t seconds. Find the smallest positive value of t for which the current is 15 amperes.

69. Electrical Circuits If the voltage given by $V = V_0\sin(\omega t + \alpha)$ is impressed on a series circuit, an alternating current is produced. If $V_0 = 110$ volts, $\omega = 120\pi$ radians per second, and $\alpha = -\pi/6$, when is the voltage equal to zero?

70. Refraction of Light Consider a ray of light passing from one medium (such as air) into another medium (such as a crystal). Let ϕ be the angle of incidence and θ the angle of refraction. As shown in FIGURE 3.6.7, these angles are measured from a vertical line. According to **Snell's law**, there is a constant c that depends on the two mediums, such that $\dfrac{\sin\phi}{\sin\theta} = c$. Assume that for light passing from air into a crystal, $c = 1.437$. Find ϕ and θ such that the angle of incidence is twice the angle of refraction.

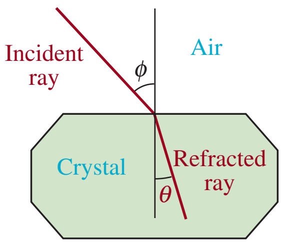


FIGURE 3.6.7 Light rays in Problem 70

71. Snow Cover On the basis of data collected from 1966 to 1980, the extent of snow cover S in the northern hemisphere, measured in millions of square kilometers, can be modeled by the function

$$S(w) = 25 + 21\cos\frac{\pi}{26}(w - 5),$$

where w is the number of weeks past January 1.

(a) How much snow cover does this formula predict for April Fool's Day? (Round w to the nearest integer.)

(b) In which week does the formula predict the least amount of snow cover?

(c) What month does this fall in?

CONCEPTS REVIEW

You should be able to give the meaning of each of the following concepts.

Circular functions:
- unit circle
- central angle
- reference angle

Periodic functions:
- period of sine
- period of cosine
- period of tangent
- period of cotangent
- period of secant
- period of cosecant

Graphs of trignometric functions:
- cycle
- amplitude
- phase shift

Identities:
- Pythagorean
- odd-even

Special formulas:
- addition
- subtraction
- double-angle
- half-angle

Inverse trigonometric functions:
- arcsine
- arccosine
- arctangent

Graphs of inverse trigonometric functions:
- arcsine
- arccosine
- arctangent

Trigonometric equations

CHAPTER 3 Review Exercises

Answers to selected odd-numbered problems begin on page ANS-11.

A. True/False

In Problems 1–20, answer true or false.

1. If $\tan t = \frac{3}{4}$, then $\sin t = 3$ and $\cos t = 4$. ____

2. In a right triangle, if $\sin\theta = \frac{11}{61}$, then $\cot\theta = \frac{60}{11}$. ____

3. $\sec(-\pi) = \csc\left(\dfrac{3\pi}{2}\right)$ ____

4. There is no angle t such that $\sec t = \frac{1}{2}$. ____

5. $\sin(2\pi - t) = -\sin t$ ____

6. $1 + \sec^2\theta = \tan^2\theta$ ____

7. $(-2, 0)$ is an x-intercept of the graph of $y = 3\sin(\pi x/2)$. ____

8. $\left(2\pi/3, -1/\sqrt{3}\right)$ is a point on the graph of $y = \cot x$. ____

9. The range of the function $y = \csc x$ is $(-\infty, -1] \cup [1, \infty)$. ____

10. The graph of $y = \csc x$ does not intersect the y-axis. ____

11. The line $x = \pi/2$ is a vertical asymptote for the graph of $y = \tan x$. ____

12. If $\tan(x + 2\pi) = 0.3$, then $\tan x = 0.3$. ____

13. For the function $f(x) = -2\sin x$, the range is defined by $-2 \le y \le 2$. ____

14. $\sin 20x = 2\sin 10x\cos 10x$ ____

15. The graph of $y = \sin(2x - \pi/3)$ is the graph of $y = \sin 2x$ shifted $\pi/3$ units to the right. ___

16. The graphs $y = 3\sin(-2x)$ and $y = -3\cos(2x - \pi/2)$ are the same. ___

17. Since $\tan(5\pi/4) = 1$, then $\arctan(1) = 5\pi/4$. ___

18. $\tan 8\pi = \tan 9\pi$ ___

19. The function $f(x) = \arcsin x$ is not periodic. ___

20. $\arcsin\left(\frac{1}{2}\right) = 30^\circ$ ___

B. Fill in the Blanks

In Problems 1–14, fill in the blanks.

1. If $\sin u = \frac{3}{5}$, $0 < u < \pi/2$, and $\cos v = 1/\sqrt{5}$, $3\pi/2 < v < 2\pi$, then $\cos(u + v) =$ ___________.

2. The y-intercept for the graph of the function $y = 2\sec(x + \pi)$ is ___________.

3. The period of the function $y = 2\sin\frac{\pi}{3}x$ is ___________.

4. The first vertical asymptote for the graph of $y = \tan\left(x - \frac{\pi}{4}\right)$ to the right of the y-axis is ___________.

5. The phase shift for the graph of $y = 5\cos(3x - 4\pi)$ is ___________.

6. If $\sin t = \frac{1}{6}$, then $\cos\left(t - \frac{\pi}{2}\right) =$ ___________.

7. The amplitude of $y = -10\cos\left(\frac{\pi}{3}x\right)$ is ___________.

8. $\cos\left(\frac{\pi}{6} - \frac{5\pi}{4}\right) =$ ___________.

9. The exact value of $\arccos\left(\cos\frac{9\pi}{5}\right) =$ ___________.

10. The period of the function $y = \tan 4x$ is ___________.

11. The fifth x-intercept on the positive x-axis for the graph of the function $y = \sin \pi x$ is___________.

12. If $P(t) = \left(-\frac{1}{3}, \frac{2\sqrt{2}}{3}\right)$ is a point on the unit circle, then $\sin 2t =$ ___________.

13. If $\cos x = \frac{\sqrt{2}}{3}$, where $3\pi/2 < x < 2\pi$, then the exact values of $\sin\frac{x}{2} =$ ___________, $\cos\frac{x}{2} =$ ___________, $\sin 2x =$ ___________, and $\cos 2x =$ ___________.

14. From the results in Problem 13, we find $\tan\frac{x}{2} =$ ___________ and $\tan 2x =$ ___________.

C. Review Exercises

In Problems 1–4, graph the given functions. Give the amplitude, the period, and the phase shift where appropriate.

1. $y = 5(1 + \sin x)$

2. $y = -\frac{4}{3}\cos x$

3. $y = 10\cos\left(-3x + \frac{\pi}{2}\right)$

4. $y = -4\cos\left(\frac{1}{4}x - \pi\right)$

In Problems 5–10, find all solutions of the given equation in the interval $[0, 2\pi]$.

5. $\cos x \sin x - \cos x + \sin x - 1 = 0$
6. $\cos x - \sin x = 0$
7. $4\sin^2 x - 1 = 0$
8. $\sin x = 2\tan x$
9. $\cos 4x = -1$
10. $\tan x - 3\cot x = 2$

In Problems 11 and 12, find the solutions of the given equation in the interval $(-\pi/2, \pi/2)$. Round your solutions to two decimal places.

11. $3\cos 2x + \sin x = 0$
12. $\tan^4 x + \tan^2 x - 1 = 0$

In Problems 13–20, find the indicated value without using a calculator.

13. $\cos^{-1}\left(-\frac{1}{2}\right)$
14. $\arcsin(-1)$
15. $\cot\left(\cos^{-1}\frac{3}{4}\right)$
16. $\cos\left(\arcsin\frac{2}{5}\right)$
17. $\sin^{-1}(\sin\pi)$
18. $\cos(\arccos 0.42)$
19. $\sin\left(\arccos\left(\frac{5}{13}\right)\right)$
20. $\arctan(\cos\pi)$

In Problems 21 and 22, write the given expression as an algebraic expression in x.

21. $\sin(\arccos x)$
22. $\sec(\tan^{-1} x)$

In Problems 23–26, give two examples of the indicated trigonometric function such that each has the given properties.

23. Sine function with period 4 and amplitude 6
24. Cosine function with period π, amplitude 4, and phase shift $\frac{1}{2}$
25. Sine function with period $\pi/2$, amplitude 3, and phase shift $\pi/4$
26. Tangent function whose graph completes one cycle on the interval $(-\pi/8, \pi/8)$

In Problems 27–30, the given graph can be interpreted as a rigid/nonrigid transformation of the graph of $y = \sin x$ and of the graph of $y = \cos x$. Find an equation of the graph using the sine function. Then find an equation of the same graph using the cosine function.

27.

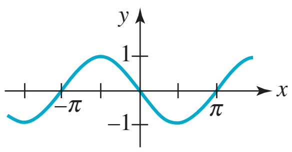


FIGURE 3.R.1 Graph for Problem 27

28.

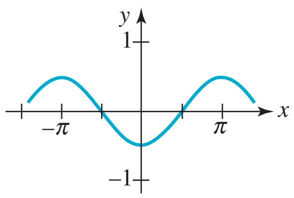


FIGURE 3.R.2 Graph for Problem 28

29.

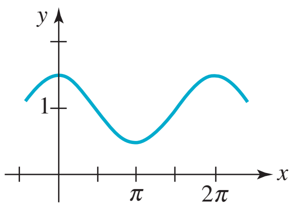


FIGURE 3.R.3 Graph for Problem 29

30.

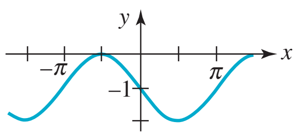


FIGURE 3.R.4 Graph for Problem 30