chapter 8 residue theory

Overview

You now have the necessary machinery to see some amazing applications of the tools we developed in the last few chapters. You will learn how Laurent expansions can give useful information concerning seemingly unrealated properties of complex functions. You will also learn how the ideas of complex analysis make the solution of very complicated integrals of real-valued functions as easy—literally—as the computation of residues. We begin with a theorem relating residues to the evaluation of complex integrals.

8.1 THE RESIDUE THEOREM

The Cauchy integral formulas given in Section 6.5 are useful in evaluating contour integrals over a simple closed contour C where the integrand has the form $\frac{f(z)}{(z-z_0)^k}$ and f is an analytic function. In this case, the singularity of the integrand is at worst a pole of order k at z_0. We begin this section by extending this result to integrals that have a finite number of isolated singularities inside the contour C. This new method can be used in cases where the integrand has an essential singularity at z_0 and is an important extension of the previous method.

Definition 8.1: Residue

Let f have a nonremovable isolated singularity at the point z_0. Then f has the Laurent series representation for all z in some punctured disk $D_R^*(z_0)$ given by $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$. The coefficient a_{-1} is called the **residue of f at z_0**. We use the notation

$$\operatorname{Res}[f, z_0] = a_{-1}.$$

■ EXAMPLE 8.1 If $f(z) = \exp\left(\frac{2}{z}\right)$, then the Laurent series of f about the point 0 has the form

$$f(z) = \exp\left(\frac{2}{z}\right) = 1 + \frac{2}{z} + \frac{2^2}{2!z^2} + \frac{2^3}{3!z^3} + \cdots,$$

and $\text{Res}[f, 0] = a_{-1} = 2$.

■ EXAMPLE 8.2 Find $\text{Res}[g, 0]$ if $g(z) = \frac{3}{2z+z^2-z^3}$.

Solution Using Example 7.7, we find that g has three Laurent series representations involving powers of z. The Laurent series valid in the punctured disk $D_1^*(0)$ is $g(z) = \sum_{n=0}^{\infty} \left[(-1)^n + \frac{1}{2^{n+1}}\right] z^{n-1}$. Computing the first few coefficients, we obtain

$$g(z) = \frac{3}{2}\frac{1}{z} - \frac{3}{4} + \frac{9}{8}z - \frac{15}{16}z^2 + \cdots.$$

Therefore, $\text{Res}[g, 0] = a_{-1} = \frac{3}{2}$.

Recall that, for a function f analytic in $D_R^*(z_0)$ and for any r with $0 < r < R$, the Laurent series coefficients of f are given by

$$a_n = \frac{1}{2\pi i} \int_{C_r^+(z_0)} \frac{f(\xi)\, d\xi}{(\xi - z_0)^{n+1}} \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots, \tag{8-1}$$

where $C_r^+(z_0)$ denotes the circle $\{z : |z - z_0| = r\}$ with positive orientation. This result gives us an important fact concerning $\text{Res}[f, z_0]$. If we set $n = -1$ in Equation (8-1) and replace $C_r^+(z_0)$ with any positively oriented simple closed contour C containing z_0, provided z_0 is the still only singularity of f that lies inside C, then we obtain

$$\int_C f(\xi)\, d\xi = 2\pi i a_{-1} = 2\pi i \,\text{Res}[f, z_0]. \tag{8-2}$$

If we are able to find the residue of f at z_0, then Equation (8-2) gives us an important tool for evaluating contour integrals.

EXAMPLE 8.3 Evaluate $\int_{C_1^+(0)} \exp\left(\frac{2}{z}\right) dz$.

Solution In Example 8.1 we showed that the residue of $f(z) = \exp\left(\frac{2}{z}\right)$ at $z_0 = 0$ is $\text{Res}[f, 0] = 2$. Using Equation (8-2), we get

$$\int_{C_1^+(0)} \exp\left(\frac{2}{z}\right) dz = 2\pi i \ \text{Res}[f, 0] = 4\pi i.$$

Theorem 8.1 (Cauchy's residue theorem) *Let D be a simply connected domain and let C be a simple closed positively oriented contour that lies in D. If f is analytic inside C and on C, except at the points $z_1, z_2, \ldots, z_n$ that lie inside C, then*

$$\int_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} Res[f, z_k].$$

The situation is illustrated in Figure 8.1.

Proof Since there are a finite number of singular points inside C, there exists an $r > 0$ such that the positively oriented circles $C_k = C_r^+(z_k)$, for $k = 1, 2, \ldots, n$, are mutually disjoint and all lie inside C. From the extended Cauchy–Goursat theorem (Theorem 6.7), it follows that

$$\int_C f(z)\, dz = \sum_{k=1}^{n} \int_{C_k} f(z)\, dz.$$

The function f is analytic in a punctured disk with center z_k that contains the circle C_k, so we can use Equation (8-2) to obtain

$$\int_{C_k} f(z)\, dz = 2\pi i \ \text{Res}[f, z_k], \qquad \text{for } k = 1, 2, \ldots, n.$$

Combining the last two equations gives the desired result.

The calculation of a Laurent series expansion is tedious in most circumstances. Since the residue at z_0 involves only the coefficient a_{-1} in the Laurent

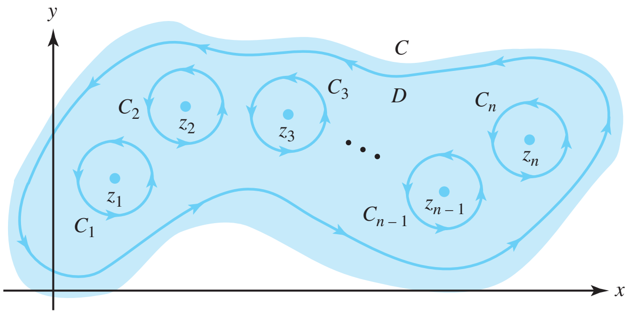

Figure 8.1 The domain D and contour C and the singular points $z_1, z_2, \ldots, z_n$ in the statement of Cauchy's residue theorem.

expansion, we seek a method to calculate the residue from special information about the nature of the singularity at z_0.

If f has a removable singularity at z_0, then $a_{-n} = 0$, for $n = 1, 2, \ldots$. Therefore, $\text{Res}[f, z_0] = 0$. Theorem 8.2 gives methods for evaluating residues at poles.

◗ Theorem 8.2 (Residues at poles)

i. If f has a simple pole at z_0, then

$$\text{Res}[f, z_0] = \lim_{z \to z_0} (z - z_0) f(z).$$

ii. If f has a pole of order 2 at z_0, then

$$\text{Res}[f, z_0] = \lim_{z \to z_0} \frac{d}{dz} (z - z_0)^2 f(z).$$

iii. If f has a pole of order k at z_0, then

$$\text{Res}[f, z_0] = \frac{1}{(k-1)!} \lim_{z \to z_0} \frac{d^{k-1}}{dz^{k-1}} (z - z_0)^k f(z).$$

Proof If f has a simple pole at z_0, then the Laurent series is

$$f(z) = \frac{a_{-1}}{z - z_0} + a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \cdots.$$

If we multiply both sides of this equation by $(z - z_0)$ and take the limit as $z \to z_0$, we obtain

$$\begin{aligned} \lim_{z \to z_0} (z - z_0) f(z) &= \lim_{z \to z_0} \left[a_{-1} + a_0 (z - z_0) + a_1 (z - z_0)^2 + \cdots \right] \\ &= a_{-1} = \text{Res}[f, z_0], \end{aligned}$$

which establishes part (i). We proceed to part (iii), as part (ii) is a special case of it. Suppose that f has a pole of order k at z_0. Then f can be written as

$$f(z) = \frac{a_{-k}}{(z-z_0)^k} + \frac{a_{-k+1}}{(z-z_0)^{k-1}} + \cdots + \frac{a_{-1}}{z-z_0} + a_0 + a_1(z-z_0) + \cdots.$$

Multiplying both sides of this equation by $(z-z_0)^k$ gives

$$(z-z_0)^k f(z) = a_{-k} + \cdots + a_{-1}(z-z_0)^{k-1} + a_0(z-z_0)^k + \cdots.$$

If we differentiate both sides $k-1$ times, we get

$$\begin{aligned}\frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right] &= (k-1)!a_{-1} + k!a_0(z-z_0) \\ &\quad + \frac{(k+1)!}{2!}a_1(z-z_0)^2 + \cdots,\end{aligned}$$

and when we let $z \to z_0$, the result is

$$\lim_{z\to z_0} \frac{d^{k-1}}{dz^{k-1}}\left[(z-z_0)^k f(z)\right] = (k-1)!a_{-1} = (k-1)!\operatorname{Res}[f, z_0],$$

which establishes part (iii).

■ EXAMPLE 8.4 Find the residue of $f(z) = \frac{\pi\cot(\pi z)}{z^2}$ at $z_0 = 0$.

Solution We write $f(z) = \frac{\pi\cos(\pi z)}{z^2\sin(\pi z)}$. Because $z^2 \sin \pi z$ has a zero of order 3 at $z_0 = 0$ and $\pi\cos(\pi z_0) \neq 0$, f has a pole of order 3 at z_0. By part (iii) of Theorem 8.2, we have

$$\begin{aligned}\operatorname{Res}[f, 0] &= \frac{1}{2!}\lim_{z\to 0}\frac{d^2}{dz^2}\pi z\cot(\pi z) \\ &= \frac{1}{2}\lim_{z\to 0}\frac{d}{dz}\left[\pi\cot(\pi z) - \pi^2 z\csc^2(\pi z)\right] \\ &= \frac{1}{2}\lim_{z\to 0}\left[-\pi^2\csc^2(\pi z) - \pi^2\left\{\csc^2(\pi z) - 2\pi z\csc^2(\pi z)\cot(\pi z)\right\}\right] \\ &= \pi^2\lim_{z\to 0}(\pi z\cot(\pi z) - 1)\csc^2(\pi z) \\ &= \pi^2\lim_{z\to 0}\frac{\pi z\cos(\pi z) - \sin(\pi z)}{\sin^3(\pi z)}.\end{aligned}$$

This last limit involves an indeterminate form, which we evaluate by using L'Hôpital's rule:

$$\begin{aligned}\operatorname{Res}[f,0] &= \pi^2 \lim_{z\to 0} \frac{-\pi^2 z\sin(\pi z)+\pi\cos(\pi z)-\pi\cos(\pi z)}{3\pi\sin^2(\pi z)\cos(\pi z)} \\ &= \pi^2 \lim_{z\to 0}\frac{-\pi z}{3\sin(\pi z)\cos(\pi z)} \\ &= \frac{-\pi^2}{3}\lim_{z\to 0}\frac{\pi z}{\sin(\pi z)}\lim_{z\to 0}\frac{1}{\cos(\pi z)} = \frac{-\pi^2}{3}.\end{aligned}$$

■ EXAMPLE 8.5 Find $\int_{C_3^+(0)} \frac{1}{z^4+z^3-2z^2}\,dz$.

Solution We write the integrand as $f(z) = \frac{1}{z^2(z+2)(z-1)}$. The singularities of f that lie inside $C_3(0)$ are simple poles at the points 1 and -2, and a pole of order 2 at the origin. We compute the residues as follows:

$$\operatorname{Res}[f,0] = \lim_{z\to 0}\frac{d}{dz}\left[z^2 f(z)\right] = \lim_{z\to 0}\frac{-2z-1}{\left(z^2+z-2\right)^2} = \frac{-1}{4},$$

$$\operatorname{Res}[f,1] = \lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}\frac{1}{z^2(z+2)} = \frac{1}{3}, \quad \text{and}$$

$$\operatorname{Res}[f,-2] = \lim_{z\to -2}(z+2)f(z) = \lim_{z\to -2}\frac{1}{z^2(z-1)} = \frac{-1}{12}.$$

Finally, the residue theorem yields

$$\int_{C_3^+(0)} \frac{dz}{z^4+z^3-2z^2} = 2\pi i\left[\frac{-1}{4}+\frac{1}{3}-\frac{1}{12}\right] = 0.$$

The answer, $\int_{C_3^+(0)} \frac{dz}{z^4+z^3-2z^2} = 0$, is not at all obvious, and all the preceding calculations are required to get it.

■ EXAMPLE 8.6 Find $\int_{C_2^+(1)} \left(z^4+4\right)^{-1} dz$.

Solution The singularities of the integrand $f(z) = \dfrac{1}{z^4+4}$ that lie inside $C_2(1)$ are simple poles occurring at the points $1 \pm i$, as the points $-1 \pm i$ lie outside $C_2(1)$. Factoring the denominator is tedious, so we use a different approach.

If z_0 is any one of the singularities of f, then we can use L'Hôpital's rule to compute $\text{Res}[f, z_0]$:

$$\text{Res}\,[f, z_0] = \lim_{z \to z_0} \frac{z - z_0}{z^4 + 4} = \lim_{z \to z_0} \frac{\frac{d}{dz}(z - z_0)}{\frac{d}{dz}(z^4 + 4)} = \lim_{z \to z_0} \frac{1}{4z^3} = \frac{1}{4z_0^3}.$$

Since $z_0^4 = -4$, we can simplify this expression further to yield $\text{Res}[f, z_0] = -\frac{1}{16} z_0$. Hence $\text{Res}[f, 1+i] = \frac{-1-i}{16}$, and $\text{Res}[f, 1-i] = \frac{-1+i}{16}$. We now use the residue theorem to get

$$\int_{C_2^+(1)} \frac{dz}{z^4 + 4} = 2\pi i \left(\frac{-1-i}{16} + \frac{-1+i}{16} \right) = \frac{-\pi i}{4}.$$

The theory of residues can be used to expand the quotient of two polynomials into its *partial fraction* representation.

■ EXAMPLE 8.7 Let $P(z)$ be a polynomial of degree at most 2. Show that if a, b, and c are distinct complex numbers, then

$$f(z) = \frac{P(z)}{(z-a)(z-b)(z-c)} = \frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c},$$

where

$$A = \text{Res}\,[f, a] = \frac{P(a)}{(a-b)(a-c)},$$
$$B = \text{Res}\,[f, b] = \frac{P(b)}{(b-a)(b-c)}, \quad \text{and}$$
$$C = \text{Res}\,[f, c] = \frac{P(c)}{(c-a)(c-b)}.$$

Solution It will suffice to prove that $A = \text{Res}[f, a]$. We expand f in its Laurent series about the point a by writing the three terms $\frac{A}{z-a}$, $\frac{B}{z-b}$, and $\frac{C}{z-c}$ in their Laurent series about the point a and adding them. The term $\frac{A}{z-a}$ is itself a one-term Laurent series about the point a. The term $\frac{B}{z-b}$ is analytic at the point a, and its Laurent series is actually a Taylor series given by

$$\frac{B}{z-b} = \frac{-B}{(b-a)} \frac{1}{(1 - \frac{z-a}{b-a})} = -\sum_{n=0}^{\infty} \frac{B}{(b-a)^{n+1}} (z-a)^n,$$

which is valid for $|z-a| < |b-a|$. Likewise, the expansion of the term $\frac{C}{z-c}$ is

$$\frac{C}{z-c} = -\sum_{n=0}^{\infty} \frac{C}{(c-a)^{n+1}} (z-a)^n,$$

which is valid for $|z-a| < |c-a|$. Thus, the Laurent series of f about the point a is

$$f(z) = \frac{A}{z-a} - \sum_{n=0}^{\infty} \left[\frac{B}{(b-a)^{n+1}} + \frac{C}{(c-a)^{n+1}} \right] (z-a)^n,$$

which is valid for $|z-a| < R$, where $R = \min\{|b-a|, |c-a|\}$. Therefore, $A = \text{Res}[f,a]$, and calculation reveals that

$$\text{Res}[f,a] = A = \lim_{z\to a} \frac{P(z)}{(z-b)(z-c)} = \frac{P(a)}{(a-b)(a-c)}.$$

■ **EXAMPLE 8.8** Express $f(z) = \frac{3z+2}{z(z-1)(z-2)}$ in partial fractions.

Solution Computing the residues, we obtain

$$\text{Res}[f,0] = 1, \qquad \text{Res}[f,1] = -5, \quad \text{and} \quad \text{Res}[f,2] = 4.$$

Example 8.7 gives us

$$\frac{3z+2}{z(z-1)(z-2)} = \frac{1}{z} - \frac{5}{z-1} + \frac{4}{z-2}.$$

Remark 8.1 If a repeated root occurs, then the process is similar, and we can easily show that if $P(z)$ has degree of at most 2, then

$$f(z) = \frac{P(z)}{(z-a)^2(z-b)} = \frac{A}{(z-a)^2} + \frac{B}{z-a} + \frac{C}{z-b},$$

where $A = \text{Res}[(z-a)f(z), a]$, $B = \text{Res}[f,a]$, and $C = \text{Res}[f,b]$. ■

■ **EXAMPLE 8.9** Express $f(z) = \frac{z^2+3z+2}{z^2(z-1)}$ in partial fractions.

Solution Using the previous remark, we have

$$f(z) = \frac{A}{(z-a)^2} + \frac{B}{z-a} + \frac{C}{z-b},$$

where

$$A = \text{Res}\left[zf(z), 0\right] = \lim_{z\to 0} \frac{z^2+3z+2}{z-1} = -2,$$

$$B = \text{Res}\left[f, 0\right] = \lim_{z\to 0} \frac{d}{dz}\frac{z^2+3z+2}{z-1}$$

$$= \lim_{z\to 0} \frac{(2z+3)(z-1)-\left(z^2+3z+2\right)}{(z-1)^2} = -5, \quad \text{and}$$

$$C = \text{Res}\left[f, 1\right] = \lim_{z\to 1} \frac{z^2+3z+2}{z^2} = 6.$$

Thus,

$$\frac{z^2+3z+2}{z^2(z-1)} = \frac{-2}{z^2} - \frac{5}{z} + \frac{6}{z-1}.$$

EXERCISES FOR SECTION 8.1

1. Find $\text{Res}[f, 0]$ for

(a) $f(z) = z^{-1}\exp z$.

(b) $f(z) = z^{-3}\cosh 4z$.

(c) $f(z) = \csc z$.

(d) $f(z) = \dfrac{z^2+4z+5}{z^2+z}$.

(e) $f(z) = \cot z$.

(f) $f(z) = z^{-3}\cos z$.

(g) $f(z) = z^{-1}\sin z$.

(h) $f(z) = \dfrac{z^2+4z+5}{z^3}$.

(i) $f(z) = \exp\left(1+\frac{1}{z}\right)$.

(j) $f(z) = z^4\sin\left(\frac{1}{z}\right)$.

(k) $f(z) = z^{-1}\csc z$.

(l) $f(z) = z^{-2}\csc z$.

(m) $f(z) = \dfrac{\exp(4z)-1}{\sin^2 z}$.

(n) $f(z) = z^{-1}\csc^2 z$.

2. Let f and g have an isolated singularity at z_0. Show that the formula $\operatorname{Res}[f+g, z_0] = \operatorname{Res}[f, z_0] + \operatorname{Res}[g, z_0]$ holds true.

3. Evaluate

(a) $\displaystyle\int_{C_1^+(-1+i)} \frac{dz}{z^4+4}$.

(b) $\displaystyle\int_{C_2^+(i)} \frac{dz}{z\left(z^2-2z+2\right)}$.

(c) $\displaystyle\int_{C_2^+(0)} \frac{\exp z\ dz}{z^3+z}$.

(d) $\displaystyle\int_{C_2^+(0)} \frac{\sin z\ dz}{4z^2-\pi^2}$.

(e) $\displaystyle\int_{C_2^+(0)} \frac{\sin z\ dz}{z^2+1}$.

(f) $\displaystyle\int_{C_1^+(0)} \frac{dz}{z^2 \sin z}$.

(g) $\displaystyle\int_{C_1^+(0)} \frac{dz}{z \sin^2 z}$.

4. Let f and g be analytic at z_0. If $f(z_0) \neq 0$ and g has a simple zero at z_0, then show that $\operatorname{Res}\left[\frac{f}{g}, z_0\right] = \dfrac{f(z_0)}{g'(z_0)}$.

5. Find $\int_C (z-1)^{-2}\left(z^2+4\right)^{-1} dz$ when

(a) $C = C_1^+(1)$.

(b) $C = C_4^+(0)$.

6. Find $\int_C \left(z^6+1\right)^{-1} dz$ when

(a) $C = C_{\frac{1}{2}}^+(i)$.

(b) $C = C_1^+\left(\dfrac{1+i}{2}\right)$. *Hint*: If z_0 is a singularity of $f(z) = \dfrac{1}{z^6+1}$, show that $\operatorname{Res}[f, z_0] = -\frac{1}{6} z_0$.

7. Find $\int_C \left(3z^4 + 10z^2 + 3\right)^{-1} dz$ when

(a) $C = C_1^+\left(i\sqrt{3}\right)$.

(b) $C = C_1^+\left(\frac{i}{\sqrt{3}}\right)$.

8. Find $\int_C \left(z^4 - z^3 - 2z^2\right)^{-1} dz$ when

(a) $C = C_{\frac{1}{2}}^+(0)$.

(b) $C = C_{\frac{3}{2}}^+(0)$.

9. Use residues to find the partial fraction representations of

(a) $\dfrac{1}{z^2 + 3z + 2}$.

(b) $\dfrac{3z - 3}{z^2 - z - 2}$.

(c) $\dfrac{z^2 - 7z + 4}{z^2(z+4)}$.

(d) $\dfrac{10z}{(z^2+4)(z^2+9)}$.

(e) $\dfrac{2z^2 - 3z - 1}{(z-1)^3}$.

(f) $\dfrac{z^3 + 3z^2 - z + 1}{z(z+1)^2(z^2+1)}$.

10. Let f be analytic in a simply connected domain D, and let C be a simple closed positively oriented contour in D. If z_0 is the only zero of f in D and z_0 lies interior to C, then show that $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz = k$, where k is the order of the zero at z_0.

11. Let f be analytic at the points $0, \pm 1, \pm 2, \ldots$. If $g(z) = \pi f(z) \cot \pi z$, then show that $\text{Res}[g, n] = f(n)$ for $n = 0, \pm 1, \pm 2, \ldots$.

8.2 TRIGONOMETRIC INTEGRALS

As indicated at the beginning of this chapter, we can evaluate certain definite *real* integrals with the aid of the residue theorem. One way to do this is by interpreting the definite integral as the parametric form of an integral of an analytic function along a simple closed contour.

Suppose that we want to evaluate an integral of the form

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta, \tag{8-3}$$

where $F(u, v)$ is a function of the two real variables u and v. Consider the unit circle $C_1(0)$ with parametrization

$$C_1^+(0) : z = \cos\theta + i\sin\theta = e^{i\theta}, \qquad \text{for } 0 \le \theta \le 2\pi,$$

which gives us the symbolic differentials

$$dz = (-\sin\theta + i\cos\theta)\, d\theta = ie^{i\theta} d\theta \quad \text{and}$$

$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}. \tag{8-4}$$

Combining $z = \cos\theta + i\sin\theta$ with $\frac{1}{z} = \cos\theta - i\sin\theta$, we obtain

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{and} \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right). \tag{8-5}$$

Using the substitutions for $\cos\theta$, $\sin\theta$, and $d\theta$ in Expression (8-3) transforms the definite integral into the contour integral

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \int_{C_1^+(0)} f(z)\, dz,$$

where the new integrand is $f(z) = \frac{F\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)}{iz}$.

Suppose that f is analytic inside and on the unit circle $C_1(0)$, except at the points $z_1, z_2, \ldots, z_n$ that lie interior to $C_1(0)$. Then the residue theorem gives

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = 2\pi i \sum_{k=1}^{n} \text{Res}[f, z_k]. \tag{8-6}$$

The situation is illustrated in Figure 8.2.

■ EXAMPLE 8.10 Evaluate $\int_0^{2\pi} \frac{1}{1+3\cos^2\theta} d\theta$ by using complex analysis.

Solution Using Substitutions (8-4) and (8-5), we transform the integral to

$$\int_{C_1^+(0)} \frac{1}{1 + 3\left(\frac{z+z^{-1}}{2}\right)^2} \frac{dz}{iz} = \int_{C_1^+(0)} \frac{-i4z}{3z^4 + 10z^2 + 3}\, dz = \int_{C_1^+(0)} f(z)\, dz,$$

where $f(z) = \frac{-i4z}{3z^4+10z^2+3}$. The singularities of f are poles located at the points where $3(z^2)^2 + 10(z^2) + 3 = 0$. Using the quadratic formula, we see that the

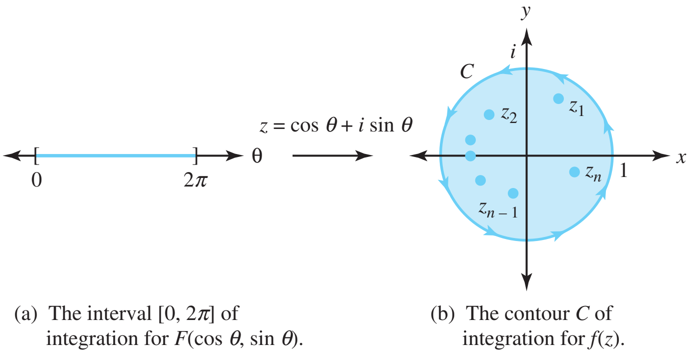

Figure 8.2 The change of variables from a definite integral on $[0, 2\pi]$ to a contour integral around C.

singular points satisfy the relation $z^2 = \frac{-10 \pm \sqrt{100-36}}{6} = \frac{-5 \pm 4}{3}$. Hence the only singularities that lie inside the unit circle are simple poles corresponding to the solutions of $z^2 = -\frac{1}{3}$, which are the two points $z_1 = \frac{i}{\sqrt{3}}$ and $z_2 = -\frac{i}{\sqrt{3}}$. We use Theorem 8.2 and L'Hôpital's rule to get the residues at z_k, for $k = 1, 2$:

$$\begin{aligned} \operatorname{Res}[f, z_k] &= \lim_{z \to z_k} \frac{-i4z(z - z_k)}{3z^4 + 10z^2 + 3} \\ &= \lim_{z \to z_k} \frac{-i4(2z - z_k)}{12z^3 + 20z} \\ &= \frac{-i4z_k}{12z_k^3 + 20z_k} \\ &= \frac{-i}{3z_k^2 + 5}. \end{aligned}$$

As $z_k = \frac{\pm i}{\sqrt{3}}$ and $z_k^2 = -\frac{1}{3}$, the residues are given by $\operatorname{Res}[f, z_k] = -\frac{i}{3\left(-\frac{1}{3}\right)+5} = -\frac{i}{4}$. We now use Equation (8-6) to compute the value of the integral:

$$\int_0^{2\pi} \frac{d\theta}{1 + 3\cos^2\theta} = 2\pi i \left(\frac{-i}{4} + \frac{-i}{4} \right) = \pi.$$

■ **EXAMPLE 8.11** Evaluate $\int_0^{2\pi} \frac{1}{1+3\cos^2 t}\, dt$ by using a computer algebra system.

Solution Using a variety of software packages we can obtain the antiderivative of $\frac{1}{1+3\cos^2 t}$. Many of them give $\int \frac{1}{1+3\cos^2 t}\, dt = \frac{-\operatorname{Arctan}(2\cot t)}{2} = g(t)$. Since $\cot 0$ and $\cot 2\pi$ are not defined, the computations for both $g(0)$ and $g(2\pi)$ are indeterminate. The graph $s = g(t)$ shown in Figure 8.3 reveals another

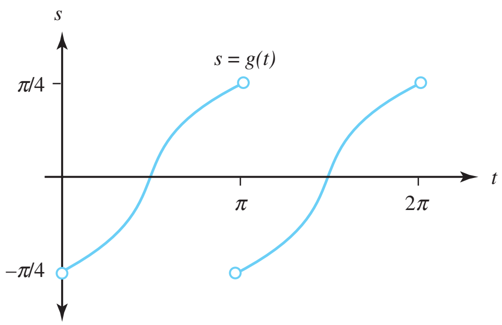

Figure 8.3 Graph of $g(t) = \int \frac{1}{1+3\cos^2 t}dt = \frac{-\text{Arctan}\,(2\cot t)}{2}$.

problem: The integrand $\frac{1}{1+3\cos^2 t}$ is a continuous function for all t, but the function g has a discontinuity at π. This condition appears to be a violation of the fundamental theorem of calculus, which asserts that the integral of a continuous function must be differentiable and hence continuous. The problem is that $g(t)$ is not an antiderivative of $\frac{1}{1+3\cos^2 t}$ for *all* t in the interval $[0, 2\pi]$. Oddly, it is the antiderivative at all points *except* 0, π, and 2π, which you can verify by computing $g'(t)$ and showing that it equals $\frac{1}{1+3\cos^2 t}$ whenever $g(t)$ is defined.

The integration algorithm used by computer algebra systems here (the Risch–Norman algorithm) gives the antiderivative $g(t) = \frac{-\text{Arctan}(2\cot t)}{2}$, and we must take great care in using this information.

We get the proper value of the integral by using $g(t)$ on the open subintervals $(0, \pi)$ and $(\pi, 2\pi)$ where it is continuous and taking appropriate limits:

$$\begin{aligned}\int_0^{2\pi} \frac{dt}{1+3\cos^2 t} &= \int_0^{\pi} \frac{dt}{1+3\cos^2 t} + \int_\pi^{2\pi} \frac{dt}{1+3\cos^2 t} \\ &= \lim_{\substack{t\to\pi^- \\ s\to 0^+}} \int_s^t \frac{dt}{1+3\cos^2 t} + \lim_{\substack{t\to 2\pi^- \\ s\to \pi^+}} \int_s^t \frac{dt}{1+3\cos^2 t} \\ &= \lim_{t\to\pi^-} g(t) - \lim_{s\to 0^+} g(s) + \lim_{t\to 2\pi^-} g(t) - \lim_{s\to\pi^+} g(s) \\ &= \frac{\pi}{4} - \frac{-\pi}{4} + \frac{\pi}{4} - \frac{-\pi}{4} = \pi.\end{aligned}$$

■ EXAMPLE 8.12

Evaluate $\int_0^{2\pi} \frac{\cos 2\theta}{5-4\cos\theta}\, d\theta$.

Solution For values of z that lie on the unit circle $C_1(0)$, we have

$$z^2 = \cos 2\theta + i\sin 2\theta \quad \text{and} \quad z^{-2} = \cos 2\theta - i\sin 2\theta.$$

We solve for $\cos 2\theta$ and $\sin 2\theta$ to obtain the substitutions

$$\cos 2\theta = \frac{1}{2}\left(z^2 + z^{-2}\right) \quad \text{and} \quad \sin 2\theta = \frac{1}{2i}\left(z^2 - z^{-2}\right).$$

Using the identity for $\cos 2\theta$ along with Substitutions (8-4) and (8-5), we rewrite the integral as

$$\int_{C_1^+(0)} \frac{\frac{1}{2}\left(z^2 + z^{-2}\right)}{5 - 4\left(\frac{z+z^{-1}}{2}\right)} \frac{dz}{iz} = \int_{C_1^+(0)} \frac{i\left(z^4+1\right)}{2z^2(z-2)(2z-1)}\, dz = \int_{C_1^+(0)} f(z)\, dz,$$

where $f(z) = \frac{i(z^4+1)}{2z^2(z-2)(2z-1)}$. The singularities of f lying inside $C_1^+(0)$ are poles located at the points 0 and $\frac{1}{2}$. We use Theorem 8.2 to get the residues:

$$\begin{aligned}\text{Res}[f,0] &= \lim_{z\to 0} \frac{d}{dz} z^2 f(z) = \lim_{z\to 0} \frac{d}{dz} i \frac{\left(z^4+1\right)}{2\left(2z^2 - 5z + 2\right)} \\ &= \lim_{z\to 0} i \frac{4z^3\left(2z^2-5z+2\right) - (4z-5)\left(z^4+1\right)}{2\left(2z^2-5z+2\right)^2} = \frac{5i}{8}\end{aligned}$$

and

$$\text{Res}\left[f, \frac{1}{2}\right] = \lim_{z\to\frac{1}{2}} \left(z - \frac{1}{2}\right) f(z) = \lim_{z\to\frac{1}{2}} \frac{i\left(z^4+1\right)}{4z^2(z-2)} = -\frac{17i}{24}.$$

Therefore, we conclude that

$$\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5 - 4\cos\theta} = 2\pi i\left(\frac{5i}{8} - \frac{17i}{24}\right) = \frac{\pi}{6}.$$

EXERCISES FOR SECTION 8.2

Use residues to find

1. $\int_0^{2\pi} \frac{1}{3\cos\theta + 5} d\theta$.

2. $\int_0^{2\pi} \frac{1}{4\sin\theta + 5} d\theta$.

3. $\int_0^{2\pi} \frac{1}{15\sin^2\theta + 1} d\theta$.

4. $\int_0^{2\pi} \frac{1}{5\cos^2\theta + 4} d\theta$.

5. $\int_0^{2\pi} \frac{\sin^2 \theta}{5 + 4\cos\theta} d\theta.$

6. $\int_0^{2\pi} \frac{\sin^2 \theta}{5 - 3\cos\theta} d\theta.$

7. $\int_0^{2\pi} \frac{1}{(5 + 3\cos\theta)^2} d\theta.$

8. $\int_0^{2\pi} \frac{1}{(5 + 4\cos\theta)^2} d\theta.$

9. $\int_0^{2\pi} \frac{\cos 2\theta}{5 + 3\cos\theta} d\theta.$

10. $\int_0^{2\pi} \frac{\cos 2\theta}{13 - 12\cos\theta} d\theta.$

11. $\int_0^{2\pi} \frac{1}{(1 + 3\cos^2\theta)^2} d\theta.$

12. $\int_0^{2\pi} \frac{1}{(1 + 8\cos^2\theta)^2} d\theta.$

13. $\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta} d\theta.$

14. $\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 3\cos 2\theta} d\theta.$

15. $\int_0^{2\pi} \frac{1}{a\cos\theta + b\sin\theta + d} d\theta,$ where a, b, and d are real and $a^2 + b^2 < d^2$.

16. $\int_0^{2\pi} \frac{1}{a\cos^2\theta + b\sin^2\theta + d} d\theta,$ where a, b, and d are real and $a > d$ and $b > d$.

8.3 IMPROPER INTEGRALS OF RATIONAL FUNCTIONS

An important application of the theory of residues is the evaluation of certain types of improper integrals. We let f be a continuous function of the real variable x on the interval $0 \le x < \infty$. Recall from calculus that the improper integral f over $[0, \infty)$ is defined by

$$\int_0^\infty f(x)\,dx = \lim_{b\to\infty} \int_0^b f(x)\,dx,$$

provided the limit exists. If f is defined for all real x, then the integral of f over $(-\infty, \infty)$ is defined by

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{a\to-\infty} \int_{a}^{0} f(x)\,dx + \lim_{b\to\infty} \int_{0}^{b} f(x)\,dx, \tag{8-7}$$

provided both limits exist. If the integral in Equation (8-7) exists, we can obtain its value by taking a single limit:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx. \tag{8-8}$$

For some functions the limit on the right side of Equation (8-8) exists, but the limit on the right side of Equation (8-7) doesn't exist.

■ **EXAMPLE 8.13** $\lim_{R\to\infty} \int_{-R}^{R} x\,dx = \lim_{R\to\infty} \left[\frac{R^2}{2} - \frac{(-R)^2}{2}\right] = 0$, but we know from Equation (8-7) that the improper integral of $f(x) = x$ over $(-\infty, \infty)$ doesn't exist. Therefore, we can use Equation (8-8) to extend the notion of the value of an improper integral, as Definition 8.2 indicates.

Definition 8.2: Cauchy principal value

Let $f(x)$ be a continuous real-valued function for all x. The **Cauchy principal value** (P.V.) of the integral $\int_{-\infty}^{\infty} f(x)\,dx$ is defined by

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx,$$

provided the limit exists.

Example 8.13 shows that P.V. $\int_{-\infty}^{\infty} x\,dx = 0$.

■ **EXAMPLE 8.14** The Cauchy principal value of $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$ is

$$\begin{aligned}
\text{P.V.} \int_{-\infty}^{\infty} \frac{1}{x^2+1}dx &= \lim_{R\to\infty} \int_{-R}^{R} \frac{1}{x^2+1}dx \\
&= \lim_{R\to\infty} [\text{Arctan}\,R - \text{Arctan}\,(-R)] \\
&= \frac{\pi}{2} - \frac{-\pi}{2} = \pi.
\end{aligned}$$

If $f(x) = \frac{P(x)}{Q(x)}$, where P and Q are polynomials, then f is called a **rational function**. In calculus you probably learned techniques for integrating certain types of rational functions. We now show how to use the residue theorem to obtain the Cauchy principal value of the integral of f over $(-\infty, \infty)$.

Theorem 8.3 *Let $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials of degree m and n, respectively. If $Q(x) \neq 0$ for all real x and $n \geq m+2$, then*

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}\left[\frac{P}{Q}, z_j\right],$$

where $z_1, z_2, \ldots, z_{k-1}, z_k$ are the poles of $\frac{P}{Q}$ that lie in the upper half-plane. The situation is illustrated in Figure 8.4.

Proof There are a finite number of poles of $\frac{P}{Q}$ that lie in the upper half-plane, so we can find a real number R such that the poles all lie inside the contour C, which consists of the segment $-R \leq x \leq R$ of the x-axis together with the upper semicircle C_R of radius R shown in Figure 8.4. By properties of integrals,

$$\int_{-R}^{R} \frac{P(x)}{Q(x)}\, dx = \int_C \frac{P(z)}{Q(z)}\, dz - \int_{C_R} \frac{P(z)}{Q(z)}\, dz.$$

Using the residue theorem, we rewrite this equation as

$$\int_{-R}^{R} \frac{P(x)}{Q(x)}\, dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}\left[\frac{P}{Q}, z_j\right] - \int_{C_R} \frac{P(z)}{Q(z)}\, dz. \tag{8-9}$$

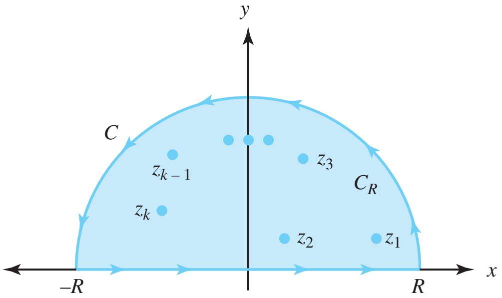

Figure 8.4 The poles $z_1, z_2, \ldots, z_{k-1}, z_k$ of $\frac{P}{Q}$ that lie in the upper half-plane.

Our proof will be complete if we can show that $\int_{C_R} \frac{P(z)}{Q(z)}\, dz$ tends to zero as $R \to \infty$. Suppose that

$$P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0 \quad \text{and}$$
$$Q(z) = b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0.$$

Then

$$\frac{zP(z)}{Q(z)} = \frac{z^{m+1}\left(a_m + a_{m-1}z^{-1} + \cdots + a_1 z^{-m+1} + a_0 z^{-m}\right)}{z^n\left(b_n + b_{n-1}z^{-1} + \cdots + b_1 z^{-n+1} + b_0 z^{-n}\right)},$$

so

$$\begin{aligned}\lim_{|z|\to\infty} \frac{zP(z)}{Q(z)} &= \lim_{|z|\to\infty} \frac{z^{m+1}\left(a_m + a_{m-1}z^{-1} + \cdots + a_1 z^{-m+1} + a_0 z^{-m}\right)}{z^n\left(b_n + b_{n-1}z^{-1} + \cdots + b_1 z^{-n+1} + b_0 z^{-n}\right)} \\ &= \lim_{|z|\to\infty} \frac{z^{m+1}}{z^n} \lim_{|z|\to\infty} \frac{a_m + a_{m-1}z^{-1} + \cdots + a_1 z^{-m+1} + a_0 z^{-m}}{b_n + b_{n-1}z^{-1} + \cdots + b_1 z^{-n+1} + b_0 z^{-n}}.\end{aligned}$$

Since $n \geq m+2$, this limit reduces to $0\left(\frac{a_m}{b_n}\right) = 0$. Therefore, for any $\varepsilon > 0$, we may choose R large enough so that $\left|\frac{zP(z)}{Q(z)}\right| < \frac{\varepsilon}{\pi}$ whenever z lies on C_R. But this means that

$$\left|\frac{P(z)}{Q(z)}\right| < \frac{\varepsilon}{\pi |z|} = \frac{\varepsilon}{\pi R} \tag{8-10}$$

whenever z lies on C_R. Using the ML inequality (Theorem 6.3) and the result of Inequality (8-10), we get

$$\left|\int_{C_R} \frac{P(z)}{Q(z)} dz\right| \leq \int_{C_R} \frac{\varepsilon}{\pi R} |dz| = \frac{\varepsilon}{\pi R} \pi R = \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_{R\to\infty} \int_{C_R} \frac{P(z)}{Q(z)}\, dz = 0. \tag{8-11}$$

If we let $R \to \infty$ and combine Equations (8-9) and (8-11), we arrive at the desired conclusion.

■ **EXAMPLE 8.15** Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)}$.

Solution We write the integrand as $f(z) = \frac{1}{(z+i)(z-i)(z+2i)(z-2i)}$. We see that f has simple poles at the points i and $2i$ in the upper half-plane. Computing the residues, we obtain

$$\operatorname{Res}[f, i] = \frac{-i}{6} \quad \text{and} \quad \operatorname{Res}[f, 2i] = \frac{i}{12}.$$

Using Theorem 8.3, we conclude that

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = 2\pi i \left(\frac{-i}{6} + \frac{i}{12} \right) = \frac{\pi}{6}.$$

■ EXAMPLE 8.16 Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3}$.

Solution The integrand $f(z) = \frac{1}{(z^2+4)^3}$ has a pole of order 3 at the point $2i$, which is the only singularity of f in the upper half-plane. Computing the residue, we get

$$\begin{aligned} \operatorname{Res}[f, 2i] &= \frac{1}{2} \lim_{z \to 2i} \frac{d^2}{dz^2} \frac{1}{(z+2i)^3} \\ &= \frac{1}{2} \lim_{z \to 2i} \frac{d}{dz} \frac{-3}{(z+2i)^4} \\ &= \frac{1}{2} \lim_{z \to 2i} \frac{12}{(z+2i)^5} = \frac{-3i}{512}. \end{aligned}$$

Therefore, $\int_{-\infty}^{\infty} \frac{dx}{(x^2+4)^3} = 2\pi i \left(\frac{-3i}{512} \right) = \frac{3\pi}{256}$.

EXERCISES FOR SECTION 8.3

Use residues to evaluate

1. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2+16)^2}$.

2. $\int_{-\infty}^{\infty} \frac{dx}{x^2+16}$.

3. $\int_{-\infty}^{\infty} \frac{x\, dx}{(x^2+9)^2}$.

4. $\int_{-\infty}^{\infty} \frac{x+3}{(x^2+9)^2}\, dx$.

5. $\int_{-\infty}^{\infty} \frac{2x^2+3}{(x^2+9)^2}\, dx$.

6. $\int_{-\infty}^{\infty} \frac{dx}{x^4+4}$.

7. $\int_{-\infty}^{\infty} \frac{x^2 dx}{x^4 + 4}$.

8. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + 4)^2}$.

9. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + 1)^2 (x^2 + 4)}$.

10. $\int_{-\infty}^{\infty} \frac{x + 2}{(x^2 + 4)(x^2 + 9)}\, dx$.

11. $\int_{-\infty}^{\infty} \frac{3x^2 + 2}{(x^2 + 4)(x^2 + 9)} dx$.

12. $\int_{-\infty}^{\infty} \frac{dx}{x^6 + 1}$.

13. $\int_{-\infty}^{\infty} \frac{x^4 dx}{x^6 + 1}$.

14. $\int_{-\infty}^{\infty} \frac{dx}{(x^2 + a^2)(x^2 + b^2)}$, where $a > 0$ and $b > 0$.

15. $\int_{-\infty}^{\infty} \frac{x^2 dx}{(x^2 + a^2)^3}$, where $a > 0$.

8.4 IMPROPER INTEGRALS INVOLVING TRIGONOMETRIC FUNCTIONS

Let P and Q be polynomials of degree m and n, respectively, where $n \geq m + 1$. We can show (but omit the proof) that if $Q(x) \neq 0$ for all real x, then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos x \, dx \quad \text{and} \quad \text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin x \, dx$$

are convergent improper integrals. You may encounter integrals of this type in the study of Fourier transforms and Fourier integrals. We now show how to evaluate them.

Particularly important is our use of the identities

$$\cos(\alpha x) = \text{Re}[\exp(i\alpha x)] \quad \text{and} \quad \sin(\alpha x) = \text{Im}[\exp(i\alpha x)],$$

where α is a positive real number. The crucial step in the proof of Theorem 8.4 wouldn't hold if we were to use $\cos(\alpha z)$ and $\sin(\alpha z)$ instead of $\exp(i\alpha z)$, as you will see when you get to Lemma 8.1.

Theorem 8.4 *Let P and Q be polynomials with real coefficients of degree m and n, respectively, where $n \geq m+1$ and $Q(x) \neq 0$, for all real x. If $\alpha > 0$ and*

$$f(z) = \frac{\exp(i\alpha z) P(z)}{Q(z)}, \tag{8-12}$$

then

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos(\alpha x)\, dx = -2\pi \sum_{j=1}^{k} \text{Im}(\text{Res}[f, z_j]) \quad \textit{and} \tag{8-13}$$

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin(\alpha x)\, dx = 2\pi \sum_{j=1}^{k} \text{Re}(\text{Res}[f, z_j]), \tag{8-14}$$

where $z_1, z_2, \ldots, z_{k-1}, z_k$ are the poles of f that lie in the upper half-plane and $\text{Re}(\text{Res}[f, z_j])$ and $\text{Im}(\text{Res}[f, z_j])$ are the real and imaginary parts of $\text{Res}[f, z_j]$, respectively.

The proof of Theorem 8.4 is similar to the proof of Theorem 8.3. Before turning to the proof, we illustrate how to use Theorem 8.4.

■ EXAMPLE 8.17

Evaluate P.V. $\int_{-\infty}^{\infty} \frac{x \sin x\, dx}{x^2+4}$.

Solution The function f in Equation (8-12) is $f(z) = \frac{\exp(iz)z}{z^2+4}$, which has a simple pole at the point $2i$ in the upper half-plane. Calculating the residue yields

$$\text{Res}[f, 2i] = \lim_{z \to 2i} \frac{\exp(iz)\, z}{z+2i} = \frac{2ie^{-2}}{4i} = \frac{1}{2e^2}.$$

Using Equation (8-14) gives

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x\, dx}{x^2+4} = 2\pi \text{Re}(\text{Res}[f, 2i]) = \frac{\pi}{e^2}.$$

■ EXAMPLE 8.18

Evaluate P.V. $\int_{-\infty}^{\infty} \frac{\cos x\, dx}{x^4+4}$.

Solution The function f in Equation (8-12) is $f(z) = \frac{\exp(iz)}{z^4+4}$, which has simple poles at the points $z_1 = 1+i$ and $z_2 = -1+i$ in the upper half-plane. We get the residues with the aid of L'Hôpital's rule:

$$\begin{aligned}\operatorname{Res}[f,1+i] &= \lim_{z\to 1+i}\frac{(z-1-i)\exp(iz)}{z^4+4}\\ &= \lim_{z\to 1+i}\frac{[1+i(z-1-i)]\exp(iz)}{4z^3}\\ &= \frac{\exp(-1+i)}{4(1+i)^3}\\ &= \frac{\sin 1-\cos 1-i(\cos 1+\sin 1)}{16e}.\end{aligned}$$

Similarly,

$$\operatorname{Res}[f,-1+i] = \frac{\cos 1-\sin 1-i(\cos 1+\sin 1)}{16e}.$$

Using Equation (8-13), we get

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{\cos x\,dx}{x^4+4} &= -2\pi\left[\operatorname{Im}(\operatorname{Res}[f,1+i])+\operatorname{Im}(\operatorname{Res}[f,-1+i])\right]\\ &= \frac{\pi(\cos 1+\sin 1)}{4e}.\end{aligned}$$

We are almost ready to give the proof of Theorem 8.4, but first we need one preliminary result.

Lemma 8.1 (Jordan's lemma) Suppose that P and Q are polynomials of degree m and n, respectively, where $n \geq m+1$. If C_R is the upper semicircle $z = Re^{i\theta}$, for $0 \leq \theta \leq \pi$, then

$$\lim_{R\to\infty}\int_{C_R}\frac{\exp(iz)P(z)}{Q(z)}\,dz = 0.$$

Proof From $n \geq m+1$, it follows that $\left|\frac{P(z)}{Q(z)}\right| \to 0$ as $|z| \to \infty$. Therefore, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\left|\frac{P(z)}{Q(z)}\right| < \frac{\varepsilon}{\pi} \tag{8-15}$$

whenever $|z| \geq R_\varepsilon$. Using the ML inequality (Theorem 6.3) together with Inequality (8-15), we get

$$\left|\int_{C_R}\frac{\exp(iz)P(z)}{Q(z)}\,dz\right| \leq \int_{C_R}\frac{\varepsilon}{\pi}\left|e^{iz}\right||dz|, \tag{8-16}$$

provided $R \geq R_{\varepsilon}$. The parametrization of C_R leads to the equation

$$|dz| = R\, d\theta \quad \text{and} \quad \left|e^{iz}\right| = e^{-y} = e^{-R\sin\theta}. \tag{8-17}$$

Using the trigonometric identity $\sin(\pi - \theta) = \sin\theta$ and Equations (8-17), we express the integral on the right side of Inequality (8-16) as

$$\int_{C_R} \frac{\varepsilon}{\pi}\left|e^{iz}\right| |dz| = \frac{\varepsilon}{\pi}\int_0^{\pi} e^{-R\sin\theta} R\, d\theta = \frac{2\varepsilon}{\pi}\int_0^{\frac{\pi}{2}} e^{-R\sin\theta} R\, d\theta. \tag{8-18}$$

On the interval $0 \leq \theta \leq \frac{\pi}{2}$ we can use the inequality

$$0 \leq \frac{2\theta}{\pi} \leq \sin\theta.$$

We combine this inequality with Inequality (8-16) and Equation (8-18) to conclude that, for $R \geq R_{\varepsilon}$,

$$\left|\int_{C_R} \frac{\exp(iz)\,P(z)\,dz}{Q(z)}\right| \leq \frac{2\varepsilon}{\pi}\int_0^{\frac{\pi}{2}} e^{\frac{-2R\theta}{\pi}} R\, d\theta$$
$$= -\varepsilon e^{\frac{-2R\theta}{\pi}}\Big|_{\theta=0}^{\theta=\frac{\pi}{2}} = \varepsilon\left(1 - e^{-R}\right) < \varepsilon.$$

Because $\varepsilon > 0$ is arbitrary, our proof is complete. ■

We now turn to the proof of our main theorem.

Proof of Theorem 8.4 Let C be the contour that consists of the segment $-R \leq x \leq R$ of the real axis together with the upper semicircle C_R parametrized by $z = Re^{i\theta}$, for $0 \leq \theta \leq \pi$. Using properties of integrals, we have

$$\int_{-R}^{R} \frac{\exp(i\alpha x)\,P(x)\,dx}{Q(x)} = \int_C \frac{\exp(i\alpha z)\,P(z)\,dz}{Q(z)} - \int_{C_R} \frac{\exp(i\alpha z)\,P(z)\,dz}{Q(z)}.$$

If R is sufficiently large, all the poles $z_1, z_2, \ldots, z_k$ of f will lie inside C, and we can use the residue theorem to obtain

$$\int_{-R}^{R} \frac{\exp(i\alpha x)\,P(x)\,dx}{Q(x)} = 2\pi i\sum_{j=1}^{k} \operatorname{Res}[f, z_j] - \int_{C_R} \frac{\exp(i\alpha z)\,P(z)\,dz}{Q(z)}. \tag{8-19}$$

Since α is a positive real number, the change of variables $\zeta = \alpha z$ shows that the conclusion of Jordan's lemma holds for the integrand $\frac{\exp(i\alpha z)P(z)}{Q(z)}$. Hence we let $R \to \infty$ in Equation (8-19) to obtain

$$\begin{aligned} \text{P.V.} \int_{-\infty}^{\infty} \frac{[\cos(\alpha x) + i \sin(\alpha x)] P(x)\, dx}{Q(x)} &= 2\pi i \sum_{j=1}^{k} \text{Res}[f, z_j] \\ &= -2\pi \sum_{j=1}^{k} \text{Im}(\text{Res}[f, z_j]) \\ &\quad + 2\pi i \sum_{j=1}^{k} \text{Re}(\text{Res}[f, z_j]). \end{aligned}$$

Equating the real and imaginary parts of this equation gives us Equations (8-13) and (8-14), which completes the proof.

EXERCISES FOR SECTION 8.4

Use residues to find the Cauchy principal value of

1. $\int_{-\infty}^{\infty} \frac{\cos x\ dx}{x^2+9}$ and $\int_{-\infty}^{\infty} \frac{\sin x\ dx}{x^2+9}$.

2. $\int_{-\infty}^{\infty} \frac{x \cos x\ dx}{x^2+9}$ and $\int_{-\infty}^{\infty} \frac{x \sin x\ dx}{x^2+9}$.

3. $\int_{-\infty}^{\infty} \frac{x \sin x\ dx}{(x^2+4)^2}$.

4. $\int_{-\infty}^{\infty} \frac{\cos x\ dx}{(x^2+4)^2}$.

5. $\int_{-\infty}^{\infty} \frac{\cos x\ dx}{(x^2+4)(x^2+9)}$.

6. $\int_{-\infty}^{\infty} \frac{\cos x\ dx}{(x^2+1)(x^2+4)}$.

7. $\int_{-\infty}^{\infty} \frac{\cos x\ dx}{x^2-2x+5}$.

8. $\int_{-\infty}^{\infty} \frac{\cos x\ dx}{x^2-4x+5}$.

9. $\int_{-\infty}^{\infty} \frac{x \sin x\ dx}{x^4+4}$.

10. $\int_{-\infty}^{\infty} \frac{x^3 \sin x\ dx}{x^4+4}$.

11. $\int_{-\infty}^{\infty} \dfrac{\cos 2x \; dx}{x^2 + 2x + 2}$.

12. $\int_{-\infty}^{\infty} \dfrac{x^3 \sin 2x \; dx}{x^4 + 4}$.

13. Why do you need to use the exponential function when evaluating improper integrals involving the sine and cosine functions?

8.5 INDENTED CONTOUR INTEGRALS

If f is continuous on the interval $b < x \leq c$, but discontinuous at b, then the improper integral of f over $[b, c]$ is defined by

$$\int_b^c f(x)\,dx = \lim_{r \to b^+} \int_r^c f(x)\,dx,$$

provided the limit exists. Similarly, if f is continuous on the interval $a \leq x < b$, but discontinuous at b, then the improper integral of f over $[a, b]$ is defined by

$$\int_a^b f(x)\,dx = \lim_{R \to b^-} \int_a^R f(x)\,dx,$$

provided the limit exists. For example,

$$\int_0^9 \frac{dx}{2\sqrt{x}} = \lim_{r \to 0^+} \int_r^9 \frac{dx}{2\sqrt{x}} = \lim_{r \to 0^+} \left(\sqrt{x}\,\Big|_{x=r}^{x=9}\right) = 3 - \lim_{r \to 0^+} \sqrt{r} = 3.$$

If we let f be continuous for all values of x in the interval $[a, c]$, except at the value $x = b$, where $a < b < c$, then the Cauchy principal value of f over $[a, c]$ is defined by

$$\text{P.V.} \int_a^c f(x)\,dx = \lim_{r \to 0^+} \left[\int_a^{b-r} f(x)\,dx + \int_{b+r}^c f(x)\,dx \right],$$

provided the limit exists.

■ EXAMPLE 8.19

$$\text{P.V.} \int_{-1}^8 \frac{dx}{x^{\frac{1}{3}}} = \lim_{r \to 0^+} \left[\int_{-1}^{-r} \frac{dx}{x^{\frac{1}{3}}} + \int_r^8 \frac{dx}{x^{\frac{1}{3}}} \right].$$

Evaluating the integrals and computing limits give

$$\lim_{r \to 0^+} \left[\frac{3}{2} r^{\frac{2}{3}} - \frac{3}{2} + 6 - \frac{3}{2} r^{\frac{2}{3}} \right] = \frac{9}{2}.$$

In this section we show how to use residues to evaluate the Cauchy principal value of the integral of f over $(-\infty, \infty)$ when the integrand f has simple poles on the x-axis. We state our main results and then look at some examples before giving proofs.

Theorem 8.5 *Let $f(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials with real coefficients of degree m and n, respectively, and $n \geq m + 2$. If Q has simple zeros at the points $t_1, t_2, \ldots, t_l$ on the x-axis, then*

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)\,dx}{Q(x)} = 2\pi i \sum_{j=1}^{k} \operatorname{Res}[f, z_j] + \pi i \sum_{j=1}^{l} \operatorname{Res}[f, t_j], \qquad (8\text{-}20)$$

where $z_1, z_2, \ldots, z_k$ are the poles of f that lie in the upper half-plane.

Theorem 8.6 *Let P and Q be polynomials of degree m and n, respectively, where $n \geq m + 1$, and let Q have simple zeros at the points $t_1, t_2, \ldots, t_l$ on the x-axis. If α is a positive real number and if $f(z) = \frac{\exp(i\alpha z)P(z)}{Q(z)}$, then*

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \cos \alpha x \; dx = -2\pi \sum_{j=1}^{k} \operatorname{Im}(\operatorname{Res}[f, z_j]) - \pi \sum_{j=1}^{l} \operatorname{Im}(\operatorname{Res}[f, t_j]) \qquad (8\text{-}21)$$

and

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)} \sin \alpha x \; dx = 2\pi \sum_{j=1}^{k} \operatorname{Re}(\operatorname{Res}[f, z_j]) + \pi \sum_{j=1}^{l} \operatorname{Re}(\operatorname{Res}[f, t_j]) \qquad (8\text{-}22)$$

where $z_1, z_2, \ldots, z_k$ are the poles of f that lie in the upper half-plane.

Remark 8.2 The formulas in these theorems give the Cauchy principal value of the integral, which pays special attention to the manner in which any limits are taken. They are similar to those in Sections 8.3 and 8.4, except here we add one-half the value of each residue at the points $t_1, t_2, \ldots, t_l$ on the x-axis. ■

■ **EXAMPLE 8.20** Evaluate P.V. $\int_{-\infty}^{\infty} \frac{x\,dx}{x^3-8}$ by using complex analysis.

Solution The integrand

$$f(z) = \frac{z}{z^3 - 8} = \frac{z}{(z-2)\left(z + 1 + i\sqrt{3}\right)\left(z + 1 - i\sqrt{3}\right)}$$

has simple poles at the points $t_1 = 2$ on the x-axis and $z_1 = -1 + i\sqrt{3}$ in the upper half-plane. By Theorem 8.5,

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x\,dx}{x^3 - 8} = 2\pi i \text{Res}\,[f, z_1] + \pi i \text{Res}\,[f, t_1]$$
$$= 2\pi i \frac{-1 - i\sqrt{3}}{12} + \pi i \frac{1}{6} = \frac{\pi\sqrt{3}}{6}.$$

■ EXAMPLE 8.21 Evaluate P.V. $\int_{-\infty}^{\infty} \frac{t\,dx}{t^3-8}$ by using a computer algebra system.

Solution A variety of computer algebra systems give the indefinite integral

$$\int \frac{t\,dt}{t^3 - 8} = \frac{\text{Arctan}\frac{1+t}{\sqrt{3}}}{2\sqrt{3}} + \frac{\text{Log}\,(t-2)}{6} + \frac{\text{Log}\,(t^2 + 2t + 4)}{12} = g\,(t)\,.$$

However, for real numbers, we should write the second term as $\frac{\text{Log}\left[(t-2)^2\right]}{12}$ and use the equivalent formula:

$$g\,(t) = \frac{\text{Arctan}\frac{1+t}{\sqrt{3}}}{2\sqrt{3}} + \frac{\text{Log}\left[(t-2)^2\right]}{12} + \frac{\text{Log}\,(t^2 + 2t + 4)}{12}.$$

This antiderivative has the property that $\lim_{t \to 2} g\,(t) = -\infty$, as shown in Figure 8.5. We also compute

$$\lim_{t \to \infty} g\,(t) = \frac{\pi\sqrt{3}}{12} \quad \text{and} \quad \lim_{t \to -\infty} g\,(t) = \frac{-\pi\sqrt{3}}{12},$$

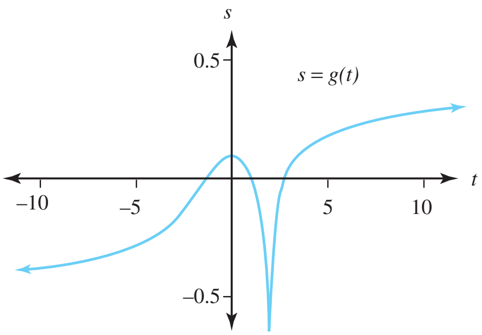

Figure 8.5 Graph of $s = g\,(t) = \int \frac{t\,dt}{t^3 - 8}$.

and the Cauchy principal limit at $t = 2$ as $r \to 0$ is

$$\lim_{r \to 0^+} [g(2+r) - g(2-r)] = 0.$$

Therefore, the Cauchy principal value of the improper integral is

$$\begin{aligned} \text{P.V.} \int_{-\infty}^{\infty} \frac{t\ dt}{t^3 - 8} &= \lim_{r \to 0^+} \left[\int_{-\infty}^{2-r} \frac{t\ dt}{t^3 - 8} + \int_{2+r}^{\infty} \frac{t\ dt}{t^3 - 8} \right] \\ &= \lim_{t \to \infty} g(t) - \lim_{r \to 0^+} [g(2+r) - g(2-r)] - \lim_{t \to -\infty} g(t) \\ &= \frac{\pi\sqrt{3}}{12} - 0 + \frac{\pi\sqrt{3}}{12} = \frac{\pi\sqrt{3}}{6}. \end{aligned}$$

■ EXAMPLE 8.22 Evaluate P.V. $\int_{-\infty}^{\infty} \frac{\sin x\ dx}{(x-1)(x^2+4)}$.

Solution The integrand $f(z) = \frac{\exp(iz)}{(z-1)(z^2+4)}$ has simple poles at the points $t_1 = 1$ on the x-axis and $z_1 = 2i$ in the upper half-plane. By Theorem 8.6,

$$\begin{aligned} \text{P.V.} \int_{-\infty}^{\infty} \frac{\sin x\ dx}{(x-1)(x^2+4)} &= 2\pi \text{Re}(\text{Res}[f, z_1]) + \pi \text{Re}(\text{Res}[f, t_1]) \\ &= 2\pi \text{Re}\left(\frac{-2+i}{20e^2}\right) + \pi \text{Re}\left(\frac{\cos 1 + i \sin 1}{5}\right) \\ &= \frac{\pi}{5}\left(\cos 1 - \frac{1}{e^2}\right). \end{aligned}$$

The proofs of Theorems 8.5 and 8.6 depend on the following result.

Lemma 8.2 Suppose that f has a simple pole at the point t_0 on the x-axis. If C_r is the contour $C_r : z = t_0 + re^{i\theta}$, for $0 \le \theta \le \pi$, then

$$\lim_{r \to 0} \int_{C_r} f(z)\, dz = i\pi \text{Res}[f, t_0].$$

Proof The Laurent series for f at $z = t_0$ has the form

$$f(z) = \frac{\text{Res}[f, t_0]}{z - t_0} + g(z), \qquad \text{(8-23)}$$

where g is analytic at $z = t_0$. Using the parametrization of C_r and Equation (8-23), we get

$$\int_{C_r} f(z)\,dz = \operatorname{Res}[f,t_0]\int_0^{\pi}\frac{ire^{i\theta}d\theta}{re^{i\theta}} + ir\int_0^{\pi} g\left(t_0+re^{i\theta}\right)e^{i\theta}d\theta$$
$$= i\pi\operatorname{Res}[f,t_0] + ir\int_0^{\pi} g\left(t_0+re^{i\theta}\right)e^{i\theta}d\theta. \tag{8-24}$$

As g is continuous at t_0, there is an $M > 0$ so that $\left|g\left(t_0+re^{i\theta}\right)\right| \le M$, and

$$\left|\lim_{r\to 0} ir\int_0^{\pi} g\left(t_0+re^{i\theta}\right)e^{i\theta}d\theta\right| \le \lim_{r\to 0} r\int_0^{\pi} M\,d\theta = \lim_{r\to 0} r\pi M = 0.$$

Combining this inequality with Equation (8-24) gives the conclusion we want. ■

Proof of Theorems 8.5 and 8.6 Since f has only a finite number of poles, we can choose r small enough that the semicircles

$$C_j : z = t_j + re^{i\theta}, \qquad \text{for } 0 \le \theta \le \pi \quad \text{and} \quad j = 1, 2, \ldots, l,$$

are disjoint and the poles $z_1, z_2, \ldots, z_k$ of f in the upper half-plane lie above them, as shown in Figure 8.6.

Let R be large enough so that the poles of f in the upper half-plane lie under the semicircle $C_R : z = Re^{i\theta}$, for $0 \le \theta \le \pi$, and the poles of f on the x-axis lie in the interval $-R \le x \le R$. Let C be the simple closed positively

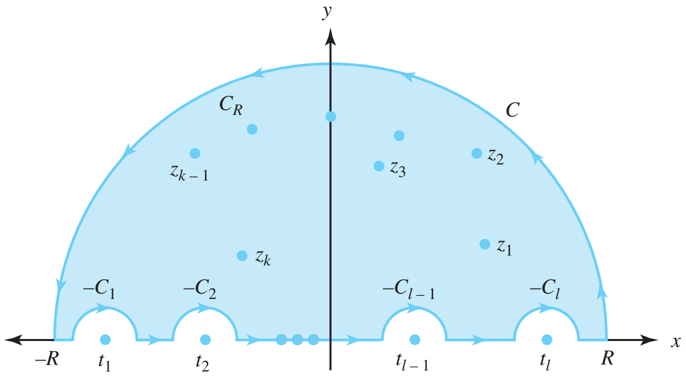

Figure 8.6 The poles $t_1, t_2, \ldots, t_l$ of f that lie on the x-axis and the poles $z_1, z_2, \ldots, z_k$ that lie above the semicircles $C_1, C_2, \ldots, C_l$.

oriented contour that consists of C_R and $-C_1$, $-C_2$, ..., $-C_l$ and the segments of the real axis that lie between the semicircles shown in Figure 8.6. The residue theorem gives $\int_C f(z)\,dz = 2\pi i \sum_{j=1}^{k} \text{Res}[f, z_j]$, which we rewrite as

$$\int_{I_R} f(x)\,dx = 2\pi i \sum_{j=1}^{k} \text{Res}\,[f, z_j] + \sum_{j=1}^{l} \int_{C_j} f(z)\,dz - \int_{C_R} f(z)\,dz, \tag{8-25}$$

where I_R is the portion of the interval $-R \le x \le R$ that lies outside the intervals $(t_j - r, t_j + r)$ for $j = 1, 2, \ldots, l$. Using the same techniques that we used in Theorems 8.3 and 8.4 yields

$$\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0. \tag{8-26}$$

If we let $R \to \infty$ and $r \to 0$ in Equation (8-25) and use the results of Equation (8-26) and Lemma 8.2, we obtain

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{j=1}^{k} \text{Res}\,[f, z_j] + \pi i \sum_{j=1}^{l} \text{Res}\,[f, t_j]\,. \tag{8-27}$$

If f is the function given in Theorem 8.5, then Equation (8-27) becomes Equation (8-20). If f is the function given in Theorem 8.6, then equating the real and imaginary parts of Equation (8-27) results in Equations (8-21) and (8-22), respectively, and with these results our proof is complete.

EXERCISES FOR SECTION 8.5

Use residues to compute

1. P.V. $\int_{-\infty}^{\infty} \frac{dx}{x(x-1)(x-2)}$.

2. P.V. $\int_{-\infty}^{\infty} \frac{dx}{x^3+x}$.

3. P.V. $\int_{-\infty}^{\infty} \frac{x\,dx}{x^3+1}$.

4. P.V.$\int_{-\infty}^{\infty} \frac{dx}{x^3+1}$.

5. P.V. $\int_{-\infty}^{\infty} \frac{x^2\,dx}{x^4-1}$.

6. P.V. $\int_{-\infty}^{\infty} \frac{x^4\,dx}{x^6-1}$.

7. P.V. $\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x}$.

8. P.V. $\int_{-\infty}^{\infty} \frac{\cos x\,dx}{x^2-x}$.

9. P.V. $\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x\left(\pi^2-x^2\right)}$.

10. P.V. $\int_{-\infty}^{\infty} \frac{\cos x\,dx}{\pi^2-4x^2}$.

11. P.V. $\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x\left(x^2+1\right)}$.

12. P.V. $\int_{-\infty}^{\infty} \frac{x\cos x\,dx}{x^2+3x+2}$.

13. P.V. $\int_{-\infty}^{\infty} \frac{\sin x\,dx}{x\left(1-x^2\right)}$.

14. P.V. $\int_{-\infty}^{\infty} \frac{\cos x\,dx}{a^2-x^2}$.

15. P.V. $\int_{-\infty}^{\infty} \frac{\sin^2 x\,dx}{x^2}$.
Hint: Use trigonometric identity $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

16. P.V. $\int_{0}^{\infty} \frac{1}{x^3+1}\,dx$.
Hint: Use the contour $C = L_1 + C_R - L_2$ shown in Figure 8.7.

17. P.V. $\int_{0}^{\infty} \frac{x}{x^3+1}\,dx$.
Hint: Use the contour $C = L_1 + C_R - L_2$ shown in Figure 8.7.

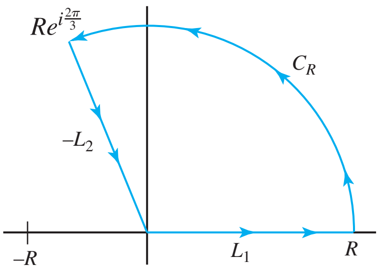

Figure 8.7 The contour $C = L_1 + C_R - L_2$ for Exercises 16 and 17.

8.6 INTEGRANDS WITH BRANCH POINTS

We now show how to evaluate certain improper real integrals involving the integrand $x^\alpha \frac{P(x)}{Q(x)}$. The complex function z^α is multivalued, so we must first specify the branch to be used.

Let α be a real number with $0 < \alpha < 1$. In this section, we use the branch of z^α corresponding to the branch of the logarithm $\log_0$ (see Equation (5-20)) as follows:

$$z^\alpha = e^{\alpha[\log_0(z)]} = e^{\alpha(\ln|z| + i \arg_0 z)} = e^{\alpha(\ln r + i\theta)} = r^\alpha (\cos \alpha\theta + i \sin \alpha\theta), \tag{8-28}$$

where $z = re^{i\theta} \neq 0$ and $0 < \theta \leq 2\pi$. Note that this is not the traditional principal branch of z^a and that, as defined, the function z^a is analytic in the domain $\{re^{i\theta} : r > 0,\, 0 < \theta < 2\pi\}$.

Theorem 8.7 *Let P and Q be polynomials of degree m and n, respectively, where $n \geq m + 2$. If $Q(x) \neq 0$, for $x > 0$, Q has a zero of order at most 1 at the origin, and $f(z) = \frac{z^\alpha P(z)}{Q(z)}$, where $0 < \alpha < 1$, then*

$$\text{P.V.} \int_0^\infty \frac{x^\alpha P(x)\, dx}{Q(x)} = \frac{2\pi i}{1 - e^{i\alpha 2\pi}} \sum_{j=1}^{k} \text{Res}[f, z_j],$$

where $z_1, z_2, \ldots, z_k$ are the nonzero poles of $\frac{P}{Q}$.

Proof Let C denote the simple closed positively oriented contour that consists of the portions of the circles $C_r(0)$ and $C_R(0)$ and the horizontal segments joining them, as shown in Figure 8.8. We select a small value of r and a large value of R so that the nonzero poles $z_1, z_2, \ldots, z_k$ of $\frac{P}{Q}$ lie inside C. Using the residue theorem, we write

$$\int_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} \text{Res}[f, z_j]. \tag{8-29}$$

If we let $r \to 0$ in Equation (8-29), the integrand $f(z)$ on the upper horizontal line of Figure 8.8 approaches $\frac{x^\alpha P(x)}{Q(x)}$, where x is a real number; however, because of the branch we chose for z^a (see Equation (8-28)), the integrand $f(z)$ on the lower horizontal line approaches $\frac{x^\alpha e^{i\alpha 2\pi} P(x)}{Q(x)}$. Therefore,

$$\lim_{r\to 0} \int_C f(z)\, dz = \int_0^R \frac{x^\alpha P(x)}{Q(x)}\, dx + \int_R^0 \frac{x^\alpha e^{i\alpha 2\pi} P(x)}{Q(x)}\, dx + \int_{C_R^+(0)} f(z)\, dz. \tag{8-30}$$

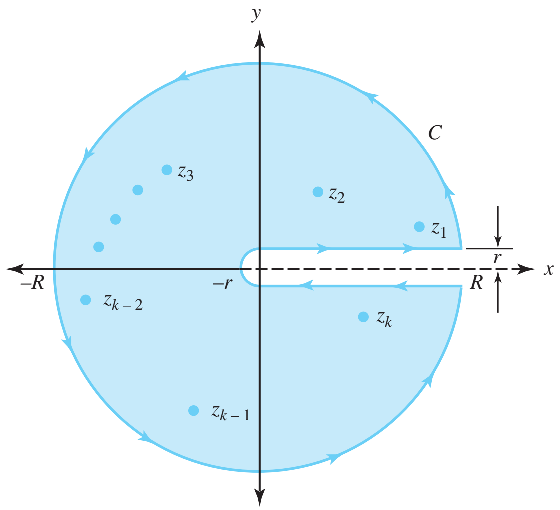

Figure 8.8 The contour C that encloses the nonzero poles $z_1, z_2, \ldots, z_k$ of $\frac{P}{Q}$.

It is here that we need the function Q to have a zero of order at most 1 at the origin. Otherwise, the first two integrals on the right side of Equation (8-30) would not necessarily converge. Combining this result with Equation (8-29) gives

$$\int_0^R \frac{x^\alpha P(x)}{Q(x)}\,dx - \int_0^R \frac{x^\alpha e^{i\alpha 2\pi} P(x)}{Q(x)}\,dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}[f, z_j] - \int_{C_R^+(0)} f(z)\,dz,$$

so

$$\int_0^R \frac{x^a P(x)}{Q(x)}\,dx\left(1 - e^{ia2\pi}\right) = 2\pi i \sum_{j=1}^{k} \operatorname{Res}[f, z_j] - \int_{C_R^+(0)} f(z)\,dz,$$

which we rewrite as

$$\int_0^R \frac{x^\alpha P(x)\,dx}{Q(x)} = \frac{2\pi i}{1 - e^{i\alpha 2\pi}} \sum_{j=1}^{k} \operatorname{Res}[f, z_j] - \frac{1}{1 - e^{i\alpha 2\pi}} \int_{C_R^+(0)} f(z)\,dz. \tag{8-31}$$

Using the ML inequality (Theorem 6.3) gives

$$\lim_{R\to\infty} \int_{C_R^+(0)} f(z)\,dz = 0. \tag{8-32}$$

The argument is essentially the same as that used to establish Equation (8-11), and we omit the details. If we combine Equations (8-31) and (8-32) and let $R \to \infty$, we arrive at the desired result.

■ **EXAMPLE 8.23** Evaluate P.V. $\int_0^\infty \frac{x^\alpha}{x(x+1)}\, dx$, where $0 < a < 1$.

Solution The function $f(z) = \frac{z^a}{z(z+1)}$ has a nonzero pole at the point -1, and the denominator has a zero of order at most 1 (in fact, exactly 1) at the origin. Using Theorem 8.7, we have

$$\int_0^\infty \frac{x^a}{x(x+1)}\, dx = \frac{2\pi i}{1-e^{ia2\pi}}\text{Res}[f,-1] = \frac{2\pi i}{1-e^{ia2\pi}}\left(\frac{e^{ia\pi}}{-1}\right)$$
$$= \frac{\pi}{\frac{e^{ia\pi}-e^{-ia\pi}}{2i}} = \frac{\pi}{\sin a\pi}.$$

We can apply the preceding ideas to other multivalued functions.

■ **EXAMPLE 8.24** Evaluate P.V. $\int_0^\infty \frac{\ln x}{x^2+a^2}\, dx$, where $a > 0$.

Solution We use the function $f(z) = \frac{\log_{-\frac{\pi}{2}} z}{z^2+a^2}$. Recall that

$$\log_{-\frac{\pi}{2}} z = \ln|z| + i \arg_{-\frac{\pi}{2}} z = \ln r + i\theta,$$

where $z = re^{i\theta} \neq 0$ and $-\frac{\pi}{2} < \theta \leq \frac{3\pi}{2}$. The path C of integration will consist of the segments $[-R,-r]$ and $[r,R]$ of the x-axis together with the upper semicircles $C_r : z = re^{i\theta}$ and $C_R : z = Re^{i\theta}$, for $0 \leq \theta \leq \pi$, as shown in Figure 8.9.

We chose the branch $\log_{-\frac{\pi}{2}}$ because it is analytic on C and its interior—hence so is the function f. This choice enables us to apply the residue theorem properly (see the hypotheses of Theorem 8.1), and we get

$$\int_C f(z)\, dz = 2\pi i \text{Res}[f, ai] = \frac{\pi \ln a}{a} + i\frac{\pi^2}{2a}.$$

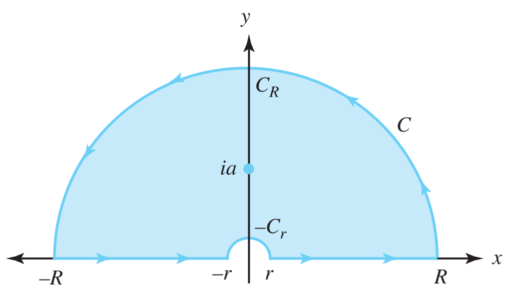

Figure 8.9 The contour C for the integrand $f(z) = \frac{\log_{-\frac{\pi}{2}} z}{z^2+a^2}$.

Keeping in mind the branch of logarithm that we're using, we then have

$$\begin{aligned}\int_C f(z)\,dz &= \int_{-R}^{-r} f(x)\,dx + \int_{-C_r} f(z)\,dz + \int_r^R f(x)\,dx + \int_{C_R} f(z)\,dz \\ &= \int_{-R}^{-r} \frac{\ln|x| + i\pi}{x^2+a^2}\,dx + \int_{-C_r} f(z)\,dz \\ &\quad + \int_r^R \frac{\ln x}{x^2+a^2}\,dx + \int_{C_R} f(z)\,dz \\ &= \frac{\pi \ln a}{a} + i\frac{\pi^2}{2a}. \end{aligned} \tag{8-33}$$

If $R^2 > a^2$, then by the ML inequality (Theorem 6.3)

$$\begin{aligned}\left|\int_{C_R} f(z)\,dz\right| &= \left|\int_0^{\pi} \frac{\ln R + i\theta}{R^2 e^{i2\theta} + a^2} iRe^{i\theta}\,d\theta\right| \\ &\leq \frac{R(\ln R + \pi)\pi}{R^2 - a^2},\end{aligned}$$

and L'Hôpital's rule yields $\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0$. A similar computation shows that $\lim_{r\to 0^+} \int_{C_r} f(z)\,dz = 0$. We use these results when we take corresponding limits in Equations (8-33) to get

$$\text{P.V.}\left(\int_{-\infty}^{0} \frac{\ln|x| + i\pi}{x^2+a^2}\,dx + \int_0^{\infty} \frac{\ln x}{x^2+a^2}\,dx\right) = \frac{\pi \ln a}{a} + i\frac{\pi^2}{2a}.$$

Equating the real parts in this equation gives

$$\text{P.V.} \int_0^{\infty} \frac{\ln x}{x^2+a^2}\,dx = \frac{\pi \ln a}{2a}.$$

Remark 8.3 The theory of this section is not purely esoteric. Many applications of contour integrals surface in government and industry worldwide. Many years ago, for example, a briefing was given at the Korean Institute for Defense Analysis (KIDA) in which a sophisticated problem was analyzed by means of a contour integral whose path of integration was virtually identical to that given in Figure 8.8. ■

EXERCISES FOR SECTION 8.6

Use residues to compute

1. P.V. $\int_0^{\infty} \frac{dx}{x^{\frac{2}{3}}(1+x)}$.

2. P.V. $\int_0^{\infty} \frac{dx}{x^{\frac{1}{2}}(1+x)}$.

3. P.V. $\int_0^\infty \dfrac{x^{\frac{1}{2}}\ dx}{(1+x)^2}$.

4. P.V. $\int_0^\infty \dfrac{x^{\frac{1}{2}}\ dx}{1+x^2}$.

5. P.V. $\int_0^\infty \dfrac{\ln\left(x^2+1\right)\ dx}{x^2+1}$. *Hint*: Use the integrand $f(z) = \frac{\log(z+i)}{z^2+1}$.

6. P.V. $\int_0^\infty \dfrac{\ln x\ dx}{(1+x^2)^2}$.

7. P.V. $\int_0^\infty \frac{(\ln x)^2}{x^2+1}\,dx$.

8. P.V. $\int_0^\infty \frac{x^{1/2}\ln x}{x^2+1}\,dx$.

9. P.V. $\int_0^\infty \frac{\ln x}{x^2+2^2}\,dx$.

10. Carry out the following computations:

(a) For $f(z) = \frac{z^{1/3}}{z^3(z+1)}$, show that $\text{Res}[f,-1] = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

(b) Use part (a) and $\alpha = \frac{1}{3}$ to verify that $\frac{2\pi i}{1-e^{i\alpha 2\pi}}\text{Res}[f,-1] = \frac{2\sqrt{3}}{3}\pi$.

(c) Can you conclude that P.V. $\int_0^\infty \frac{x^{1/3}}{x^3(x+1)}\,dx = \frac{2\sqrt{3}}{3}\pi$? Justify your answer.

11. Carry out the following computations:

(a) For $f(z) = \frac{z^{4/3}}{z+1}$, show that $\text{Res}[f,-1] = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.

(b) Use part (a) and $\alpha = \frac{4}{3}$ to verify that $\frac{2\pi i}{1-e^{i\alpha 2\pi}}\text{Res}[f,-1] = \frac{2\sqrt{3}}{3}\pi$.

(c) Can you say that P.V. $\int_0^\infty \frac{x^{4/3}}{x+1}\,dx = \frac{2\sqrt{3}}{3}\pi$? Justify your answer.

12. P.V. $\int_0^\infty \frac{1}{x^{1/2}(x+1)^2}\,dx$.

13. P.V. $\int_0^\infty \frac{1}{x^{1/2}(1+x^2)}\,dx$.

14. P.V. $\int_0^\infty \frac{x^{1/3}}{(x+1)^2}\,dx$.

15. P.V. $\int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx$.

16. P.V. $\int_0^\infty \frac{x^{1/3}\ln x}{x^2+1}\,dx$ and P.V. $\int_0^\infty \frac{x^{1/3}}{x^2+1}\,dx$.
Hint: Use the complex integrand $f(z) = \frac{z^{1/3}\text{Log}z}{z^2+1}$.

17. P.V. $\int_0^\infty \dfrac{\ln(1+x)}{x^{1+a}}$, where $0 < a < 1$.

18. P.V. $\int_0^\infty \frac{\ln x \, dx}{(x+a)^2}$, where $a > 0$.

19. P.V. $\int_{-\infty}^\infty \frac{\sin x}{x} \, dx$. *Hint*: Use the integrand $f(z) = \frac{\exp(iz)}{z}$ and the contour C in Figure 8.9. Let $r \to 0$ and $R \to \infty$.

20. P.V. $\int_{-\infty}^\infty \frac{\sin^2 x}{x^2} \, dx$. *Hint*: Use the integrand $f(z) = \frac{1 - \exp(i2z)}{z^2}$ and the contour C in Figure 8.9. Let $r \to 0$ and $R \to \infty$.

21. The Fresnel integrals $\int_0^\infty \cos(x^2) \, dx$ and $\int_0^\infty \sin(x^2) \, dx$ are important in the study of optics. Use the integrand $f(z) = \exp(-z^2)$ and the contour C shown in Figure 8.10, and let $R \to \infty$ to get the value of these integrals. Use the fact from calculus that $\int_0^\infty e^{-x^2} dx = \sqrt{\frac{\pi}{2}}$.

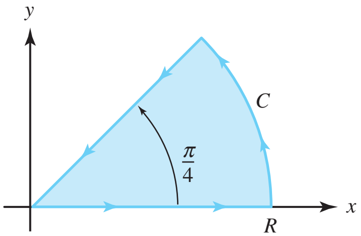

Figure 8.10 For Exercise 21.

8.7 THE ARGUMENT PRINCIPLE AND ROUCHÉ'S THEOREM

We now derive two results based on Cauchy's residue theorem. They have important practical applications and pertain only to functions all of whose isolated singularities are poles.

> **Definition 8.3: Meromorphic function**
>
> A function f is said to be **meromorphic** in a domain D provided the only singularities of f are isolated poles and removable singularities.

We make three important observations relating to this definition.

- Analytic functions are a special case of meromorphic functions.
- Rational functions $f(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials, are meromorphic in the entire complex plane.

◆ By definition, meromorphic functions have no essential singularities.

Suppose that f is analytic at each point on a simple closed contour C and f is meromorphic in the domain that is the interior of C. We assert without proof that Theorem 7.13 can be extended to meromorphic functions so that f has at most a finite number of zeros that lie inside C. Since the function $g(z) = \frac{1}{f(z)}$ is also meromorphic, it can have only a finite number of zeros inside C, and so f can have at most a finite number of poles that lie inside C.

Theorem 8.8, known as the argument principle, is useful in determining the number of zeros and poles that a function has.

Theorem 8.8 (Argument principle) *Suppose that f is meromorphic in the simply connected domain D and that C is a simple closed positively oriented contour in D such that f has no zeros or poles for $z \in C$. Then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = Z_f - P_f, \tag{8-34}$$

where Z_f is the number of zeros of f that lie inside C and P_f is the number of poles of f that lie inside C.

Proof Let $a_1, a_2, \ldots, a_{Z_f}$ be the zeros of f inside C counted according to multiplicity and let $b_1, b_2, \ldots, b_{P_f}$ be the poles of f inside C counted according to multiplicity. Then $f(z)$ has the representation

$$f(z) = \frac{(z-a_1)(z-a_2)\cdots(z-a_{Z_f})}{(z-b_1)(z-b_2)\cdots(z-b_{P_f})}g(z),$$

where g is analytic and nonzero on C and inside C. An elementary calculation shows that

$$\frac{f'(z)}{f(z)} = \frac{1}{(z-a_1)} + \frac{1}{(z-a_2)} + \cdots + \frac{1}{(z-a_{Z_f})} - \frac{1}{(z-b_1)} - \frac{1}{(z-b_2)} - \cdots - \frac{1}{(z-b_{P_f})} + \frac{g'(z)}{g(z)}. \tag{8-35}$$

According to Corollary 6.1, we have

$$\int_C \frac{dz}{(z-a_j)} = 2\pi i, \qquad \text{for } j = 1, 2, \ldots, Z_f, \quad \text{and}$$
$$\int_C \frac{dz}{(z-b_k)} = 2\pi i, \qquad \text{for } k = 1, 2, \ldots, P_f.$$

The function $\frac{g'(z)}{g(z)}$ is analytic inside and on C, so the Cauchy–Goursat theorem gives $\int_C \frac{g'(z)}{g(z)}\,dz = 0$. These facts lead to the conclusion of our theorem if we integrate both sides of Equation (8-35) over C.

Corollary 8.1 Suppose that f is analytic in the simply connected domain D. Let C be a simple closed positively oriented contour in D such that for $z \in C$, $f(z) \neq 0$. Then

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = Z_f,$$

where Z_f is the number of zeros of f that lie inside C. ■

Remark 8.4 Certain feedback control systems in engineering must be stable. A test for stability involves the function $G(z) = 1 + F(z)$, where F is a rational function. If G does not have any zeros in the region $\{z : \mathrm{Re}(z) \geq 0\}$, then the system is stable. We determine the number of zeros of G by writing $F(z) = \frac{P(z)}{Q(z)}$, where P and Q are polynomials with no common zero. Then $G(z) = \frac{Q(z)+P(z)}{Q(z)}$, and we can check for the zeros of $Q(z)+P(z)$ by using Theorem 8.8. We select a value R so that $G(z) \neq 0$ for $\{z : |z| > R\}$ and then integrate along the contour consisting of the right half of the circle $C_R(0)$ and the line segment between iR and $-iR$. This method is known as the *Nyquist stability criterion*. ■

Why do we label Theorem 8.8 as the *argument principle*? The answer lies with a fascinating application known as the **winding number**. Recall that a branch of the logarithm function, $\log_\alpha$, is defined by

$$\log_\alpha z = \ln|z| + i\arg_\alpha z = \ln r + i\phi,$$

where $z = re^{i\phi} \neq 0$ and $\alpha < \phi \leq \alpha + 2\pi$. Loosely speaking, suppose that for some branch of the logarithm, the composite function $\log_\alpha(f(z))$ were analytic in a simply connected domain D containing the contour C. This would imply that $\log_\alpha(f(z))$ is an antiderivative of the function $\frac{f'(z)}{f(z)}$ for all $z \in D$. Theorems 6.9 and 8.8 would then tell us that, as z winds around the curve C, the quantity $\log_\alpha(f(z)) = \ln|f(z)| + i\arg_\alpha f(z)$ would change by $2\pi i(Z_f - P_f)$. Since $2\pi i(Z_f - P_f)$ is purely imaginary, this result tells us that $\arg_a f(z)$ would change by $2\pi(Z_f - P_f)$ radians. In other words, as z winds around C, the integral $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$ would count how many times the curve $f(C)$ winds around the origin.

Unfortunately, we can't always claim that $\log_\alpha(f(z))$ is an antiderivative of the function $\frac{f'(z)}{f(z)}$ for all $z \in D$. If it were, the Cauchy–Goursat theorem

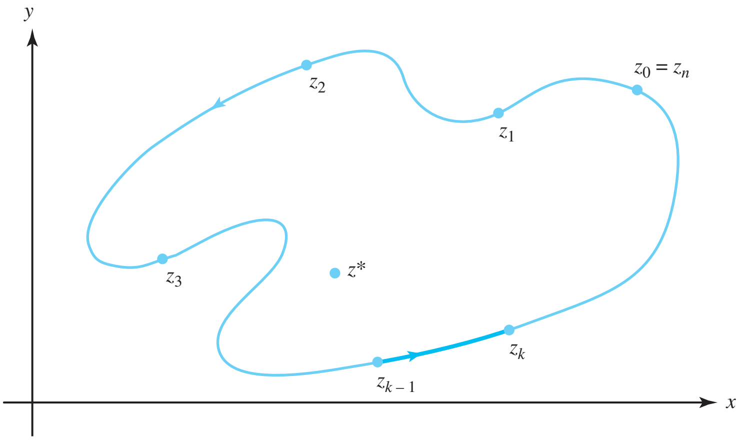

Figure 8.11 The points z_k on the contour C that winds around z^*.

would imply that $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = 0$. Nevertheless, the heuristics that we gave—indicating that $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz$ counts how many times the curve $f(C)$ winds around the origin—still hold true, as we now demonstrate.

Suppose that $C : z(t) = x(t) + iy(t)$ for $a \le t \le b$ is a simple closed contour and that we let $a = t_0 < t_1 < \cdots < t_n = b$ be a partition of the interval $[a, b]$. For $k = 0, 1, \ldots, n$, we let $z_k = z(t_k)$ denote the corresponding points on C, where $z_0 = z_n$. If z^* lies inside C, then the curve $C : z(t)$ winds around z^* once as t goes from a to b, as shown in Figure 8.11.

Now suppose that a function f is analytic at each point on C and meromorphic inside C. Then $f(C)$ is a closed curve in the w plane that passes through the points $w_k = f(z_k)$, for $k = 0, 1, \ldots, n$, where $w_0 = w_n$. We can choose subintervals $[t_{k-1}, t_k]$ small enough so that, on the portion of $f(C)$ between w_{k-1} and w_k, we can define a continuous branch of the logarithm

$$\log_{\alpha_k} w = \ln|w| + i \arg_{\alpha_k} w = \ln\rho + i\phi,$$

where $w = \rho e^{i\phi}$ and $\alpha_k < \phi < \alpha_k + 2\pi$, as shown in Figure 8.12. Then

$$\log_{\alpha_k} f(z_k) - \log_{\alpha_k} f(z_{k-1}) = \ln\rho_k - \ln\rho_{k-1} + i\Delta\phi_k,$$

where $\Delta\phi_k = \phi_k - \phi_{k-1}$ measures in radians the amount that the portion of the curve $f(C)$ between w_k and w_{k-1} winds around the origin. With small enough subintervals $[t_{k-1}, t_k]$, the angles α_{k-1} and α_k might be different, but the values $\arg_{\alpha_{k-1}} w_{k-1}$ and $\arg_{\alpha_k} w_{k-1}$ will be the same, so that $\log_{\alpha_{k-1}} w_{k-1} = \log_{\alpha_k} w_{k-1}$.

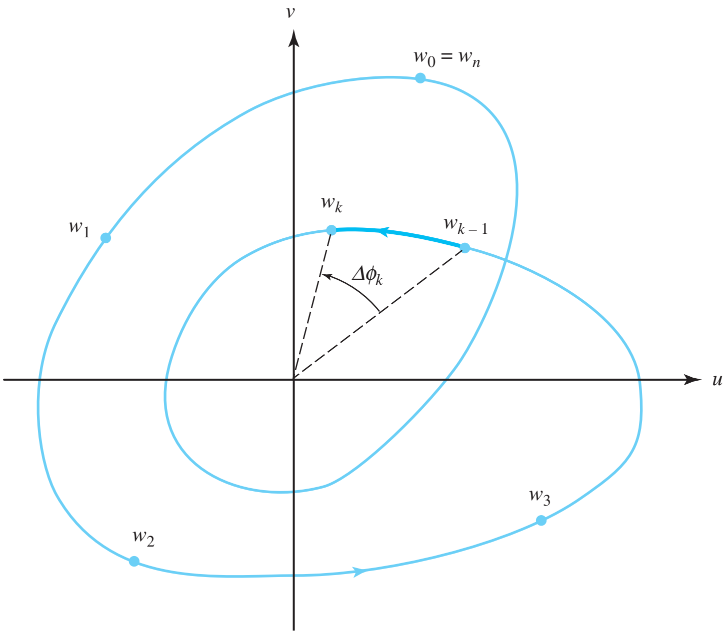

Figure 8.12 The points w_k on the contour $f(C)$ that winds around 0.

We can now show why $\int_C \frac{f'(z)}{f(z)}\,dz$ counts the number of times that $f(C)$ winds around the origin. We parametrize $C: z(t)$, for $a \le t \le b$, and choose the appropriate branches of $\log_{\alpha_k} w$, giving

$$\begin{aligned}\int_C \frac{f'(z)}{f(z)}\,dz &= \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} \frac{f'(z(t))}{f(z(t))} z'(t)\,dt \\ &= \sum_{k=1}^{n} \left(\log_{\alpha_k}[f(z(t_k))] - \log_{\alpha_k}[f(z(t_{k-1}))]\right) \\ &= \sum_{k=1}^{n} \left(\log_{\alpha_k} w_k - \log_{\alpha_k} w_{k-1}\right),\end{aligned}$$

which we rewrite as

$$\int_C \frac{f'(z)}{f(z)}\,dz = \sum_{k=1}^{n} [\ln \rho_k - \ln \rho_{k-1}] + i \sum_{k=1}^{n} \Delta\phi_k. \tag{8-36}$$

When we use the fact that $\rho_0 = \rho_n$, the first summation in Equation (8-36) vanishes. The summation of the quantities $\Delta\phi_k$ expresses the accumulated radian measure of $f(C)$ around the origin. Therefore, when we divide both

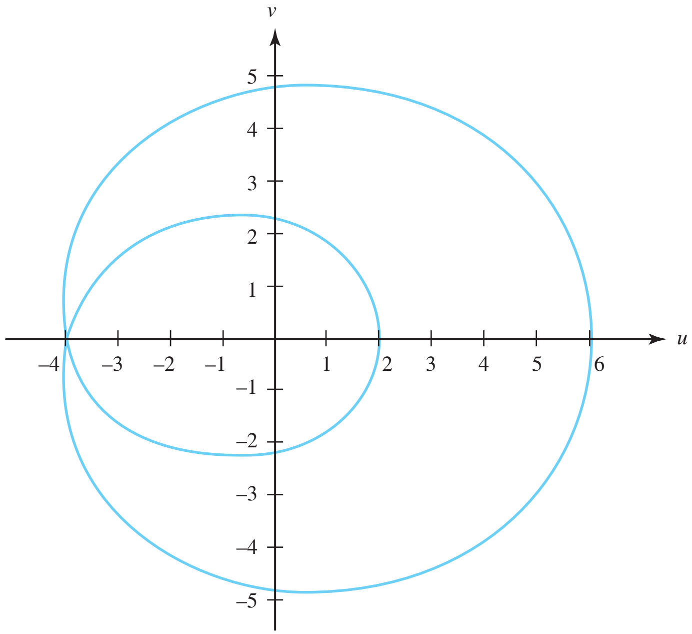

Figure 8.13 The image curve $f(C_2(0))$ under $f(z) = z^2 + z$.

sides of Equation (8-36) by $2\pi i$, its right side becomes an integer (by Theorem 8.8) that must count the number of times $f(C)$ winds around the origin.

■ EXAMPLE 8.25 The image of the circle $C_2(0)$ under $f(z) = z^2 + z$ is the curve $\{(x, y) = (4\cos 2t + 2\cos t, 4\sin 2t + 2\sin t) : 0 < t < 2\pi\}$ shown in Figure 8.13. Note that the image curve $f(C_2(0))$ winds twice around the origin. We check this by computing $\frac{1}{2\pi i}\int_{C_2^+(0)} \frac{f'(z)}{f(z)} dz = \frac{1}{2\pi i}\int_{C_2^+(0)} \frac{2z+1}{z^2+z} dz$. The residues of the integrand are at 0 and -1. Thus,

$$\frac{1}{2\pi i}\int_{C_2^+(0)} \frac{2z+1}{z^2+z} dz = \operatorname{Res}\left[\frac{2z+1}{z^2+z}, 0\right] + \operatorname{Res}\left[\frac{2z+1}{z^2+z}, -1\right]$$
$$= 1 + 1 = 2.$$

Finally, we note that if $g(z) = f(z) - a$, then $g'(z) = f(z)$, and thus we can generalize what we've just said to compute how many times the curve $f(C)$ winds around the point a. Theorem 8.9 summarizes our discussion.

Theorem 8.9 (Winding numbers) *Suppose that f is meromorphic in the simply connected domain D. If C is a simple closed positively oriented contour in D such that for $z \in C$, $f(z) \neq 0$ and $f(z) \neq \infty$, then*

$$W(f(C), a) = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z) - a}\, dz,$$

known as the **winding number of $f(C)$ about a,** *counts the number of times the curve $f(C)$ winds around the point a. If $a = 0$, the integral counts the number of times the curve $f(C)$ winds around the origin.*

Remark 8.5 Letting $f(z) = z$ in Theorem 8.9 gives

$$W(C, a) = \frac{1}{2\pi i} \int_C \frac{1}{z - a}\, dz = \begin{cases} 1 & \text{if } a \text{ is inside } C, \quad \text{or} \\ 0 & \text{if } a \text{ is outside } C, \end{cases}$$

which counts the number of times the curve C winds around the point a. If C is not a simple closed curve, but crosses itself perhaps several times, we can show (but omit the proof) that $W(C, a)$ still gives the number of times the curve C winds around the point a. Thus, *winding number* is indeed an appropriate term. ■

We close this section with a result that will help us gain information about the location of the zeros and poles of meromorphic functions.

Theorem 8.10 (Rouché's theorem) *Suppose that f and g are meromorphic functions defined in the simply connected domain D, that C is a simply closed contour in D, and that f and g have no zeros or poles for $z \in C$. If the strict inequality $|f(z) + g(z)| < |f(z)| + |g(z)|$ holds for all $z \in C$, then $Z_f - P_f = Z_g - P_g$.*

Proof Because g has no zeros or poles on C, we may legitimately divide both sides of the inequality $|f(z) + g(z)| < |f(z)| + |g(z)|$ by $|g(z)|$ to get

$$\left| \frac{f(z)}{g(z)} + 1 \right| < \left| \frac{f(z)}{g(z)} \right| + 1, \qquad \text{for all } z \in C. \tag{8-37}$$

For $z \in C$, $\frac{f(z)}{g(z)}$ cannot possibly be zero or any positive real number, as that would contradict Inequality (8-37). This means that C^*, the image of the curve C under the mapping $\frac{f}{g}$, does not contain the interval $[0, \infty)$, and so the function defined by

$$w(z) = \log_0 \left(\frac{f(z)}{g(z)} \right) = \ln \left| \frac{f(z)}{g(z)} \right| + i \arg_0 \left(\frac{f(z)}{g(z)} \right) = \ln r + i\phi,$$

where $\frac{f(z)}{g(z)} = re^{i\phi} \neq 0$ and $0 < \phi \leq 2\pi$, is analytic in a simply connected domain D^* that contains C^*. We calculate

$$w'(z) = \frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)},$$

so $w(z) = \log_0\left(\frac{f(z)}{g(z)}\right)$ is an antiderivative of $\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}$, for all $z \in D^*$. As C^* is a closed curve in D^*, Theorem 6.9 gives $\int_{C^*}\left(\frac{f'(z)}{f(z)} - \frac{g'(z)}{g(z)}\right) dz = 0$. According to Theorem 8.8, then

$$\int_{C^*} \frac{f'(z)}{f(z)}dz - \int_{C^*} \frac{g'(z)}{g(z)}dz = (Z_f - P_f) - (Z_g - P_g) = 0,$$

which completes the proof.

Corollary 8.2 Suppose that f and g are analytic functions defined in the simply connected domain D, that C is a simple closed contour in D, and that f and g have no zeros for $z \in C$. If the strict inequality $|f(z) + g(z)| < |f(z)| + |g(z)|$ holds for all $z \in C$, then $Z_f = Z_g$. ■

Remark 8.6 Theorem 8.10 is usually stated with the requirement that f and g satisfy the condition $|f(z) + g(z)| < |g(z)|$, for $z \in C$. The improved theorem that we gave was discovered by Irving Glicksberg (see the *American Mathematical Monthly*, 83 (1976), pp. 186–187). The weaker version is adequate for most purposes, however, as the following examples illustrate. ■

■ EXAMPLE 8.26 Consider the polynomial $g(z) = z^4 - 7z - 1$ and show that all four of its zeros lie in the disk $D_2(0) = \{z : |z| < 2\}$

Solution Let $f(z) = -z^4$. Then $f(z) + g(z) = -7z - 1$, and at points on the circle $C_2(0) = \{z : |z| = 2\}$ we have the relation

$$|f(z) + g(z)| \leq |-7z| + |-1| = 7(2) + 1 < 16 = |f(z)|.$$

Of course, if $|f(z) + g(z)| < |f(z)|$, then as we indicated in Remark 8.6 we certainly have $|f(z) + g(z)| < |f(z)| + |g(z)|$, so that the conditions for applying Corollary 8.2 are satisfied on the circle $C_2(0)$. The function f has a zero of order 4 at the origin, so g must have four zeros inside $D_2(0)$.

■ EXAMPLE 8.27 Show that the polynomial $g(z) = z^4 - 7z - 1$ has one zero in the disk $D_1(0)$.

Solution Let $f(z) = 7z + 1$, then $f(z) + g(z) = z^4$. At points on the circle $C_1(0) = \{z : |z| = 1\}$ we have the relation

$$|f(z) + g(z)| = |z^4| = 1 < 6 = |7| - |1| \leq |7z - 1| = |f(z)|.$$

The function f has one zero at the point $-\frac{1}{7}$ in the disk $D_1(0)$, and the hypotheses of Corollary 8.2 hold on the circle $C_1(0)$. Therefore, g has one zero inside $D_1(0)$.

EXERCISES FOR SECTION 8.7

1. Let $f(z) = z^5 - z$. Find the number of times the image $f(C)$ winds around the origin if

(a) $C = C_{\frac{1}{2}}(0)$.

(b) C is the rectangle with vertices $\pm\frac{1}{2} \pm 3i$.

(c) $C = C_2(0)$.

(d) $C = C_{\frac{5}{4}}(i)$.

2. Show that four of the five roots of the equation $z^5 + 15z + 1 = 0$ belong to the annulus $A\left(\frac{3}{2}, 2, 0\right) = \left\{z : \frac{3}{2} < |z| < 2\right\}$.

3. Let $g(z) = z^5 + 4z - 15$.

(a) Show that there are no zeros in $D_1(0)$.

(b) Show that there are five zeros in $D_2(0)$. *Hint*: Consider $f(z) = -z^5$.
Remark: A factorization of the polynomial using numerical approximations for the coefficients is

$$(z - 1.546)\left(z^2 - 1.340z + 2.857\right)\left(z^2 + 2.885z + 3.397\right).$$

4. Let $g(z) = z^3 + 9z + 27$.

(a) Show that there are no zeros in $D_2(0)$.

(b) Show that there are three zeros in $D_4(0)$.
Remark: A factorization of the polynomial using numerical approximations for the coefficients is

$$(z + 2.047)\left(z^2 - 2.047z + 13.19\right).$$

5. Let $g(z) = z^5 + 6z^2 + 2z + 1$.

(a) Show that there are two zeros in $D_1(0)$.

(b) Show that there are five zeros in $D_2(0)$.

6. Let $g(z) = z^6 - 5z^4 + 10$.

(a) Show that there are no zeros in $|z| < 1$.

(b) Show that there are four zeros in $|z| < 2$.

(c) Show that there are six zeros in $|z| < 3$.

7. Let $g(z) = 3z^3 - 2iz^2 + iz - 7$.

(a) Show that there are no zeros in $|z| < 1$.

(b) Show that there are three zeros in $|z| < 2$.

8. Use Rouché's theorem to prove the fundamental theorem of algebra. *Hint*: For the polynomial $g(z) = a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n$, let $f(z) = -a_n z^n$. Show that, for points z on the circle $C_R(0)$,

$$\left| \frac{f(z) + g(z)}{f(z)} \right| < \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n| R},$$

and conclude that the right side of this inequality is less than 1 when R is large.

9. Suppose that $h(z)$ is analytic and nonzero and $|h(z)| < 1$ for $z \in D_1(0)$. Prove that the function $g(z) = h(z) - z^n$ has n zeros inside the unit circle $C_1(0)$.

10. Suppose that $f(z)$ is analytic inside and on the simple closed contour C. If $f(z)$ is a one-to-one function at points z on C, then prove that $f(z)$ is one-to-one inside C. *Hint*: Consider the image of C.