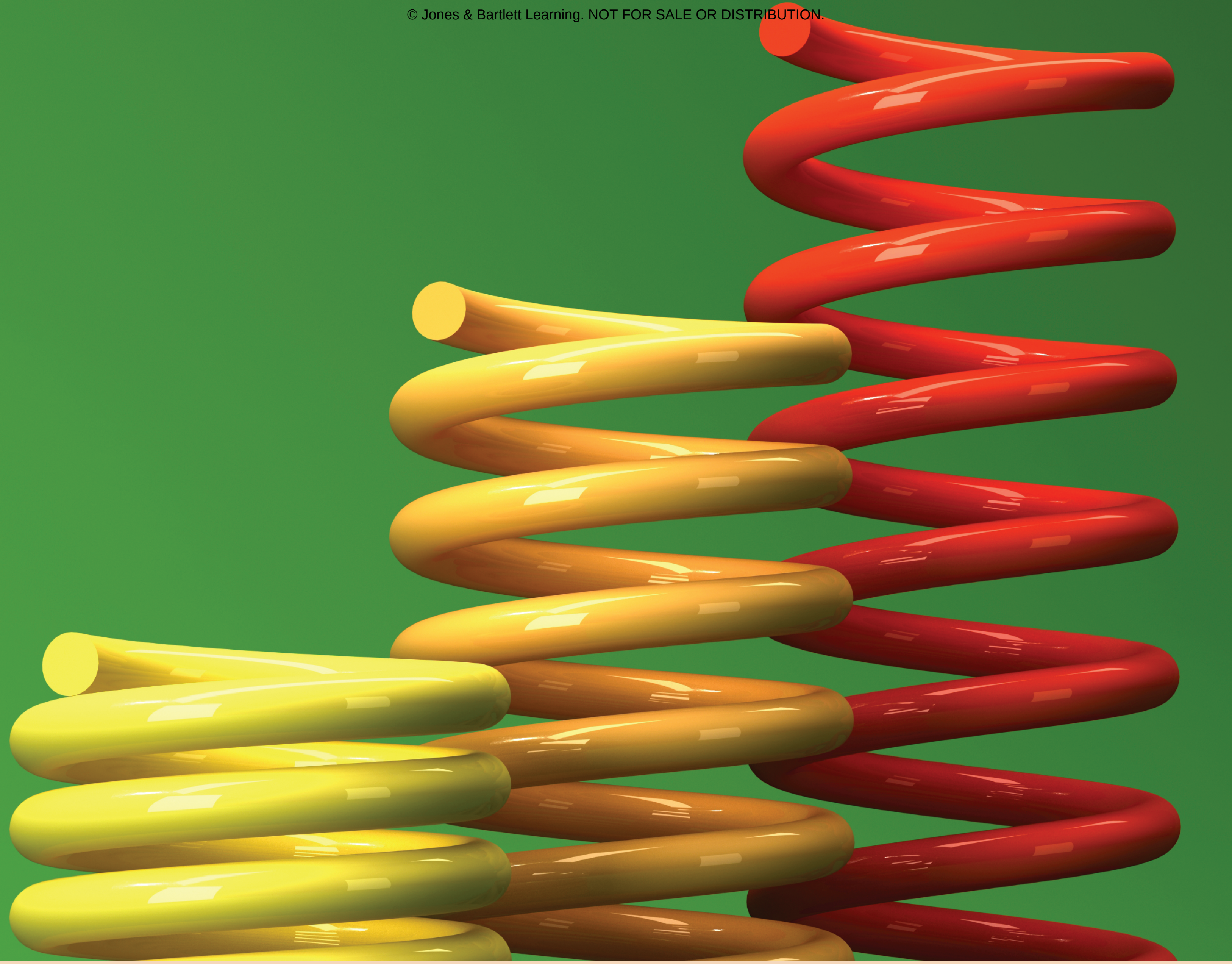
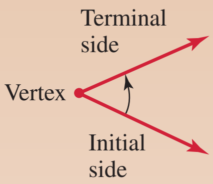


4 Trigonometric Functions

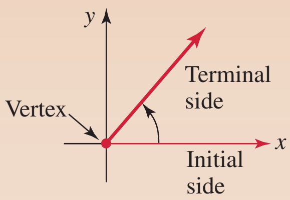
Chapter Contents

4.1 Angles and Their Measurement

INTRODUCTION We begin our study of trigonometry by discussing angles and two methods of measuring them: degrees and radians. As we will see it is the radian measure of an angle that enables us to define trigonometric functions on sets of real numbers.



(a) Two half-rays



(b) Standard position

FIGURE 4.1.1 Initial and terminal sides of an angle

Angles An **angle** is formed by two half-rays, or half-lines, which have a common endpoint, called the **vertex**. We designate one ray the **initial side** of the angle and the other the **terminal side**. It is useful to consider the angle as having been formed by a rotation from the initial side to the terminal side as shown in FIGURE 4.1.1(a). An angle is said to be in **standard position** if its vertex is placed at the origin of a rectangular coordinate system with its initial side coinciding with the positive x-axis, as shown in Figure 4.1.1(b).

Degree Measure The **degree measure** of an angle is based on the assignment of 360 degrees (written 360°) to the angle formed by one complete counterclockwise rotation, as shown in FIGURE 4.1.2. Other angles are then measured in terms of a 360° angle, with a 1° angle being formed by $\frac{1}{360}$ of a complete rotation. If the rotation is counterclockwise, the measure will be *positive*; if clockwise, the measure is *negative*. For example, the angle in FIGURE 4.1.3(a) obtained by one-fourth of a complete counterclockwise rotation will be

$$\tfrac{1}{4}(360^\circ) = 90^\circ.$$

Shown in Figure 4.1.3(b) is the angle formed by three-fourths of a complete clockwise rotation. This angle has measure

$$\tfrac{3}{4}(-360^\circ) = -270^\circ.$$

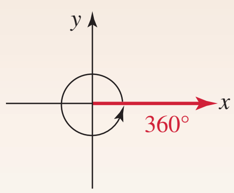
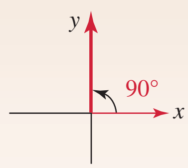
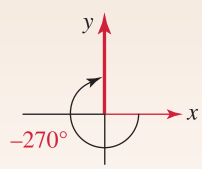


FIGURE 4.1.2 Angle of 360 degrees



(a) 90° angle



(b) –270° angle

FIGURE 4.1.3 Positive measure in (a); negative measure in (b)

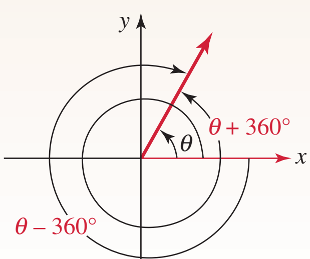


FIGURE 4.1.4 Three coterminal angles

Coterminal Angles Comparison of Figure 4.1.3(a) with Figure 4.1.3(b) shows that the terminal side of a 90° angle coincides with the terminal side of a $-270°$ angle. When two angles in standard position have the same terminal sides we say they are **coterminal**. For example, the angles θ, $\theta + 360°$, and $\theta - 360°$ shown in FIGURE 4.1.4 are coterminal. In fact, the addition of any integer multiple of 360° to a given angle results in a coterminal angle. Conversely, any two coterminal angles have degree measures that differ by an integer multiple of 360°.

EXAMPLE 1 Angles and Coterminal Angles

For a 960° angle:

(a) Locate the terminal side and sketch the angle.
(b) Find a coterminal angle between 0° and 360°.
(c) Find a coterminal angle between $-360°$ and 0°.

Solution (a) We first determine how many full rotations are made in forming this angle. Dividing 960 by 360 we obtain a quotient of 2 and a remainder of 240. Equivalently we can write

$$960 = 2(360) + 240.$$

Thus, this angle is formed by making two counterclockwise rotations before completing $\frac{240}{360} = \frac{2}{3}$ of another rotation. As illustrated in FIGURE 4.1.5(a), the terminal side of 960° lies in the third quadrant.

(b) Figure 4.1.5(b) shows that the angle 240° is coterminal with a 960° angle.

(c) Figure 4.1.5(c) shows that the angle −120° is coterminal with a 960° angle.

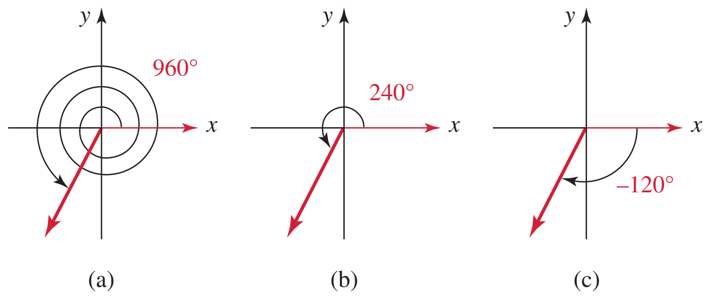


FIGURE 4.1.5 Angles in (b) and (c) are coterminal with the angle in (a)

Minutes and Seconds With calculators it is convenient to represent fractions of degrees by decimals, such as 42.23°. Traditionally, however, fractions of degrees were expressed in **minutes** and **seconds**, where

$$1° = 60 \text{ minutes (written } 60')^* \tag{1}$$

and

$$1' = 60 \text{ seconds (written } 60''). \tag{2}$$

For example, an angle of 7 degrees, 30 minutes, and 5 seconds is expressed as $7°30'5''$. Some calculators have a special DMS key for converting an angle given in decimal degrees to Degrees, Minutes, and Seconds (DMS notation), and vice versa. The following example shows how to perform these conversions by hand.

EXAMPLE 2 Using (1) and (2)

Convert:

(a) 86.23° to degrees, minutes, and seconds,
(b) $17°47'13''$ to decimal notation.

Solution In each case we will use (1) and (2).

(a) Since 0.23° represents $\frac{23}{100}$ of 1° and $1° = 60'$, we have

$$\begin{aligned} 86.23° &= 86° + 0.23° \\ &= 86° + (0.23)(60') \\ &= 86° + 13.8'. \end{aligned}$$

Now $13.8' = 13' + 0.8'$, so we must convert $0.8'$ to seconds. Since $0.8'$ represents $\frac{8}{10}$ of $1'$ and $1' = 60''$, we have

$$\begin{aligned} 86° + 13' + 0.8' &= 86° + 13' + (0.8)(60'') \\ &= 86° + 13' + 48''. \end{aligned}$$

Hence, $86.23° = 86°13'48''$.

---

*The use of the number 60 as a base dates back to the Babylonians. Another example of the use of this base in our culture in the measurement of time (1 hour = 60 minutes and 1 minute = 60 seconds).

(b) Since $1° = 60'$, it follows that $1' = \left(\frac{1}{60}\right)°$. Similarly, $1'' = \left(\frac{1}{60}\right)' = \left(\frac{1}{3600}\right)°$. Thus we have

$$\begin{aligned} 17°47'13'' &= 17° + 47' + 13'' \\ &= 17° + 47\left(\tfrac{1}{60}\right)° + 13\left(\tfrac{1}{3600}\right)° \\ &\approx 17° + 0.7833° + 0.0036°. \end{aligned}$$

Thus we see that $17°47'13'' \approx 17.7869°$.

Radian Measure Another measure for angles is **radian measure**, which is generally used in almost all applications of trigonometry that involve calculus. The radian measure of an angle θ is based on the length of an arc on a circle. If we place the vertex of the angle θ at the center of a circle of radius r, then θ is called a **central angle**. As we know, an angle θ in standard position can be viewed as having been formed by the initial side rotating from the positive x-axis to the terminal side. The region formed by the initial and terminal sides with a central angle θ is called a **circular sector**. As shown in FIGURE 4.1.6(a), if the initial side of θ traverses a distance s along the circumference of the circle, then the **radian measure of θ** is defined by

$$\theta = \frac{s}{r}. \tag{3}$$

In the case when the terminal side of θ traverses an arc of length s along the circumference of the circle equal to the radius r of the circle, then we see from (3) that the measure of the angle θ is **1 radian**. See Figure 4.1.6(b).

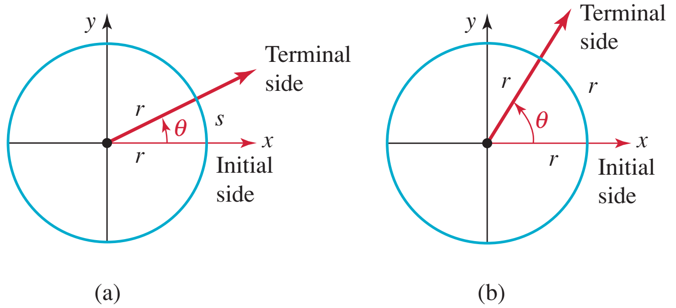


FIGURE 4.1.6 Central angle in (a); angle of 1 radian in (b)

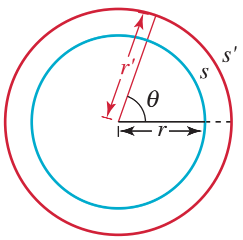


FIGURE 4.1.7 Concentric circles

The definition given in (3) does not depend on the size of the circle. To see this, all we need do is to draw another circle centered at the vertex of θ of radius $r'$and subtended arc length s'. See FIGURE 4.1.7. Because the two circular sectors are similar the ratios s/r and s'/r' are equal. Therefore, regardless of which circle we use, we obtain the same radian measure for θ.

In equation (3) any convenient unit of length may be used for s and r, but the same unit must be used for *both* s and r. Thus,

$$\theta\,(\text{in radians}) = \frac{s\,(\text{units of length})}{r\,(\text{units of length})}$$

appears to be a "dimensionless" quantity. For example, if $s = 4$ in. and $r = 2$ in., then the radian measure of the angle is

$$\theta = \frac{4 \text{ in.}}{2 \text{ in.}} = 2,$$

where 2 is simply a real number. This is the reason why sometimes the word *radians* is omitted when an angle is measured in radians. We will come back to this idea in Section 4.2.

One complete rotation of the initial side of θ will traverse an arc equal in length to the circumference of the circle $2\pi r$. It follows from (3) that

$$\text{one rotation} = \frac{s}{r} = \frac{2\pi r}{r} = 2\pi \text{ radians.}$$

We have the same convention as before: An angle formed by a counterclockwise rotation is considered positive, whereas an angle formed by a clockwise rotation is negative. In FIGURE 4.1.8 we illustrate angles in standard position of $\pi/2$, $-\pi/2$, π, and 3π radians, respectively. Note that the angle of $\pi/2$ radians shown in 4.1.8(a) is obtained by one-fourth of a complete counterclockwise rotation; that is

$$\frac{1}{4}(2\pi \text{ radians}) = \frac{\pi}{2} \text{ radians.}$$

The angle shown in Figure 4.1.8(b), obtained by one-fourth of a complete clockwise rotation, is $-\pi/2$ radians. The angle shown in Figure 4.1.8(c) is coterminal with the angle shown in Figure 4.1.8(d). In general, the addition of any integer multiple of 2π radians to an angle measured in radians results in a coterminal angle. Conversely, any two coterminal angles measured in radians will differ by an integer multiple of 2π.

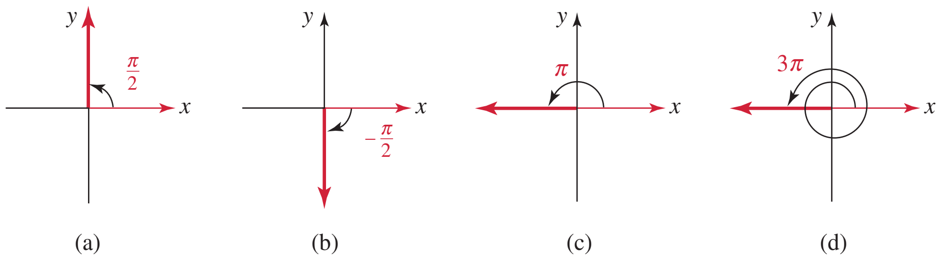


FIGURE 4.1.8 Angles measured in radians

EXAMPLE 3 A Coterminal Angle

Find an angle between 0 and 2π radians that is coterminal with $\theta = 11\pi/4$ radians. Sketch the angle.

Solution Since $2\pi < 11\pi/4 < 3\pi$, we subtract the equivalent of one rotation, or 2π radians, to obtain

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} = \frac{3\pi}{4}.$$

Alternatively, we can proceed as in part (a) of Example 1 and divide: $11\pi/4 = 2\pi + 3\pi/4$. Thus, an angle of $3\pi/4$ radians is coterminal with θ, as illustrated in FIGURE 4.1.9.

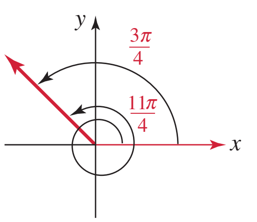


FIGURE 4.1.9 Coterminal angles in Example 3

Conversion Formulas Although many scientific calculators have keys that convert between degree and radian measure, there is an easy way to remember the relationship between the two measures. Since the circumference of a unit circle is 2π, one complete rotation has measure 2π radians as well as 360°. It follows that $360° = 2\pi$ radians or

$$180° = \pi \text{ radians.} \tag{4}$$

If we interpret (4) as $180(1°) = \pi(1 \text{ radian})$, then we obtain the following two formulas for converting between degree and radian measure.

Conversion Between Degrees and Radians

$$1^\circ = \frac{\pi}{180} \text{ radian} \tag{5}$$

$$1 \text{ radian} = \left(\frac{180}{\pi}\right)^\circ \tag{6}$$

Using a calculator to carry out the divisions in (5) and (6), we find that

$$1^\circ \approx 0.0174533 \text{ radian} \quad \text{and} \quad 1 \text{ radian} \approx 57.29578^\circ.$$

Although we will continue to use the terms *radian* and *radians* it is common to use the abbreviation *rad* for both words.

EXAMPLE 4 Conversion Between Degrees and Radians

Convert:

(a) 20° to radians **(b)** $7\pi/6$ radians to degrees **(c)** 2 radians to degrees.

Solution (a) To convert from degrees to radians we use (5):

$$20^\circ = 20(1^\circ) = 20 \cdot \left(\frac{\pi}{180} \text{ radian}\right) = \frac{\pi}{9} \text{ radian}.$$

(b) To convert from radians to degrees we use (6):

$$\frac{7\pi}{6} \text{ radians} = \frac{7\pi}{6} \cdot (1 \text{ radian}) = \frac{7\pi}{6}\left(\frac{180}{\pi}\right)^\circ = 210^\circ.$$

(c) We again use (6):

$$2 \text{ radians} = 2 \cdot (1 \text{ radian}) = 2 \cdot \left(\frac{180}{\pi}\right)^\circ = \left(\frac{360}{\pi}\right)^\circ \approx \overbrace{114.59^\circ}^{\text{approximate answer rounded to two decimal places}}.$$

Table 4.1.1 provides the radian and degree measure of the most commonly used angles.

TABLE 4.1.1

Degrees	0	30	45	60	90	180
Radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π

Terminology You may recall from geometry that a 90° angle is called a **right angle** and a 180° angle is called a **straight angle**. In radian measure, $\pi/2$ is a right angle and π is a straight angle. An **acute angle** has measure between 0° and 90° (or between 0 and $\pi/2$ radians); and an **obtuse angle** has measure between 90° and 180° (or between $\pi/2$ and π radians). Two acute angles are said to be **complementary** if their sum is 90° (or $\pi/2$ radians). Two positive angles are **supplementary** if their sum is 180° (or π radians). The angle 180° (or π radians) is a **straight angle**. An angle whose terminal side coincides with a coordinate axis is called a **quadrantal angle**. For example, 90° (or $\pi/2$ radians) is a quadrantal angle. A triangle that contains a right angle is called a **right triangle**. The lengths a, b, and c of the sides of a right triangle satisfy the Pythagorean theorem $a^2 + b^2 = c^2$, where c is the length of the side opposite the right angle (the hypotenuse).

EXAMPLE 5 Complementary and Supplementary Angles

(a) Find the angle that is complementary to $\theta = 74.23°$.
(b) Find the angle that is supplementary to $\phi = \pi/3$ radians.

Solution (a) Since two acute angles are complementary if their sum is 90°, we find the angle that is complementary to $\theta = 74.23°$ is

$$90° - \theta = 90° - 74.23° = 15.77°.$$

(b) Since two positive angles are supplementary if their sum is π radians, we find the angle that is supplementary to $\phi = \pi/3$ radians is

$$\pi - \phi = \pi - \frac{\pi}{3} = \frac{3\pi}{3} - \frac{\pi}{3} = \frac{2\pi}{3} \text{ radians.}$$

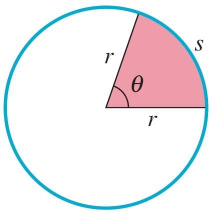


FIGURE 4.1.10 Length of arc s determined by a central angle θ

Arc Length In many applications it is necessary to find the length s of the arc subtended by a central angle θ in a circle of radius r. See FIGURE 4.1.10. From the definition of radian measure given in (3),

$$\theta \text{ (in radians)} = \frac{s}{r}.$$

By multiplying both sides of the last equation by r we obtain the **arc length formula** $s = r\theta$. We summarize the result.

THEOREM 4.1.1 Arc Length Formula

For a circle of radius r, a central angle of θ radians subtends an **arc of length**

$$s = r\theta \tag{7}$$

EXAMPLE 6 Finding Arc Length

Find the arc length subtended by a central angle of **(a)** 2 radians in a circle of radius 6 inches, **(b)** 30° in a circle of radius 12 feet.

Solution (a) From the arc length formula (7) with $\theta = 2$ radians and $r = 6$ inches, we have $s = r\theta = 2 \cdot 6 = 12$. So the arc length is 12 inches.

Students often apply the arc length formula incorrectly by using degree measure. Remember $s = r\theta$ is valid only if θ is measured in radians.

(b) We must first express 30° in radians. Recall that $30° = \pi/6$ radians. Then from the arc length formula (7) we have $s = r\theta = (12)(\pi/6) = 2\pi$. So the arc length is $2\pi \approx 6.28$ feet.

Area of a Circular Sector The area of, say, a *quarter* circle is the fractional amount *one-fourth* of the total area πr^2, that is, the area is $\frac{1}{4}\pi r^2$. This reasoning carries over in finding the area of any circular sector. If θ in radians is the central angle of the circular sector shown in Figure 4.1.10, then the area A of the sector is simply the fractional amount $\theta/2\pi$ of the total area of the circle. Thus

$$A = \frac{\theta}{2\pi}(\pi r^2) = \frac{1}{2}r^2\theta.$$

Note for the quarter circle example, $\theta = \pi/2$ and the fractional amount of the total area is $(\pi/2)/2\pi = \frac{1}{4}$. We summarize this result in the next theorem.

THEOREM 4.1.2 Area of a Circular Sector

For a circle of radius r, the area A of a circular sector with central angle θ measured in radians is given by

$$A = \frac{1}{2}r^2\theta \tag{8}$$

EXAMPLE 7 **Area of a Circular Sector**


© Tobik/ShutterStock, Inc.

A 14-inch pizza is usually cut into eight slices (or pieces)

A 14-inch pizza is cut into 8 slices. Let us assume that the pizza is a perfect circle and that the slices are exactly the same size. Find the area of one slice.

Solution One slice of pizza is a circular sector with radius $r = 7$ inches. The central angle of the sector is $360°/8 = 45°$. We convert this central angle from degree measure to radian measure:

$$45° = 45 \cdot \frac{\pi}{180} \approx \frac{\pi}{4} \text{ radian.}$$

Then by (8) the area of the sector, or slice, is

$$A = \frac{1}{2} \cdot 7^2 \cdot \frac{\pi}{4} = \frac{49}{2} \cdot \frac{\pi}{4} = \frac{49\pi}{8} \text{ in}^2 \approx 19.24 \text{ in}^2.$$

Exercises 4.1

Answers to selected odd-numbered problems begin on page ANS-14.

In Problems 1–16, draw the given angle in standard position. Bear in mind that the lack of a degree symbol (°) in an angular measurement indicates that the angle is measured in radians.

1. $60°$	**2.** $-120°$	**3.** $135°$	**4.** $150°$
5. $1140°$	**6.** $-315°$	**7.** $-240°$	**8.** $-210°$
9. $\frac{\pi}{3}$	**10.** $\frac{5\pi}{4}$	**11.** $\frac{7\pi}{6}$	**12.** $-\frac{2\pi}{3}$
13. $-\frac{\pi}{6}$	**14.** -3π	**15.** 3	**16.** 4

In Problems 17–20, express the given angle in decimal notation.

17. $10°39'17''$	**18.** $143°7'2''$	**19.** $5°10'$	**20.** $10°25'$

In Problems 21–24, express the given angle in terms of degrees, minutes, and seconds.

21. $210.78°$	**22.** $15.45°$	**23.** $30.81°$	**24.** $110.5°$

In Problems 25–32, convert the given angle from degrees to radians.

25. $10°$	**26.** $15°$	**27.** $75°$	**28.** $215°$
29. $270°$	**30.** $-120°$	**31.** $-230°$	**32.** $540°$

In Problems 33–40, convert the given angle from radians to degrees.

33. $\frac{2\pi}{9}$	**34.** $\frac{11\pi}{6}$	**35.** $\frac{2\pi}{3}$	**36.** $\frac{7\pi}{12}$
37. $\frac{5\pi}{4}$	**38.** 7π	**39.** 3.1	**40.** 12

In Problems 41–44, find the angle between 0° and 360° that is coterminal with the given angle.

41. $875°$ **42.** $400°$ **43.** $-610°$ **44.** $-150°$

45. Find the angle between $-360°$ and $0°$ that is coterminal with the angle in Problem 41.

46. Find the angle between $-360°$ and $0°$ that is coterminal with the angle in Problem 43.

In Problems 47–52, find the angle between 0 and 2π that is coterminal with the given angle.

47. $-\frac{9\pi}{4}$ **48.** $\frac{17\pi}{2}$ **49.** 5.3π

50. $-\frac{9\pi}{5}$ **51.** -4 **52.** 7.5

53. Find the angle between -2π and 0 radians that is coterminal with the angle in Problem 47.

54. Find the angle between -2π and 0 radians that is coterminal with the angle in Problem 49.

In Problems 55–62, find an angle that is **(a)** complementary and **(b)** supplementary to the given angle, or state why no such angle can be found.

55. $48.25°$ **56.** $93°$ **57.** $98.4°$ **58.** $63.08°$

59. $\frac{\pi}{4}$ **60.** $\frac{\pi}{6}$ **61.** $\frac{2\pi}{3}$ **62.** $\frac{5\pi}{6}$

In Problems 63 and 64, find both the degree and the radian measures of the angle formed by the given rotation. Refer to Figures 4.1.2 and 4.1.3.

63. three-fifths of a counterclockwise rotation

64. five and one-eighth clockwise rotations

In Problems 65–68, find the measure of a central angle θ in a circle of radius r that subtends an arc length s. Give θ in **(a)** radians and **(b)** degrees.

65. $r = 5$ ft, $s = 7.5$ ft **66.** $r = 10$ in, $s = 36$ in

67. $r = 9$ m, $s = 15$ m **68.** $r = 20$ cm, $s = 90$ cm

In Problems 69–72, find the arc length s subtended by a central angle θ in a circle of radius r.

69. $\theta = 3$ radians, $r = 5$ in **70.** $\theta = 1.5$ radians, $r = 4$ cm

71. $\theta = 30°$, $r = 2$ m **72.** $\theta = 15°$, $r = 6$ ft

In Problems 73–76, find the area of the circular sector having the given radius r and central angle θ.

73. $r = 3$ ft, $\theta = 7.2$ radians **74.** $r = 18$ in, $\theta = \frac{2\pi}{3}$ radians

75. $r = 6$ m, $\theta = 30°$ **76.** $r = 12$ cm, $\theta = 75°$

In Problems 77 and 78, the light red region in the given figure is portion of a circle. Find the area A of the region if θ is measured in **(a)** radians, and **(b)** degrees.

77.

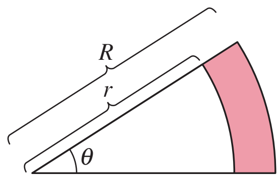


FIGURE 4.1.11 Region in Problem 77

78.

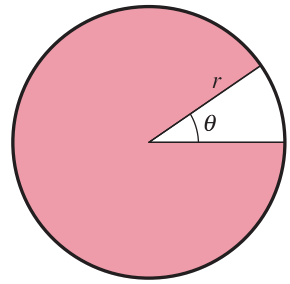
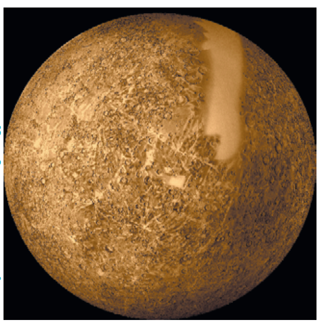


FIGURE 4.1.12 Region in Problem 78

Applications

FIGURE 4.1.13 Analog clock in Problems 79–82



Courtesy of Mariner 10, Astrogeology Team, and USGS

Planet Mercury in Poblem 84

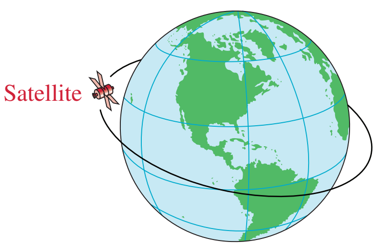


FIGURE 4.1.14 Satellite in Problem 85

79. Analog Clock Consider the analog clock shown in FIGURE 4.1.13. What are the degree and the radian measures of the angle between two adjacent hour tick marks on the clock face?

80. What are the degree and the radian measures of the angle between two adjacent minute tick marks on the analog clock face in Figure 4.1.13?

81. What are the degree and the radian measures of the smallest positive angle formed by the hands of the analog clock in Figure 4.1.13 at **(a)** 8:00, **(b)** 2:00, and **(c)** 7:30?

82. What are degree and the radian measures of the angle through which the minute hand on the analog clock in Figure 4.1.13 rotates in **(a)** $\frac{3}{4}$ hour, and **(b)** 3.5 hours?

83. Planet Earth The Earth rotates on its axis once every 24 hours. How long does it take the Earth to rotate through an angle of **(a)** 240° and **(b)** $\pi/6$ radians?

84. Planet Mercury The planet Mercury completes one rotation on its axis every 59 days. Through what angle (measured in degrees) does it rotate in **(a)** 1 day, **(b)** 1 hour, and **(c)** 1 minute?

85. Angular and Linear Speed If we divide (7) by time t we get the relationship $v = r\omega$, where $v = s/t$ is called the **linear speed** of a point on the circumference of a circle and $\omega = \theta/t$ is called the **angular speed** of the point. A communications satellite is placed in a circular geosynchronous orbit 35,786 km above the surface of the Earth. The time it takes the satellite to make one full revolution around the Earth is 23 hours, 56 minutes, 4 seconds and the radius of the Earth is 6378 km. See FIGURE 4.1.14.

(a) What is the angular speed of the satellite in rad/s?

(b) What is the linear speed of the satellite in km/s?

86. Pendulum Clock A clock pendulum is 1.3 m long and swings back and forth along a 15-cm arc. Find **(a)** the central angle and **(b)** the area of the sector through which the pendulum sweeps in one swing.

87. Sailing at Sea A nautical mile is defined as the arc length subtended on the surface of the Earth by an angle of measure 1 minute. If the diameter of the Earth is 7927 miles, find how many statute (land) miles there are in a nautical mile.

88. Circumference of the Earth Around 230 B.C.E. Eratosthenes calculated the circumference of the Earth from the following observations. At noon on the longest day of the year, the Sun was directly overhead in Syene, while it was inclined 7.2° from the vertical in Alexandria. He believed the two cities to be on

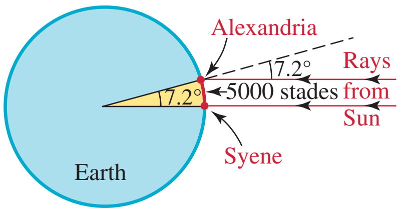


FIGURE 4.1.15 Earth in Problem 88

Yo-Yo in Problems 89 and 90

the same longitudinal line and assumed that the rays of the Sun are parallel. Thus he concluded that the arc from Syene to Alexandria was subtended by a central angle of 7.2° at the center of the Earth. See FIGURE 4.1.15. At that time the distance from Syene to Alexandria was measured as 5000 stades. If one stade $=$ 559 feet, find the circumference of the Earth in **(a)** stades and **(b)** miles. Show that Eratosthenes' data gives a result that is within 7% of the correct value if the polar diameter of the Earth is 7900 miles (to the nearest mile).

89. Circular Motion of a Yo-Yo A yo-yo is whirled around in a circle at the end of its 100-cm string.

(a) If it makes six revolutions in 4 seconds, find its rate of turning, or angular speed, in radians per second.

(b) Find the speed at which the yo-yo travels in centimeters per second; that is, its linear speed.

90. More Yo-Yos If there is a knot in the yo-yo string described in Problem 79 at a point 40 cm from the yo-yo, find **(a)** the angular speed of the knot and **(b)** the linear speed.

91. Locating a Cell Phone As shown in the accompanying photo, many cell phone antennas have a triangular shape. The reception sectors α, β, and γ corresponding to the three sides of the antenna are shown in FIGURE 4.1.16. A cell tower with a triangular antenna is located at the common center of the circles in the figure; the circles have a radius 1 mile, 2 miles, 3 miles, and so on. Suppose a cell phone signal is detected in the β sector approximately 5.3 miles from the antenna. Determine the area of the circular band where the cell phone is located bounded between radius 5 miles and radius 6 miles. (A more precise location of a cell phone can be obtained using either triangulation between three cell towers or GPS.)

Triangular cell phone antenna

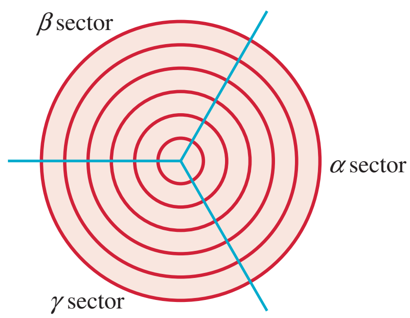


FIGURE 4.1.16 Reception sectors in Problem 91

92. Circular Motion of a Car Tire As shown in FIGURE 4.1.17 the diameter of a car tire is 26 inches. Suppose the car is driven 1.5 miles. What is the corresponding radian measure of the angle through which the tire turns?

93. Circular Motion of a Car Tire An automobile with 26-in. diameter tires is traveling at a rate of 55 mi/h.

(a) Find the number of revolutions per minute that its tires are making.

(b) Find the angular speed of its tires in radians per minute.

94. Diameter of the Moon The average distance from the Earth to the Moon as given by NASA is 238,855 miles. If the angle subtended by the Moon at the eye of an observer on Earth is 0.52°, then what is the approximate diameter of the Moon? FIGURE 4.1.18 is not to scale.

FIGURE 4.1.17 Car tire in Problems 92 and 93

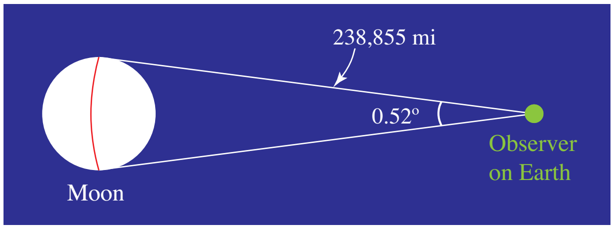


FIGURE 4.1.18 The curved red arc represents the approximate diameter of the Moon

For Discussion

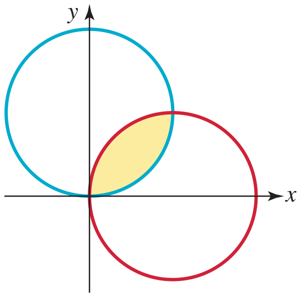


FIGURE 4.1.19 Intersecting circles in Problem 95

95. Intersection of Circles Each of the circles in FIGURE 4.1.19 has its center on an axis, passes through the origin, and has radius r.

(a) Construct a right triangle in the figure and then use that triangle to express the area A of the intersection of the circles, the yellow region in the figure, as a function of r.

(b) Find the area of the intersection of the circles

$$x^2 + (y - 5)^2 = 25 \quad \text{and} \quad (x - 5)^2 + y^2 = 25.$$

4.2 The Sine and Cosine Functions

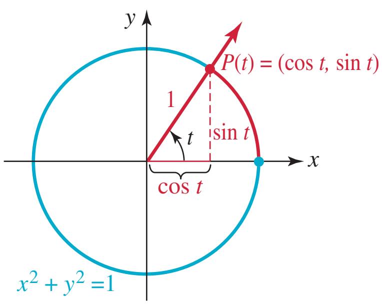


FIGURE 4.2.1 Coordinates of $P(t)$ are $(\cos t, \sin t)$

INTRODUCTION Originally, the trigonometric functions were defined using angles in right triangles. A more modern approach, and one that is used in calculus, is to define the trigonometric functions on sets of real numbers. As we will see, the radian measure for angles is key in making these definitions.

Trigonometric Functions For each real number t there corresponds an angle of t radians in standard position. As shown in FIGURE 4.2.1 we denote the point of intersection of the terminal side of the angle t with the **unit circle** by $P(t)$. The x and y coordinates of this point give us the values of the six basic trigonometric functions. The y-coordinate of $P(t)$ is called the **sine of *t***, while the x-coordinate of $P(t)$ is called the **cosine of *t***.

DEFINITION 4.2.1 Sine and Cosine Functions

Let t be any real number and $P(t) = (x, y)$ be the point of intersection of the unit circle with the terminal side of the angle of t radians in standard position. Then, the **sine of *t***, denoted $\sin t$, and the **cosine of *t***, denoted $\cos t$, are

$$\sin t = y \tag{1}$$

and

$$\cos t = x \tag{2}$$

Since to each real number t there corresponds a unique point $P(t) = (\cos t, \sin t)$, we have just defined two functions – the sine function and the cosine function – each with domain the set R of real numbers. Four additional trigonometric functions are defined in terms of the coordinates of $P(t) = (x, y)$.

DEFINITION 4.2.2 Tangent, Cotangent, Secant, and Cosecant Functions

The **tangent, cotangent, secant,** and **cosecant functions** of the real number t are

$$\tan t = \frac{y}{x}, \quad x \neq 0 \tag{3}$$

$$\cot t = \frac{x}{y}, \quad y \neq 0 \tag{4}$$

$$\sec t = \frac{1}{x}, \quad x \neq 0 \tag{5}$$

and

$$\csc t = \frac{1}{y}, \quad y \neq 0 \tag{6}$$

Using $\sin t = y$ and $\cos t = x$ in (3)–(6) of Definition 4.2.2 we obtain the important identities:

$$\tan t = \frac{\sin t}{\cos t} \qquad \cot t = \frac{\cos t}{\sin t} \tag{7}$$

$$\sec t = \frac{1}{\cos t} \qquad \csc t = \frac{1}{\sin t}. \tag{8}$$

Because of the role played by the unit circle in Definitions 4.2.1 and 4.2.2, the six trigonometric functions are referred to as the **circular functions**.

For the remainder of this section and the next we are going to examine the sine and cosine functions in detail. We will come back to the tangent, cotangent, secant, and cosecant functions in Section 4.4.

A number of properties of the sine and cosine functions follow from the fact that $P(t) = (\cos t, \sin t)$ lies on the unit circle. For instance, the coordinates of $P(t)$ must satisfy the equation of the circle:

$$x^2 + y^2 = 1. \tag{9}$$

Substituting $x = \cos t$ and $y = \sin t$ gives an important relationship between the sine and the cosine called the **Pythagorean identity**:

$$(\cos t)^2 + (\sin t)^2 = 1.$$

From now on we will follow two standard practices in writing this identity: $(\cos t)^2$ and $(\sin t)^2$ will be written as $\cos^2 t$ and $\sin^2 t$, respectively, and the $\sin^2 t$ term will be written first.

THEOREM 4.2.1 Pythagorean Identity

For all real numbers t,

$$\sin^2 t + \cos^2 t = 1 \tag{10}$$

Again, if $P(x, y)$ denotes a point on the unit circle (9), it follows that the coordinates of P must satisfy the inequalities $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Because $x = \cos t$ and $y = \sin t$ we have the following bounds on the values of the sine and cosine functions.

THEOREM 4.2.2 Bounds on the Values of Sine and Cosine

For all real numbers t,

$$-1 \le \sin t \le 1 \quad \text{and} \quad -1 \le \cos t \le 1$$

The foregoing inequalities can also be expressed as $|\sin t| \le 1$ and $|\cos t| \le 1$. Thus, for example, there is no real number t such that $\sin t = \frac{3}{2}$.

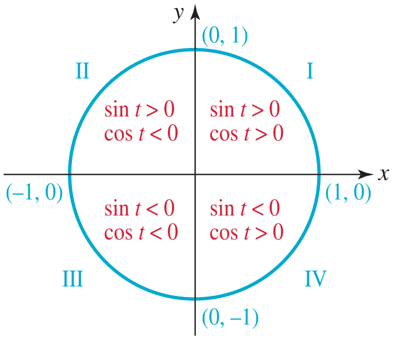


FIGURE 4.2.2 Algebraic signs of $\sin t$ and $\cos t$ in the four quadrants

Domain and Range From the preceding observations we have the sine and cosine functions $f(t) = \sin t$ and $g(t) = \cos t$ each with **domain** the set R of real numbers and **range** the interval $[-1, 1]$.

Signs of the Circular Functions The signs of the function values $\sin t$ and $\cos t$ are determined by the quadrant in which the point $P(t)$ lies, and conversely. For example, if $\sin t$ and $\cos t$ are both negative, then the point $P(t)$ and terminal side of the corresponding angle of t radians must lie in quadrant III. FIGURE 4.2.2 displays the signs of the cosine and sine functions in each of the four quadrants.

EXAMPLE 1 Using the Pythagorean Identity

Given that $\cos t = \frac{1}{3}$ and that $P(t)$ is a point in the fourth quadrant, find $\sin t$.

Solution Substitution of $\cos t = \frac{1}{3}$ into the Pythagorean identity (10) gives $\sin^2 t + \left(\frac{1}{3}\right)^2 = 1$ or $\sin^2 t = \frac{8}{9}$. Since $\sin t$ is the y-coordinate of $P(t)$, a point in the fourth quadrant, we must take the negative square root for $\sin t$:

$$\sin t = -\sqrt{\frac{8}{9}} = -\frac{2\sqrt{2}}{3}.$$

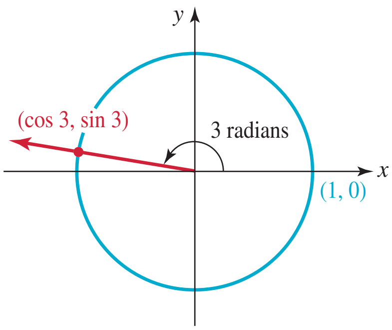


FIGURE 4.2.3 The point $P(3)$

EXAMPLE 2 Sine and Cosine of a Real Number

Use a calculator to approximate $\sin 3$ and $\cos 3$ and give a geometric interpretation of these values.

Solution From a calculator set in *radian mode*, we obtain $\cos 3 \approx -0.9899925$ and $\sin 3 \approx 0.1411200$. These values represent the x- and y-coordinates, respectively, of the point of intersection of the terminal side of the angle of 3 radians in standard position with the unit circle. As shown in FIGURE 4.2.3, this point lies in the second quadrant because $\pi/2 < 3 < \pi$. This would also be expected in view of Figure 4.2.2 since $\cos 3$, the x-coordinate, is *negative* and $\sin 3$, the y-coordinate, is *positive*.

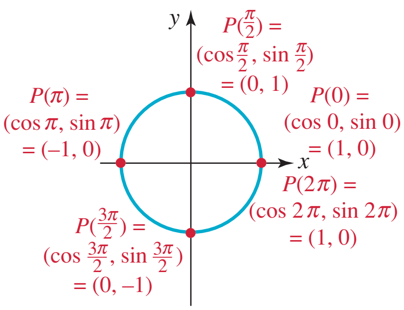


FIGURE 4.2.4 Sine and cosine values for quadrantal angles

Values Corresponding to Unit Circle Intercepts As shown in FIGURE 4.2.4, the x- and y-intercepts of the unit circle give us the values of the sine and cosine functions for the real numbers corresponding to **quadrantal angles** listed next.

Values of the Sine and Cosine

For $t = 0$:	$\sin 0 = 0$	and	$\cos 0 = 1$
For $t = \frac{\pi}{2}$:	$\sin\frac{\pi}{2} = 1$	and	$\cos\frac{\pi}{2} = 0$
For $t = \pi$:	$\sin \pi = 0$	and	$\cos \pi = -1$
For $t = \frac{3\pi}{2}$:	$\sin\frac{3\pi}{2} = -1$	and	$\cos\frac{3\pi}{2} = 0$

Periodicity In Section 4.1 we saw that for any real number t, the angles of t radians and $t \pm 2\pi$ radians are coterminal. Thus they determine the same point (x, y) on the unit circle. Therefore

$$\sin t = \sin(t \pm 2\pi) \qquad \text{and} \qquad \cos t = \cos(t \pm 2\pi). \tag{11}$$

In other words, the sine and cosine functions repeat their values every 2π units. It also follows that for any integer n:

$$\sin(t + 2n\pi) = \sin t \qquad \text{and} \qquad \cos(t + 2n\pi) = \cos t. \tag{12}$$

DEFINITION 4.2.3 Periodic Functions

A nonconstant function f is said to be **periodic** if there is a positive number p such that

$$f(t) = f(t + p) \tag{13}$$

for every t in the domain of f. If p is the smallest positive number for which (13) is true, then p is called the **period** of the function f.

The equations in (11) imply that the sine and the cosine functions are periodic with period $p \le 2\pi$. To see that the period of $\sin t$ is actually 2π, we observe that there is only one point on the unit circle with y-coordinate 1, namely, $P(\pi/2) = (\cos(\pi/2), \sin(\pi/2)) = (0, 1)$. Therefore,

$$\sin t = 1 \qquad \text{only for} \qquad t = \frac{\pi}{2}, \frac{\pi}{2} \pm 2\pi, \frac{\pi}{2} \pm 4\pi,$$

and so on. Thus the smallest possible positive value of p is 2π.

THEOREM 4.2.3 Period of the Sine and Cosine

The sine and cosine functions are periodic with **period 2π**. Therefore,

$$\sin(t + 2\pi) = \sin t \qquad \text{and} \qquad \cos(t + 2\pi) = \cos t \tag{14}$$

for every real number t.

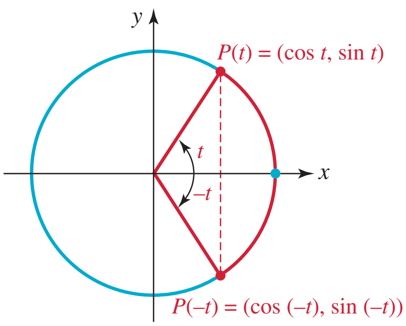


FIGURE 4.2.5 Coordinates of $P(t)$ and $P(-t)$

Even–Odd Properties The symmetry of the unit circle endows the circular functions with several additional properties. For any real number t, the points $P(t)$ and $P(-t)$ on the unit circle are located on the terminal side of an angle of t and $-t$ radians, respectively. These two points will always be symmetric with respect to the x-axis. FIGURE 4.2.5 illustrates the situation for a point $P(t)$ lying in the first quadrant: The x-coordinates of the two points are identical; however, the y-coordinates have equal magnitudes but opposite signs. The same symmetries will hold regardless of which quadrant contains $P(t)$. Thus, for any real number t, $\cos(-t) = \cos t$ and $\sin(-t) = -\sin t$. Applying the definitions of **even** and **odd functions** from Section 2.2 we have the following result.

THEOREM 4.2.4 Even and Odd Functions

The cosine function is **even** and the sine function is **odd**. That is, for every real number t,

$$\cos(-t) = \cos t \qquad \text{and} \qquad \sin(-t) = -\sin t \tag{15}$$

The following additional properties of the sine and cosine functions can be verified by considering the symmetries of appropriately chosen points on the unit circle.

THEOREM 4.2.5 Additional Properties

$$\cos\left(\frac{\pi}{2} - t\right) = \sin t \quad \text{and} \quad \sin\left(\frac{\pi}{2} - t\right) = \cos t \tag{16}$$

$$\cos(t + \pi) = -\cos t \quad \text{and} \quad \sin(t + \pi) = -\sin t \tag{17}$$

$$\cos(\pi - t) = -\cos t \quad \text{and} \quad \sin(\pi - t) = \sin t \tag{18}$$

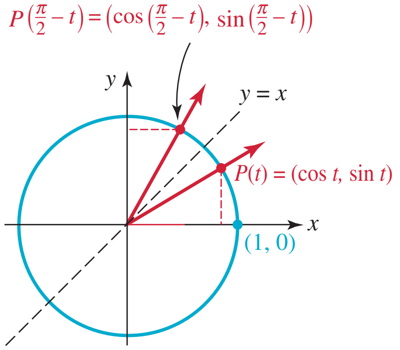


FIGURE 4.2.6 Geometric justification of (16) in Theorem 4.2.5

For example, to justify the properties in (16) of Theorem 4.2.5 for $0 < t < \pi/2$, consider FIGURE 4.2.6. Since the points $P(t)$ and $P(\pi/2 - t)$ are symmetric with respect to the line $y = x$, we can obtain the coordinates of $P(\pi/2 - t)$, by interchanging the coordinates of $P(t)$. Thus,

$$\cos t = \sin\left(\frac{\pi}{2} - t\right) \quad \text{and} \quad \sin t = \cos\left(\frac{\pi}{2} - t\right).$$

The special properties of the sine and cosine functions in Theorem 4.2.5 will become quite useful as soon as we determine additional values for $\sin t$ and $\cos t$ in the interval $[0, 2\pi)$. Using results from plane geometry we will now find the values of the sine and cosine functions for $t = \pi/6$, $t = \pi/4$, and $t = \pi/3$.

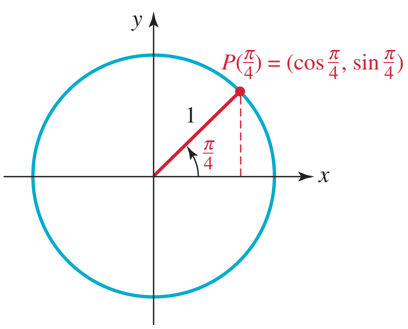


FIGURE 4.2.7 The point $P(\pi/4)$

□ **Finding $\sin(\pi/4)$ and $\cos(\pi/4)$** We draw an angle of $\pi/4$ radians (45°) in standard position and locate and label $P(\pi/4) = (\cos(\pi/4), \sin(\pi/4))$ on the unit circle. As shown in FIGURE 4.2.7, we form a right triangle by dropping a perpendicular from $P(\pi/4)$ to the x-axis. Since the sum of the angles in any triangle is π radians (180°), the third angle of this triangle is also $\pi/4$ radians, hence the triangle is isosceles. Therefore the coordinates of $P(\pi/4)$ are equal; that is, $\cos(\pi/4) = \sin(\pi/4)$. It follows from the Pythagorean identity (10)

$$\sin^2\frac{\pi}{4} + \cos^2\frac{\pi}{4} = 1 \qquad \text{that} \qquad 2\cos^2\frac{\pi}{4} = 1.$$

Dividing by 2 and taking the square root, we obtain $\cos(\pi/4) = \pm\sqrt{2}/2$. Since $P(\pi/4)$ lies in the first quadrant, both coordinates must be positive. So we have found the (equal) coordinates of $P(\pi/4)$:

$$\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2} \quad \text{and} \quad \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}.$$

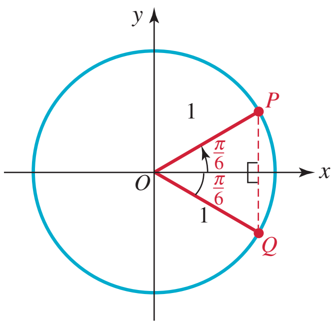


FIGURE 4.2.8 The point $P(\pi/6)$

□ **Finding $\sin(\pi/6)$ and $\cos(\pi/6)$** We construct two angles of $\pi/6$ radians (30°) in the first and fourth quadrants, as shown in FIGURE 4.2.8, and label the points of intersection with the unit circle $P(\pi/6)$ and Q, respectively. By drawing perpendicular line segments from P and Q to the x-axis, we obtain two *congruent* right triangles because each triangle has a hypotenuse of length 1 and angles of 30°, 60°, and 90°. Since the 90° angles form a straight angle, these two right triangles form an *equilateral* triangle $\triangle POQ$ with sides of length 1. Since $\sin(\pi/6)$ is equal to half of the vertical side of $\triangle POQ$, we have

$$\sin\frac{\pi}{6} = \frac{1}{2}.$$

From this result and the Pythagorean identity (10) we find the value of $\cos(\pi/6)$:

$$\left(\frac{1}{2}\right)^2 + \cos^2\frac{\pi}{6} = 1 \qquad \text{implies} \qquad \cos^2\frac{\pi}{6} = \frac{3}{4}$$

We take the positive square root here because $P(\pi/6)$ lies in the first quadrant.

or

$$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

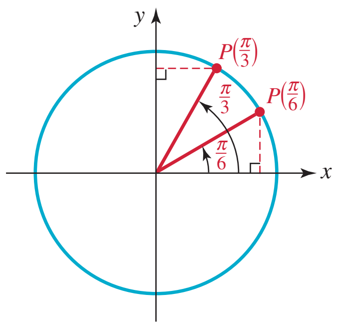


FIGURE 4.2.9 The point $P(\pi/3)$

Finding $\sin(\pi/3)$ and $\cos(\pi/3)$ We draw angles of $\pi/6$ and $\pi/3$ in standard position and locate and label the points $P(\pi/6)$ and $P(\pi/3)$, as shown in **FIGURE 4.2.9**. We then construct two congruent 30°-60°-90° triangles by dropping perpendiculars to the x- and y-axes, respectively. It follows from the congruence of these triangles that

$$\cos\frac{\pi}{3} = \sin\frac{\pi}{6} = \frac{1}{2} \qquad \text{and} \qquad \sin\frac{\pi}{3} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

The foregoing results also follow from (16) of Theorem 4.2.5 with $t = \pi/6$.

We summarize the values of the sine and cosine functions corresponding to the basic fractional multiples of π that we have determined so far.

Values of the Sine and Cosine (Continued)

For $t = \dfrac{\pi}{6}$: $\quad \sin\dfrac{\pi}{6} = \dfrac{1}{2} \quad$ and $\quad \cos\dfrac{\pi}{6} = \dfrac{\sqrt{3}}{2}$

For $t = \dfrac{\pi}{4}$: $\quad \sin\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2} \quad$ and $\quad \cos\dfrac{\pi}{4} = \dfrac{\sqrt{2}}{2}$

For $t = \dfrac{\pi}{3}$: $\quad \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2} \quad$ and $\quad \cos\dfrac{\pi}{3} = \dfrac{1}{2}$

Reference Angle As we noted at the beginning of this section, for each real number t there is a unique angle of t radians in standard position that determines the point $P(t)$, with coordinates $(\cos t, \sin t)$, on the unit circle. As shown in **FIGURE 4.2.10**, the terminal side of any angle of t radians (with $P(t)$ not on an axis) will form an acute angle with the x-axis. We can then locate an angle of t' radians in the first quadrant that is congruent to this acute angle. The angle of t' radians is called the **reference angle** for t. Because of the symmetry of the unit circle, the coordinates of $P(t')$ will be equal *in absolute value* to the respective coordinates of $P(t)$. Hence

$$\sin t = \pm\sin t' \qquad \text{and} \qquad \cos t = \pm\cos t'.$$

As the following examples will show, reference angles can be used to find the trigonometric function values of any integer multiple of $\pi/6$, $\pi/4$, and $\pi/3$.

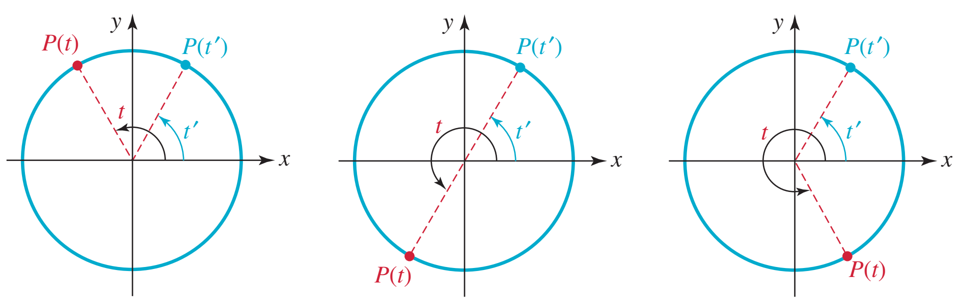


FIGURE 4.2.10 Reference angle t' is an acute angle

EXAMPLE 3 Using Reference Angles

Find exact values of $\sin t$ and $\cos t$ for the given real number t.

(a) $t = 5\pi/3$ **(b)** $t = -3\pi/4$

Solution In each part we begin by finding the reference angle corresponding to the given value of t.

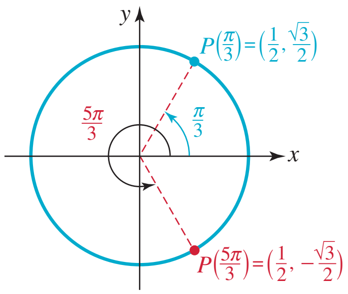


FIGURE 4.2.11 Reference angle in part (a) of Example 3

(a) From FIGURE 4.2.11 we find that an angle of $t = 5\pi/3$ radians determines a point $P(5\pi/3)$ in the fourth quadrant and has the reference angle $t' = \pi/3$ radians. After adjusting the signs of the coordinates of $P(\pi/3) = \left(1/2, \sqrt{3}/2\right)$ to obtain the fourth quadrant point $P(5\pi/3) = \left(1/2, -\sqrt{3}/2\right)$, we find that

$$\sin\frac{5\pi}{3} = -\underbrace{\sin\frac{\pi}{3} = -\frac{\sqrt{3}}{2} \quad \text{and} \quad \cos\frac{5\pi}{3} = \cos\frac{\pi}{3}}_{\text{reference angle}} = \frac{1}{2}.$$

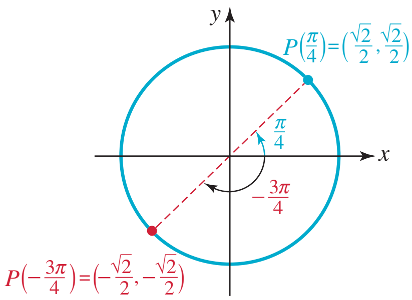


FIGURE 4.2.12 Reference angle in part (b) of Example 3

(b) The point $P(-3\pi/4)$ lies in the third quadrant and has reference angle $\pi/4$, as shown in FIGURE 4.2.12. Therefore,

$$\sin\left(-\frac{3\pi}{4}\right) = -\sin\frac{\pi}{4} = -\frac{\sqrt{2}}{2} \quad \text{and} \quad \cos\left(-\frac{3\pi}{4}\right) = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}.$$

Sometimes, in order to find the trigonometric values of multiples of our basic fractions of π we must use periodicity or the even-odd function properties in addition to reference numbers.

EXAMPLE 4 Using Periodicity and a Reference Angle

Find exact values of $\sin t$ and $\cos t$ for $t = 29\pi/6$.

Solution Since $29\pi/6$ is greater than 2π, we rewrite $29\pi/6$ as an integer multiple of 2π plus a number less than 2π:

$$\frac{29\pi}{6} = 4\pi + \frac{5\pi}{6} = 2(2\pi) + \frac{5\pi}{6}.$$

From the periodicity equations (12) with $n = 2$ and $t = 5\pi/6$ we know that $\sin(29\pi/6) = \sin(5\pi/6)$ and $\cos(29\pi/6) = \cos(5\pi/6)$. Next we see from FIGURE 4.2.13 that the reference angle for $5\pi/6$ is $\pi/6$. Since $P(5\pi/6)$ is a second quadrant point, we have

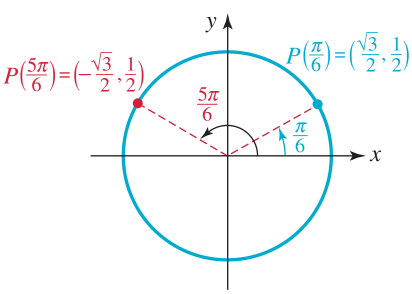


FIGURE 4.2.13 Reference angle in Example 4

$$\sin\frac{29\pi}{6} = \sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$$

and

$$\cos\frac{29\pi}{6} = \cos\frac{5\pi}{6} = -\cos\frac{\pi}{6} = -\frac{\sqrt{3}}{2}.$$

EXAMPLE 5 Using the Even–Odd Properties

Find exact values of $\sin t$ and $\cos t$ for $t = -\pi/6$.

Solution Since sine is an odd function and cosine is an even function,

See (15) in Theorem 4.2.4.

$$\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2} \quad \text{and} \quad \cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

This problem could also have been solved by using a reference angle.

Trigonometric Functions of Angles In this section we have defined sine and cosine functions of the real number t by using the coordinates of a point $P(t)$ on the unit circle. It is now possible to define the **trigonometric functions of any angle θ**. For any angle θ, we simply let

$$\sin\theta = \sin t \quad \text{and} \quad \cos\theta = \cos t,$$

where the real number t is the radian measure of θ. As mentioned in Section 4.1, it is common to omit the word radians when measuring an angle. So we write $\sin(\pi/6)$ for both the sine of the real number $\pi/6$ and for the sine of the angle of $\pi/6$ radians. Furthermore, since the values of the trigonometric functions are determined by the coordinates of the point $P(t)$ on the unit circle, it really does not matter whether θ is measured in radians or in degrees. For example, regardless of whether we are given $\theta = \pi/6$ radians or $\theta = 30°$, the point on the unit circle corresponding to this angle in standard position is $(\sqrt{3}/2, 1/2)$. Thus,

$$\sin\frac{\pi}{6} = \sin 30° = \frac{1}{2} \quad \text{and} \quad \cos\frac{\pi}{6} = \cos 30° = \frac{\sqrt{3}}{2}.$$

Exercises 4.2 Answers to selected odd-numbered problems begin on page ANS-14.

1. Given that $\cos t = -\frac{2}{5}$ and that $P(t)$ is a point in the second quadrant, find $\sin t$.
2. Given that $\sin t = \frac{1}{4}$ and that $P(t)$ is a point in the second quadrant, find $\cos t$.
3. Given that $\sin t = -\frac{2}{3}$ and that $P(t)$ is a point in the third quadrant, find $\cos t$.
4. Given that $\cos t = \frac{3}{4}$ and that $P(t)$ is a point in the fourth quadrant, find $\sin t$.
5. If $\sin t = -\frac{2}{7}$, find all possible values of $\cos t$.
6. If $\cos t = \frac{3}{10}$, find all possible values of $\sin t$.
7. If $\cos t = -0.2$, find all possible values of $\sin t$.
8. If $\sin t = 0.4$, find all possible values of $\cos t$.
9. If $2\sin t - \cos t = 0$, find all possible values of $\sin t$ and $\cos t$.
10. If $3\sin t - 2\cos t = 0$, find all possible values of $\sin t$ and $\cos t$.

In Problems 11–14, find the exact value of **(a)** $\sin t$ and **(b)** $\cos t$ for the given value of t. Do not use a calculator.

11. $t = -\pi/2$
12. $t = 3\pi$
13. $t = 8\pi$
14. $t = -3\pi/2$

In Problems 15–26, for the given value of t determine the reference angle t' and the exact values of $\sin t$ and $\cos t$. Do not use a calculator.

15. $t = 2\pi/3$
16. $t = 4\pi/3$
17. $t = 5\pi/4$
18. $t = 3\pi/4$
19. $t = 11\pi/6$
20. $t = 7\pi/6$

21. $t = -\pi/4$
22. $t = -7\pi/4$
23. $t = -5\pi/6$
24. $t = -11\pi/6$
25. $t = -5\pi/3$
26. $t = -2\pi/3$

In Problems 27–32, find the given trigonometric function value. Do not use a calculator.

27. $\sin(-11\pi/3)$
28. $\cos(17\pi/6)$
29. $\cos(-7\pi/4)$
30. $\sin(-19\pi/2)$
31. $\cos 5\pi$
32. $\sin(23\pi/3)$

In Problems 33–38, justify the given statement with one of the properties of the trigonometric functions.

33. $\sin\pi = \sin 3\pi$
34. $\cos(\pi/4) = \sin(\pi/4)$
35. $\sin(-3 - \pi) = -\sin(3 + \pi)$
36. $\cos 16.8\pi = \cos 14.8\pi$
37. $\cos 0.43 = \cos(-0.43)$
38. $\sin(2\pi/3) = \sin(\pi/3)$

In Problems 39–46, find the given trigonometric function value. Do not use a calculator.

39. $\sin 135°$
40. $\cos 150°$
41. $\cos 210°$
42. $\sin 270°$
43. $\cos 330°$
44. $\sin(-180°)$
45. $\sin(-60°)$
46. $\cos(-300°)$

In Problems 47–50, find all angles t, where $0 \le t < 2\pi$, that satisfy the given condition.

47. $\sin t = 0$
48. $\cos t = -1$
49. $\cos t = \sqrt{2}/2$
50. $\sin t = \frac{1}{2}$

In Problems 51–54, find all angles θ, where $0° \le \theta < 360°$, that satisfy the given condition.

51. $\cos\theta = \sqrt{3}/2$
52. $\sin\theta = -\frac{1}{2}$
53. $\sin\theta = -\sqrt{2}/2$
54. $\cos\theta = 1$

Applications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Free throw

55. Free Throw Under certain conditions the maximum height y attained by a basketball released from a height h at an angle θ measured from the horizontal with an initial velocity v_0 is given by $y = h + (v_0^2\sin^2\theta)/2g$, where g is the acceleration due to gravity. Compute the maximum height reached by a free throw if $h = 2.15$ m, $v_0 = 8$ m/s, $\theta = 64.47°$, and $g = 9.81$ m/s^2.

56. Putting the Shot The range of a shot put released from a height h above the ground with an initial velocity v_0 at an angle θ to the horizontal can be approximated by

$$R = \frac{v_0\cos\theta}{g}\left(v_0\sin\theta + \sqrt{v_0^2\sin^2\theta + 2gh}\right),$$

where g is the acceleration due to gravity. If $v_0 = 13.7$ m/s, $\theta = 40°$, and $g = 9.81$ m/s^2, compare the ranges achieved for the release heights **(a)** $h = 2.0$ m and **(b)** $h = 2.4$ m. **(c)** Explain why an increase in h yields an increase in R if the other parameters are held fixed. **(d)** What does this imply about the advantage that height gives a shot putter?

57. Acceleration Due to Gravity Because of its rotation the Earth bulges at the equator and is flattened at the poles. As a consequence, the acceleration due to gravity is

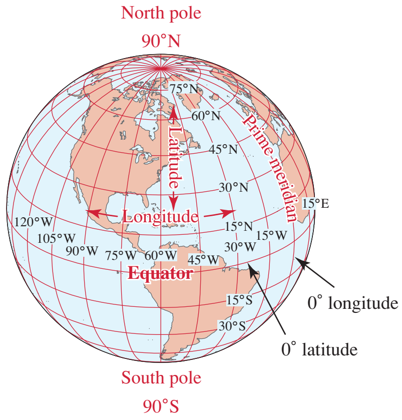


FIGURE 4.2.14 Latitude in Problem 57

not a constant 980 cm/s^2, but varies with latitude. As shown in FIGURE 4.2.14, the latitude of a point on the Earth is an angle ϕ measured (usually in degrees, minutes, seconds) north (N) or south (S) from the equatorial plane. Based on satellite studies, a mathematical model for the acceleration due to gravity g_{lat} is given by

$$g_{\text{lat}} = 978.0309 + 5.18552\sin^2\phi - 0.00570\sin^2 2\phi.$$

(a) Find g_{lat} at Mexico City, Mexico (19.42° N), Los Angeles, CA (34.05° N), New York City, NY (40.70° N), and Fairbanks, AK (64.83° N).
(b) At what latitude is g_{lat} a minimum? A maximum?

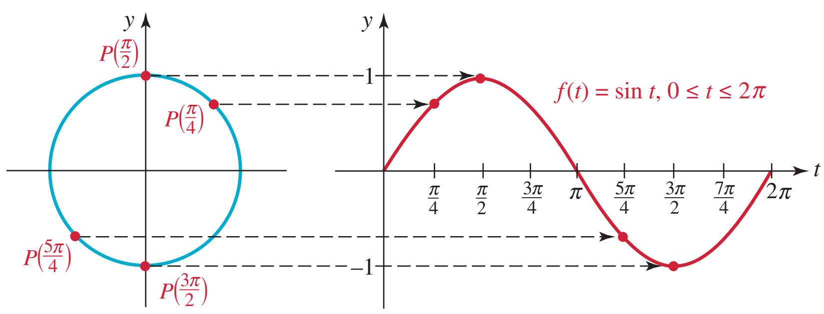
For Discussion

58. Discuss how it is possible to determine without a calculator that the point $P(6) = (\cos 6, \sin 6)$ lies in the fourth quadrant.
59. Discuss how it is possible to determine without the aid of a calculator that both $\sin 4$ and $\cos 4$ are negative.
60. Is there a real number t satisfying $3\sin t = 5$? Explain why or why not.
61. Is there an angle θ satisfying $\cos\theta = -2$? Explain why or why not.
62. Suppose f is a periodic function with period p. Show that $F(x) = f(ax)$, $a > 0$, is periodic with period p/a.

4.3 Graphs of Sine and Cosine Functions

INTRODUCTION One way to further your understanding of the trigonometric functions is to examine their graphs. In this section we consider the graphs of the sine and cosine functions.

Graphs of Sine and Cosine In Section 4.2 we saw that the domain of the sine function $f(t) = \sin t$ is the set of real numbers $(-\infty, \infty)$ and the interval $[-1, 1]$ is its range. Since the sine function has period 2π, we begin by sketching its graph on the interval $[0, 2\pi]$. We obtain a rough sketch of the graph given in FIGURE 4.3.1(b) by considering various positions of the point $P(t)$ on the unit circle, as shown in Figure 4.3.1(a). As t varies from 0 to $\pi/2$, the value $\sin t$ increases from 0 to its maximum value 1. But as t varies from $\pi/2$ to $3\pi/2$, the value $\sin t$ decreases from 1 to its minimum value -1. We note that $\sin t$ changes from positive to negative at $t = \pi$. For t between $3\pi/2$ and 2π, we see that the corresponding values of $\sin t$ increase from -1 to 0. The graph of *any* periodic function over an interval of length equal



(a) Unit circle (b) One cycle of sine graph

FIGURE 4.3.1 Points $P(t)$ on a circle corresponding to points on the graph to its period is said to be one **cycle of its graph**. In the case of the sine function, the graph over the interval $[0, 2\pi]$ in Figure 4.3.1(b) is one cycle of the graph of $f(t) = \sin t$.

Note: Change of symbols

From this point on we will revert to the traditional symbols x and y when graphing trigonometric functions. Thus, $f(t) = \sin t$ will either be written $f(x) = \sin x$ or simply $y = \sin x$.

The graph of a periodic function is easily obtained by repeatedly drawing one cycle of its graph. In other words, the graph of $y = \sin x$ on, say, the intervals $[-2\pi, 0]$ and $[2\pi, 4\pi]$ is the same as that given in Figure 4.3.1(b). Recall from Theorem 4.2.4 of Section 4.2 that the sine function is an odd function since $f(-x) = \sin(-x) = -\sin x = -f(x)$. In other words, if (x, y) is a point on the graph of f, then so is $(-x, -y)$. Thus, from Theorem 2.2.1 it follows that the graph of $y = \sin x$ shown in **FIGURE 4.3.2** is symmetric with respect to the origin.

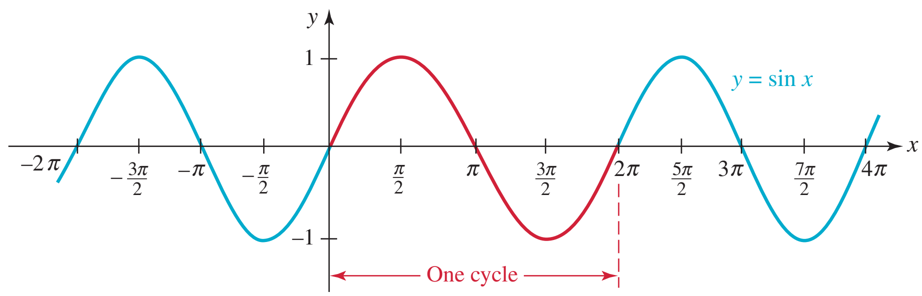


FIGURE 4.3.2 Graph of $y = \sin x$

By working again with the unit circle we can obtain one cycle of the graph of the cosine function $g(x) = \cos x$ on the interval $[0, 2\pi]$. In contrast to the graph of $f(x) = \sin x$ where $f(0) = f(2\pi) = 0$, for the cosine function we have $g(0) = g(2\pi) = 1$. **FIGURE 4.3.3** shows one cycle (in red) of $y = \cos x$ on $[0, 2\pi]$ along with the extension of that cycle (in blue) to the adjacent intervals $[-2\pi, 0]$ and $[2\pi, 4\pi]$. We see from this figure that the graph of the cosine function is symmetric with respect to the y-axis. This is a consequence of g being an even function:

$$g(-x) = \cos(-x) = \cos x = g(x).$$

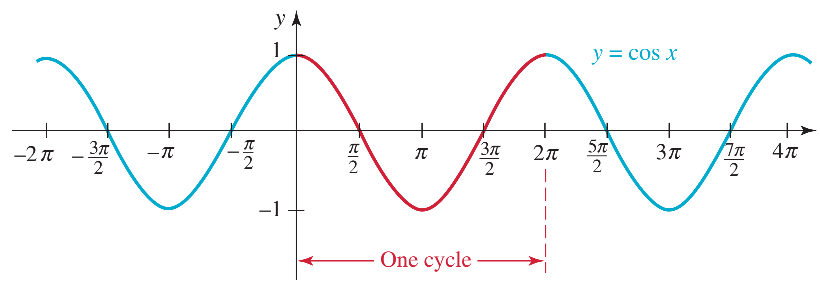


FIGURE 4.3.3 Graph of $y = \cos x$

Intercepts In this and subsequent courses in mathematics it is important that you know the x-coordinates of the x-intercepts of the sine and cosine graphs—in other words, the zeros of $f(x) = \sin x$ and $g(x) = \cos x$. From the sine graph in Figure 4.3.2 we see that the zeros of the sine function, or the numbers for which $\sin x = 0$, are

$$x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots .$$

These numbers are integer multiples of π. From the cosine graph in Figure 4.3.3 we see that $\cos x = 0$ when

$$x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots.$$

These numbers are odd-integer multiples of $\pi/2$.

We summarize the foregoing discussion.

Properties of the Sine and Cosine Functions

- The **domain** of $f(x) = \sin x$ and of $g(x) = \cos x$ is the set of all real numbers $(-\infty, \infty)$.
- The **range** of $f(x) = \sin x$ and of $g(x) = \cos x$ is the interval $[-1, 1]$ on the y-axis.
- The **period** of $f(x) = \sin x$ and of $g(x) = \cos x$ is the number 2π.
- The sine function f is an **odd function**, and so its graph is symmetric with respect to the origin.
- The cosine function g is an **even function**, and so its graph is symmetric with respect to the y-axis.
- The **zeros of the sine function** f are the numbers

$$x = n\pi,\ n = 0, \pm 1, \pm 2, \ldots. \tag{1}$$

- The **zeros of the cosine function** g are the numbers

$$x = (2n + 1)\pi/2,\ n = 0, \pm 1, \pm 2, \ldots. \tag{2}$$

Note in (2) that if n is an integer, then $2n + 1$ is an odd integer. Using the distributive law, the zeros of the cosine function are often written as $x = \pi/2 + n\pi$.

Variation of the Graphs As we did in Chapters 2 and 3 we can obtain variations of the basic sine and cosine graphs through rigid and nonrigid transformations. For the remainder of the discussion in this section we will consider graphs of functions of the form

$$y = A\sin(Bx + C) + D \qquad \text{or} \qquad y = A\cos(Bx + C) + D, \tag{3}$$

where A, B, C, and D are real constants.

Graphs of $y = A\sin x$ and $y = A\cos x$ We begin by considering the special cases of (3):

$$y = A\sin x \qquad \text{and} \qquad y = A\cos x.$$

The multiple A can be either positive or negative, but does not affect the period of the function; in other words, the **period** of both $y = A \sin x$ and $y = A\cos x$ is 2π. For $|A| > 1$ graphs of these functions can be interpreted as a **vertical stretch** of the graphs of $y = \sin x$ or $y = \cos x$; when $0 < |A| < 1$ the graphs are a **vertical compression** of the basic sine or cosine graphs. The number $|A|$ is called the **amplitude** of the functions or of their graphs. For example, as FIGURE 4.3.4 shows we obtain the graph of $y = 2\sin x$ by stretching the graph of $y = \sin x$ vertically by a factor of 2. Note that the maximum and minimum values of $y = 2\sin x$ occur at the same x-values as the maximum and minimum values of $y = \sin x$. In general, the maximum distance from any point on the graph of $y = A\sin x$ or $y = A\cos x$ to the x-axis is $|A|$. The amplitude of the basic functions $y = \sin x$ and $y = \cos x$ is $|A| = 1$. In general, if a periodic function f is continuous, then over a closed interval of length equal to its period, f has both a maximum value M and a minimum value m. The amplitude is defined by

$$\text{amplitude} = \tfrac{1}{2}[M - m]. \tag{4}$$

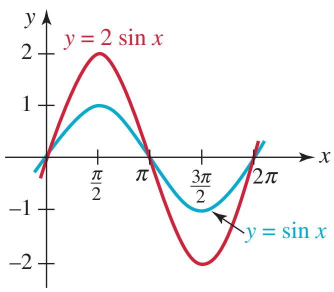


FIGURE 4.3.4 Vertical stretch of the graph of $y = \sin x$

As the next example shows, when $A < 0$ the graphs are also reflected in the x-axis.

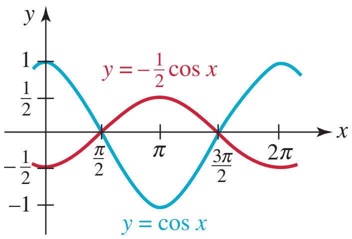


FIGURE 4.3.5 Graph of function in Example 1

EXAMPLE 1 Vertically Compressed/Reflected Graph

Graph $y = -\frac{1}{2}\cos x$.

Solution With the identification $A = -\frac{1}{2}$ we see that the amplitude of the function is $|A| = |-\frac{1}{2}| = \frac{1}{2}$. Because $|A| < 1$ and A is negative, the graph of $y = -\frac{1}{2}\cos x$ is the graph of $y = \cos x$ compressed vertically by a factor of $\frac{1}{2}$ and then reflected in the x-axis. The graph of $y = -\frac{1}{2}\cos x$ on the interval $[0, 2\pi]$ is shown in red in FIGURE 4.3.5.

Vertical Stretch/Compression

- The graphs of the functions

$$y = A\sin x \quad \text{and} \quad y = A\cos x$$

have amplitude $|A|$ and period 2π. The graphs of $y = A\sin x$ and $y = A\cos x$ are the graphs of $y = \sin x$ and $y = \cos x$ stretched vertically if $|A| > 1$, and compressed vertically if $0 < |A| < 1$.

Graphs of $y = A\sin x + D$ and $y = A\cos x + D$ The graphs of

$$y = A\sin x + D \qquad \text{and} \qquad y = A\cos x + D$$

are the graphs of $y = A\sin x$ and $y = A\cos x$ shifted vertically, up for $D > 0$ and down for $D < 0$. The amplitude of the graph of either $y = A\sin x + D$ or $y = A\cos x + D$ is still $|A|$.

EXAMPLE 2 Vertically Shifted Sine Graph

Graph $y = 1 + 2\sin x$.

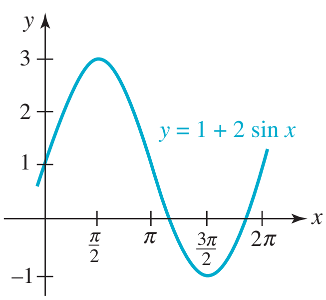


FIGURE 4.3.6 Graph of function in Example 2

Solution The graph of $y = 1 + 2\sin x$ is the graph of $y = 2\sin x$ given in Figure 4.3.4 shifted up 1 unit. From the identification $A = 2$, the amplitude is $|A| = 2$. Because the maximum 2 and minimum -2 of $y = 2\sin x$ occur, respectively, at $x = \pi/2$ and $x = 3\pi/2$, a rigid upward translation of the graph does not change the latter numbers but increases the maximum and minimum by 1 unit. We see in FIGURE 4.3.6 that the maximum of $y = 1 + 2\sin x$ is 3 at $x = \pi/2$ and the minimum of $y = 1 + 2\sin x$ is -1 at $x = 3\pi/2$. Using the function values $M = 3$ and $m = -1$, we can verify the amplitude of $y = 1 + 2\sin x$ using (4):

$$\tfrac{1}{2}[M - m] = \tfrac{1}{2}[3 - (-1)] = 2.$$

Note that the range $[-1, 3]$ of the function $y = 1 + 2\sin x$ is the range $[-2, 2]$ of $y = 2\sin x$ shifted up 1 unit on the y-axis.

Graphs of $y = A\sin Bx$ and $y = A\cos Bx$ We now consider the graph of $y = A\sin Bx$ and $y = A\cos Bx$. Throughout the discussion we may assume that $B > 0$. Because 2π is the period of both $y = A\sin x$ and $y = A\cos x$ a cycle of the graphs of $y = A\sin Bx$ and $y = A\cos Bx$ begins at $x = 0$ and will start to repeat its values when $Bx = 2\pi$. Dividing the last equality by B shows that the **period** of each of the functions

$y = \sin Bx$ and $y = \cos Bx$ is $2\pi/B$ and so the graph of each function over the interval $[0, 2\pi/B]$ is one **cycle** of its graph. If $0 < B < 1$, then the period $2\pi/B$ is greater than 2π, and we can we can characterize the cycle of either $y = \sin Bx$ or $y = \cos Bx$ on $[0, 2\pi/B]$ as a **horizontal stretch** of the graphs of $y = \sin x$ and $y = \cos x$ on the interval $[0, 2\pi]$. On the other hand, if $B > 1$, then the period $2\pi/B$ is less than 2π, and so the graphs on $[0, 2\pi/B]$ can be interpreted as a **horizontal compression** of the graphs of the functions $y = \sin x$ and $y = \cos x$ on $[0, 2\pi]$. Finally, we can easily find the x-coordinates of the x-intercepts of the graphs of $y = A \sin Bx$ and $y = A \cos Bx$ by replacing the symbol x by Bx in (1) and (2) and solving for x.

The next example illustrates these concepts.

EXAMPLE 3 Horizontally Compressed Sine Graph

Careful here: $\sin 2x \neq 2 \sin x$

With the identification $B = 2$ the period of the sine function $y = \sin 2x$ is $2\pi/B = 2\pi/2 = \pi$. Therefore one cycle of the graph is completed on the interval $[0, \pi]$. FIGURE 4.3.7 shows that two cycles of the graph of $y = \sin 2x$ (in red) are completed on the interval $[0, 2\pi]$ whereas the graph of $y = \sin x$ (in blue) has completed only one cycle. Interpreted in terms of transformations, the graph of $y = \sin 2x$ on $[0, \pi]$ is a horizontal compression of the graph of $y = \sin x$ on $[0, 2\pi]$.

To find the x-intercepts of the graph of $y = \sin 2x$ we solve the equation $\sin 2x = 0$. From (1) with x replaced by $2x$, we get

$$2x = n\pi \quad \text{or} \quad x = \tfrac{1}{2}n\pi.$$

By letting $n = 0, \pm 1, \pm 2, \pm 3, \pm 4, \ldots$, the zeros of $y = \sin 2x$ are $x = 0, \pm\frac{1}{2}\pi, \pm\pi, \pm\frac{3}{2}\pi, \pm 2\pi$, and so on. As we see in Figure 4.3.7, the x-intercepts on the nonnegative x-axis are the points

$$(0, 0), (\pi/2, 0), (\pi, 0), (3\pi/2, 0), (2\pi, 0), \ldots.$$

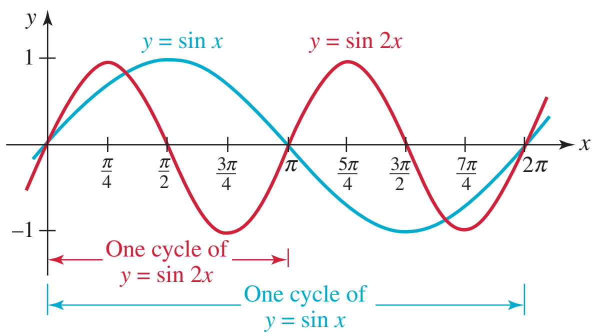


FIGURE 4.3.7 Graph of function in Example 3

Horizontal Stretch/Compression

- The graphs of the functions

$$y = A \sin Bx \quad \text{and} \quad y = A \cos Bx$$

for $B > 0$ have amplitude $|A|$ and period $2\pi/B$. The graphs of $y = A \sin Bx$ and $y = A \cos Bx$ are the graphs of $y = A \sin x$ and $y = A \cos x$ stretched horizontally if $0 < B < 1$, and compressed horizontally if $B > 1$.

EXAMPLE 4 Horizontally Compressed Cosine Graph

Find the period of $y = \cos 4x$ and graph the function.

Solution Since $B = 4$, we see that the period of $y = \cos 4x$ is $2\pi/4 = \pi/2$. We conclude that the graph of $y = \cos 4x$ is the graph of $y = \cos x$ compressed horizontally. To graph the function, we draw one cycle of the cosine graph with amplitude 1 on the interval $[0, \pi/2]$ and then use periodicity to extend the graph. FIGURE 4.3.8 shows four complete cycles of $y = \cos 4x$ (in red) and one cycle of $y = \cos x$ (in blue) on $[0, 2\pi]$. Notice that $y = \cos 4x$ attains its minimum at $x = \pi/4$ since $\cos 4(\pi/4) = \cos \pi = -1$ and its maximum at $x = \pi/2$ since $\cos 4(\pi/2) = \cos 2\pi = 1$.

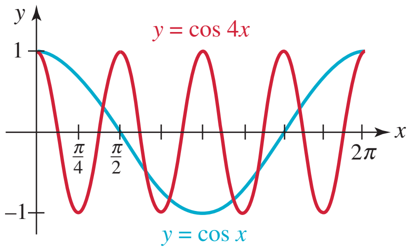


FIGURE 4.3.8 Graph of function in Example 4

In the case when $B < 0$ in either $y = A \sin Bx$ or $y = A \cos Bx$, we can use the even/odd properties, (15) of Section 4.2, to rewrite the function with positive B. This is illustrated in the next example.

EXAMPLE 5 Horizontally Stretched Sine Graph

Find the amplitude and period of $y = \sin(-\frac{1}{2}x)$. Graph the function.

Solution Since we require $B > 0$, we use $\sin(-x) = -\sin x$ to rewrite the function as

$$y = \sin(-\tfrac{1}{2}x) = -\sin \tfrac{1}{2}x.$$

With the identification $A = -1$, the amplitude is seen to be $|A| = |-1| = 1$. Now with $B = \frac{1}{2}$ we find that the period is $2\pi/\frac{1}{2} = 4\pi$. Hence we can interpret the cycle of $y = -\sin \frac{1}{2}x$ on $[0, 4\pi]$ as a horizontal stretch and a reflection in the x-axis (because $A < 0$) of the cycle of $y = \sin x$ on $[0, 2\pi]$. FIGURE 4.3.9 shows that on the interval $[0, 4\pi]$ the graph of $y = -\sin \frac{1}{2}x$ (in red) completes one cycle whereas the graph of $y = \sin x$ (in blue) completes two cycles.

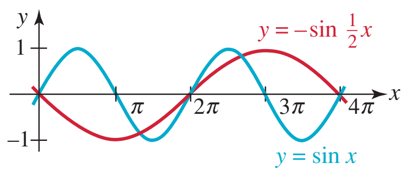


FIGURE 4.3.9 Graph of function in Example 5

Graphs of $y = A\sin(Bx + C)$ and $y = A\cos(Bx + C)$ We have seen that the basic graphs of $y = \sin x$ and $y = \cos x$ can be stretched or compressed vertically:

$$y = A \sin x \quad \text{and} \quad y = A \cos x,$$

shifted vertically:

$$y = A \sin x + D \quad \text{and} \quad y = A \cos x + D,$$

and stretched or compressed horizontally:

$$y = A \sin Bx + D \quad \text{and} \quad y = A \cos Bx + D.$$

The graphs of

$$y = A \sin(Bx + C) \quad \text{and} \quad y = A \cos(Bx + C)$$

are the graphs of $y = A \sin Bx$ and $y = A \cos Bx$ shifted horizontally. And finally, the graphs of

$$y = A \sin(Bx + C) + D \quad \text{and} \quad y = A \cos(Bx + C) + D$$

are the graphs of $y = A \sin(Bx + C)$ and $y = A \cos(Bx + C)$ shifted vertically.

Since the last case is straightforward, we are going to focus on the graphs of $y = A \sin(Bx + C)$ and $y = A \cos(Bx + C)$ in the remaining discussion. For example, we know from Section 2.2 that the graph of $y = \cos(x - \pi/2)$ is the basic cosine graph shifted $\pi/2$ units to the right. In FIGURE 4.3.10 the graph of $y = \cos(x - \pi/2)$ (in red) on the interval $[0, 2\pi]$ is one cycle of $y = \cos x$ on the interval $[-\pi/2, 3\pi/2]$

(in blue) shifted horizontally $\pi/2$ units to the right. Similarly, the graphs of $y = \sin(x + \pi/2)$ and $y = \sin(x - \pi/2)$ are the basic sine graphs shifted $\pi/2$ units to the left and to the right, respectively. See **FIGURES 4.3.11** and **4.3.12**.

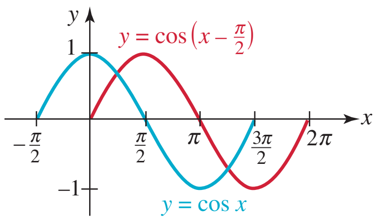


FIGURE 4.3.10 Horizontally shifted cosine graph

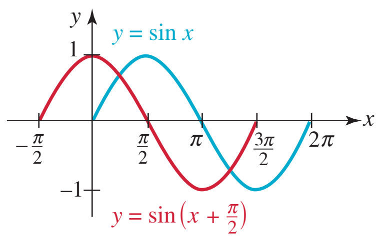


FIGURE 4.3.11 Horizontally shifted sine graph

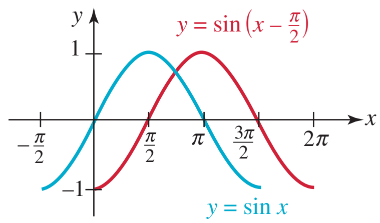


FIGURE 4.3.12 Horizontally shifted sine graph

By comparing the red graphs in Figures 4.3.10–4.3.12 with the graphs in Figures 4.3.2 and 4.3.3 we see that:

- the cosine graph shifted $\pi/2$ units to the right is the sine graph,
- the sine graph shifted $\pi/2$ units to the left is the cosine graph, and
- the sine graph shifted $\pi/2$ units to the right is the cosine graph reflected in the x-axis.

In other words, we have graphically verified the identities

$$\cos\left(x - \frac{\pi}{2}\right) = \sin x, \quad \sin\left(x + \frac{\pi}{2}\right) = \cos x, \quad \text{and} \quad \sin\left(x - \frac{\pi}{2}\right) = -\cos x. \tag{5}$$

Everything said in this paragraph also holds for $y = A\cos(Bx + C)$.

For convenience we now consider the graph of $y = A\sin(Bx + C)$, for $B > 0$. Since the values of $\sin(Bx + C)$ range from –1 to 1, it follows that $A\sin(Bx + C)$ varies between $-A$ and A. That is, the **amplitude** of $y = A\sin(Bx + C)$ is $|A|$. Also, as $Bx + C$ varies from 0 to 2π, the graph will complete one cycle. By solving $Bx + C = 0$ and $Bx + C = 2\pi$, we find that one cycle is completed as x varies from $-C/B$ to $(2\pi - C)/B$. Therefore, the function $y = A\sin(Bx + C)$ has the **period**

$$\frac{2\pi - C}{B} - \left(-\frac{C}{B}\right) = \frac{2\pi}{B}.$$

Moreover, if $f(x) = A\sin Bx$, then

$$f\left(x + \frac{C}{B}\right) = A\sin B\left(x + \frac{C}{B}\right) = A\sin(Bx + C). \tag{6}$$

The result in (6) shows that the graph of $y = A\sin(Bx + C)$ can be obtained by shifting the graph of $f(x) = A\sin Bx$ horizontally a distance $|C|/B$. The number C/B is called the **phase shift** of the graph of $y = A\sin(Bx + C)$. If $C/B < 0$ the shift is to the right whereas if $C/B > 0$, the shift is to the left.

EXAMPLE 6 Equation of Shifted Cosine Graph

The graph of $y = 10\cos 4x$ is shifted $\pi/12$ units to the right. Find its equation.

Solution We first identify $f(x) = 10\cos 4x$ and $B = 4$. Because we want to shift the graph of f to the right the phase shift is $C/B = -\pi/12$. Then the analogue of (6) for the cosine function is:

$$f\left(x - \frac{\pi}{12}\right) = 10\cos 4\left(x - \frac{\pi}{12}\right) \quad \text{or} \quad y = 10\cos\left(4x - \frac{\pi}{3}\right).$$

As a practical matter, if we are given the equation $y = A\sin(Bx + C)$ (or $y = A\cos(Bx + C)$), then the phase shift C/B of the graph of can be obtained by factoring the number B from $Bx + C$:

$$y = A\sin(Bx + C) = A\sin B\left(x + \frac{C}{B}\right).$$

The foregoing information is summarized next.

Horizontally Shifted Sine and Cosine Graphs

- The graphs of the functions

$$y = A\sin(Bx + C) \text{ and } y = A\cos(Bx + C), B > 0, \tag{7}$$

are the graphs of $y = A\sin Bx$ and $y = A\cos Bx$ shifted horizontally by $|C|/B$ units. The number C/B is called the **phase shift** of the graph.
- The horizontal shift is to the right if $C/B < 0$ and to left if $C/B > 0$.
- The **amplitude** of each function in (7) is $|A|$ and the **period** of each function is $2\pi/B$.
- The **range** of each function in (7) is the interval $[-|A|, |A|]$ on the y-axis.

The numbers $|A|$ and $2\pi/B$ are also referred to as the amplitude and period *of the graphs* of the functions in (7).

EXAMPLE 7 Horizontally Shifted Sine Graph

Graph $y = 3\sin(2x - \pi/3)$.

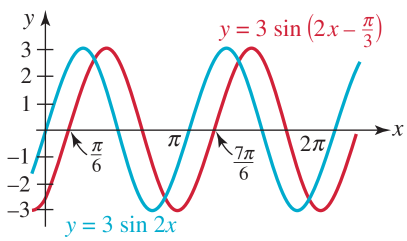


FIGURE 4.3.13 Graph of function in Example 7

Solution For purposes of comparison we will first graph $y = 3\sin 2x$. The amplitude of $y = 3\sin 2x$ is $|A| = 3$ and its period is $2\pi/B = 2\pi/2 = \pi$. Thus one cycle of $y = 3\sin 2x$ is completed on the interval $[0, \pi]$. We extend this graph to the adjacent interval $[\pi, 2\pi]$ as shown in blue in FIGURE 4.3.13. Next, we rewrite $y = 3\sin(2x - \pi/3)$ by factoring 2 from $2x - \pi/3$:

$$y = 3\sin\left(2x - \frac{\pi}{3}\right) = 3\sin 2\left(x - \frac{\pi}{6}\right).$$

From the last form we see that the phase shift is $-\pi/6 < 0$. This means that the graph of the given function, shown in red in Figure 4.3.13, is obtained by shifting the graph of $y = 3\sin 2x$ to the right $|C|/B = |(-\pi/3)|/2 = \pi/6$ units. Remember, this means that if (x, y) is a point on the blue graph, then $(x + \pi/6, y)$ is the corresponding point on the red graph. For example, $x = 0$ and $x = \pi$ are the x-coordinates of two x-intercepts of the blue graph. Thus $x = 0 + \pi/6 = \pi/6$ and $x = \pi + \pi/6 = 7\pi/6$ are x-coordinates of the x-intercepts of the red or shifted graph. These numbers are indicated by the arrows in Figure 4.3.13. Note that one cycle of the red graph is completed on the interval $[\pi/6, 7\pi/6]$.

EXAMPLE 8 Horizontally Shifted Graphs

Determine the amplitude, the period, the phase shift, and the direction of horizontal shift for each of the following functions.

(a) $y = 15\cos\left(5x - \dfrac{3\pi}{2}\right)$ **(b)** $y = -8\sin\left(2x + \dfrac{\pi}{4}\right)$

Solution **(a)** We first make the identifications $A = 15$, $B = 5$, and $C = -3\pi/2$. Thus the amplitude is $|A| = 15$ and the period is $2\pi/B = 2\pi/5$. The phase shift can be computed either by $C/B = (-3\pi/2)/5 = -3\pi/10$ or by rewriting the function as

$$y = 15\cos 5\left(x - \frac{3\pi}{10}\right).$$

The last form indicates that the graph of $y = 15\cos(5x - 3\pi/2)$ is the graph of $y = 15\cos 5x$ shifted $3\pi/10$ units to the right.

(b) Since $A = -8$ the amplitude is $|A| = |-8| = 8$. With $B = 2$ the period is $2\pi/B = 2\pi/2 = \pi$. By factoring 2 from $2x + \pi/4$, we see from

$$y = -8\sin\left(2x + \frac{\pi}{4}\right) = -8\sin 2\left(x + \frac{\pi}{8}\right)$$

that the phase shift is $\pi/8 > 0$ and so the graph of $y = -8\sin(2x + \pi/4)$ is the graph of $y = -8\sin 2x$ shifted $\pi/8$ units to the left.

EXAMPLE 9 Horizontally Shifted Cosine Graph

Graph $y = 2\cos(\pi x + \pi)$.

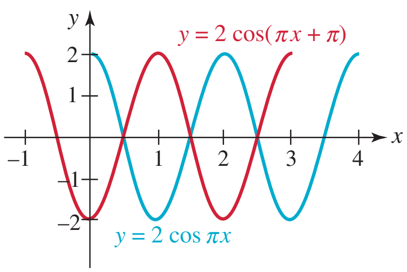


FIGURE 4.3.14 Graph of function in Example 9

Solution The amplitude of $y = 2\cos \pi x$ is $|A| = 2$ and the period is $2\pi/\pi = 2$. Thus one cycle of $y = 2\cos \pi x$ is completed on the interval $[0, 2]$. In FIGURE 4.3.14 two cycles of the graph of $y = 2\cos \pi x$ (in blue) are shown. The x-intercepts of this graph correspond to the values of x for which $\cos \pi x = 0$. By (2), this implies $\pi x = (2n + 1)\pi/2$ or $x = (2n + 1)/2$, n an integer. In other words, for $n = 0, -1, 1, -2, 2, -3, \ldots$ we get $x = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}$, and so on. Now by rewriting the given function as

$$y = 2\cos \pi(x + 1)$$

we see the phase shift is $1 > 0$. Thus the graph of $y = 2\cos(\pi x + \pi)$ (in red) in Figure 4.3.14, is obtained by shifting the graph of $y = 2\cos \pi x$ to the left 1 unit. This means that the x-intercepts are the same for both graphs.

EXAMPLE 10 Alternating Current

The current I (in amperes) in a wire of an alternating-current circuit is given by $I(t) = 30\sin 120\pi t$, where t is time measured in seconds. Sketch one cycle of the graph. What is the maximum value of the current?

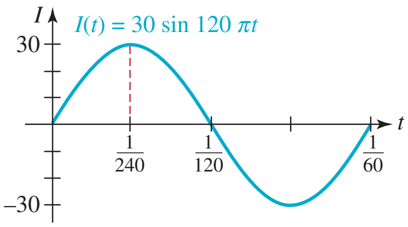


FIGURE 4.3.15 Graph of current in Example 10

Solution The graph has amplitude 30 and period $2\pi/120\pi = \frac{1}{60}$. Therefore, we sketch one cycle of the basic sine curve on the interval $\left[0, \frac{1}{60}\right]$, as shown in FIGURE 4.3.15. From the figure it is evident that the maximum value of the current is $I = 30$ amperes and occurs at $t = \frac{1}{240}$ since

$$I\left(\frac{1}{240}\right) = 30\sin\left(120\pi \cdot \frac{1}{240}\right) = 30\sin\frac{\pi}{2} = 30.$$

Exercises 4.3

Answers to selected odd-numbered problems begin on page ANS-15.

In Problems 1–6, use the techniques of shifting, stretching, compressing, and reflecting to sketch at least one cycle of the graph of the given function.

1. $y = \frac{1}{2} + \cos x$ **2.** $y = -1 + \cos x$ **3.** $y = 2 - \sin x$
4. $y = 3 + 3\sin x$ **5.** $y = -2 + 4\cos x$ **6.** $y = 1 - 2\sin x$

In Problems 7–10, the given figure shows one cycle of a sine or cosine graph. From the figure, determine A and D and write an equation of the form $y = A\sin x + D$ or $y = A\cos x + D$ for the graph.

7.

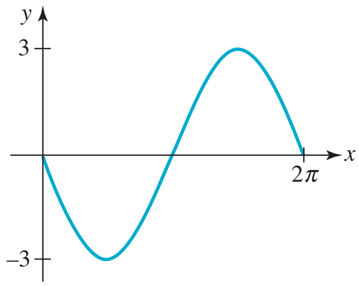


FIGURE 4.3.16 Graph for Problem 7

8.

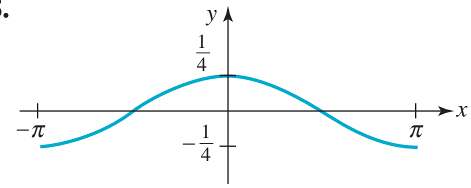


FIGURE 4.3.17 Graph for Problem 8

9.

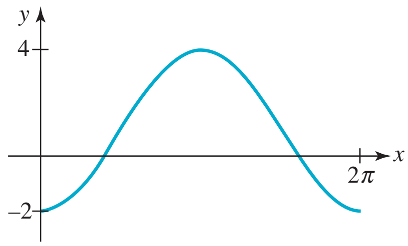


FIGURE 4.3.18 Graph for Problem 9

10.

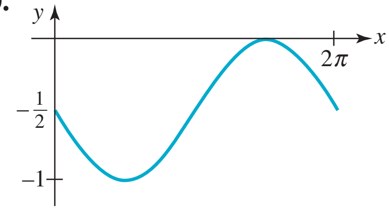


FIGURE 4.3.19 Graph for Problem 10

In Problems 11–16, use (1) and (2) of this section to find the x-intercepts for the graph of the given function. Do not graph.

11. $y = \sin \pi x$

12. $y = -\cos 2x$

13. $y = 10\cos\dfrac{x}{2}$

14. $y = 3\sin(-5x)$

15. $y = \sin\left(x - \dfrac{\pi}{4}\right)$

16. $y = \cos(2x - \pi)$

In Problems 17 and 18, find the x-intercepts of the graph of the given function on the interval $[0, 2\pi]$. Then find all intercepts using periodicity.

17. $y = -1 + \sin x$

18. $y = 1 - 2\cos x$

In Problems 19–24, the given figure shows one cycle of a sine or cosine graph. From the figure, determine A and B and write an equation of the form $y = A\sin Bx$ or $y = A\cos Bx$ for the graph.

19.

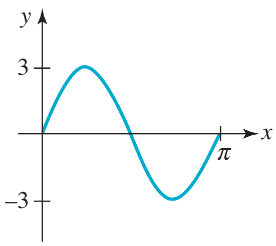


FIGURE 4.3.20 Graph for Problem 19

20.

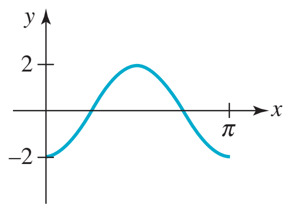


FIGURE 4.3.21 Graph for Problem 20

21.

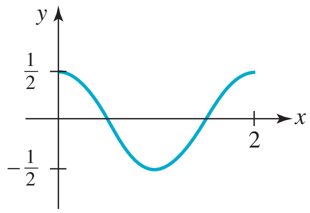


FIGURE 4.3.22 Graph for Problem 21

22.

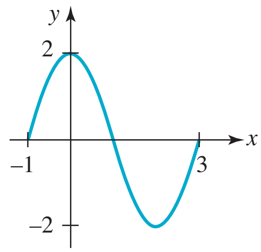


FIGURE 4.3.23 Graph for Problem 22

23.

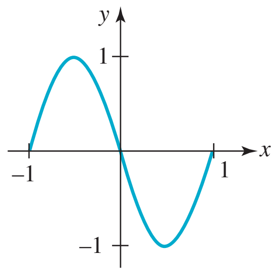


FIGURE 4.3.24 Graph for Problem 23

24.

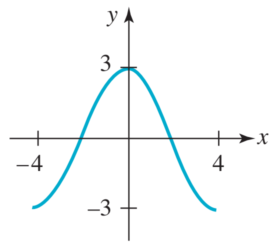


FIGURE 4.3.25 Graph for Problem 24

In Problems 25–30, find the amplitude and period of the given function. Sketch at least one cycle of the graph.

25. $y = 4 \sin \pi x$

26. $y = -5 \sin \dfrac{x}{2}$

27. $y = -3 \cos 2\pi x$

28. $y = \dfrac{5}{2} \cos 4x$

29. $y = -2 \sin(-2x)$

30. $y = 5 \cos\left(-\dfrac{\pi}{2}x\right)$

In Problems 31–36, **(a)** sketch one cycle of the graph of the given function. **(b)** Find the amplitude $|A|$ by inspection of the function. **(c)** Find the maximum value M and the minimum value m of the function on the interval in part (a). **(d)** Then use (4) to verify the amplitude $|A|$ of the function. **(e)** Give the range of each function.

31. $y = 3 - 4\cos x$

32. $y = -3 + 3\sin x$

33. $y = -1 + \sin \dfrac{\pi}{2}x$

34. $y = -1 - 5\cos \pi x$

35. $y = 3 - 2\sin \pi x$

36. $y = 1 - 4\sin \dfrac{2}{3}x$

In Problems 37 and 38, from the given figure determine A, B, and D and write an equation of the form $y = A\sin Bx + D$ or $y = A\cos Bx + D$ for the graph.

37.

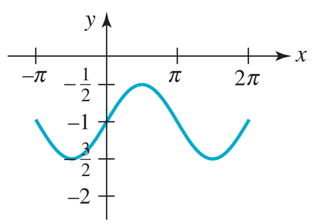


FIGURE 4.3.26 Graph for Problem 37

38.

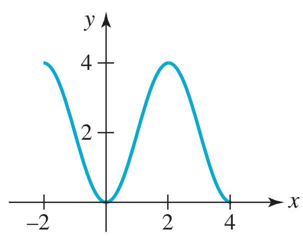


FIGURE 4.3.27 Graph for Problem 38

In Problems 39–48, find the amplitude, period, and phase shift of the given function. Sketch at least one cycle of the graph.

39. $y = \sin\left(x - \dfrac{\pi}{6}\right)$

40. $y = \sin\left(3x - \dfrac{\pi}{4}\right)$

41. $y = \cos\left(x + \dfrac{\pi}{4}\right)$

42. $y = -2\cos\left(2x - \dfrac{\pi}{6}\right)$

43. $y = 4\cos\left(2x - \dfrac{3\pi}{2}\right)$

44. $y = 3\sin\left(2x + \dfrac{\pi}{4}\right)$

45. $y = 3\sin\left(\dfrac{x}{2} - \dfrac{\pi}{3}\right)$

46. $y = -\cos\left(\dfrac{x}{2} - \pi\right)$

47. $y = -4\sin\left(\dfrac{\pi}{3}x - \dfrac{\pi}{3}\right)$

48. $y = 2\cos\left(-2\pi x - \dfrac{4\pi}{3}\right)$

In Problems 49 and 50, write an equation of the function whose graph is described in words.

49. The graph of $y = \cos x$ is vertically stretched up by a factor of 3 and shifted down by 5 units. One cycle of $y = \cos x$ on $[0, 2\pi]$ is compressed to $[0, \pi/3]$ and then the compressed cycle is shifted horizontally $\pi/4$ units to the left.

50. One cycle of $y = \sin x$ on $[0, 2\pi]$ is stretched to $[0, 8\pi]$ and then the stretched cycle is shifted horizontally $\pi/12$ units to the right. The graph is also compressed vertically by a factor of $\frac{3}{4}$ and then reflected in the x-axis.

In Problems 51–54, find horizontally shifted sine and cosine functions so that each function satisfies the given conditions. Graph the functions.

51. Amplitude 3, period $2\pi/3$, shifted by $\pi/3$ units to the right

52. Amplitude $\frac{2}{3}$, period π, shifted by $\pi/4$ units to the left

53. Amplitude 0.7, period 0.5, shifted by 4 units to the right

54. Amplitude $\frac{5}{4}$, period 4, shifted by $1/2\pi$ units to the left

In Problems 55 and 56, graphically verify the given identity.

55. $\cos(x + \pi) = -\cos x$

56. $\sin(x + \pi) = -\sin x$

Applications

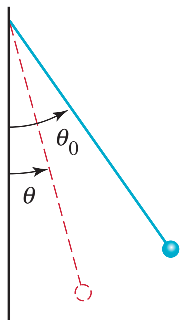


FIGURE 4.3.28 Pendulum in Problem 57

57. Pendulum The angular displacement θ of a pendulum from the vertical at time t seconds is given by $\theta(t) = \theta_0 \cos \omega t$, where θ_0 is the initial displacement at $t = 0$ seconds. See FIGURE 4.3.28. For $\omega = 2$ rad/s and $\theta_0 = \pi/10$, sketch two cycles of the resulting function.

58. Fahrenheit Temperature Suppose that

$$T(t) = 50 + 10\sin\frac{\pi}{12}(t - 8),$$

$0 \le t \le 24$, is a mathematical model of the Fahrenheit temperature at t hours after midnight on a certain day of the week.

(a) What is the temperature at 8 A.M.?
(b) At what time(s) does $T(t) = 60$?
(c) Sketch the graph of T.
(d) Find the maximum and minimum temperatures and the times at which they occur.

59. Depth of Water The depth d of water at the entrance to a small harbor at time t is modeled by a function of the form

$$d(t) = A\sin B\left(t - \frac{\pi}{2}\right) + C,$$

where A is one-half the difference between the high- and low-tide depths; $2\pi/B$, $B > 0$, is the tidal period; and C is the average depth. Assume that the tidal period is 12 hours, the depth at high tide is 18 feet, and the depth at low tide is 6 feet. Sketch two cycles of the graph of d.

60. Hours of Daylight The number H of daylight hours per day in various locations in the world can be modeled by a function of the form

$$H(t) = A\sin B(t - C) + D,$$

where the variable t represents the number of days in a year corresponding to a specific calendar date (for example, February 1 corresponds to $t = 32$ days), and A, B, C, and D are positive constants. In this problem we construct a model for Los Angeles, CA for the year 2017 (not a leap year) using data obtained from the U.S. Naval Observatory, Washington, D.C.

(a) Find the amplitude A if 14.43 is the maximum number of daylight hours at the summer solstice and if 9.88 is the minimum number of daylight hours at the winter solstice.

(b) Find B if the function $H(t)$ is to have the period 365 days.

(c) For Los Angeles in the year 2017, we choose $C = 79$. Explain the significance of this number. [*Hint*: C has the same units as t.]

(d) Find D if the number of daylight hours at the vernal equinox for 2017 is 12.14 and occurs on March 20.

(e) What does the model $H(t)$ predict to be the number of daylight hours on January 1? On June 21? On August 1? On December 21?

(f) Using a graphing utility to obtain the graph of $H(t)$ on the interval $[0, 365]$.

Sunset in Los Angeles

Calculator/Computer Problems

In Problems 61–64, use a graphing utility to investigate whether the given function is periodic.

61. $f(x) = \sin\left(\dfrac{1}{x}\right)$

62. $f(x) = \dfrac{1}{\sin 2x}$

63. $f(x) = 1 + (\cos x)^2$

64. $f(x) = x\sin x$

For Discussion

In Problems 65 and 66, describe in words how you would obtain the graph of the given function by starting with the graph of $y = \sin x$ (Problem 65) and the graph of $y = \cos x$ (Problem 66).

65. $y = 5 + 3\sin(2x - \pi)$

66. $y = -6 + \frac{1}{4}\cos(\frac{1}{2}x + \pi)$

In Problems 67 and 68, find the period of the given function.

67. $f(x) = \sin\frac{1}{2}x\sin 2x$

68. $f(x) = \sin\frac{3}{2}x + \cos\frac{5}{2}x$

In Problems 69 and 70, discuss and then sketch the graph of the given function.

69. $f(x) = |\sin x|$

70. $f(x) = |\cos x|$

4.4 Other Trigonometric Functions

INTRODUCTION Recall that the remaining four trigonometric functions are the **tangent**, **cotangent**, **secant**, and **cosecant functions** and are denoted, in turn, as $\tan x$, $\cot x$, $\sec x$, and $\csc x$. We saw in Section 4.2 that by using (1) and (2) of Definition 4.2.1 in (3)–(6) of Definition 4.2.2 we can express these four new functions in terms of $\sin x$ and $\cos x$:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \tag{1}$$

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x}. \tag{2}$$

Domain and Range Because the functions in (1) and (2) are quotients, we know from Definition 2.6.1 that the **domain** of each function consists of the set of real numbers *except* those numbers for which the denominator is zero. We have seen in (2) of Section 4.3 that $\cos x = 0$ for $x = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$, and so

- the domain of $\tan x$ and of $\sec x$ is $\{x \mid x \neq (2n + 1)\pi/2, n = 0, \pm 1, \pm 2, \ldots\}$.

Similarly, from (1) of Section 4.3, $\sin x = 0$ for $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$, and so it follows that

- the domain of $\cot x$ and of $\csc x$ is $\{x \mid x \neq n\pi, n = 0, \pm 1, \pm 2, \ldots\}$.

We know that the values of the sine and cosine are bounded, that is, $|\sin x| \leq 1$ and $|\cos x| \leq 1$. From these last inequalities we have

$$|\sec x| = \left|\frac{1}{\cos x}\right| = \frac{1}{|\cos x|} \geq 1 \tag{3}$$

and

$$|\csc x| = \left|\frac{1}{\sin x}\right| = \frac{1}{|\sin x|} \geq 1. \tag{4}$$

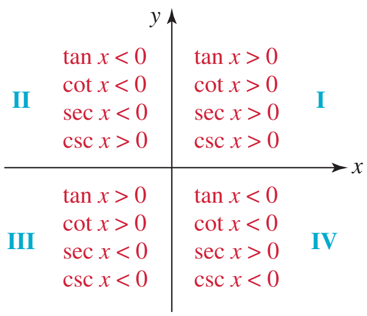


FIGURE 4.4.1 Signs of $\tan x$, $\cot x$, $\sec x$, and $\csc x$, in the four quadrants

Recall that an inequality such as (3) means that $\sec x \geq 1$ or $\sec x \leq -1$. Hence the range of the secant function is $(-\infty, -1] \cup [1, \infty)$. The inequality in (4) implies that the cosecant function has the same range $(-\infty, -1] \cup [1, \infty)$. When we consider the graphs of the tangent and cotangent functions we will see that they have the same range: $(-\infty, \infty)$.

If we interpret x as an angle, then FIGURE 4.4.1 illustrates the algebraic signs of the tangent, cotangent, secant, and cosecant functions in each of the four quadrants. This is easily verified using the signs of the sine and cosine functions displayed in Figure 4.2.2.

EXAMPLE 1 Example 5 of Section 4.2 Revisited

Find $\tan x$, $\cot x$, $\sec x$, and $\csc x$ for $x = -\pi/6$.

Solution In Example 5 of Section 4.2 we saw that

$$\sin\left(-\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} = -\frac{1}{2} \quad \text{and} \quad \cos\left(-\frac{\pi}{6}\right) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}.$$

Therefore, by the definitions in (1) and (2):

$$\tan\left(-\frac{\pi}{6}\right) = \frac{-1/2}{\sqrt{3}/2} = -\frac{1}{\sqrt{3}}, \quad \cot\left(-\frac{\pi}{6}\right) = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3},$$
$$\sec\left(-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}/2} = \frac{2}{\sqrt{3}}, \quad \csc\left(-\frac{\pi}{6}\right) = \frac{1}{-1/2} = -2.$$

TABLE 4.4.1

x	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\tan x$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	–
$\cot x$	–	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0
$\sec x$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	–
$\csc x$	–	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1

Table 4.4.1 summarizes some important values of the tangent, cotangent, secant, and cosecant and was constructed using values of the sine and cosine from Section 4.2. A dash in the table indicates that the trigonometric function is not defined at that particular value of x.

Identities The tangent is related to the secant by a useful identity. If we divide the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1 \tag{5}$$

by $\cos^2 x$, we see that

$$\frac{\sin^2 x}{\cos^2 x} + \frac{\cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}. \tag{6}$$

Similarly, dividing (5) by $\sin^2 x$ yields an identity relating the cotangent with the cosecant:

$$\frac{\sin^2 x}{\sin^2 x} + \frac{\cos^2 x}{\sin^2 x} = \frac{1}{\sin^2 x}. \tag{7}$$

Using the laws of exponents,

$$\frac{\sin^2 x}{\cos^2 x} = \left(\frac{\sin x}{\cos x}\right)^2 = \tan^2 x, \qquad \frac{1}{\cos^2 x} = \left(\frac{1}{\cos x}\right)^2 = \sec^2 x,$$
$$\frac{\cos^2 x}{\sin^2 x} = \left(\frac{\cos x}{\sin x}\right)^2 = \cot^2 x, \qquad \frac{1}{\sin^2 x} = \left(\frac{1}{\sin x}\right)^2 = \csc^2 x,$$

we see that (6) and (7) can be written in a simpler manner:

$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x.$$

Since the last two identities are direct consequences of $\sin^2 x + \cos^2 x = 1$ they too are called **Pythagorean identities**.

Finally, note that the tangent and cotangent function are related by the **reciprocal identity**

$$\cot x = \frac{\cos x}{\sin x} = \frac{1}{\dfrac{\sin x}{\cos x}} = \frac{1}{\tan x}.$$

Summary For future reference, especially for the work in the next section, we pause here to summarize a small collection of identities that are so basic to the study of trigonometry that they are known collectively as the **fundamental identities**. You should firmly commit these identities to memory.

Fundamental Trigonometric Identities

Pythagorean identities:

$$\sin^2 x + \cos^2 x = 1 \tag{8}$$

$$1 + \tan^2 x = \sec^2 x \tag{9}$$

$$1 + \cot^2 x = \csc^2 x \tag{10}$$

Quotient identities:

$$\tan x = \frac{\sin x}{\cos x} \qquad \cot x = \frac{\cos x}{\sin x} \tag{11}$$

Reciprocal identities:

$$\sec x = \frac{1}{\cos x} \qquad \csc x = \frac{1}{\sin x} \qquad \cot x = \frac{1}{\tan x} \tag{12}$$

EXAMPLE 2 Using a Pythagorean Identity

Given that $\csc x = -5$ and $3\pi/2 < x < 2\pi$, determine the values of $\tan x$ and $\cot x$.

Solution We first compute $\cot x$. It follows from (10) that

$$\cot^2 x = \csc^2 x - 1.$$

For $3\pi/2 < x < 2\pi$, we see from Figure 4.4.1 that $\cot x$ must be negative and so we take the negative square root:

$$\cot x = -\sqrt{\csc^2 x - 1} = -\sqrt{(-5)^2 - 1} = -\sqrt{24} = -2\sqrt{6}.$$

Using $\cot x = 1/\tan x$, we have

$$\tan x = \frac{1}{\cot x} = \frac{1}{-2\sqrt{6}} \overset{\text{rationalizing the denominator}}{\downarrow}= -\frac{\sqrt{6}}{12}.$$

In Example 2, given the information $\csc x = -5$ and $3\pi/2 < x < 2\pi$, we could easily find the values of the remaining five trigonometric functions. One way of proceeding would be to use $\csc x = 1/\sin x$ to find $\sin x = 1/\csc x = -\frac{1}{5}$. Then we use $\sin^2 x + \cos^2 x = 1$ to find $\cos x$. After we have found $\cos x$, the remaining three trigonometric functions can be obtained from (1) and (2).

Periodicity Because the sine and cosine functions are 2π periodic, each of the functions in (1) and (2) have a period 2π. But from (17) of Theorem 4.2.5 we have

Also see Problems 55 and 56 in Exercises 4.3.

$$\tan(x + \pi) = \frac{\sin(x + \pi)}{\cos(x + \pi)} = \frac{-\sin x}{-\cos x} = \tan x. \tag{13}$$

Thus (13) implies that $\tan x$ and $\cot x$ are periodic with a period $p \le \pi$. In the case of the tangent function, $\tan x = 0$ only if $\sin x = 0$, that is, only if $x = 0, \pm\pi, \pm 2\pi$, and so on. Therefore, the smallest positive number p for which $\tan(x + p) = \tan x$ is $p = \pi$. The cotangent function has the same period since it is the reciprocal of the tangent function.

THEOREM 4.4.1 Period of the Tangent and Cotangent

The tangent and cotangent functions are periodic with **period π**. Therefore,

$$\tan(x+\pi) = \tan x \quad \text{and} \quad \cot(x+\pi) = \cot x \tag{14}$$

for every real number x for which the functions are defined.

THEOREM 4.4.2 Period of the Secant and Cosecant

The secant and cosecant functions are periodic with **period 2π**. Therefore,

$$\sec(x+2\pi) = \sec x \quad \text{and} \quad \csc(x+2\pi) = \csc x \tag{15}$$

for every real number x for which the functions are defined.

Even-Odd Properties Because the cosine function is even and the sine function is odd, each of the remaining four trigonometric functions is either even or odd.

THEOREM 4.4.3 Even and Odd Functions

The tangent, cotangent, and cosecant functions are **odd functions**, whereas the secant function is an **even function**. That is,

$$\tan(-x) = -\tan x \quad \text{and} \quad \cot(-x) = -\cot x \tag{16}$$

$$\sec(-x) = \sec x \quad \text{and} \quad \csc(-x) = -\csc x \tag{17}$$

for every real number x for which the functions are defined,

PROOF: We prove the first entries in (16) and (17). Because $\cos(-x) = \cos x$ and $\sin(-x) = -\sin x$,

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = \frac{-\sin x}{\cos x} = -\frac{\sin x}{\cos x} = -\tan x,$$

$$\sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos x} = \sec x$$

proves, in turn, that $\tan x$ is an odd function and $\sec x$ is an even function. ∎

This is a good time to review (7) of Section 3.5.

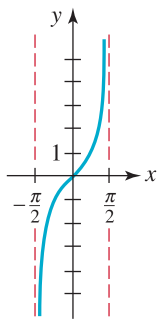


FIGURE 4.4.2 One cycle of the graph of $y = \tan x$

Graphs of $y = \tan x$ and $y = \cot x$ The numbers that make the denominators of $\tan x$, $\cot x$, $\sec x$, and $\csc x$ equal to zero correspond to vertical asymptotes of their graphs. For example, we encourage you to verify, using a calculator, that

$$\tan x \to -\infty \text{ as } x \to -\frac{\pi}{2}^{+} \quad \text{and} \quad \tan x \to \infty \text{ as } x \to \frac{\pi}{2}^{-}.$$

In other words, $x = -\pi/2$ and $x = \pi/2$ are vertical asymptotes. The graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$ given in FIGURE 4.4.2 is one **cycle** of the graph of $y = \tan x$. Using periodicity we extend the cycle in Figure 4.4.2 to adjacent intervals of length π, as shown in FIGURE 4.4.3. The x-intercepts of the graph of the tangent function are $(0, 0)$, $(\pm\pi, 0)$, $(\pm 2\pi, 0)$, . . . , and the vertical asymptotes of the graph are $x = \pm\pi/2$, $\pm 3\pi/2$, $\pm 5\pi/2$, ...

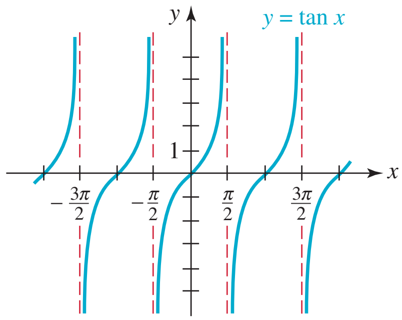


FIGURE 4.4.3 Graph of $y = \tan x$

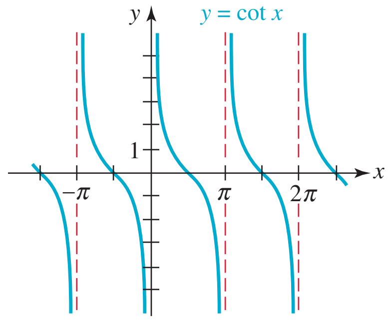


FIGURE 4.4.4 Graph of $y = \cot x$

The graph of $y = \cot x$ is similar to the graph of the tangent function and is given in FIGURE 4.4.4. In this case, the graph of $y = \cot x$ on the interval $(0, \pi)$ is one **cycle** of the graph of $y = \cot x$. The x-intercepts of the graph of the cotangent function are $(\pm\pi/2, 0)$, $(\pm 3\pi/2, 0)$, $(\pm 5\pi/2, 0)$, . . . , and the vertical asymptotes of the graph are $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Because $y = \tan x$ and $y = \cot x$ are odd functions, their graphs are symmetric with respect to the origin.

Graphs of $y = \sec x$ and $y = \csc x$ For both $y = \sec x$ and $y = \csc x$ we know that $|y| \geq 1$, and so no portion of their graphs can appear in the horizontal strip $-1 < y < 1$ of the Cartesian plane. Hence the graphs of $y = \sec x$ and $y = \csc x$ have no x-intercepts. Both $y = \sec x$ and $y = \csc x$ have period 2π. The vertical asymptotes for the graph of $y = \sec x$ are the same as $y = \tan x$, namely, $x = \pm\pi/2, \pm 3\pi/2, \pm 5\pi/2, \ldots$. Because $y = \sec x$ is an even function, its graph is symmetric with respect to the y-axis. On the other hand, the vertical asymptotes for the graph of $y = \csc x$ are the same as $y = \cot x$, namely, $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$. Because $y = \csc x$ is an odd function, its graph is symmetric with respect to the origin. One cycle of the graph of $y = \sec x$ on $[0, 2\pi]$ is extended to the interval $[-2\pi, 0]$ by periodicity (or y-axis symmetry) in FIGURE 4.4.5. Similarly, in FIGURE 4.4.6 we extend one cycle of $y = \csc x$ on $(0, 2\pi)$ to the interval $(-2\pi, 0)$ by periodicity (or origin symmetry).

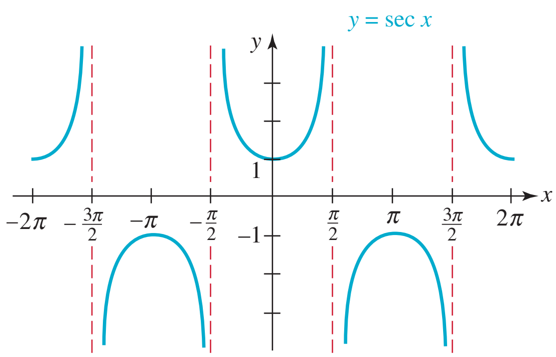


FIGURE 4.4.5 Graph of $y = \sec x$

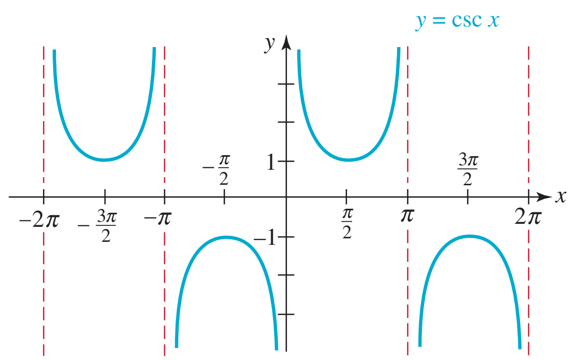


FIGURE 4.4.6 Graph of $y = \csc x$

Transformations and Graphs Similar to the sine and cosine graphs, rigid and nonrigid transformations can be applied to the graphs of $y = \tan x$, $y = \cot x$, $y = \sec x$, and $y = \csc x$. For example, a function such as $y = A\tan(Bx + C) + D$ can be analyzed in the following manner:

vertical stretch/compression/reflection ↓   vertical shift ↓

$$y = A\tan(Bx + C) + D \tag{18}$$

↑ horizontal stretch/compression by changing period   ↑ horizontal shift

If $B > 0$, then the period of

$$y = A\tan(Bx + C) \quad \text{and} \quad y = A\cot(Bx + C) \text{ is } \pi/B, \tag{19}$$

whereas the period of

$$y = A\sec(Bx + C) \quad \text{and} \quad y = A\csc(Bx + C) \text{ is } 2\pi/B. \tag{20}$$

Of the six trigonometric functions, only the sine and cosine functions have an amplitude.

As we see in (18), the number A in each case can be interpreted as either a vertical stretch or a compression of a graph. However, you should be aware of the fact that the functions in (19) and (20) have no amplitude, because none of the functions has a maximum *and* a minimum value.

EXAMPLE 3 Comparison of Graphs

Find the period, x-intercepts, and vertical asymptotes for the graph of $y = \tan 2x$. Graph the function on $[0, \pi]$.

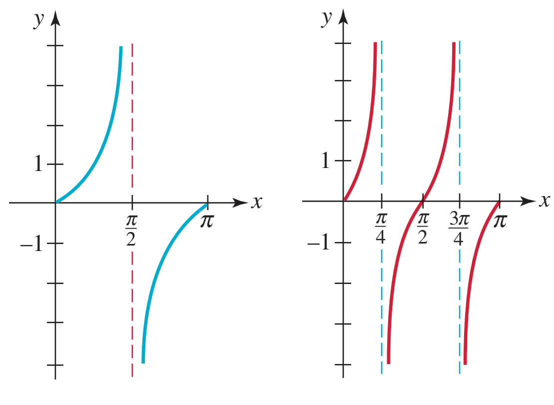
Solution With the identification $B = 2$, we see from (19) that the period is $\pi/2$. Since $\tan 2x = \sin 2x/\cos 2x$, the x-intercepts of the graph occur at the zeros of $\sin 2x$. From (1) of Section 4.3, $\sin 2x = 0$ for

$$2x = n\pi \quad \text{so that} \quad x = \tfrac{1}{2}n\pi,\ n = 0, \pm 1, \pm 2, \ldots .$$

That is, $x = 0, \pm\pi/2, \pm 2\pi/2 = \pi, \pm 3\pi/2, \pm 4\pi/2 = 2\pi$, and so on. The x-intercepts are $(0, 0), (\pm\pi/2, 0), (\pm\pi, 0), (\pm 3\pi/2, 0), \ldots$. The vertical asymptotes of the graph occur at zeros of $\cos 2x$. From (2) of Section 4.3, the numbers for which $\cos 2x = 0$ are found in the following manner:

$$2x = (2n + 1)\frac{\pi}{2} \quad \text{so that} \quad x = (2n + 1)\frac{\pi}{4},\ n = 0, \pm 1, \pm 2, \ldots .$$

That is, the vertical asymptotes are $x = \pm\pi/4, \pm 3\pi/4, \pm 5\pi/4, \ldots$. On the interval $[0, \pi]$, the graph of $y = \tan 2x$ has three intercepts $(0, 0)$, $(\pi/2, 0)$, and $(\pi, 0)$ and two vertical asymptotes $x = \pi/4$ and $x = 3\pi/4$. In FIGURE 4.4.7, we have compared the graphs of $y = \tan x$ and $y = \tan 2x$ on the interval. The graph of $y = \tan 2x$ is a horizontal compression of the graph of $y = \tan x$.



(a) $y = \tan x$ on $[0, \pi]$ (b) $y = \tan 2x$ on $[0, \pi]$

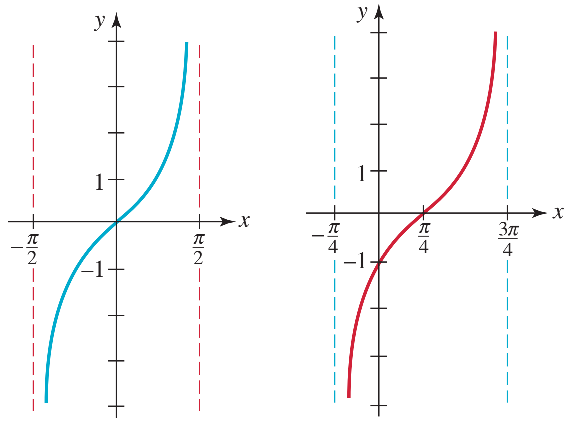
FIGURE 4.4.7 Graph of function in Example 3

EXAMPLE 4 Comparison of Graphs

Compare one cycle of the graphs of $y = \tan x$ and $y = \tan(x - \pi/4)$.

Solution The graph of $y = \tan(x - \pi/4)$ is the graph of $y = \tan x$ shifted horizontally $\pi/4$ units to the right. The intercept $(0, 0)$ for the graph of $y = \tan x$ is shifted to $(\pi/4, 0)$ on the graph of $y = \tan(x - \pi/4)$. The vertical asymptotes $x = -\pi/2$ and

$x = \pi/2$ for the graph of $y = \tan x$ are shifted to $x = -\pi/4$ and $x = 3\pi/4$ for the graph of $y = \tan(x - \pi/4)$. In FIGURES 4.4.8(a) and 4.4.8(b) we see, respectively, that a cycle of the graph of $y = \tan x$ on the interval $(-\pi/2, \pi/2)$ is shifted to the right to yield a cycle of the graph of $y = \tan(x - \pi/4)$ on the interval $(-\pi/4, 3\pi/4)$.



(a) Cycle of $y = \tan x$ on $(-\pi/2, \pi/2)$

(b) Cycle of $y = \tan(x - \pi/4)$ on $(-\pi/4, 3\pi/4)$

FIGURE 4.4.8 Graphs of functions in Example 4

As we did in the analysis of the graphs of $y = A\sin(Bx + C)$ and $y = A\cos(Bx + C)$, we can determine the amount of horizontal shift for graphs of functions such as $y = A\tan(Bx + C)$ and $y = A\sec(Bx + C)$ by factoring the number $B > 0$ from $Bx + C$.

EXAMPLE 5 Two Shifts and Two Compressions

Graph $y = 2 - \frac{1}{2}\sec(3x - \pi/2)$.

Solution Let's break down the analysis of the graph into four parts, namely, by transformations.

(*i*) One cycle of the graph of $y = \sec x$ occurs on $[0, 2\pi]$. Since the period of $y = \sec 3x$ is $2\pi/3$, one cycle of its graph occurs on the interval $[0, 2\pi/3]$. In other words, the graph of $y = \sec 3x$ is a horizontal compression of the graph of $y = \sec x$. Since $\sec 3x = 1/\cos 3x$, the vertical asymptotes occur at the zeros of $\cos 3x$. Using (2) of Section 4.3, we find

$$3x = (2n + 1)\frac{\pi}{2} \qquad \text{or} \qquad x = (2n + 1)\frac{\pi}{6},\ n = 0, \pm 1, \pm 2, \ldots.$$

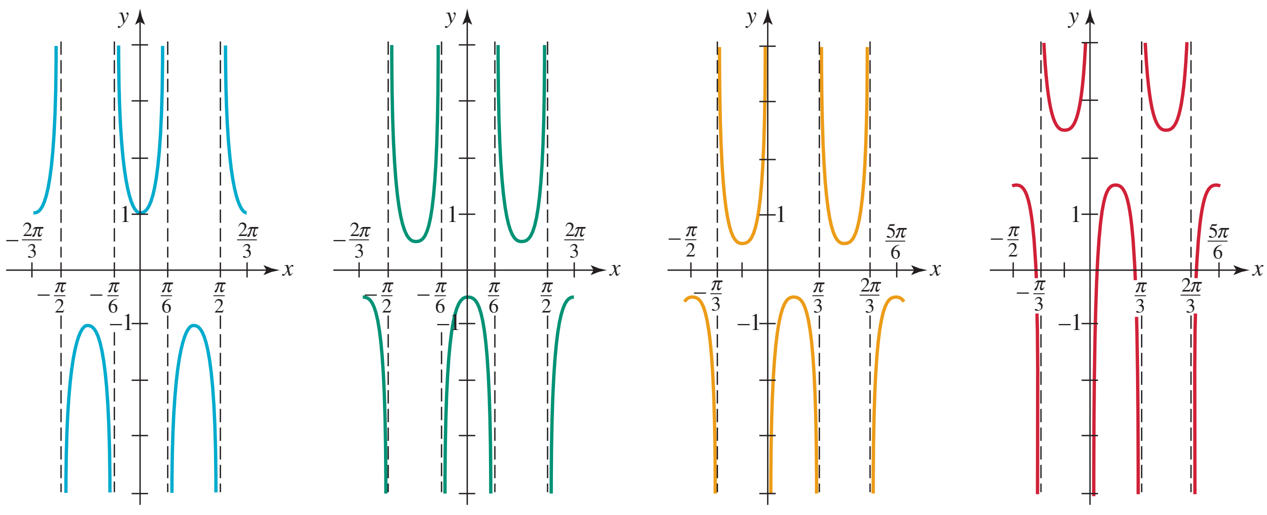
FIGURE 4.4.9(a) shows two cycles of the graph $y = \sec 3x$; one cycle on $[-2\pi/3, 0]$ and another on $[0, 2\pi/3]$. Within those intervals the vertical asymptotes are $x = -\pi/2$, $x = -\pi/6$, $x = \pi/6$, and $x = \pi/2$.

(*ii*) The graph of $y = -\frac{1}{2}\sec 3x$ is the graph of $y = \sec 3x$ compressed vertically by a factor of $\frac{1}{2}$ and then reflected in the x-axis. See Figure 4.4.9(b).

(*iii*) By factoring 3 from $3x - \pi/2$, we see from

$$y = -\tfrac{1}{2}\sec\left(3x - \frac{\pi}{2}\right) = -\tfrac{1}{2}\sec 3\left(x - \frac{\pi}{6}\right)$$

that the graph of $y = -\frac{1}{2}\sec(3x - \pi/2)$ is the graph of $y = -\frac{1}{2}\sec 3x$ shifted $\pi/6$ units to the right. By shifting the two intervals $[-2\pi/3, 0]$ and $[0, 2\pi/3]$ in Figure 4.4.9(b)



(a) Horizontal compression (b) Vertical compression and reflection in x-axis (c) Horizontal shift (d) Vertical shift

FIGURE 4.4.9 Graph of function in Example 5

to the right $\pi/6$ units, we see in Figure 4.4.9(c) two cycles of $y = -\frac{1}{2}\sec(3x - \pi/2)$ on the intervals $[-\pi/2, \pi/6]$ and $[\pi/6, 5\pi/6]$. The vertical asymptotes $x = -\pi/2$, $x = -\pi/6$, $x = \pi/6$, and $x = \pi/2$ shown in Figure 4.4.9(b) are shifted to $x = -\pi/3$, $x = 0$, $x = \pi/3$, and $x = 2\pi/3$. Observe that the y-intercept $\left(0, -\frac{1}{2}\right)$ in Figure 4.4.9(b) is now moved to $\left(\pi/6, -\frac{1}{2}\right)$ in Figure 4.4.9(c).

(*iv*) Finally, we obtain the graph $y = 2 - \frac{1}{2}\sec(3x - \pi/2)$ in Figure 4.4.9(d) by shifting the graph of $y = -\frac{1}{2}\sec(3x - \pi/2)$ in Figure 4.4.9(c) upward 2 units.

Exercises 4.4

Answers to selected odd-numbered problems begin on page ANS-16.

In Problems 1 and 2, complete the given table.

1.

x	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\tan x$												
$\cot x$												

2.

x	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{7\pi}{6}$	$\frac{5\pi}{4}$	$\frac{4\pi}{3}$	$\frac{3\pi}{2}$	$\frac{5\pi}{3}$	$\frac{7\pi}{4}$	$\frac{11\pi}{6}$	2π
$\sec x$												
$\csc x$												

In Problems 3–18, find the indicated value without the use of a calculator.

3. $\cot\frac{13\pi}{6}$ **4.** $\csc\left(-\frac{3\pi}{2}\right)$ **5.** $\tan\frac{9\pi}{2}$ **6.** $\sec 7\pi$

7. $\csc\left(-\frac{\pi}{3}\right)$ **8.** $\cot\left(-\frac{13\pi}{3}\right)$ **9.** $\tan\frac{23\pi}{4}$ **10.** $\tan\left(-\frac{5\pi}{6}\right)$

11. $\sec\frac{10\pi}{3}$ **12.** $\cot\frac{17\pi}{6}$ **13.** $\csc 5\pi$ **14.** $\sec\frac{29\pi}{4}$

15. $\sec(-120°)$ **16.** $\tan 405°$ **17.** $\csc 495°$ **18.** $\cot(-720°)$

In Problems 19–26, use the given information to find the values of the remaining five trigonometric functions.

19. $\tan x = -2, \quad \pi/2 < x < \pi$

20. $\cot x = \frac{1}{2}, \quad \pi < x < 3\pi/2$

21. $\csc x = \frac{4}{3}, \quad 0 < x < \pi/2$

22. $\sec x = -5, \quad \pi/2 < x < \pi$

23. $\sin x = \frac{1}{3}, \quad \pi/2 < x < \pi$

24. $\cos x = -1/\sqrt{5}, \quad \pi < x < 3\pi/2$

25. $\cos x = \frac{12}{13}, \quad 3\pi/2 < x < 2\pi$

26. $\sin x = \frac{4}{5}, \quad 0 < x < \pi/2$

27. If $3\cos x = \sin x$, find all values of $\tan x$, $\cot x$, $\sec x$, and $\csc x$.

28. If $\csc x = \sec x$, find all values of $\tan x$, $\cot x$, $\sin x$, and $\cos x$.

In Problems 29–36, find the period, x-intercepts, and the vertical asymptotes of the given function. Sketch at least one cycle of the graph.

29. $y = \tan \pi x$

30. $y = \tan \dfrac{x}{2}$

31. $y = \cot 2x$

32. $y = -\cot \dfrac{\pi x}{3}$

33. $y = \tan\left(\dfrac{x}{2} - \dfrac{\pi}{4}\right)$

34. $y = \dfrac{1}{4}\cot\left(x - \dfrac{\pi}{2}\right)$

35. $y = -1 + \cot \pi x$

36. $y = \tan\left(x + \dfrac{5\pi}{6}\right)$

In Problems 37–44, find the period and the vertical asymptotes of the given function. Sketch at least one cycle of the graph.

37. $y = -\sec x$

38. $y = 2\sec \dfrac{\pi x}{2}$

39. $y = 3\csc \pi x$

40. $y = -2\csc \dfrac{x}{3}$

41. $y = \sec\left(3x - \dfrac{\pi}{2}\right)$

42. $y = \csc(4x + \pi)$

43. $y = 3 + \csc\left(2x + \dfrac{\pi}{2}\right)$

44. $y = -1 + \sec(x - 2\pi)$

In Problems 45 and 46, use the graphs of $y = \tan x$ and $y = \sec x$ to find numbers A and C for which the given equality is true.

45. $\cot x = A\tan(x + C)$

46. $\csc x = A\sec(x + C)$

Calculus-Related Problems

In Problems 47–56, use the Pythagorean identities (8)–(10) and the indicated substitution to rewrite the given algebraic expression as a trigonometric expression without radicals. Assume that $a > 0$.

47. $\sqrt{a^2 - x^2}; \quad x = a\cos\theta,\ 0 \le \theta \le \pi$

48. $\sqrt{a^2 + x^2}; \quad x = a\tan\theta,\ -\pi/2 < \theta < \pi/2$

49. $\sqrt{x^2 - a^2}; \quad x = a\sec\theta,\ 0 \le \theta < \pi/2$

50. $\sqrt{16 - 25x^2}; \quad x = \frac{4}{5}\sin\theta,\ -\pi/2 \le \theta \le \pi/2$

51. $\dfrac{x}{\sqrt{9 - x^2}}; \quad x = 3\sin\theta,\ -\pi/2 < \theta < \pi/2$

52. $x^2\sqrt{x^2 - 4}; \quad x = 2\sec\theta,\ 0 \le \theta < \pi/2$

53. $\dfrac{\sqrt{x^2-3}}{x^2}$; $x = \sqrt{3}\sec\theta,\ 0 \le \theta < \pi/2$

54. $(36 + x^2)^{3/2}$; $x = 6\tan\theta,\ -\pi/2 < \theta < \pi/2$

55. $\dfrac{1}{\sqrt{7+x^2}}$; $x = \sqrt{7}\tan\theta,\ -\pi/2 < \theta < \pi/2$

56. $\dfrac{\sqrt{5-x^2}}{x}$; $x = \sqrt{5}\cos\theta,\ 0 \le \theta \le \pi$

For Discussion

57. Use a calculator in radian mode to compare the values of $\tan(1.57)$ and $\tan(1.58)$. Explain the difference in these values.

58. Use a calculator in radian mode to compare the values of $\cot(3.14)$ and $\cot(3.15)$.

59. Explain why there are no real numbers x satisfying the equation $9\csc x = 1$.

60. For which real numbers x is **(a)** $\sin x \le \csc x$? **(b)** $\sin x < \csc x$?

61. For which real numbers x is **(a)** $\sec x \le \cos x$? **(b)** $\sec x < \cos x$?

62. Discuss and then sketch the graphs of $y = |\sec x|$ and $y = |\csc x|$.

4.5 Sum and Difference Formulas

INTRODUCTION There are *many* identities involving trigonometric functions. A **trigonometric identity** is an equation or formula involving only trigonometric functions that is valid for all angles measured in degrees or radians or for real numbers for which both sides of the equality are defined. The most basic of all these identities—the Pythagorean identities, the Quotient identities, and Reciprocal identities—were discussed in Section 4.4. In this section we develop some additional identities that are of particular importance in courses in mathematics and science.

Sum and Difference Formulas The **sum** and **difference formulas** for the cosine and sine functions are identities that reduce $\cos(x_1 + x_2)$, $\cos(x_1 - x_2)$, $\sin(x_1 + x_2)$, and $\sin(x_1 - x_2)$ to expressions that involve $\cos x_1$, $\cos x_2$, $\sin x_1$, and $\sin x_2$. We will derive the formula for $\cos(x_1 - x_2)$ first, and then we will use that result to obtain the others.

For convenience, let us suppose that x_1 and x_2 represent angles measured in radians. As shown in FIGURE 4.5.1(a), let d denote the distance between $P(x_1)$ and $P(x_2)$. If we place the angle $x_1 - x_2$ in standard position as shown in Figure 4.5.1(b), then d is

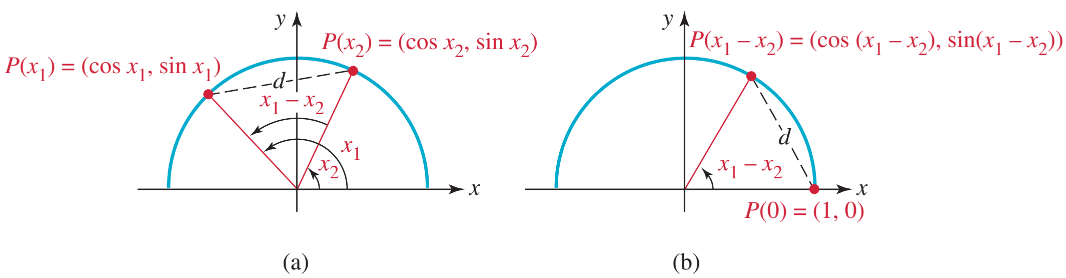


FIGURE 4.5.1 The difference of two angles

also the distance between $P(x_1 - x_2)$ and $P(0)$. Equating the squares of these distances gives

$$(\cos x_1 - \cos x_2)^2 + (\sin x_1 - \sin x_2)^2 = (\cos(x_1 - x_2) - 1)^2 + \sin^2(x_1 - x_2)$$

or
$$\cos^2 x_1 - 2\cos x_1 \cos x_2 + \cos^2 x_2 + \sin^2 x_1 - 2\sin x_1 \sin x_2 + \sin^2 x_2$$
$$= \cos^2(x_1 - x_2) - 2\cos(x_1 - x_2) + 1 + \sin^2(x_1 - x_2).$$

In view of the Pythagorean identity (8) of Section 4.4,

$$\cos^2 x_1 + \sin^2 x_1 = 1, \quad \cos^2 x_2 + \sin^2 x_2 = 1, \quad \cos^2(x_1 - x_2) + \sin^2(x_1 - x_2) = 1,$$

and so the preceding equation simplifies to

$$\cos(x_1 - x_2) = \cos x_1 \cos x_2 + \sin x_1 \sin x_2.$$

This last result can be put to work immediately to find the cosine of the sum of two angles. Since $x_1 + x_2$ can be rewritten as the difference $x_1 - (-x_2)$,

$$\begin{aligned}\cos(x_1 + x_2) &= \cos(x_1 - (-x_2)) \\ &= \cos x_1 \cos(-x_2) + \sin x_1 \sin(-x_2).\end{aligned}$$

By the even–odd identities, $\cos(-x_2) = \cos x_2$ and $\sin(-x_2) = -\sin x_2$, it follows that the last line is the same as

$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2.$$

The two results just obtained are summarized in the next theorem.

THEOREM 4.5.1 Sum and Difference Formulas for the Cosine

For all real numbers x_1 and x_2,

$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2 \tag{1}$$
$$\cos(x_1 - x_2) = \cos x_1 \cos x_2 + \sin x_1 \sin x_2 \tag{2}$$

EXAMPLE 1 Cosine of a Sum

Evaluate $\cos(7\pi/12)$.

Solution We have no way of evaluating $\cos(7\pi/12)$ directly. However, observe that

$$\frac{7\pi}{12} \text{ radians} = 105° = 60° + 45° = \frac{\pi}{3} + \frac{\pi}{4}.$$

Because $7\pi/12$ radians is a second-quadrant angle, we know that the value of $\cos(7\pi/12)$ is negative. Proceeding, the sum formula (1) of the Theorem 4.5.1 gives

$$\begin{aligned}\cos\frac{7\pi}{12} = \cos\left(\frac{\pi}{3} + \frac{\pi}{4}\right) &= \cos\frac{\pi}{3}\cos\frac{\pi}{4} - \sin\frac{\pi}{3}\sin\frac{\pi}{4} \\ &= \frac{1}{2}\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(1 - \sqrt{3}).\end{aligned}$$

Using $\sqrt{2}\sqrt{3} = \sqrt{6}$, this result can also be written as $\cos(7\pi/12) = (\sqrt{2} - \sqrt{6})/4$. Since $\sqrt{6} > \sqrt{2}$, we see that $\cos(7\pi/12) < 0$, as expected.

To obtain the corresponding sum/difference identities for the sine function we will make use of two identities:

See (5) in Section 4.3.

$$\cos\left(x - \frac{\pi}{2}\right) = \sin x \qquad \text{and} \qquad \sin\left(x - \frac{\pi}{2}\right) = -\cos x. \tag{3}$$

These identities were first discovered in Section 4.3 by shifting the graphs of the cosine and sine. However, both results in (3) can now be proved using (2):

$$\cos\left(x - \frac{\pi}{2}\right) = \cos x \cos\frac{\pi}{2} + \sin x \sin\frac{\pi}{2} = \cos x \cdot 0 + \sin x \cdot 1 = \sin x,$$

$$\cos x = \cos\Big(\underbrace{\frac{\pi}{2} - \frac{\pi}{2}}_{\text{zero}} + x\Big) = \cos\left(\frac{\pi}{2} - \left(\frac{\pi}{2} - x\right)\right) \overset{\text{by (2) of Theorem 4.5.1}}{=} \sin\left(\frac{\pi}{2} - x\right) = -\sin\left(x - \frac{\pi}{2}\right).$$

Now from the first equation in (3), the sine of the sum $x_1 + x_2$ can be written

$$\begin{aligned}
\sin(x_1 + x_2) &= \cos\left((x_1 + x_2) - \frac{\pi}{2}\right) \\
&= \cos\left(x_1 + \left(x_2 - \frac{\pi}{2}\right)\right) \\
&= \cos x_1 \cos\left(x_2 - \frac{\pi}{2}\right) - \sin x_1 \sin\left(x_2 - \frac{\pi}{2}\right) && \leftarrow \text{by (1) of Theorem 4.5.1} \\
&= \cos x_1 \sin x_2 - \sin x_1(-\cos x_2). && \leftarrow \text{by (3)}
\end{aligned}$$

The last line is traditionally written as

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2.$$

To obtain the sine of the difference $x_1 - x_2$, we use again $\cos(-x_2) = \cos x_2$ and $\sin(-x_2) = -\sin x_2$:

$$\begin{aligned}
\sin(x_1 - x_2) = \sin(x_1 + (-x_2)) &= \sin x_1 \cos(-x_2) + \cos x_1 \sin(-x_2) \\
&= \sin x_1 \cos x_2 - \cos x_1 \sin x_2.
\end{aligned}$$

THEOREM 4.5.2 Sum and Difference Formulas for the Sine

For all real numbers x_1 and x_2,

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2 \tag{4}$$

$$\sin(x_1 - x_2) = \sin x_1 \cos x_2 - \cos x_1 \sin x_2 \tag{5}$$

EXAMPLE 2 Sine of a Sum

Evaluate $\sin(7\pi/12)$.

Solution We proceed as in Example 1, except we use the sum formula (4) of Theorem 4.5.2:

$$\begin{aligned}
\sin\frac{7\pi}{12} = \sin\left(\frac{\pi}{3} + \frac{\pi}{4}\right) &= \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4} \\
&= \frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} + \frac{1}{2}\frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{4}(1 + \sqrt{3}).
\end{aligned}$$

As in Example 1, the result can be rewritten as $\sin(7\pi/12) = (\sqrt{2} + \sqrt{6})/4$.

Since we know the value of $\cos(7\pi/12)$ from Example 1, we can also compute the value of $\sin(7\pi/12)$ using the Pythagorean identity (8) of Section 4.4:

$$\sin^2\frac{7\pi}{12} + \cos^2\frac{7\pi}{12} = 1.$$

We solve for $\sin(7\pi/12)$ and take the positive square root:

$$\begin{aligned} \sin\frac{7\pi}{12} &= \sqrt{1 - \cos^2\frac{7\pi}{12}} = \sqrt{1 - \left[\frac{\sqrt{2}}{4}(1 - \sqrt{3})\right]^2} \\ &= \sqrt{\frac{4 + 2\sqrt{3}}{8}} = \frac{\sqrt{2 + \sqrt{3}}}{2}. \end{aligned} \tag{6}$$

Although the number in (6) does not look like the result obtained in Example 2, the values are the same. See Problem 77 in Exercises 4.5.

There are sum and difference formulas for the tangent function as well. We can derive the sum formula using the sum formulas for the sine and cosine as follows:

$$\tan(x_1 + x_2) = \frac{\sin(x_1 + x_2)}{\cos(x_1 + x_2)} = \frac{\sin x_1 \cos x_2 + \cos x_1 \sin x_2}{\cos x_1 \cos x_2 - \sin x_1 \sin x_2}. \tag{7}$$

We now divide the numerator and denominator of (7) by $\cos x_1 \cos x_2$ (assuming that x_1 and x_2 are such that $\cos x_1 \cos x_2 \neq 0$),

$$\tan(x_1 + x_2) = \frac{\dfrac{\sin x_1}{\cos x_1}\dfrac{\cos x_2}{\cos x_2} + \dfrac{\cos x_1}{\cos x_1}\dfrac{\sin x_2}{\cos x_2}}{\dfrac{\cos x_1}{\cos x_1}\dfrac{\cos x_2}{\cos x_2} - \dfrac{\sin x_1}{\cos x_1}\dfrac{\sin x_2}{\cos x_2}} = \frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2}. \tag{8}$$

The derivation of the difference formula for $\tan(x_1 - x_2)$ is obtained in a similar manner. We summarize the two results.

THEOREM 4.5.3 Sum and Difference Formulas for the Tangent

For real numbers x_1 and x_2 for which the functions are defined,

$$\tan(x_1 + x_2) = \frac{\tan x_1 + \tan x_2}{1 - \tan x_1 \tan x_2} \tag{9}$$

$$\tan(x_1 - x_2) = \frac{\tan x_1 - \tan x_2}{1 + \tan x_1 \tan x_2} \tag{10}$$

EXAMPLE 3 Tangent of a Difference

Evaluate $\tan(\pi/12)$.

Solution If we think of $\pi/12$ as an angle in radians, then

$$\frac{\pi}{12} \text{ radians} = 15° = 45° - 30° = \frac{\pi}{4} - \frac{\pi}{6} \text{ radians.}$$

It follows from formula (10) of Theorem 4.5.3:

You should rework this example using $\pi/12 = \pi/3 - \pi/4$ to see that the result is the same.

$$\tan\frac{\pi}{12} = \tan\left(\frac{\pi}{4} - \frac{\pi}{6}\right) = \frac{\tan\frac{\pi}{4} - \tan\frac{\pi}{6}}{1 + \tan\frac{\pi}{4}\tan\frac{\pi}{6}}$$

$$= \frac{1 - \frac{1}{\sqrt{3}}}{1 + 1 \cdot \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$$

$$= \frac{\sqrt{3} - 1}{\sqrt{3} + 1} \cdot \frac{\sqrt{3} - 1}{\sqrt{3} - 1} \quad \leftarrow \text{rationalizing the denominator}$$

$$= \frac{(\sqrt{3} - 1)^2}{2} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}.$$

Strictly speaking, we really do not need the identities for $\tan(x_1 \pm x_2)$, since we can always compute $\sin(x_1 \pm x_2)$ and $\cos(x_1 \pm x_2)$ using (1), (2), (4), (5) and then proceed as in (7), that is, form the quotient $\sin(x_1 \pm x_2)/\cos(x_1 \pm x_2)$.

Double-Angle Formulas Many useful trigonometric formulas can be derived from the sum and difference formulas. The **double-angle formulas** for the cosine and sine functions express the cosine and sine of $2x$ in terms of the cosine and sine of x.

If we set $x_1 = x_2 = x$ in (1) and use $\cos(x + x) = \cos 2x$, then

$$\cos 2x = \cos x \cos x - \sin x \sin x = \cos^2 x - \sin^2 x.$$

Similarly, by setting $x_1 = x_2 = x$ in (4) and using $\sin(x + x) = \sin 2x$, then

$$\sin 2x = \underbrace{\sin x \cos x + \cos x \sin x}_{\text{these two terms are equal}} = 2\sin x \cos x.$$

We summarize the last two results along with the double-angle formula for the tangent function.

THEOREM 4.5.4 Double-Angle Formulas

For a real number x for which the functions are defined,

$$\cos 2x = \cos^2 x - \sin^2 x \tag{11}$$

$$\sin 2x = 2\sin x \cos x \tag{12}$$

$$\tan 2x = \frac{2\tan x}{1 - \tan^2 x} \tag{13}$$

EXAMPLE 4 Using the Double-Angle Formulas

If $\sin x = -\frac{1}{4}$ and $\pi < x < 3\pi/2$, find the exact values of $\cos 2x$ and $\sin 2x$.

Solution First, we compute $\cos x$ using $\sin^2 x + \cos^2 x = 1$. Since $\pi < x < 3\pi/2$, $\cos x < 0$, and so we choose the negative square root:

$$\cos x = -\sqrt{1 - \sin^2 x} = -\sqrt{1 - \left(-\frac{1}{4}\right)^2} = -\frac{\sqrt{15}}{4}.$$

From the double-angle formula (11) of Theorem 4.5.4,

$$\begin{aligned}\cos 2x &= \cos^2 x - \sin^2 x\\ &= \left(-\frac{\sqrt{15}}{4}\right)^2 - \left(-\frac{1}{4}\right)^2\\ &= \frac{15}{16} - \frac{1}{16} = \frac{14}{16} = \frac{7}{8}.\end{aligned}$$

Finally, from the double-angle formula (12),

$$\sin 2x = 2\sin x \cos x = 2\left(-\frac{1}{4}\right)\left(-\frac{\sqrt{15}}{4}\right) = \frac{\sqrt{15}}{8}.$$

The formula in (11) has two useful alternative forms. By (8) of Section 4.4, we know that $\sin^2 x = 1 - \cos^2 x$. Substituting the last expression into (11) yields $\cos 2x = \cos^2 x - (1 - \cos^2 x)$ or

$$\cos 2x = 2\cos^2 x - 1. \tag{14}$$

On the other hand, if we substitute $\cos^2 x = 1 - \sin^2 x$ into (11) we get

$$\cos 2x = 1 - 2\sin^2 x. \tag{15}$$

Half-Angle Formulas The alternative forms (14) and (15) of the double-angle formula (11) are the source of two **half-angle formulas**. Solving (14) and (15) for $\cos^2 x$ and $\sin^2 x$ gives, respectively,

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \qquad \text{and} \qquad \sin^2 x = \frac{1}{2}(1 - \cos 2x). \tag{16}$$

By replacing the symbol x in (16) by $x/2$ and using $2(x/2) = x$, we obtain the half-angle formulas for the cosine and sine functions.

THEOREM 4.5.5 Half-Angle Formulas

For a real number x for which the functions are defined,

$$\cos^2\frac{x}{2} = \frac{1}{2}(1 + \cos x) \tag{17}$$

$$\sin^2\frac{x}{2} = \frac{1}{2}(1 - \cos x) \tag{18}$$

$$\tan^2\frac{x}{2} = \frac{1 - \cos x}{1 + \cos x} \tag{19}$$

EXAMPLE 5 Using the Half-Angle Formulas

Find the exact values of $\cos(5\pi/8)$ and $\sin(5\pi/8)$.

Solution If we let $x = 5\pi/4$, then $x/2 = 5\pi/8$ and formulas (17) and (18) of Theorem 4.5.5 yield, respectively,

$$\cos^2\frac{5\pi}{8} = \frac{1}{2}\left(1 + \cos\frac{5\pi}{4}\right) = \frac{1}{2}\left[1 + \left(-\frac{\sqrt{2}}{2}\right)\right] = \frac{2 - \sqrt{2}}{4},$$

and

$$\sin^2\frac{5\pi}{8} = \frac{1}{2}\left(1 - \cos\frac{5\pi}{4}\right) = \frac{1}{2}\left[1 - \left(-\frac{\sqrt{2}}{2}\right)\right] = \frac{2 + \sqrt{2}}{4}.$$

Because $5\pi/8$ radians is a second-quadrant angle, $\cos(5\pi/8) < 0$ and $\sin(5\pi/8) > 0$. Therefore, we take the negative square root for the value of the cosine,

$$\cos\frac{5\pi}{8} = -\sqrt{\frac{2-\sqrt{2}}{4}} = -\frac{\sqrt{2-\sqrt{2}}}{2},$$

and the positive square root for the value of the sine,

$$\sin\frac{5\pi}{8} = \sqrt{\frac{2+\sqrt{2}}{4}} = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

If want the exact value of, say, $\tan(5\pi/8)$ we can use the results of Example 5 or formula (19) with $x = 5\pi/4$. Either way, the result is the same

$$\tan\frac{5\pi}{8} = -\frac{\sqrt{2+\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = -1-\sqrt{2}. \qquad \leftarrow \left\{\begin{array}{l}\text{rationalizing}\\ \text{the denominator}\end{array}\right.$$

Reducing Powers of Sine and Cosine As discussed in Section 3.7, the principal topics of study in calculus are *derivatives* and *integrals* of functions. Trigonometric identities are especially useful in integral calculus. To evaluate integrals of powers of the sine and cosine, specifically $\cos^n x$ and $\sin^n x$, where $n \geq 2$ is an even positive integer, you would use the half-angle formulas in the form given in (16). The idea is to express $\cos^n x$ and $\sin^n x$ in terms of one or more of cosine functions raised to the first power. The next example illustrates the idea.

EXAMPLE 6 Reducing a Power

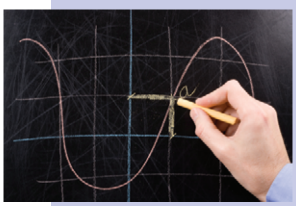
Express $\sin^4 x$ in terms of first-powers of cosine functions.

Solution We first use the laws of exponents to rewrite $\sin^4 x$ as $(\sin^2 x)^2$ and then use the second identity in (16) to replace $\sin^2 x$ in terms of $\cos 2x$:

$$\begin{aligned}
\sin^4 x &= (\sin^2 x)^2 && \leftarrow \left\{\begin{array}{l}\text{replace } \sin^2 x \text{ using}\\ \text{second formula in (16)}\end{array}\right.\\
&= \left[\frac{1}{2}(1-\cos 2x)\right]^2 && \leftarrow \text{expand}\\
&= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\cos^2 2x && \leftarrow \left\{\begin{array}{l}\text{now replace } \cos^2 2x \text{ using}\\ \text{the first formula in (16)}\\ \text{with } x \text{ replaced by } 2x\end{array}\right.\\
&= \frac{1}{4} - \frac{1}{2}\cos 2x + \frac{1}{4}\left[\frac{1}{2}(1+\cos 4x)\right]\\
&= \frac{3}{8} - \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x.
\end{aligned}$$

As required, the last line contains only first-powers of cosine functions of multiples of the original independent variable x.

NOTES FROM THE CLASSROOM



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(*i*) Should you memorize all the identities presented in this section? You should consult your instructor about this, but in the opinion of the authors, you should at the very least memorize formulas (1), (2), (4), (5), (11), and (12).

(*ii*) When you enroll in a calculus course, check the title of your text. If it has the words *Early Transcendentals* in its title, then your knowledge of the graphs and properties of the trigonometric functions will come into play almost immediately.

(*iii*) At some point in your study of integral calculus you may be required to evaluate integrals of products such as

$$\sin 2x \sin 5x \qquad \text{and} \qquad \sin 10x \cos 4x.$$

One way of doing this is to use the sum/difference formulas to devise an identity that converts these products into either a sum of sines or a sum of cosines. This will be the topic of discussion in the next section.

Exercises 4.5

Answers to selected odd-numbered problems begin on page ANS-16.

In Problems 1–22, use a sum or difference formula to find the exact value of the given trigonometric function. Do not use a calculator.

1. $\cos\dfrac{\pi}{12}$

2. $\sin\dfrac{\pi}{12}$

3. $\sin 75°$

4. $\cos 75°$

5. $\sin\dfrac{7\pi}{12}$

6. $\cos\dfrac{11\pi}{12}$

7. $\tan\dfrac{5\pi}{12}$

8. $\cos\left(-\dfrac{5\pi}{12}\right)$

9. $\sin\left(-\dfrac{\pi}{12}\right)$

10. $\tan\dfrac{11\pi}{12}$

11. $\sin\dfrac{11\pi}{12}$

12. $\tan\dfrac{7\pi}{12}$

13. $\cos 165°$

14. $\sin 165°$

15. $\tan 165°$

16. $\cos 195°$

17. $\sin 195°$

18. $\tan 195°$

19. $\cos 345°$

20. $\sin 345°$

21. $\cos\dfrac{13\pi}{12}$

22. $\tan\dfrac{17\pi}{12}$

In Problems 23–28, use a double-angle formula to write the given expression as a single trigonometric function of twice the angle.

23. $2\cos\beta\sin\beta$

24. $\cos^2 2t - \sin^2 2t$

25. $1 - 2\sin^2\dfrac{\pi}{5}$

26. $2\cos^2\dfrac{19}{2}x - 1$

27. $\dfrac{\tan 3t}{1 - \tan^2 3t}$

28. $2\sin\dfrac{y}{2}\cos\dfrac{y}{2}$

In Problems 29–34, use the given information to find **(a)** $\cos 2x$, **(b)** $\sin 2x$, and **(c)** $\tan 2x$.

29. $\sin x = \sqrt{2}/3, \quad \pi/2 < x < \pi$

30. $\cos x = \sqrt{3}/5, \quad 3\pi/2 < x < 2\pi$

31. $\tan x = \frac{1}{2}, \quad \pi < x < 3\pi/2$

32. $\csc x = -3, \quad \pi < x < 3\pi/2$

33. $\sec x = -\frac{13}{5}, \quad \pi/2 < x < \pi$

34. $\cot x = \frac{4}{3}, \quad 0 < x < \pi/2$

In Problems 35–44, use a half-angle formula to find the exact value of the given trigonometric function. Do not use a calculator.

35. $\cos(\pi/12)$

36. $\sin(\pi/8)$

37. $\sin(3\pi/8)$

38. $\tan(\pi/12)$

39. $\cos 67.5°$

40. $\sin 15°$

41. $\tan 105°$
42. $\cot 157.5°$
43. $\csc(13\pi/12)$
44. $\sec(-3\pi/8)$

In Problems 45–50, use the given information to find **(a)** $\cos(x/2)$, **(b)** $\sin(x/2)$, and **(c)** $\tan(x/2)$.

45. $\sin x = \frac{12}{13}, \quad \pi/2 < x < \pi$
46. $\cos x = \frac{4}{5}, \quad 3\pi/2 < x < 2\pi$
47. $\tan x = 2, \quad \pi < x < 3\pi/2$
48. $\csc x = 9, \quad 0 < x < \pi/2$
49. $\sec x = \frac{3}{2}, \quad 0 < x < 90°$
50. $\cot x = -\frac{1}{4}, \quad 90° < x < 180°$

In Problems 51–60, verify the given identity.

51. $\sin 4x = 4\cos x(\sin x - 2\sin^3 x)$
52. $\cos 3x = 4\cos^3 x - 3\cos x$
53. $(\sin x + \cos x)^2 = 1 + \sin 2x$
54. $\cos 2x = \cos^4 x - \sin^4 x$
55. $\cot 2x = \frac{1}{2}(\cot x - \tan x)$
56. $\sec 2x = \dfrac{1}{2\cos^2 x - 1}$
57. $\dfrac{2\tan x}{1 + \tan^2 x} = \sin 2x$
58. $\dfrac{\cot x - \tan x}{\cot x + \tan x} = \cos 2x$
59. $\tan\dfrac{x}{2} = \dfrac{1 - \cos x}{\sin x}$
60. $\tan\dfrac{x}{2} = \dfrac{\sin x}{1 + \cos x}$

In Problems 61–64, proceed as in Example 6 and use (16) to rewrite the given function in terms of first powers of cosine functions.

61. $\sin^2 5x$
62. $\cos^2 2x$
63. $\cos^4 x$
64. $\sin^4 3x$

In Problems 65 and 66, proceed as in Problems 61–64 to rewrite the given function in terms of first powers of cosine functions. You will also need (7) of Section 1.5 and the identity in Problem 52 written in the form

$$\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x.$$

65. $\sin^6 x$
66. $\cos^6 \frac{1}{2}x$

In Problems 67–70, rewrite the given function as a single trigonometric function involving no products or squares. Give the amplitude and period of the function.

67. $y = 4\cos^2 x - 2$
68. $y = \sin(x/2)\cos(x/2)$
69. $y = 2\sin 2x \cos 2x$
70. $y = 5\cos^2 4x - 5\sin^2 4x$

71. If $P(x_1)$ and $P(x_2)$ are points in quadrant II on the terminal side of the angles x_1 and x_2, respectively, with $\cos x_1 = -\frac{1}{3}$ and $\sin x_2 = \frac{2}{3}$, find **(a)** $\sin(x_1 + x_2)$, **(b)** $\cos(x_1 + x_2)$, **(c)** $\sin(x_1 - x_2)$, and **(d)** $\cos(x_1 - x_2)$.

72. If x_1 is a quadrant II angle, x_2 is a quadrant III angle, $\sin x_1 = \frac{8}{17}$, and $\tan x_2 = \frac{3}{4}$, find **(a)** $\sin(x_1 + x_2)$, **(b)** $\sin(x_1 - x_2)$, **(c)** $\cos(x_1 + x_2)$, and **(d)** $\cos(x_1 - x_2)$.

Applications

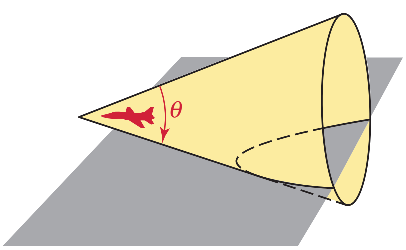


FIGURE 4.5.2 Airplane in Problem 73

73. Mach Number The ratio of the speed of an airplane to the speed of sound is called the Mach number M of the plane. If $M > 1$, the plane makes sound waves that form a (moving) cone, as shown in FIGURE 4.5.2. A sonic boom is heard at the intersection of the cone with the ground. If the vertex angle of the cone is θ, then

$$\sin\frac{\theta}{2} = \frac{1}{M}.$$

If $\theta = \pi/6$, find the exact value of the Mach number.

74. Cardiovascular Branching A mathematical model for blood flow in a large blood vessel predicts that the optimal values of the angles θ_1 and θ_2, which represent the (positive) angles of the smaller daughter branches (vessels) with respect to the axis of the parent branch, are given by

$$\cos\theta_1 = \frac{A_0^2 + A_1^2 - A_2^2}{2A_0A_1} \quad \text{and} \quad \cos\theta_2 = \frac{A_0^2 - A_1^2 + A_2^2}{2A_0A_2},$$

where A_0 is the cross-sectional area of the parent branch and A_1 and A_2 are the cross-sectional areas of the daughter branches. See FIGURE 4.5.3. Let $\psi = \theta_1 + \theta_2$ be the junction angle, as shown in the figure.

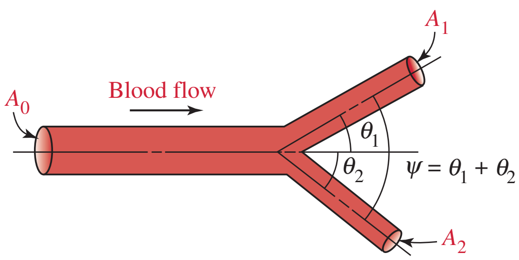


FIGURE 4.5.3 Branching of a large blood vessel in Problem 74

(a) Show that

$$\cos\psi = \frac{A_0^2 - A_1^2 - A_2^2}{2A_1A_2}.$$

(b) Show that for the optimal values of θ_1 and θ_2, the cross-sectional area of the daughter branches, $A_1 + A_2$, is greater than or equal to that of the parent branch. Therefore, the blood must slow down in the daughter branches.

75. Range of a Projectile We saw in Problem 56 of Exercises 4.2 that if a projectile, such as a shot put, is released from a height h, upward at an angle θ with velocity v_0, the range R at which it strikes the ground is given by

$$R = \frac{v_0\cos\theta}{g}\left(v_0\sin\theta + \sqrt{v_0^2\sin^2\theta + 2gh}\right),$$

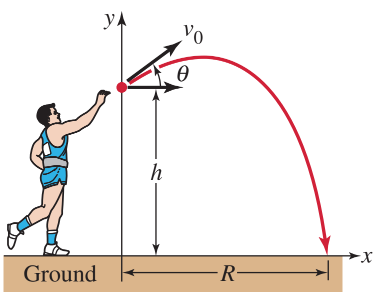


FIGURE 4.5.4 Projectile in Problem 75

where g is the acceleration due to gravity. See FIGURE 4.5.4.

(a) Show that when $h = 0$ the range of the projectile is

$$R = \frac{v_0^2\sin 2\theta}{g}.$$

(b) It can be shown that the maximum range R_{max} is achieved when the angle θ satisfies the equation

$$\cos 2\theta = \frac{gh}{v_0^2 + gh}.$$

Show that maximum range is

$$R_{max} = \frac{v_0\sqrt{v_0^2 + 2gh}}{g}$$

by using the expressions for R and $\cos 2\theta$ and the half-angle formulas for the sine and the cosine with $t = 2\theta$.

For Discussion

76. Discuss: Why would you expect to get an error message from your calculator when you try to evaluate

$$\frac{\tan 35^\circ + \tan 55^\circ}{1 - \tan 35^\circ \tan 55^\circ}?$$

77. In Example 2 we showed that $\sin(7\pi/12) = \frac{1}{4}(\sqrt{2} + \sqrt{6})$. Following the example, we then showed that $\sin(7\pi/12) = \frac{1}{2}\sqrt{2 + \sqrt{3}}$. Demonstrate that these two answers are equivalent.

78. Discuss: How would you find a formula that expresses $\sin 3\theta$ in terms of $\sin\theta$? Carry out your ideas.

79. In Problem 71, in what quadrants do $P(x_1 + x_2)$ and $P(x_1 - x_2)$ lie?

80. In Problem 72, in which quadrant does the terminal side of $x_1 + x_2$ lie? The terminal side of $x_1 - x_2$?

4.6 Product-to-Sum and Sum-to-Product Formulas

INTRODUCTION There are instances, especially in integral calculus, where it is necessary to convert a product of sine and cosine functions to a sum of these functions. Moreover, in solving trigonometric equations we may find it convenient to convert a sum of sine and cosine functions into a product of these functions. In the discussion that follows we establish trigonometric identities or formulas that do the job.

Reduction of a Product to a Sum The **product-to-sum formulas** given in the next theorem are direct consequences of the sum and difference formulas for the cosine and sine functions in Section 4.5.

THEOREM 4.6.1 Product-to-Sum Formulas

For all real numbers x_1 and x_2,

$$\sin x_1 \sin x_2 = \tfrac{1}{2}[\cos(x_1 - x_2) - \cos(x_1 + x_2)] \tag{1}$$
$$\cos x_1 \cos x_2 = \tfrac{1}{2}[\cos(x_1 - x_2) + \cos(x_1 + x_2)] \tag{2}$$
$$\sin x_1 \cos x_2 = \tfrac{1}{2}[\sin(x_1 + x_2) + \sin(x_1 - x_2)] \tag{3}$$

PROOF: To prove (1), we use (1) and (2) of Theorem 4.5.1:

$$\cos(x_1 - x_2) = \cos x_1 \cos x_2 + \sin x_1 \sin x_2 \tag{4}$$
$$\cos(x_1 + x_2) = \cos x_1 \cos x_2 - \sin x_1 \sin x_2. \tag{5}$$

Subtracting (5) from (4) yields

$$\cos(x_1 - x_2) - \cos(x_1 + x_2) = 2\sin x_1 \sin x_2.$$

And so,

$$\sin x_1 \sin x_2 = \tfrac{1}{2}[\cos(x_1 - x_2) - \cos(x_1 + x_2)]$$

which is (1). Similarly, by adding (4) and (5) we get

$$\cos(x_1 - x_2) + \cos(x_1 + x_2) = 2\cos x_1 \cos x_2,$$

which in turn, yields formula (2):

$$\cos x_1 \cos x_2 = \tfrac{1}{2}[\cos(x_1 - x_2) + \cos(x_1 + x_2)].$$

Formula (3) follows analogously by adding the sum and difference formulas for the sine, (4) and (5) in Theorem 4.5.2 of Section 4.5. ∎

Although we do not feel that it is necessary to memorize (1)–(3) in Theorem 4.6.1, you should listen to what your instructor requires. By remembering the *procedure* just illustrated in the proof of Theorem 4.6.1 each of these formulas can be derived on the spot.

EXAMPLE 1 Using (2) of Theorem 4.6.1

Use a product-to-sum formula to rewrite the product $\cos 2\theta \cos 3\theta$ as a sum.

Solution From formula (2) of Theorem 4.6.1 with the identifications $x_1 = 2\theta$ and $x_2 = 3\theta$, we obtain

$$\begin{aligned}\cos 2\theta \cos 3\theta &= \tfrac{1}{2}[\cos(2\theta - 3\theta) + \cos(2\theta + 3\theta)]\\ &= \tfrac{1}{2}[\cos(-\theta) + \cos 5\theta] \qquad \leftarrow \cos(-\theta) = \cos\theta\\ &= \tfrac{1}{2}[\cos\theta + \cos 5\theta].\end{aligned}$$

EXAMPLE 2 Using (3) of Theorem 4.6.1

Use a product-to-sum formula to find the exact value of the product $\sin 45° \cos 15°$.

Solution Using formula (3) of Theorem 4.6.1 with $x_1 = 45°$ and $x_2 = 15°$, we have

$$\begin{aligned}\sin 45° \cos 15° &= \tfrac{1}{2}[\sin(45° + 15°) + \sin(45° - 15°)]\\ &= \tfrac{1}{2}[\sin 60° + \sin 30°].\end{aligned}$$

Because $\sin 60° = \frac{1}{2}\sqrt{3}$ and $\sin 30° = \frac{1}{2}$ we observe that the exact value of the given product is

$$\sin 45° \cos 15° = \tfrac{1}{2}[\sin 60° + \sin 30°] = \tfrac{1}{2}\left(\tfrac{1}{2}\sqrt{3} + \tfrac{1}{2}\right) = \tfrac{1}{4}\left(\sqrt{3} + 1\right).$$

Reduction of a Sum to a Product The results in Theorem 4.6.1 can now be used to derive the **sum-to-product formulas**.

THEOREM 4.6.2 Sum-to-Product Formulas

For all real numbers x_1 and x_2,

$$\sin x_1 + \sin x_2 = 2\sin\frac{x_1 + x_2}{2}\cos\frac{x_1 - x_2}{2} \tag{6}$$

$$\sin x_1 - \sin x_2 = 2\cos\frac{x_1 + x_2}{2}\sin\frac{x_1 - x_2}{2} \tag{7}$$

$$\cos x_1 + \cos x_2 = 2\cos\frac{x_1 + x_2}{2}\cos\frac{x_1 - x_2}{2} \tag{8}$$

$$\cos x_1 - \cos x_2 = -2\sin\frac{x_1 + x_2}{2}\sin\frac{x_1 - x_2}{2} \tag{9}$$

PROOF: By replacing, in turn, the symbols x_1 and x_2 in (1) of Theorem 4.6.1 by

$$\frac{x_1 + x_2}{2} \quad \text{and} \quad \frac{x_1 - x_2}{2}, \tag{10}$$

we get

$$\sin\frac{x_1+x_2}{2}\sin\frac{x_1-x_2}{2} = \frac{1}{2}\left[\cos\left(\frac{x_1+x_2}{2}-\frac{x_1-x_2}{2}\right)-\cos\left(\frac{x_1+x_2}{2}+\frac{x_1-x_2}{2}\right)\right]$$
$$= \frac{1}{2}[\cos x_2 - \cos x_1].$$

Multiplying the last expression by -2 yields

$$-2\sin\frac{x_1+x_2}{2}\sin\frac{x_1-x_2}{2} = \cos x_1 - \cos x_2,$$

which is formula (9). In a similar manner, each of the remaining product-to-sum formulas together with the substitutions in (10) yields one of the sum-to-product formulas.

EXAMPLE 3 Using (9) of Theorem 4.6.2

Use a sum-to-product formula to rewrite the sum $\cos t - \cos 5t$ as a product.

Solution We use formula (9) of Theorem 4.6.2 with $x_1 = t$ and $x_2 = 5t$:

$$\cos t - \cos 5t = -2\sin\left(\frac{t+5t}{2}\right)\sin\left(\frac{t-5t}{2}\right)$$
$$= -2\sin 3t\sin(-2t) \qquad \leftarrow \sin(-2t) = -\sin 2t$$
$$= 2\sin 3t \sin 2t$$

EXAMPLE 4 Using (6) of Theorem 4.6.2

Use a sum-to-product formula to find the exact value of the sum $\sin 75° + \sin 15°$.

Solution In this case we use formula (6) of Theorem 4.6.2 with $x_1 = 75°$ and $x_2 = 15°$:

$$\sin 75° + \sin 15° = 2\sin\left(\frac{75° + 15°}{2}\right)\cos\left(\frac{75° - 15°}{2}\right)$$
$$= 2\sin 45° \cos 30°.$$

Because $\sin 45° = \frac{1}{2}\sqrt{2}$ and $\cos 30° = \frac{1}{2}\sqrt{3}$ the exact value of the given sum

$$\sin 75° + \sin 15° = 2\sin 45°\cos 30° = 2\left(\tfrac{1}{2}\sqrt{2}\right)\left(\tfrac{1}{2}\sqrt{3}\right) = \tfrac{1}{2}\sqrt{6}.$$

Exercises 4.6

Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–12, use a product-to-sum formula in Theorem 4.6.1 to write the given product as a sum of cosines or a sum of sines.

1. $\cos 4\theta \cos 3\theta$

2. $\sin\frac{3t}{2}\cos\frac{t}{2}$

3. $\sin 2x \sin 5x$

4. $\sin 10x \cos 4x$

5. $\cos\frac{4x}{3}\cos\frac{x}{3}$

6. $-\sin t \sin 2t$

7. $\sin 8x \cos 12x$
8. $\sin \pi\theta \cos 7\pi\theta$
9. $2\cos 3\beta \sin \beta$
10. $6 \sin \alpha \sin 4\alpha$
11. $2\sin\left(x + \frac{\pi}{4}\right)\sin\left(x - \frac{\pi}{4}\right)$
12. $2\sin\left(t + \frac{\pi}{2}\right)\cos\left(t - \frac{\pi}{2}\right)$

In Problems 13–18, use a product-to-sum formula in Theorem 4.6.1 to find the exact value of the given expression. Do not use a calculator.

13. $\cos\frac{5\pi}{12}\sin\frac{\pi}{12}$
14. $\sin\frac{5\pi}{8}\cos\frac{\pi}{8}$
15. $\sin 75° \sin 15°$
16. $\cos 15° \cos 45°$
17. $\sin 97.5° \sin 52.5°$
18. $\sin 105° \cos 195°$

In Problems 19–30, use a sum-to-product-formula in Theorem 4.6.2 to write the given sum as a product of cosines, a product of sines, or a product of a sine and a cosine.

19. $\sin y - \sin 5y$
20. $\cos 3\theta - \cos \theta$
21. $\cos\frac{9x}{2} - \cos\frac{x}{2}$
22. $\sin\frac{x}{2} - \sin\frac{3x}{2}$
23. $\cos 2x + \cos 6x$
24. $\sin 5t + \sin 3t$
25. $\sin \omega_1 t + \sin \omega_2 t$
26. $\frac{1}{2}(\cos 2\alpha + \cos 2\beta)$
27. $-\frac{1}{2}\cos t + \frac{1}{2}\cos 5t$
28. $\sin(\theta + \pi) + \sin(\theta - \pi)$
29. $\sin\left(t + \frac{\pi}{2}\right) + \sin\left(t - \frac{\pi}{2}\right)$
30. $\cos\left(t + \frac{\pi}{2}\right) - \cos\left(t - \frac{\pi}{2}\right)$

In Problems 31–36, use a sum-to-product-formula in Theorem 4.6.2 to find the exact value of the given expression. Do not use a calculator.

31. $\sqrt{2}\sin\frac{13\pi}{12} + \sqrt{2}\sin\frac{5\pi}{12}$
32. $\sin\frac{\pi}{12} - \sin\frac{5\pi}{12}$
33. $\cos 105° - \cos 15°$
34. $\cos 15° + \cos 75°$
35. $\sin 195° + \sin 105°$
36. $2\cos 195° - 2\cos 105°$

Applications

37. **Sound Wave** A note produced by a certain musical instrument results in a sound wave described by

$$f(t) = 0.03 \sin 500\pi t + 0.03 \sin 1000\pi t,$$

where $f(t)$ is the difference between atmospheric pressure and air pressure in dynes per square centimeter at the eardrum after t seconds. Express f as the product of a sine and a cosine function.

38. **Beats** If two piano wires struck by the same key are slightly out of tune, the difference between the atmospheric pressure and air pressure at the eardrum can be represented by the function

$$f(t) = a\cos 2\pi b_1 t + a\cos 2\pi b_2 t,$$

where the value of the constant b_1 is close to the value of constant b_2. The variations in loudness that occur are called **beats**. See FIGURE 4.6.1. The two strings can be tuned to the same frequency by tightening one of them while sounding both until the beats disappear.

(a) Use a sum formula to write $f(t)$ as a product.

(b) Show that $f(t)$can be considered a cosine function with period $2/(b_1 + b_2)$ and variable amplitude $2a\cos \pi(b_1 - b_2)t$.

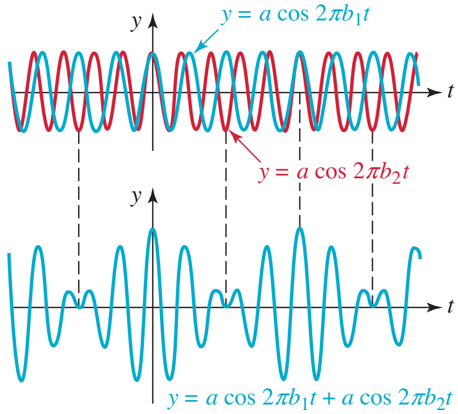


FIGURE 4.6.1 Graph for Problem 38

(c) Use a graphing utility to obtain the graph of f in the case $2\pi b_1 = 5$, $2\pi b_2 = 4$, and $a = \frac{1}{2}$.

39. Alternating Current The term $\sin \omega t \sin(\omega t + \phi)$ is encountered in the derivation of an expression for the power in an alternating-current circuit. Show that this term can be written as $\frac{1}{2}[\cos \phi - \cos(2\omega t + \phi)]$.

For Discussion

40. If $x_1 + x_2 + x_3 = \pi$, then show that

$$\sin 2x_1 + \sin 2x_2 + \sin 2x_3 = 4 \sin x_1 \sin x_2 \sin x_3.$$

41. Write as a product of cosines: $1 + \cos 2t + \cos 4t + \cos 6t$.

42. Simplify: $2 \cos 2t \cos t - \cos 3t$.

4.7 Inverse Trigonometric Functions

INTRODUCTION Although we can find the values of the trigonometric functions of real numbers or angles, in many applications we must do the reverse: Given the value of a trigonometric function, find a corresponding angle or number. This suggests we consider inverse trigonometric functions. Before we define the inverse trigonometric functions, let's recall from Section 2.8 some of the properties of a one-to-one function f and its inverse f^{-1}.

Recall, a function f is one-to-one if every y in its range corresponds to exactly one x in its domain.

Properties of Inverse Functions If $y = f(x)$ is a one-to-one function, then there is a unique inverse function f^{-1} with the following properties:

Properties of Inverse Functions

- The domain of f^{-1} = range of f.
- The range of f^{-1} = domain of f.
- $y = f(x)$ is equivalent to $x = f^{-1}(y)$.
- The graphs of f and f^{-1} are reflections in the line $y = x$.
- $f(f^{-1}(x)) = x$ for x in the domain of f^{-1}.
- $f^{-1}(f(x)) = x$ for x in the domain of f.

Inspection of the graphs of the various trigonometric functions clearly shows that *none* of these functions are one-to-one. In Section 2.8 we discussed the fact that if a function f is not one-to-one, it may be possible to restrict the function to a portion of its domain where it is

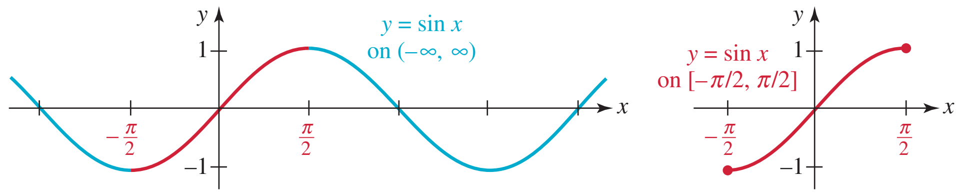
See Example 8 in Section 2.8.

one-to-one. Then we can define an inverse for f on that restricted domain. Normally, when we restrict the domain, we make sure to preserve the entire range of the original function.

Arcsine Function From FIGURE 4.7.1(a) we see that the function $y = \sin x$ on the closed interval $[-\pi/2, \pi/2]$ takes on all values in its range $[-1, 1]$. Notice that any horizontal line drawn to intersect the red portion of the graph can do so at most once. Thus the sine function on this restricted domain is one-to-one and has an inverse. There are two commonly used notations to denote the inverse of the function shown in Figure 4.7.1(b):

$$\arcsin x \qquad \text{or} \qquad \sin^{-1} x,$$

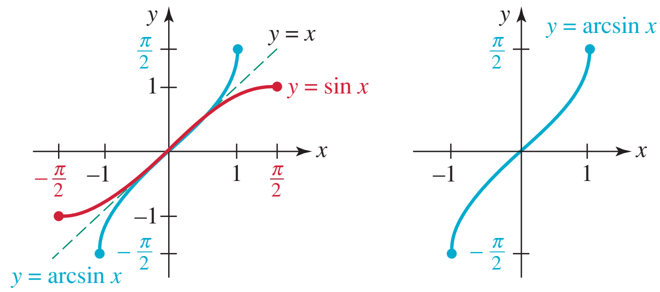
and are read **arcsine of *x*** and **inverse sine of *x***, respectively.



(a) Not a one-to-one function (b) A one-to-one function

FIGURE 4.7.1 Restricting the domain of $y = \sin x$ to produce a one-to-one function

In FIGURE 4.7.2(a) we have reflected the portion of the graph of $y = \sin x$ on the interval $[-\pi/2, \pi/2]$ (the red graph in Figure 4.7.1(b)) about the line $y = x$ to obtain the graph of $y = \arcsin x$ (in blue). For clarity, we have reproduced this blue graph in Figure 4.7.2(b). As this curve shows, the domain of the arcsine function is $[-1, 1]$ and the range is $[-\pi/2, \pi/2]$.



(a) (b)

FIGURE 4.7.2 Graph of $y = \arcsin x$ is the blue curve

Proceeding as in (6) of Section 2.8, the inverse of $y = \sin x$, $-\pi/2 \le x \le \pi/2$, is obtained by interchanging the symbols x and y, that is, $y = \arcsin x$ is defined implicitly by

$$x = \sin y, \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

The following definition summarizes the discussion.

DEFINITION 4.7.1 Arcsine Function

The **arcsine function**, or **inverse sine function**, is defined by

$$y = \arcsin x \quad \text{if and only if} \quad x = \sin y \tag{1}$$

where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$.

In words:

The arcsine of the number x is that number y (or radian-measured angle) satisfying $-\pi/2 \le y \le \pi/2$ *whose sine is x.*

Note of Caution:
$(\sin x)^{-1} = \dfrac{1}{\sin x} \neq \sin^{-1}x$

When using the notation $\sin^{-1}x$ it is important to realize that "–1" is not an exponent; rather, it denotes an inverse function. The notation arcsin x has an advantage over the notation $\sin^{-1}x$ in that there is no "-1" and hence no potential for misinterpretation; moreover, the prefix "arc" refers to an angle—*the* angle whose sine is x. But since $y = \arcsin x$ and $y = \sin^{-1}x$ are used interchangeably in calculus and in applications, we will continue to alternate their use so that you become comfortable with both notations.

EXAMPLE 1 Evaluating the Inverse Sine Function

Find **(a)** $\arcsin\frac{1}{2}$, **(b)** $\sin^{-1}\left(-\frac{1}{2}\right)$, and **(c)** $\sin^{-1}(-1)$.

Solution (a) If we let $y = \arcsin\frac{1}{2}$, then by (1) we must find the number y (or radian-measured angle) that satisfies $\sin y = \frac{1}{2}$ *and* $-\pi/2 \le y \le \pi/2$. Since $\sin(\pi/6) = \frac{1}{2}$ and $\pi/6$ satisfies the inequality $-\pi/2 \le y \le \pi/2$ it follows that $y = \pi/6$.

(b) If we let $y = \sin^{-1}\left(-\frac{1}{2}\right)$, then $\sin y = -\frac{1}{2}$. Since we must choose y such that $-\pi/2 \le y \le \pi/2$, we find that $y = -\pi/6$.

(c) Letting $y = \sin^{-1}(-1)$, we have that $\sin y = -1$ and $-\pi/2 \le y \le \pi/2$.

Hence $y = -\pi/2$.

Read this paragraph several times.

In parts (b) and (c) of Example 1 we were careful to choose y so that $-\pi/2 \le y \le \pi/2$. For example, it is a common error to think that because $\sin(3\pi/2) = -1$, then necessarily $\sin^{-1}(-1)$ can be taken to be $3\pi/2$. Remember: If $y = \sin^{-1}x$, then y is subject to the restriction $-\pi/2 \le y \le \pi/2$ and $3\pi/2$ does not satisfy this inequality.

EXAMPLE 2 Evaluating a Composition

Without using a calculator, find $\tan\left(\sin^{-1}\frac{1}{4}\right)$.

Solution We must find the tangent of the angle of t radians with sine equal to $\frac{1}{4}$, that is, $\tan t$, where $t = \sin^{-1}\frac{1}{4}$. The angle t is shown in **FIGURE 4.7.3**. Since

$$\tan t = \frac{\sin t}{\cos t} = \frac{\frac{1}{4}}{\cos t},$$

we want to determine the value of $\cos t$. From Figure 4.7.3 and the Pythagorean identity $\sin^2 t + \cos^2 t = 1$, we see that

$$\left(\frac{1}{4}\right)^2 + \cos^2 t = 1 \qquad \text{or} \qquad \cos t = \frac{\sqrt{15}}{4}.$$

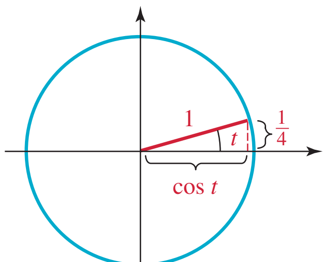


FIGURE 4.7.3 The angle $t = \sin^{-1}\frac{1}{4}$ in Example 2

Hence we have

$$\tan t = \frac{1/4}{\sqrt{15}/4} = \frac{1}{\sqrt{15}} = \frac{\sqrt{15}}{15},$$

and so $$\tan\left(\sin^{-1}\frac{1}{4}\right) = \tan t = \frac{\sqrt{15}}{15}.$$

Arccosine Function If we restrict the domain of the cosine function to the closed interval $[0, \pi]$, the resulting function is one-to-one and thus has an inverse. We denote this inverse by

$$\arccos x \qquad \text{or} \qquad \cos^{-1} x.$$

By interchanging the symbols x and y in $y = \cos x$, $0 \le x \le \pi$, the inverse function $y = \arccos x$ is defined implicitly by

$$x = \cos y,\ 0 \le y \le \pi.$$

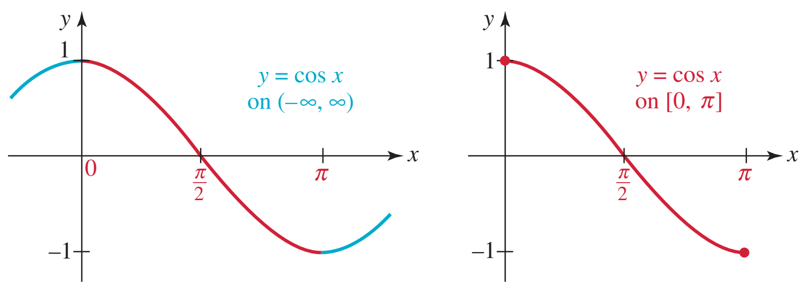
DEFINITION 4.7.2 Arccosine Function

The **arccosine function**, or **inverse cosine function**, is defined by

$$y = \arccos x \qquad \text{if and only if} \qquad x = \cos y \tag{2}$$

where $-1 \le x \le 1$ and $0 \le y \le \pi$.

The graphs shown in FIGURE 4.7.4 illustrate how the function $y = \cos x$ restricted to the interval $[0, \pi]$ becomes a one-to-one function. The inverse of the function shown in Figure 4.7.4(b) is $y = \arccos x$.



(a) Not a one-to-one function (b) A one-to-one function

FIGURE 4.7.4 Restricting the domain of $y = \cos x$ to produce a one-to-one function

By reflecting the graph of the one-to-one function in Figure 4.7.4(b) in the line $y = x$ we obtain the graph of $y = \arccos x$ shown in FIGURE 4.7.5. Note that the figure clearly shows that the domain and range of $y = \arccos x$ are $[-1, 1]$ and $[0, \pi]$, respectively.

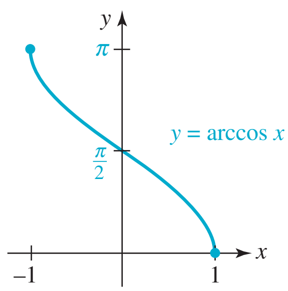


FIGURE 4.7.5 Graph of $y = \arccos x$

EXAMPLE 3 Evaluating the Inverse Cosine Function

Find **(a)** $\arccos\left(\sqrt{2}/2\right)$ **(b)** $\cos^{-1}\left(-\sqrt{3}/2\right)$.

Solution **(a)** If we let $y = \arccos\left(\sqrt{2}/2\right)$, then $\cos y = \sqrt{2}/2$ and $0 \le y \le \pi$. Thus $y = \pi/4$.

(b) Letting $y = \cos^{-1}\left(-\sqrt{3}/2\right)$, we have that $\cos y = -\sqrt{3}/2$, and we must find y such that $0 \le y \le \pi$. Therefore, $y = 5\pi/6$ since $\cos(5\pi/6) = -\sqrt{3}/2$.

EXAMPLE 4 Evaluating the Compositions of Functions

Write $\sin(\cos^{-1} x)$ as an algebraic expression in x.

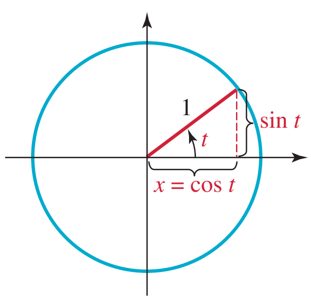


FIGURE 4.7.6 The angle $t = \cos^{-1} x$ in Example 4

Solution In FIGURE 4.7.6 we have constructed an angle of t radians with cosine equal to x. Then $t = \cos^{-1} x$, or $x = \cos t$, where $0 \le t \le \pi$. Now to find $\sin(\cos^{-1} x) = \sin t$, we use the identity $\sin^2 t + \cos^2 t = 1$. Thus

$$\begin{aligned} \sin^2 t + x^2 &= 1 \\ \sin^2 t &= 1 - x^2 \\ \sin t &= \sqrt{1 - x^2} \\ \sin(\cos^{-1} x) &= \sqrt{1 - x^2}. \end{aligned}$$

We use the positive square root of $1 - x^2$, since the range of $\cos^{-1} x$ is $[0, \pi]$, and the sine of an angle t in the first or second quadrant is positive.

Arctangent Function If we restrict the domain of $\tan x$ to the open interval $(-\pi/2, \pi/2)$, then the resulting function is one-to-one and thus has an inverse. This inverse is denoted by

$$\arctan x \qquad \text{or} \qquad \tan^{-1} x.$$

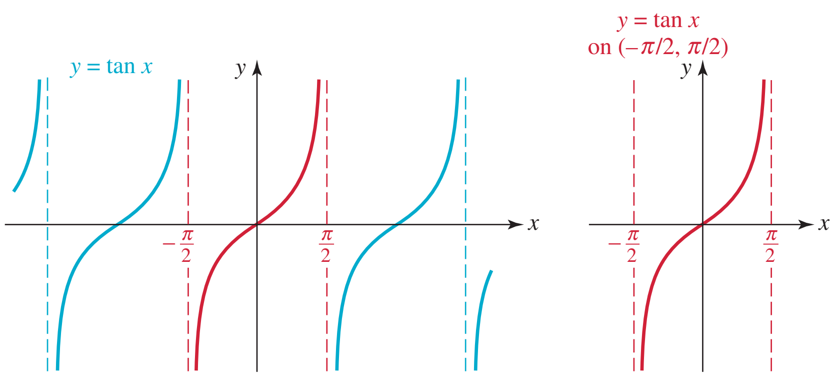
DEFINITION 4.7.3 Arctangent Function

The **arctangent**, or **inverse tangent**, function is defined by

$$y = \arctan x \qquad \text{if and only if} \qquad x = \tan y \tag{3}$$

where $-\infty < x < \infty$ and $-\pi/2 < y < \pi/2$.

The graphs shown in FIGURE 4.7.7 illustrate how the function $y = \tan x$ restricted to the open interval $(-\pi/2, \pi/2)$ becomes a one-to-one function.



(a) Not a one-to-one function

(b) A one-to-one function

FIGURE 4.7.7 Restricting the domain of $y = \tan x$ to produce a one-to-one function

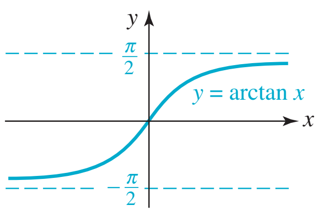


FIGURE 4.7.8 Graph of $y = \arctan x$

By reflecting the graph of the one-to-one function in Figure 4.7.7(b) in the line $y = x$ we obtain the graph of $y = \arctan x$ shown in FIGURE 4.7.8. We see in the figure that the domain and range of $y = \arctan x$ are, in turn, the intervals $(-\infty, \infty)$ and $(-\pi/2, \pi/2)$.

EXAMPLE 5 Evaluating the Inverse Tangent Function

Find $\tan^{-1}(-1)$.

Solution If $\tan^{-1}(-1) = y$, then $\tan y = -1$, where $-\pi/2 < y < \pi/2$. It follows that $\tan^{-1}(-1) = y = -\pi/4$.

EXAMPLE 6 Evaluating the Compositions of Functions

Without using a calculator, find $\sin\left(\arctan\left(-\frac{5}{3}\right)\right)$.

Solution If we let $t = \arctan\left(-\frac{5}{3}\right)$, then $\tan t = -\frac{5}{3}$. The Pythagorean identity $1 + \tan^2 t = \sec^2 t$ can be used to find $\sec t$:

$$1 + \left(-\frac{5}{3}\right)^2 = \sec^2 t$$
$$\sec t = \sqrt{1 + \frac{25}{9}} = \sqrt{\frac{34}{9}} = \frac{\sqrt{34}}{3}.$$

In the preceding line we take the positive square root because $t = \arctan\left(-\frac{5}{3}\right)$ is in the interval $(-\pi/2, \pi/2)$ (the range of the arctangent function) and the secant of an angle t in the first or fourth quadrant is positive. Also, from $\sec t = \sqrt{34}/3$ we find the value of $\cos t$ from the reciprocal identity:

$$\cos t = \frac{1}{\sec t} = \frac{1}{\sqrt{34}/3} = \frac{3}{\sqrt{34}}.$$

Finally, we can use the identity $\tan t = \sin t/\cos t$ in the form $\sin t = \tan t \cos t$ to compute $\sin\left(\arctan\left(-\frac{5}{3}\right)\right)$. It follows that

$$\sin t = \tan t \cos t = \left(-\frac{5}{3}\right)\left(\frac{3}{\sqrt{34}}\right) = -\frac{5}{\sqrt{34}}.$$

Properties of the Inverses Recall from Section 2.8 that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$ hold for any function f and its inverse under suitable restrictions on x. Thus for the inverse trigonometric functions, we have the following properties.

THEOREM 4.7.1 Properties of Inverse Trigonometric Functions

$$\begin{array}{rlll}
(i) & \arcsin(\sin x) = \sin^{-1}(\sin x) = x & \text{if} & -\pi/2 \le x \le \pi/2 \\
(ii) & \sin(\arcsin x) = \sin(\sin^{-1} x) = x & \text{if} & -1 \le x \le 1 \\
(iii) & \arccos(\cos x) = \cos^{-1}(\cos x) = x & \text{if} & 0 \le x \le \pi \\
(iv) & \cos(\arccos x) = \cos(\cos^{-1} x) = x & \text{if} & -1 \le x \le 1 \\
(v) & \arctan(\tan x) = \tan^{-1}(\tan x) = x & \text{if} & -\pi/2 < x < \pi/2 \\
(vi) & \tan(\arctan x) = \tan(\tan^{-1} x) = x & \text{if} & -\infty < x < \infty
\end{array}$$

EXAMPLE 7 Using the Inverse Properties

Without using a calculator, evaluate:

(a) $\sin^{-1}\left(\sin\frac{\pi}{12}\right)$ **(b)** $\cos\left(\cos^{-1}\frac{1}{3}\right)$ **(c)** $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$.

Solution In each case we use the properties of the inverse trigonometric functions given in Theorem 4.7.1.

(a) Because $\pi/12$ satisfies $-\pi/2 \le x \le \pi/2$ it follows from property (i) that

$$\sin^{-1}\left(\sin\frac{\pi}{12}\right) = \frac{\pi}{12}.$$

(b) By property (iv), $\cos\left(\cos^{-1}\frac{1}{3}\right) = \frac{1}{3}$.

(c) In this case we *cannot* apply property (*v*), because $3\pi/4$ is not in the interval $(-\pi/2, \pi/2)$. If we first evaluate $\tan(3\pi/4) = -1$, then we have

$$\tan^{-1}\left(\tan\frac{3\pi}{4}\right) = \underbrace{\tan^{-1}(-1)}_{\text{see Example 5}} = -\frac{\pi}{4}.$$

In the next section we illustrate how inverse trigonometric functions can be used to solve trigonometric equations.

Postscript—The Other Inverse Trig Functions The functions $\cot x$, $\sec x$, and $\csc x$ also have inverses when their domains are suitably restricted. See Problems 49–51 in Exercises 4.7. Because these functions are not used as often as arctan, arccos, and arcsin, most scientific calculators do not have keys for them. However, any calculator that computes arcsin, arccos, and arctan can be used to obtain values for **arccsc**, **arcsec**, and **arccot**. Unlike the fact that $\sec x = 1/\cos x$, we note that $\sec^{-1}x \neq 1/\cos^{-1}x$; rather, $\sec^{-1}x = \cos^{-1}(1/x)$ for $|x| \geq 1$. Similar relationships hold for $\csc^{-1}x$ and $\cot^{-1}x$. See Problems 56–58 in Exercises 4.7.

Exercises 4.7 Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–14, find the exact value of the given trigonometric expression. Do not use a calculator.

1. $\sin^{-1}0$

2. $\tan^{-1}\sqrt{3}$

3. $\arccos(-1)$

4. $\arcsin\dfrac{\sqrt{3}}{2}$

5. $\arccos\frac{1}{2}$

6. $\arctan(-\sqrt{3})$

7. $\sin^{-1}\left(-\dfrac{\sqrt{3}}{2}\right)$

8. $\cos^{-1}\dfrac{\sqrt{3}}{2}$

9. $\tan^{-1}1$

10. $\sin^{-1}\dfrac{\sqrt{2}}{2}$

11. $\arctan\left(-\dfrac{\sqrt{3}}{3}\right)$

12. $\arccos(-\frac{1}{2})$

13. $\sin^{-1}\left(-\dfrac{\sqrt{2}}{2}\right)$

14. $\arctan 0$

In Problems 15–32, find the exact value of the given trigonometric expression. Do not use a calculator.

15. $\sin(\cos^{-1}\frac{3}{5})$

16. $\cos(\sin^{-1}\frac{1}{3})$

17. $\tan(\arccos(-\frac{2}{3}))$

18. $\sin(\arctan\frac{1}{4})$

19. $\cos(\arctan(-2))$

20. $\tan(\sin^{-1}(-\frac{1}{6}))$

21. $\csc(\sin^{-1}\frac{3}{5})$

22. $\sec(\tan^{-1}4)$

23. $\sin(\sin^{-1}\frac{1}{5})$

24. $\cos(\cos^{-1}(-\frac{4}{5}))$

25. $\tan(\tan^{-1}1.2)$

26. $\sin(\arcsin 0.75)$

27. $\arcsin\left(\sin\dfrac{\pi}{16}\right)$

28. $\arccos\left(\cos\dfrac{2\pi}{3}\right)$

29. $\tan^{-1}(\tan \pi)$

30. $\sin^{-1}\left(\sin \dfrac{5\pi}{6}\right)$

31. $\cos^{-1}\left(\cos\left(-\dfrac{\pi}{4}\right)\right)$

32. $\arctan\left(\tan \dfrac{\pi}{7}\right)$

In Problems 33–40, write the given expression as an algebraic expression in x.

33. $\sin(\tan^{-1}x)$

34. $\cos(\tan^{-1}x)$

35. $\tan(\arcsin x)$

36. $\sec(\arccos x)$

37. $\cot(\sin^{-1}x)$

38. $\cos(\sin^{-1}x)$

39. $\csc(\arctan x)$

40. $\tan(\arccos x)$

In Problems 41–48, sketch the graph of the given function.

41. $y = \arctan|x|$

42. $y = \dfrac{\pi}{2} - \arctan x$

43. $y = |\arcsin x|$

44. $y = \sin^{-1}(x + 1)$

45. $y = 2\cos^{-1}x$

46. $y = \cos^{-1}2x$

47. $y = \arccos(x - 1)$

48. $y = \cos(\arcsin x)$

49. The **arccotangent** function can be defined by $y = \operatorname{arccot} x$ (or $y = \cot^{-1}x$) if and only if $x = \cot y$, where $0 < y < \pi$. Graph $y = \operatorname{arccot} x$, and give the domain and the range of this function.

50. The **arccosecant** function can be defined by $y = \operatorname{arccsc} x$ (or $y = \csc^{-1}x$) if and only if $x = \csc y$, where $-\pi/2 \le y \le \pi/2$ and $y \ne 0$. Graph $y = \operatorname{arccsc} x$, and give the domain and the range of this function.

51. One definition of the **arcsecant** function is $y = \operatorname{arcsec} x$ (or $y = \sec^{-1}x$) if and only if $x = \sec y$, where $0 \le y \le \pi$ and $y \ne \pi/2$. (See Problem 52 for an alternative definition.) Graph $y = \operatorname{arcsec} x$, and give the domain and the range of this function.

52. An alternative definition of the arcsecant function can be made by restricting the domain of the secant function to $[0, \pi/2) \cup [\pi, 3\pi/2)$. Under this restriction, define the arcsecant function. Graph $y = \operatorname{arcsec} x$, and give the domain and the range of this function.

53. Using the definition of the arccotangent function from Problem 49, for what values of x is it true that **(a)** $\cot(\operatorname{arccot} x) = x$ and **(b)** $\operatorname{arccot}(\cot x) = x$?

54. Using the definition of the arccosecant function from Problem 50, for what values of x is it true that **(a)** $\csc(\operatorname{arccsc} x) = x$ and **(b)** $\operatorname{arccsc}(\csc x) = x$?

55. Using the definition of the arcsecant function from Problem 51, for what values of x is it true that **(a)** $\sec(\operatorname{arcsec} x) = x$ and **(b)** $\operatorname{arcsec}(\sec x) = x$?

56. Verify that $\operatorname{arccot} x = \dfrac{\pi}{2} - \arctan x$, for all real numbers x.

57. Verify that $\operatorname{arccsc} x = \arcsin(1/x)$ for $|x| \ge 1$.

58. Verify that $\operatorname{arcsec} x = \arccos(1/x)$ for $|x| \ge 1$.

In Problems 59–64, use the results of Problems 56–58 and a calculator to find the indicated value.

59. $\cot^{-1} 0.75$

60. $\csc^{-1}(-1.3)$

61. $\operatorname{arccsc}(-1.5)$

62. $\operatorname{arccot}(-0.3)$

63. $\operatorname{arcsec}(-1.2)$

64. $\sec^{-1}2.5$

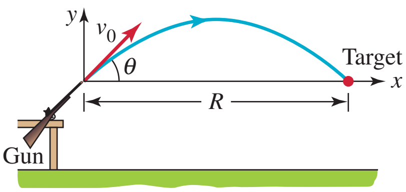


FIGURE 4.7.9 Angle of elevation in Problem 65

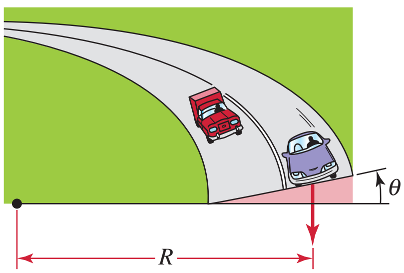


FIGURE 4.7.10 Banked curve in Problem 67

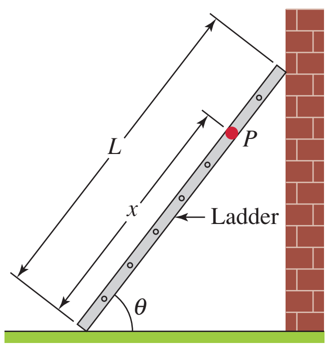


FIGURE 4.7.11 Ladder in Problem 69

Applications

65. Projectile Motion The angle of elevation θ for a gun firing a bullet to hit a target at a horizontal distance R (assuming that the target and the muzzle of the gun are at the same height) satisfies

$$R = \frac{v_0^2 \sin 2\theta}{g},$$

where v_0 is the muzzle velocity and g is the acceleration due to gravity. See FIGURE 4.7.9. Find the angle of elevation θ if the target is 800 ft from the gun and the muzzle velocity is 200 ft/s. Use $g = 32$ ft/s^2. [*Hint*: There are two solutions.]

66. Olympic Sports For the Olympic event, the hammer throw, it can be shown that the maximum distance is achieved for the release angle θ (measured from the horizontal) that satisfies

$$\cos 2\theta = \frac{gh}{v_0^2 + gh},$$

where h is the height of the hammer above the ground at release, v_0 is the initial velocity, and g is the acceleration due to gravity. For $v_0 = 13.7$ m/s and $h = 2.25$ m, find the optimal release angle. Use $g = 9.81$ m/s^2.

67. Highway Design In the design of highways and railroads, curves are banked to provide centripetal force for safety. The optimal banking angle θ is given by $\tan\theta = v^2/Rg$, where v is the speed of the vehicle, R is the radius of the curve, and g is the acceleration due to gravity. See FIGURE 4.7.10. As the formula indicates, for a given radius there is no one correct angle for all speeds. Consequently, curves are banked for the average speed of the traffic over them. Find the correct banking angle for a curve of radius 600 ft on a country road where speeds average 30 mi/h. Use $g = 32$ ft/s^2. [*Hint*: Use consistent units.]

68. Highway Design—Continued If μ is the coefficient of friction between the car and the road, then the maximum velocity v_m that a car can travel around a curve without slipping is given by $v_m^2 = gR\tan(\theta + \tan^{-1}\mu)$, where θ is the banking angle of the curve. Find v_m for the country road in Problem 67 if $\mu = 0.26$.

69. Ladder About to Slip Consider a ladder of length L leaning against a house with a load at point P a distance x measured from the bottom of the ladder. See FIGURE 4.7.11. The angle θ at which the ladder is at the verge of slipping can be shown to be related to x and the coefficient of friction c between the ladder and the ground by

$$\frac{x}{L} = \frac{c}{1 + c^2}(c + \tan\theta).$$

(a) Find θ when $c = 1$ and the load is at the top of the ladder.
(b) Find θ when $c = 0.5$ and the load is $\frac{3}{4}$ of the way up the ladder.

For Discussion

70. Using a calculator set in radian mode, evaluate $\arctan(\tan 1.8)$, $\arccos(\cos 1.8)$, and $\arcsin(\sin 1.8)$. Explain the results.

71. Using a calculator set in radian mode, evaluate $\tan^{-1}(\tan(-1))$, $\cos^{-1}(\cos(-1))$, and $\sin^{-1}(\sin(-1))$. Explain the results.

72. In Section 4.3 we saw that the graphs of $y = \sin x$ and $y = \cos x$ are related by shifting and reflecting. Justify the identity

$$\arcsin x + \arccos x = \frac{\pi}{2},$$

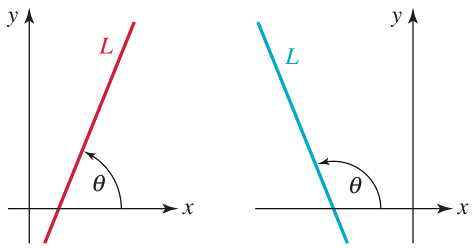
for all x in $[-1, 1]$, by finding a similar relationship between the graphs of $y = \arcsin x$ and $y = \arccos x$.

73. With a calculator set in radian mode determine which of the following inverse trigonometric evaluations result in an error message: **(a)** $\sin^{-1}(-2)$, **(b)** $\cos^{-1}(-2)$, **(c)** $\tan^{-1}(-2)$. Explain.

74. Discuss: Can any periodic function be one-to-one?

75. Show that $\arcsin\frac{3}{5} + \arcsin\frac{5}{13} = \arcsin\frac{56}{65}$. [*Hint*: See (4) of Section 4.5.]

76. The **angle of inclination of a line** is defined to be the angle θ measured counterclockwise between the line and the x-axis $0 \le \theta < \pi$ (or $0° \le \theta < 180°$). See FIGURE 4.7.12. The angle of inclination of a horizontal line is 0 and the angle of inclination of a vertical line is $\pi/2$. If L is any line with slope m (that is, not vertical), use FIGURE 4.7.13 to show that $m = \tan\theta$.



(a) θ is acute (b) θ is obtuse

FIGURE 4.7.12 Angles of inclination in Problem 76

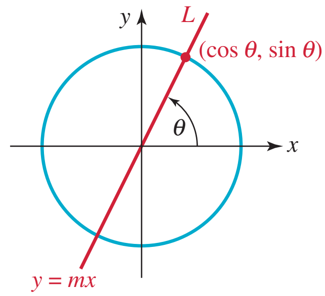


FIGURE 4.7.13 Line passing through the origin and a unit circle in Problem 76

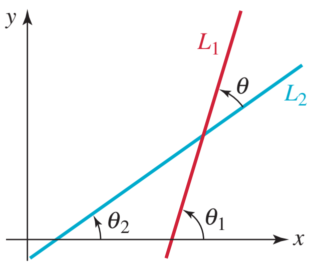


FIGURE 4.7.14 Two lines in Problem 78

77. Use Problem 76 to find the angle of inclination of the given line.
(a) $3x - 2y = 6$ **(b)** $y = -2x + 5$

78. Suppose that θ is the acute angle between the two nonperpendicular intersecting lines L_1 and L_2 in FIGURE 4.7.14. If L_1 and L_2 have slopes m_1 and m_2, respectively, then show that

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|.$$

79. Use Problem 78 to find the acute angle θ between the given pair of lines.
(a) $-x + 2y = 10,\ x + 2y = -5$ **(b)** $x + 2y = 6,\ 4x - 3y = 1$

80. **(a)** Show that

$$\tan(\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3) = 0.$$

(b) Discuss: How does the result in part (a) prove that

$$\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi?$$

4.8 Trigonometric Equations

INTRODUCTION In Sections 4.5 and 4.6 we considered **trigonometric identities**, which are equations involving trigonometric functions that are satisfied by all values of the variable for which both sides of the equality are defined. In this section we examine **conditional trigonometric equations**, that is, equations that are true for only certain values of the variable. We discuss techniques for finding those values of the variable (if any) that satisfy the equation.

We begin by considering the problem of finding all real numbers x that satisfy $\sin x = \sqrt{2}/2$. Interpreted as the x-coordinates of the points of intersection of the graphs

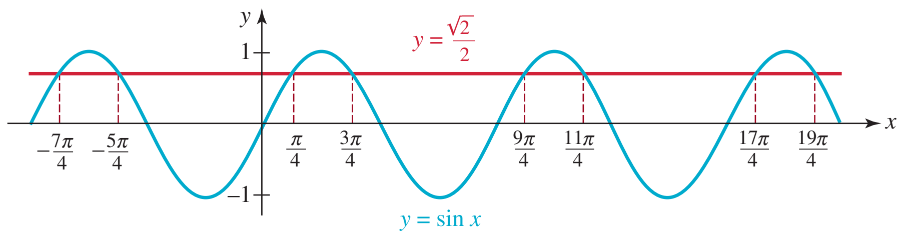


FIGURE 4.8.1 Graphs of $y = \sin x$ and $y = \frac{\sqrt{2}}{2}$

of $y = \sin x$ and $y = \sqrt{2}/2$, FIGURE 4.8.1 shows that there exists infinitely many solutions of the equation $\sin x = \sqrt{2}/2$:

$$\ldots, -\frac{7\pi}{4}, \frac{\pi}{4}, \frac{9\pi}{4}, \frac{17\pi}{4}, \ldots \tag{1}$$

$$\ldots, -\frac{5\pi}{4}, \frac{3\pi}{4}, \frac{11\pi}{4}, \frac{19\pi}{4}, \ldots. \tag{2}$$

Note that in each of the lists (1) and (2), two successive solutions differ by $2\pi = 8\pi/4$. This is a consequence of the periodicity of the sine function. It is common for trigonometric equations to have an infinite number of solutions because of the periodicity of the trigonometric functions. In general, to obtain solutions of an equation such as $\sin x = \sqrt{2}/2$, it is more convenient to use a unit circle and reference angles rather than a graph of the trigonometric function. We illustrate this approach in the following example.

EXAMPLE 1 Using the Unit Circle

Find all real numbers x satisfying $\sin x = \sqrt{2}/2$.

Solution If $\sin x = \sqrt{2}/2$, the reference angle for x is $\pi/4$ radian. Since the value of $\sin x$ is positive, the terminal side of the angle x lies in either the first or second quadrant. Thus, as shown in FIGURE 4.8.2, the only solutions between 0 and 2π are

$$x = \frac{\pi}{4} \qquad \text{and} \qquad x = \frac{3\pi}{4}.$$

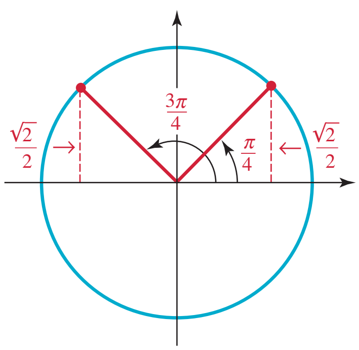


FIGURE 4.8.2 Unit circle in Example 1

Since the sine function is periodic with period 2π, all of the remaining solutions can be obtained by adding integer multiples of 2π to these solutions. The two solutions are

$$x = \frac{\pi}{4} + 2n\pi \qquad \text{and} \qquad x = \frac{3\pi}{4} + 2n\pi, \tag{3}$$

where n is an integer. The numbers that you see in (1) and (2) correspond, respectively, to letting $n = -1$, $n = 0$, $n = 1$, and $n = 2$ in the first and second formulas in (3).

When we are faced with a more complicated equation, such as

$$4\sin^2 x - 8\sin x + 3 = 0,$$

the basic approach is to solve for a single trigonometric function (in this case, it would be $\sin x$) by using methods similar to those for solving algebraic equations.

EXAMPLE 2 Solving a Trigonometric Equation by Factoring

Find all solutions of $4\sin^2 x - 8\sin x + 3 = 0$.

Solution We first observe that this is a quadratic equation in $\sin x$, and that it factors as

$$(2\sin x - 3)(2\sin x - 1) = 0.$$

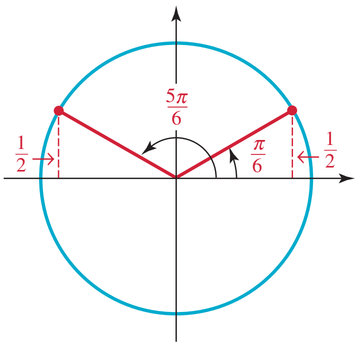


FIGURE 4.8.3 Unit circle in Example 2

This implies that either

$$\sin x = \frac{3}{2} \qquad \text{or} \qquad \sin x = \frac{1}{2}.$$

The first equation has no solution since $|\sin x| \leq 1$. As we see in **FIGURE 4.8.3** the two angles between 0 and 2π for which $\sin x$ equals $\frac{1}{2}$ are

$$x = \frac{\pi}{6} \qquad \text{and} \qquad x = \frac{5\pi}{6}.$$

Therefore, by the periodicity of the sine function, the solutions are

$$x = \frac{\pi}{6} + 2n\pi \qquad \text{and} \qquad x = \frac{5\pi}{6} + 2n\pi,$$

where n is an integer.

EXAMPLE 3 Checking for Lost Solutions

Find all solutions of

$$\sin x = \cos x. \tag{4}$$

Solution In order to work with a single trigonometric function, we divide both sides of the equation by $\cos x$ to obtain

$$\tan x = 1. \tag{5}$$

$\cos 0 = 1$, $\cos\pi = -1$, $\cos 2\pi = 1$, $\cos 3\pi = -1$, and so on. In general, $\cos n\pi = (-1)^n$, where n is an integer.

Equation (5) is equivalent to (4) *provided* that $\cos x \neq 0$. We observe that if $\cos x = 0$, then as we have seen in (2) of Section 4.3, $x = (2n + 1)\pi/2 = \pi/2 + n\pi$, for n an integer. By the sum formula for the sine,

$$\sin\left(\frac{\pi}{2} + n\pi\right) = \overbrace{\sin\frac{\pi}{2}}^{\text{see (4) of Section 4.5}}\ \overbrace{\cos n\pi}^{(-1)^n} + \cos\frac{\pi}{2}\ \overbrace{\sin n\pi}^{0} = (-1)^n \neq 0,$$

we see that these values of x do not satisfy the original equation. Thus we will find *all* the solutions to (4) by solving equation (5).

Now $\tan x = 1$ implies that the reference angle for x is $\pi/4$ radian. Since $\tan x = 1 > 0$, the terminal side of the angle of x radians can lie either in the first or in the third quadrant, as shown in **FIGURE 4.8.4**. Thus the solutions are

$$x = \frac{\pi}{4} + 2n\pi \qquad \text{and} \qquad x = \frac{5\pi}{4} + 2n\pi,$$

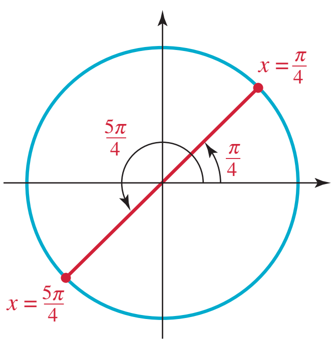


FIGURE 4.8.4 Unit circle in Example 3

where n is an integer. We can see from Figure 4.8.4 that these two sets of numbers can be written more compactly as

This also follows from the fact that $\tan x$ is π-periodic.

$$x = \frac{\pi}{4} + n\pi,$$

where n is an integer.

Losing Solutions When solving an equation, if you divide by an expression containing a variable, you may lose some solutions of the original equation. For example, in algebra a common mistake in solving equations such as $x^2 = x$ is to divide by x to obtain $x = 1$. But by writing $x^2 = x$ as $x^2 - x = 0$ or $x(x - 1) = 0$ we see that in fact $x = 0$ or $x = 1$. To prevent the loss of a solution you must determine the values that make the expression zero and check to see whether they are solutions of the original equation. Note that in Example 3, when we divided by $\cos x$, we took care to check that no solutions were lost.

Whenever possible, it is preferable to avoid dividing by a variable expression. As illustrated with the algebraic equation $x^2 = x$, this can frequently be accomplished by collecting all nonzero terms on one side of the equation and then factoring (something we could not do in Example 3). Example 4 illustrates this technique.

EXAMPLE 4 Solving a Trigonometric Equation by Factoring

Solve
$$2\sin x\cos^2 x = -\frac{\sqrt{3}}{2}\cos x. \tag{6}$$

Solution To avoid dividing by $\cos x$, we write the equation as

$$2\sin x\cos^2 x + \frac{\sqrt{3}}{2}\cos x = 0$$

and factor:
$$\cos x\left(2\sin x\cos x + \frac{\sqrt{3}}{2}\right) = 0.$$

Thus either

$$\cos x = 0 \qquad \text{or} \qquad 2\sin x\cos x + \frac{\sqrt{3}}{2} = 0.$$

Since the cosine is zero for all odd multiples of $\pi/2$, the solutions of $\cos x = 0$ are

$$x = (2n+1)\frac{\pi}{2} = \frac{\pi}{2} + n\pi,$$

where n is an integer.

See (12) in Section 4.5.

In the second equation we replace $2\sin x\cos x$ by $\sin 2x$ from the double-angle formula for the sine function to obtain an equation with a single trigonometric function:

$$\sin 2x + \frac{\sqrt{3}}{2} = 0 \qquad \text{or} \qquad \sin 2x = -\frac{\sqrt{3}}{2}.$$

Thus the reference angle for $2x$ is $\pi/3$. Since the sine is negative, the angle $2x$ must be in either the third quadrant or the fourth quadrant. As FIGURE 4.8.5 illustrates, either

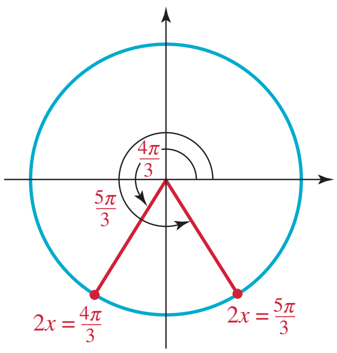


FIGURE 4.8.5 Unit circle in Example 4

$$2x = \frac{4\pi}{3} + 2n\pi \qquad \text{or} \qquad 2x = \frac{5\pi}{3} + 2n\pi.$$

Dividing by 2 gives

$$x = \frac{2\pi}{3} + n\pi \qquad \text{or} \qquad x = \frac{5\pi}{6} + n\pi.$$

Therefore, all solutions of (6) are

$$x = \frac{\pi}{2} + n\pi, \quad x = \frac{2\pi}{3} + n\pi, \quad \text{and} \quad x = \frac{5\pi}{6} + n\pi,$$

where n is an integer. ■

In Example 4 had we simplified the equation by dividing by $\cos x$ and not checked to see whether the values of x for which $\cos x = 0$ satisfied equation (6), we would have lost the solutions $x = \pi/2 + n\pi$, where n is an integer.

EXAMPLE 5 Using a Trigonometric Identity

Solve $3\cos^2 x - \cos 2x = 1$.

Solution We observe that the given equation involves both the cosine of x and the cosine of $2x$. Consequently, we use the double-angle formula for the cosine in the form

$$\cos 2x = 2\cos^2 x - 1 \qquad \leftarrow \text{See (14) of Section 4.5}$$

to replace the equation by an equivalent equation that involves $\cos x$ only. We find that

$$3\cos^2 x - (2\cos^2 x - 1) = 1 \qquad \text{becomes} \qquad \cos^2 x = 0.$$

Therefore, $\cos x = 0$, and the solutions are

$$x = (2n + 1)\frac{\pi}{2} = \frac{\pi}{2} + n\pi,$$

where n is an integer.

We are often interested in finding roots of an equation only in a specified interval.

EXAMPLE 6 Using a Trigonometric Identity

Find all solutions of the equation $\cos t - \cos 5t = 0$ in the interval $[0, 2\pi)$.

Solution In this case it is helpful to use a sum-to-product formula. In Example 3 of Section 4.6 we saw that by identifying $x_1 = t$ and $x_2 = 5t$, (9) of Theorem 4.6.2 gives

$$\cos t - \cos 5t = -2\sin\frac{t + 5t}{2}\sin\frac{t - 5t}{2} = -2\sin 3t \sin\overset{\sin(-2t) = -\sin 2t}{\overset{\downarrow}{(-2t)}} = 2\sin 3t \sin 2t$$

Replacing the sum $\cos t - \cos 5t$ in the given equation by the product $2\sin 3t \sin 2t$ gives the equivalent equation $2\sin 3t \sin 2t = 0$, or

$$\sin 3t \sin 2t = 0.$$

The last equation is satisfied if either $\sin 3t = 0$ or $\sin 2t = 0$. Then from (1) in Section 4.3 we see that $\sin 3t = 0$ implies

$$3t = n\pi,\ n = 0, 1, 2, 3, \ldots \tag{7}$$

whereas $\sin 2t = 0$ implies

$$2t = n\pi,\ n = 0, 1, 2, 3, \ldots. \tag{8}$$

If you think in terms of angles measured in radians and the unit circle, then the only angles satisfying the condition that t be in the interval $[0, 2\pi)$ correspond to $n = 0, 1, 2, 3, 4, 5$ in (7) and $n = 1, 2, 3$ in (8):

$$t = 0, \pi/3, 2\pi/3, \pi, 4\pi/3, 5\pi/3, \tag{9}$$

or

$$t = \pi/2, \pi, 3\pi/2. \tag{10}$$

The solution set of the original equation is then the union of the two sets defined by the numbers in (9) and (10), that is

$$\{0, \pi/3, \pi/2, 2\pi/3, \pi, 4\pi/3, 3\pi/2, 5\pi/3\}.$$

So far in this section we have viewed the variable in the trigonometric equation as representing either a real number or an angle measured in radians. If the variable represents an angle measured in degrees, the technique for solving is the same.

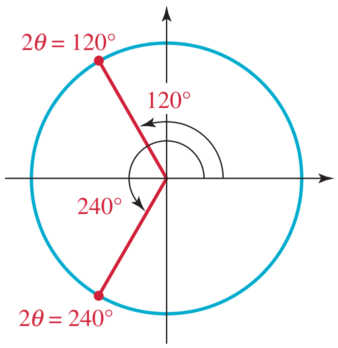


FIGURE 4.8.6 Unit circle in Example 7

EXAMPLE 7 Equation When the Angle Is in Degrees

Solve $\cos 2\theta = -\frac{1}{2}$, where θ is an angle measured in degrees.

Solution Since $\cos 2\theta = -\frac{1}{2}$, the reference angle for 2θ is 60° and the angle 2θ must be in either the second or the third quadrant. **FIGURE 4.8.6** illustrates that either $2\theta = 120°$ or $2\theta = 240°$. Any angle that is coterminal with one of these angles will also satisfy $\cos 2\theta = -\frac{1}{2}$. These angles are obtained by adding any integer multiple of 360° to 120° or to 240°:

$$2\theta = 120° + 360°n \qquad \text{or} \qquad 2\theta = 240° + 360°n,$$

where n is an integer. Dividing by 2 the last line yields the two solutions

$$\theta = 60° + 180°n \qquad \text{and} \qquad \theta = 120° + 180°n.$$

Extraneous Solutions When an equation is squared or multiplied by a variable expression, the resulting equation may not be equivalent to the original. In other words, the new equation may possess solutions that are not solutions of the given equation. Such numbers are called **extraneous solutions**. For example, the simple equation $x = 1$ has only one solution but by squaring both sides the resulting equation $x^2 = 1$ is now satisfied by $x = 1$ and $x = -1$. Thus number $x = -1$ is an extraneous solution of the original equation. The next example illustrates the same idea for trigonometric equations.

EXAMPLE 8 Extraneous Roots

Find all solutions of $1 + \tan\alpha = \sec\alpha$, where α is an angle measured in degrees.

Solution The equation does not factor, but we see that if we square both sides, we can use a fundamental identity to obtain an equation involving a single trigonometric function:

$$\begin{aligned}
(1 + \tan\alpha)^2 &= (\sec\alpha)^2 \\
1 + 2\tan\alpha + \tan^2\alpha &= \sec^2\alpha \qquad \leftarrow \text{now use (9) of Section 4.4} \\
1 + 2\tan\alpha + \tan^2\alpha &= 1 + \tan^2\alpha \\
2\tan\alpha &= 0 \\
\tan\alpha &= 0.
\end{aligned}$$

The values of α for $0° \le \alpha < 360°$ at which $\tan\alpha = 0$ are

$$\alpha = 0° \qquad \text{and} \qquad \alpha = 180°.$$

Since we squared each side of the original equation, we may have introduced extraneous solutions. Therefore, it is important that we check all solutions in the original equation. Substituting $\alpha = 0°$ into $1 + \tan\alpha = \sec\alpha$, we obtain the *true* statement $1 + 0 = 1$. But after substituting $\alpha = 180°$, we obtain the *false* statement $1 + 0 = -1$. Therefore, 180° is an extraneous solution and $\alpha = 0°$ is the only solution satisfying $0° \le \alpha < 360°$. Thus, all the solutions of the equation are given by

$$\alpha = 0° + 360°n = 360°n,$$

where n is an integer. For $n \neq 0$, these are the angles that are coterminal with 0°.

Recall from Section 2.1 that to find the x-intercepts of the graph of a function $y = f(x)$ we find the zeros of f, that is, we must solve the equation $f(x) = 0$. The following example makes use of this fact.

EXAMPLE 9 Intercepts of a Graph

Find the first three x-intercepts of the graph of $f(x) = \sin 2x \cos x$ on the positive x-axis.

Solution We must solve $f(x) = 0$, that is, $\sin 2x \cos x = 0$. It follows that either $\sin 2x = 0$ or $\cos x = 0$.

From $\sin 2x = 0$, we obtain $2x = n\pi$, where n is an integer, or $x = n\pi/2$, where n is an integer. From $\cos x = 0$, we find $x = \pi/2 + n\pi$, where n is an integer. Then for $n = 2$, $x = n\pi/2$ gives $x = \pi$, whereas for $n = 0$ and $n = 1$, $x = \pi/2 + n\pi$ gives $x = \pi/2$ and $x = 3\pi/2$, respectively. Thus the first three x-intercepts on the positive x-axis are $(\pi/2, 0)$, $(\pi, 0)$, and $(3\pi/2, 0)$.

Using Inverse Functions So far all of the trigonometric equations have had solutions that were related by reference angles to the special angles 0, $\pi/6$, $\pi/4$, $\pi/3$, or $\pi/2$. If this is not the case, we will see in the next example how to use inverse trigonometric functions and a calculator to find solutions.

EXAMPLE 10 Solving Equations Using Inverse Functions

Find the solutions of $4\cos^2 x - 3\cos x - 2 = 0$ in the interval $[0, \pi]$.

Solution We recognize that this is a quadratic equation in $\cos x$. Since the left-hand side of the equation does not readily factor, we apply the quadratic formula to obtain

$$\cos x = \frac{3 \pm \sqrt{41}}{8}.$$

At this point we can discard the value $(3 + \sqrt{41})/8 \approx 1.18$, because $\cos x$ cannot be greater than 1. We then use the inverse cosine function and a calculator to solve the remaining equation:

$$\cos x = \frac{3 - \sqrt{41}}{8} \quad \text{which implies} \quad x = \cos^{-1}\left(\frac{3 - \sqrt{41}}{8}\right) \approx 2.01.$$

Of course in Example 10, had we attempted to compute $\cos^{-1}[(3 + \sqrt{41})/8]$ with a calculator, we would have received an error message.

Exercises 4.8

Answers to selected odd-numbered problems begin on page ANS-17.

In Problems 1–6, find all solutions of the given trigonometric equation if x represents an angle measured in radians.

1. $\sin x = \sqrt{3}/2$
2. $\cos x = -\sqrt{2}/2$
3. $\sec x = \sqrt{2}$
4. $\tan x = -1$
5. $\cot x = -\sqrt{3}$
6. $\csc x = 2$

In Problems 7–12, find all solutions of the given trigonometric equation if x represents a real number.

7. $\cos x = -1$
8. $2\sin x = -1$
9. $\tan x = 0$
10. $\sqrt{3}\sec x = 2$
11. $-\csc x = 1$
12. $\sqrt{3}\cot x = 1$

In Problems 13–18, find all solutions of the given trigonometric equation if θ represents an angle measured in degrees.

13. $\csc\theta = 2\sqrt{3}/3$
14. $2\sin\theta = \sqrt{2}$
15. $1 + \cot\theta = 0$
16. $\sqrt{3}\sin\theta = \cos\theta$
17. $\sec\theta = -2$
18. $2\cos\theta + \sqrt{2} = 0$

In Problems 19–46, find all solutions of the given trigonometric equation if x is a real number and θ is an angle measured in degrees.

19. $\cos^2 x - 1 = 0$
20. $2\sin^2 x - 3\sin x + 1 = 0$
21. $3\sec^2 x = \sec x$
22. $\tan^2 x + (\sqrt{3} - 1)\tan x - \sqrt{3} = 0$
23. $2\cos^2\theta - 3\cos\theta - 2 = 0$
24. $2\sin^2\theta - \sin\theta - 1 = 0$
25. $\cot^2\theta + \cot\theta = 0$
26. $2\sin^2\theta + (2 - \sqrt{3})\sin\theta - \sqrt{3} = 0$
27. $\cos 2x = -1$
28. $\sec 2x = 2$
29. $2\sin 3\theta = 1$
30. $\tan 4\theta = -1$
31. $\cot(x/2) = 1$
32. $\csc(\theta/3) = -1$
33. $\sin 2x + \sin x = 0$
34. $\cos 2x + \sin^2 x = 1$
35. $\cos 2\theta = \sin\theta$
36. $\sin 2\theta + 2\sin\theta - 2\cos\theta = 2$
37. $\sin^4 x - 2\sin^2 x + 1 = 0$
38. $\tan^4\theta - 2\sec^2\theta + 3 = 0$
39. $\sec x\sin^2 x = \tan x$
40. $\dfrac{1 + \cos\theta}{\cos\theta} = 2$
41. $\sin\theta + \cos\theta = 1$
42. $\sin x + \cos x = 0$
43. $\sqrt{\dfrac{1 + 2\sin x}{2}} = 1$
44. $\sin x + \sqrt{\sin x} = 0$
45. $\cos\theta - \sqrt{\cos\theta} = 0$
46. $\cos\theta\sqrt{1 + \tan^2\theta} = 1$

In Problems 47–52, use a sum-to-product formula (as in Example 6) to solve the given equation on the indicated interval.

47. $\sin 6t - \sin 4t = 0,\ [0, 2\pi)$
48. $\cos 2t + \cos 3t = 0,\ [0, 2\pi)$
49. $\cos\theta - \cos 4\theta = 0,\ [-\pi, \pi)$
50. $\sin 5\alpha + \sin 3\alpha = 0,\ [0, 2\pi)$
51. $\sin 7x - \sin x - 2\sin 3x = 0,\ (-\pi, \pi)$
52. $\sin x + \cos 2x - \sin 3x = 0,\ [0, 3\pi)$

In Problems 53–60, find the first three x-intercepts of the graph of the given function on the positive x-axis.

53. $f(x) = -5\sin(3x + \pi)$
54. $f(x) = 2\cos\left(x + \dfrac{\pi}{4}\right)$
55. $f(x) = 2 - \sec\dfrac{\pi}{2}x$
56. $f(x) = 1 + \cos\pi x$
57. $f(x) = \sin x + \tan x$
58. $f(x) = 1 - 2\cos\left(x + \dfrac{\pi}{3}\right)$
59. $f(x) = \sin x - \sin 2x$
60. $f(x) = \cos x + \cos 3x$
[*Hint*: Write $3x = x + 2x$.]

In Problems 61–64, by graphing determine whether the given equation has any solutions.

61. $\tan x = x$ [*Hint*: Graph $y = \tan x$ and $y = x$ on the same coordinate axes.]
62. $\sin x = x$
63. $\cot x - x = 0$
64. $\cos x + x + 1 = 0$

In Problems 65–70, using an inverse trigonometric function find the solutions of the given equation in the indicated interval. Round your answers to two decimal places.

65. $20\cos^2 x + \cos x - 1 = 0,\quad [0, \pi]$
66. $3\sin^2 x - 8\sin x + 4 = 0,\quad [-\pi/2, \pi/2]$

67. $\tan^2 x + \tan x - 1 = 0, \quad (-\pi/2, \pi/2)$
68. $3\sin 2x + \cos x = 0, \quad [-\pi/2, \pi/2]$
69. $5\cos^3 x - 3\cos^2 x - \cos x = 0, \quad [0, \pi]$
70. $\tan^4 x - 3\tan^2 x + 1 = 0, \quad (-\pi/2, \pi/2)$

Applications

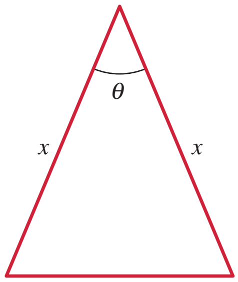


FIGURE 4.8.7 Isosceles triangle in Problem 71

71. Isosceles Triangle By methods that will be discussed in Section 4.10 it can be shown that the area of the isosceles triangle shown in FIGURE 4.8.7 is given by $A = \frac{1}{2}x^2 \sin\theta$. If the length of side x is 4, what value of θ will give a triangle with area 4?

72. Circular Motion An object travels in a circular path centered at the origin with constant angular speed. The y-coordinate of the object at any time t seconds is given by $y = 8\cos(\pi t - \pi/12)$. At what time(s) does the object cross the x-axis?

73. Mach Number Use Problem 73 in Exercises 4.5 to find the vertex angle of the cone of sound waves made by an airplane flying at Mach 2.

74. Alternating Current An electric generator produces a 60-cycle alternating current given by $I(t) = 30\sin 120\pi\left(t - \frac{7}{36}\right)$, where $I(t)$ is the current in amperes at t seconds. Find the smallest positive value of t for which the current is 15 amperes.

75. Electrical Circuits If the voltage given by $V = V_0 \sin(\omega t + \alpha)$ is impressed on a series circuit, an alternating current is produced. If $V_0 = 110$ volts, $\omega = 120\pi$ radians per second, and $\alpha = -\pi/6$, when is the voltage equal to zero?

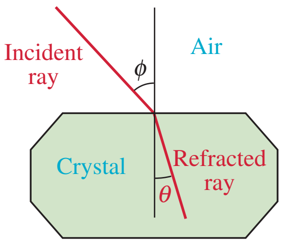


FIGURE 4.8.8 Light rays in Problem 76

76. Refraction of Light Consider a ray of light passing from one medium (such as air) into another medium (such as a crystal). Let ϕ be the angle of incidence and θ the angle of refraction. As shown in FIGURE 4.8.8, these angles are measured from a vertical line. According to **Snell's law**, there is a constant c that depends on the two mediums, such that $\dfrac{\sin\phi}{\sin\theta} = c$. Assume that for light passing from air into a crystal, $c = 1.437$. Find ϕ and θ such that the angle of incidence is twice the angle of refraction.

77. Snow Cover On the basis of data collected from 1966 to 1980, the extent of snow cover S in the northern hemisphere, measured in millions of square kilometers, can be modeled by the function

$$S(w) = 25 + 21\cos\frac{\pi}{26}(w - 5),$$

where w is the number of weeks past January 1.

(a) How much snow cover does this formula predict for April Fool's Day? (Round w to the nearest integer.)
(b) In which week does the formula predict the least amount of snow cover?
(c) What month does this fall in?

4.9 Simple Harmonic Motion

INTRODUCTION Many physical objects vibrate or oscillate in a regular manner, repeatedly moving back and forth over a definite time interval. Some examples are clock pendulums, a mass on a spring, sound waves, strings on a guitar when plucked, the human heart, tides, and alternating current. In this section we will focus on mathematical models of the undamped oscillatory motion of a mass on a spring.

Before proceeding with the main discussion we need to discuss the graph of the sum of constant multiples of $\cos Bx$ and $\sin Bx$, that is, $y = c_1\cos Bx + c_2 \sin Bx$, where c_1 and c_2 are constants.

Addition of Two Sinusoidal Functions In Section 4.3 we examined the graphs of horizontally shifted sine and cosine graphs. It turns out that any linear combination of a sine function and a cosine function of the form

$$y = c_1 \cos Bx + c_2 \sin Bx, \tag{1}$$

where c_1 and c_2 are constants, can be expressed either as a shifted sine function $y = A\sin(Bx + \phi)$, $B > 0$, or as a shifted cosine function $y = A\cos(Bx + \phi)$. Note that in (1) the sine and cosine functions sin Bx and cos Bx have the same period $2\pi/B$.

EXAMPLE 1 Addition of a Sine and a Cosine

Graph the function $y = \cos 2x - \sqrt{3}\sin 2x$.

Solution Using a graphing utility we have shown in FIGURE 4.9.1 four cycles of the graphs of $y = \cos 2x$ (in red) and $y = -\sqrt{3}\sin 2x$ (in green). It is apparent in FIGURE 4.9.2 that the period of the sum of these two functions is π, the common period of $\cos 2x$ and $\sin 2x$. Also apparent is that the blue graph is a horizontally shifted sine (or cosine) function. Although Figure 4.9.2 suggests that the amplitude of the function $y = \cos 2x - \sqrt{3}\sin 2x$ is 2, the exact phase shift of the graph is certainly is *not* apparent.

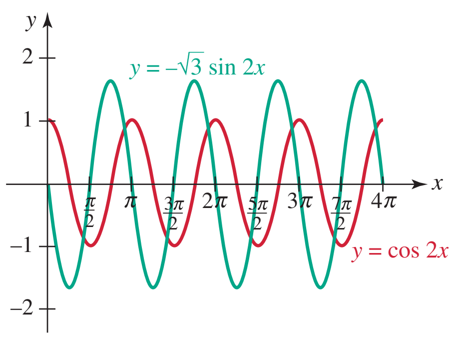


FIGURE 4.9.1 Superimposed graphs of $y = \cos 2x$ and $y = -\sqrt{3}\sin 2x$ in Example 1

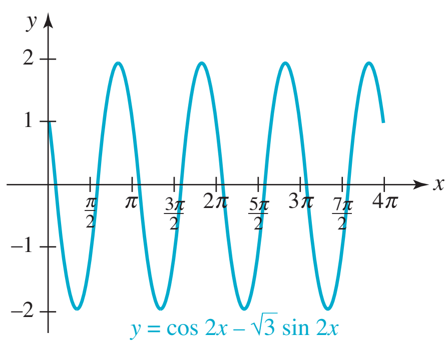


FIGURE 4.9.2 Graph of the sum $y = \cos 2x - \sqrt{3}\sin 2x$ in Example 1

The sine form $y = A\sin(Bx + \phi)$ is slightly easier to use than the cosine form $y = A\cos(Bx + \phi)$.

Reduction to a Sine Function We examine only the reduction of (1) to the form $y = A\sin(Bx + \phi)$, $B > 0$.

THEOREM 4.9.1 Reduction of (1) to (2)

For real numbers c_1, c_2, B, and x,

$$c_1 \cos Bx + c_2 \sin Bx = A\sin(Bx + \phi) \tag{2}$$

where A and ϕ are defined by

$$A = \sqrt{c_1^2 + c_2^2} \tag{3}$$

and

$$\left.\begin{aligned} \sin\phi &= \frac{c_1}{A} \\ \cos\phi &= \frac{c_2}{A} \end{aligned}\right\} \tan\phi = \frac{c_1}{c_2} \tag{4}$$

PROOF: To prove (2), we use the sum formula (4) of Section 4.5:

$$\begin{aligned} A\sin(Bx+\phi) &= A\sin Bx\cos\phi + A\cos Bx\sin\phi \\ &= (A\sin\phi)\cos Bx + (A\cos\phi)\sin Bx \\ &= c_1\cos Bx + c_2\sin Bx \end{aligned}$$

and identify $A\sin\phi = c_1$, $A\cos\phi = c_2$. Thus, $\sin\phi = c_1/A = c_1/\sqrt{c_1^2+c_2^2}$ and $\cos\phi = c_2/A = c_2/\sqrt{c_1^2+c_2^2}$.

EXAMPLE 2 Example 1 Revisited

Express $y = \cos 2x - \sqrt{3}\sin 2x$ as a single sine function.

Solution With the identifications $c_1 = 1$, $c_2 = -\sqrt{3}$, and $B = 2$, we have from (3) and (4),

$$A = \sqrt{c_1^2+c_2^2} = \sqrt{1^2+(-\sqrt{3})^2} = \sqrt{4} = 2,$$

$$\left.\begin{aligned} \sin\phi &= \frac{1}{2} \\ \cos\phi &= -\frac{\sqrt{3}}{2} \end{aligned}\right\} \tan\phi = -\frac{1}{\sqrt{3}}.$$

Although $\tan\phi = -1/\sqrt{3}$ we cannot blindly assume that $\phi = \tan^{-1}(-1/\sqrt{3})$. The value we take for ϕ must be consistent with the algebraic signs of $\sin\phi$ and $\cos\phi$. Thinking in terms of radian-measured angles, because $\sin\phi > 0$ and $\cos\phi < 0$ the terminal side of the angle ϕ lies in the second quadrant. But since the range of the inverse tangent function is the interval $(-\pi/2, \pi/2)$, $\tan^{-1}(-1/\sqrt{3}) = -\pi/6$ is a fourth-quadrant angle. The correct angle is found by using the reference angle $\pi/6$ for $\tan^{-1}(-1/\sqrt{3})$ to find the second-quadrant angle

$$\phi = \pi - \frac{\pi}{6} = \frac{5\pi}{6} \text{ radians.}$$

Therefore $y = \cos 2x - \sqrt{3}\sin 2x$ can be rewritten as

$$y = 2\sin\left(2x + \frac{5\pi}{6}\right) \quad \text{or} \quad y = 2\sin 2\left(x + \frac{5\pi}{12}\right).$$

Hence the graph of $y = \cos 2x - \sqrt{3}\sin 2x$ is the graph of $y = 2\sin 2x$, which has amplitude 2, period $2\pi/2 = \pi$, and is shifted $5\pi/12$ units to the left.

EXAMPLE 3 Example 2 Revisited

Find the first two x-intercepts on the positive x-axis of the graph of the function in Example 2.

Solution The first alternative form $y = 2\sin(2x + 5\pi/6)$ of the function in Example 2 can be used to find the x-intercepts of its graph. Recall, $\sin x = 0$ when $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. So by replacing the symbol x by $2x + 5\pi/6$, we see

$$\sin\left(2x + \frac{5\pi}{6}\right) = 0 \quad \text{implies} \quad 2x + \frac{5\pi}{6} = n\pi.$$

Solving

$$2x + \frac{5\pi}{6} = \pi \quad \text{and} \quad 2x + \frac{5\pi}{6} = 2\pi$$

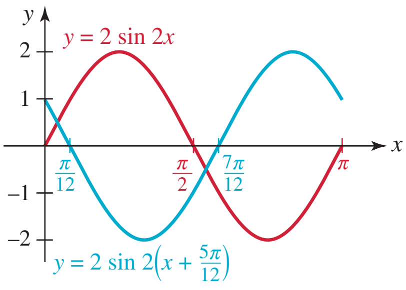


FIGURE 4.9.3 Graph of function in Example 2

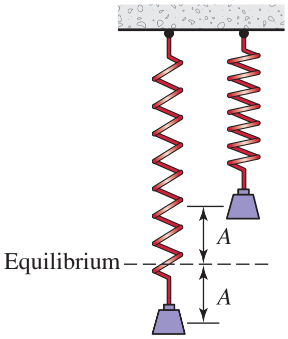


FIGURE 4.9.4 An undamped spring/mass system exhibits simple harmonic motion

yields $x = \pi/12$ and $7\pi/12$. Thus the first two intercepts on the positive x-axis are $(\pi/12, 0)$ and $(7\pi/12, 0)$. The blue graph in FIGURE 4.9.3 is the portion of the graph in Figure 4.9.2 on the interval $[0, \pi]$.

We note that the x-coordinates of the x-intercepts of $y = 2\sin(2x + 5\pi/6)$ can also be obtained by subtracting the phase shift $5\pi/12$ from the x-coordinates of the x-intercepts of the red graph of $y = 2\sin 2x$ in Figure 4.9.3.

Simple Harmonic Motion Consider the motion of a mass on a spring as shown in FIGURE 4.9.4. In the absence of frictional or damping forces, a mathematical model for the displacement (or directed distance) of the mass measured from a position called the **equilibrium position** is given by the function

$$y(t) = y_0 \cos \omega t + \frac{v_0}{\omega} \sin \omega t, \tag{5}$$

where t is time, y_0 and v_0 are, respectively, the initial $(t = 0)$ displacement and velocity of the mass.

Oscillatory motion modeled by the function (5) is said to be **simple harmonic motion**.

More precisely, we have the following definition.

DEFINITION 4.9.1 Simple Harmonic Motion

A point moving on a coordinate line whose position at time t is given by

$$y(t) = A\sin(\omega t + \phi) \quad \text{or} \quad y(t) = A\cos(\omega t + \phi) \tag{6}$$

where A, $\omega > 0$, and ϕ are constants, is said to exhibit **simple harmonic motion**.

Special cases of the trigonometric functions in (6) are $y(t) = A\sin \omega t$, $y(t) = A\cos \omega t$, and $y(t) = c_1 \cos \omega t + c_2 \sin \omega t$.

Terminology The function (5) is said to be the **equation of motion** of the mass. Also, in (5), $\omega = \sqrt{k/m}$, where k is the **spring constant** (an indicator of the stiffness of the spring), m is the **mass** attached to the spring (measured in slugs or kilograms), y_0 is the **initial displacement** of the mass (measured above or below the equilibrium position), v_0 is the **initial velocity** of the mass, t is **time** measured in seconds, and the **period** p of motion is $p = 2\pi/\omega$ seconds. The number $f = 1/p = 1/(2\pi/\omega) = \omega/2\pi$ is called the **frequency** of motion. The frequency indicates the number of cycles completed by the graph per unit time. For example, if the period of (5) is, say, $p = 2$ seconds, then we know that one cycle of the function is complete in 2 seconds. The frequency $f = 1/p = \frac{1}{2}$ means one-half of a cycle is complete in 1 second.

In the study of simple harmonic motion it is convenient to recast the equation of motion (5) as a single expression involving only the sine function:

$$y(t) = A\sin(\omega t + \phi). \tag{7}$$

The reduction of (5) to the sine function (7) can be done in exactly the same manner as illustrated in Example 2. In this situation we make the following identifications between (2) and (5):

$$c_1 = y_0, \quad c_2 = v_0/\omega, \quad A = \sqrt{c_1^2 + c_2^2}, \quad \text{and} \quad B = \omega.$$

EXAMPLE 4 Equation of Motion

(a) Find the equation of simple harmonic motion (5) for a spring mass system if $m = \frac{1}{16}$ slug, $y_0 = -\frac{2}{3}$ ft, $k = 4$ lb/ft, and $v_0 = \frac{4}{3}$ ft/s.
(b) Find the period and frequency of motion.

Solution (a) We begin with the simple harmonic motion equation (5). Since $k/m = 4/\left(\frac{1}{16}\right) = 64$, $\omega = \sqrt{k/m} = 8$, and $v_0/\omega = \left(\frac{4}{3}\right)/8 = \frac{1}{6}$, therefore (5) becomes

$$y(t) = -\tfrac{2}{3}\cos 8t + \tfrac{1}{6}\sin 8t. \tag{8}$$

(b) The period of motion is $2\pi/8 = \pi/4$ second; the frequency is $4/\pi \approx 1.27$ cycles per second.

EXAMPLE 5 Example 4 Continued

Express the equation of motion (8) as a single sine function (7).

Solution With $c_1 = -\frac{2}{3}$, $c_2 = \frac{1}{6}$, we find the amplitude of motion is

$$A = \sqrt{\left(-\tfrac{2}{3}\right)^2 + \left(\tfrac{1}{6}\right)^2} = \tfrac{1}{6}\sqrt{17} \text{ ft}.$$

Then from

$$\left.\begin{aligned} \sin\phi &= -\tfrac{2}{3}\Big/\tfrac{\sqrt{17}}{6} < 0 \\ \cos\phi &= \tfrac{1}{6}\Big/\tfrac{\sqrt{17}}{6} > 0 \end{aligned}\right\} \tan\phi = -4$$

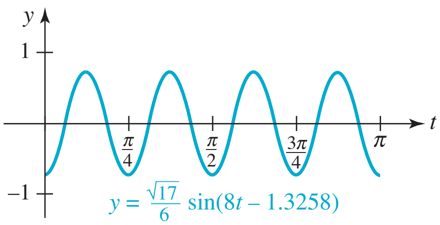


FIGURE 4.9.5 Graph of the equation of motion in Example 5

we can see from algebraic signs $\sin\phi < 0$ and $\cos\phi > 0$ that the terminal side of the angle ϕ lies in the fourth quadrant. Hence the correct value of ϕ is $\tan^{-1}(-4) \approx -1.3258$. The equation of motion is then $y(t) = \frac{1}{6}\sqrt{17}\sin(8t - 1.3258)$. As shown in **FIGURE 4.9.5**, the amplitude of motion is $A = \sqrt{17}/6 \approx 0.6872$. Since we are assuming that that is no resistance to the motion, once the spring/mass system is set in motion the model indicates it stays in motion bouncing back and forth between its maximum displacement $\sqrt{17}/6$ feet above the equilibrium position and a minimum of $-\sqrt{17}/6$ feet below the equilibrium position.

Only in the two cases, $c_1 > 0$, $c_2 > 0$ or $c_1 < 0$, $c_2 > 0$, can we use $\tan\phi$ in (4) to write $\phi = \tan^{-1}(c_1/c_2)$. (Why?) Correspondingly, ϕ is a first or a fourth-quadrant angle.

Exercises 4.9

Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–6, proceed as in Example 2 and write the given trigonometric function in the form $y = A\sin(Bx + \phi)$. Sketch the graph and give the amplitude, the period, and the phase shift.

1. $y = \cos\pi x - \sin\pi x$

2. $y = \sin\dfrac{\pi}{2}x - \sqrt{3}\cos\dfrac{\pi}{2}x$

3. $y = \sqrt{3}\sin 2x + \cos 2x$

4. $y = \sqrt{3}\cos 4x - \sin 4x$

5. $y = \dfrac{\sqrt{2}}{2}(-\sin x - \cos x)$

6. $y = \sin x + \cos x$

In Problems 7 and 8, proceed as in Example 3 and find the first two x-intercepts on the positive x-axis of the graph of the function.

7. $y = -\cos 2\pi x + \sin 2\pi x$

8. $y = \dfrac{1}{\sqrt{3}}\cos\pi x - \sin\pi x$

In Problems 9 and 10, use (2), (3), and (4) to write the left-hand side of the given equation in the form $A\sin(Bx + \phi)$. Then find the solutions of the equation in the indicated interval.

9. $-\cos 2x + \sin 2x = 1;\ [0, \pi]$ **10.** $\cos\frac{x}{2} + \sqrt{3}\sin\frac{x}{2} = 2;\ [0, 4]$

In Problems 11–14, proceed as in Examples 4 and 5 and use the given information to express the equation of simple harmonic motion (5) for a spring/mass system in the trigonometric form (7). Give the amplitude, period, and frequency of motion.

11. $m = \frac{1}{4}$ slug, $y_0 = \frac{1}{2}$ ft, $k = 1$ lb/ft, and $v_0 = \frac{3}{2}$ ft/s
12. $m = 1.6$ slug, $y_0 = -\frac{1}{3}$ ft, $k = 40$ lb/ft, and $v_0 = -\frac{5}{4}$ ft/s
13. $m = 1$ slug, $y_0 = -1$ ft, $k = 16$ lb/ft, and $v_0 = -2$ ft/s
14. $m = 2$ slug, $y_0 = -\frac{2}{3}$ ft, $k = 200$ lb/ft, and $v_0 = 5$ ft/s

15. The equation of simple harmonic motion of a spring/mass system is $y(t) = \frac{5}{2}\sin(2t - \pi/3)$. Determine the initial displacement y_0 and initial velocity v_0 of the mass. [*Hint*: Use (5).]
16. Use the equation of simple harmonic motion of the spring/mass system given in Problem 15 to find the times for which the mass passes through the equilibrium position $y = 0$.

Applications

17. Current Under certain conditions, the current $I(t)$ measured in amperes at time t in an electrical circuit is given by

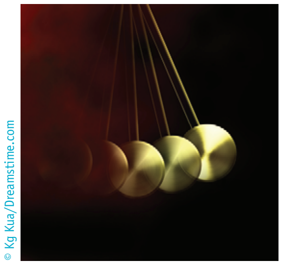
$$I(t) = I_0[\sin(\omega t + \theta)\cos\phi + \cos(\omega t + \theta)\sin\phi].$$

Express $I(t)$ as a single sine function of the form given in (7). [*Hint*: Review the sum formula in (4) of Theorem 4.5.2.]

18. More Current In a certain kind of electrical circuit, the current $I(t)$ measured in amperes at time t seconds is given by

$$I(t) = 10\cos\left(120\pi t + \frac{\pi}{3}\right).$$

(a) Change the form of the function $I(t)$ to the sine function form given in (7).
(b) Give the period of the sine function in part (a) and the phase shift. Use the sine function to sketch two cycles of the graph of $I(t)$.



Clock pendulum

19. Pendulum Motion An object that swings back and forth is called a physical pendulum. A **simple pendulum** is a special case of the physical pendulum and consists of a rod of length L with a mass m attached at one end. See FIGURE 4.9.6. If the motion of a simple pendulum takes place in a vertical plane and it is assumed that the mass of the rod is negligible and no damping forces act on the system, then it can be shown that the displacement angle θ of the pendulum as a function of time t, measured from the vertical, is given by

$$\theta(t) = \theta_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t.$$

Here θ_0 and v_0 are the initial displacement and velocity and $\omega = \sqrt{g/L}$, and $g = 32$ ft/s^2 is the acceleration due to gravity.
(a) Find $\theta(t)$ if $\theta_0 > 0$ and $v_0 = 0$ at $t = 0$.
(b) Show that the period (in seconds) of motion of a simple pendulum is

$$T = 2\pi\sqrt{\frac{L}{g}}$$

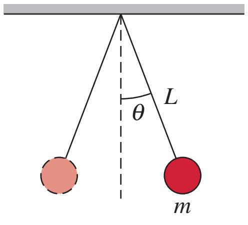


FIGURE 4.9.6 Pendulum in Problem 19

20. Pendulum Motion on the Moon Use Problem 19 to describe the motion of a simple pendulum:

(a) On the Moon where the acceleration of due to gravity is $\frac{1}{6}$ that of the Earth.
(b) If its length is increased to $4L$.
(c) If its length is decreased to $\frac{1}{4}L$.

Calculator/Computer Problems

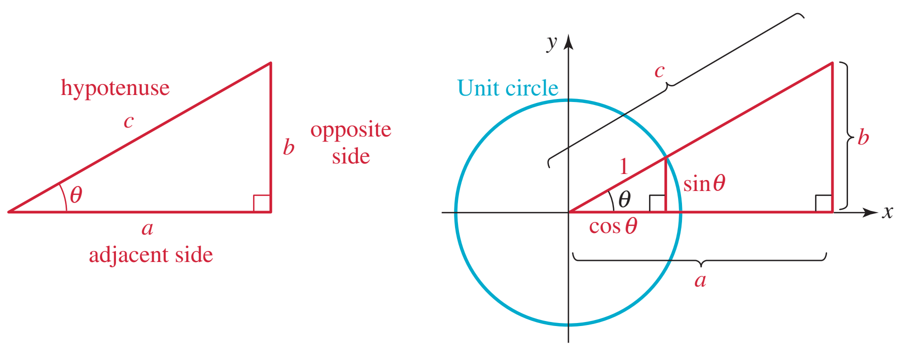
In Problems 21 and 22, use a graphing utility to obtain the graph of the given function f on the interval $[0, 2\pi]$. Use (2), (3), and (4) to write f in the form $f(x) = A\sin(Bx + \phi)$. Then find approximate solutions of indicated equation in the interval.

21. $f(x) = 3\cos 2x + 4\sin 2x;\quad f(x) = 5$
22. $f(x) = 5\cos 3x - 12\sin 3x;\quad f(x) = -13$

4.10 Right Triangle Trigonometry

INTRODUCTION The word *trigonometry* (from the Greek *trigonon* meaning "triangle" and *metria* meaning "measurement") refers to the measurement of triangles. In Section 4.2 we defined the trigonometric functions using coordinates of points on the unit circle and by using radian measure we were able to define the trigonometric functions of any angle. In this section we will show that the trigonometric functions of an acute angle in a right triangle have an equivalent definition in terms of the lengths of the sides of the triangle.

Terminology In FIGURE 4.10.1(a) we have drawn a right triangle with sides labeled a, b, and c (indicating their respective lengths) and one of the acute angles denoted by θ. From the Pythagorean theorem we know that $a^2 + b^2 = c^2$. The side opposite the right angle is called the **hypotenuse**; the remaining sides are referred to as the **legs** of the triangle. The legs labeled a and b are, in turn, said to be the side **adjacent** to the angle θ and the side **opposite** the angle θ. We will also use the abbreviations **hyp**, **adj**, and **opp** to denote the lengths of these sides.



(a) A right triangle with an acute angle θ

(b) The unit circle with angle θ placed in standard position

FIGURE 4.10.1 In (a) and (b) the right triangles are the same

If we place θ in standard position and draw a unit circle centered at the origin, we see from Figure 4.10.1(b) that there are two similar right triangles containing the same angle θ. Since corresponding sides of similar triangles are proportional, it follows that

$$\frac{\sin\theta}{1} = \frac{b}{c} = \frac{\text{opp}}{\text{hyp}} \quad \text{and} \quad \frac{\cos\theta}{1} = \frac{a}{c} = \frac{\text{adj}}{\text{hyp}}.$$

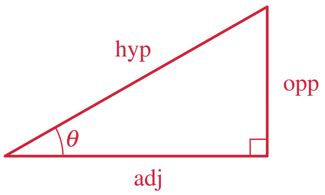


FIGURE 4.10.2 Defining the trigonometric functions of θ

Also, we have

$$\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta} = \frac{b/c}{a/c} = \frac{b}{a} = \frac{\text{opp}}{\text{adj}}.$$

Then, applying the reciprocal identities (12) in Section 4.4, each trigonometric function of θ can be written as the ratio of the lengths of the sides of a right triangle as follows. See **FIGURE 4.10.2**.

DEFINITION 4.10.1 Trigonometric Functions of θ in a Right Triangle

For an acute angle θ in a right triangle as shown in Figure 4.10.2,

$$\begin{aligned} \sin\theta &= \frac{\text{opp}}{\text{hyp}} & \cos\theta &= \frac{\text{adj}}{\text{hyp}} \\ \tan\theta &= \frac{\text{opp}}{\text{adj}} & \cot\theta &= \frac{\text{adj}}{\text{opp}} \\ \sec\theta &= \frac{\text{hyp}}{\text{adj}} & \csc\theta &= \frac{\text{hyp}}{\text{opp}} \end{aligned} \tag{1}$$

EXAMPLE 1 Values of the Six Trigonometric Functions

Find the exact values of the six trigonometric functions of the angle θ in the right triangle shown in **FIGURE 4.10.3**.

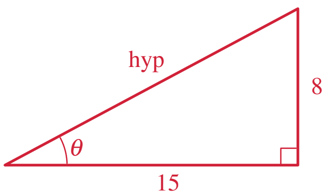


FIGURE 4.10.3 Right triangle in Example 1

Solution From Figure 4.10.3 we see that the side opposite θ has length 8 and the side adjacent has length 15. From the Pythagorean theorem the hypotenuse c is

$$c^2 = 8^2 + 15^2 = 289 \qquad \text{and so} \qquad c = \sqrt{289} = 17.$$

Thus from (1) the values of the six trigonometric functions are

$$\begin{aligned} \sin\theta &= \frac{\text{opp}}{\text{hyp}} = \frac{8}{17}, & \cos\theta &= \frac{\text{adj}}{\text{hyp}} = \frac{15}{17}, \\ \tan\theta &= \frac{\text{opp}}{\text{adj}} = \frac{8}{15}, & \cot\theta &= \frac{\text{adj}}{\text{opp}} = \frac{15}{8}, \\ \sec\theta &= \frac{\text{hyp}}{\text{adj}} = \frac{17}{15}, & \csc\theta &= \frac{\text{hyp}}{\text{opp}} = \frac{17}{8}. \end{aligned}$$

EXAMPLE 2 Using a Right Triangle Sketch

If θ is an acute angle and $\sin\theta = \frac{2}{7}$, find the values of the other trigonometric functions of θ.

Solution We sketch a right triangle with an acute angle θ satisfying $\sin\theta = \frac{2}{7}$, by making opp $= 2$ and hyp $= 7$ as shown in **FIGURE 4.10.4**. From the Pythagorean theorem we have

$$2^2 + (\text{adj})^2 = 7^2 \qquad \text{so that} \qquad (\text{adj})^2 = 7^2 - 2^2 = 45.$$

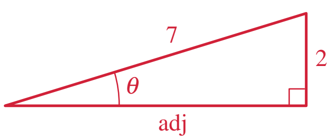


FIGURE 4.10.4 Right triangle in Example 2

Thus, $$\text{adj} = \sqrt{45} = 3\sqrt{5}.$$

The values of the remaining five trigonometric functions are obtained from the definitions in (1):

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{3\sqrt{5}}{7}, \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}} = \frac{7}{3\sqrt{5}} = \frac{7\sqrt{5}}{15},$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{2}{3\sqrt{5}} = \frac{2\sqrt{5}}{15}, \qquad \cot\theta = \frac{\text{adj}}{\text{opp}} = \frac{3\sqrt{5}}{2},$$

$$\csc\theta = \frac{\text{hyp}}{\text{opp}} = \frac{7}{2}.$$

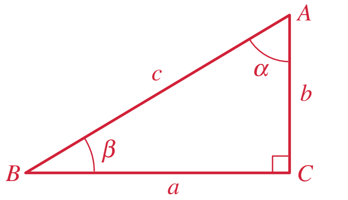


FIGURE 4.10.5 Standard labeling for a right triangle

Solving Right Triangles Applications of right triangle trigonometry in fields such as surveying and navigation involve **solving right triangles**. The expression "to solve a triangle" means that we wish to find the length of each side and the measure of each angle in the triangle. We can solve any right triangle if we know either two sides or one acute angle and one side. As the following examples will show, sketching and labeling the triangle is an essential part of the solution process. It will be our general practice to label a right triangle as shown in FIGURE 4.10.5. The three vertices will be denoted by A, B, and C, with C at the vertex of the right angle. We denote the angles at A and B by α and β and the lengths of the sides opposite these angles by a and b, respectively. The length of the side opposite the right angle at C is denoted by c.

EXAMPLE 3 Solving a Right Triangle

Solve the right triangle having a hypotenuse of length $4\sqrt{3}$ and one 60° angle.

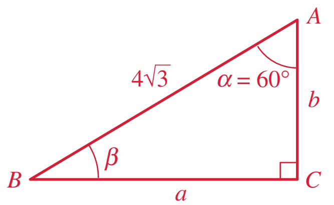


FIGURE 4.10.6 Right triangle in Example 3

Solution First we make a sketch of the triangle and label it as shown in FIGURE 4.10.6. We wish to find a, b, and β. Since α and β are complementary angles, $\alpha + \beta = 90°$ yields

$$\beta = 90° - \alpha = 90° - 60° = 30°.$$

We are given the length of the hypotenuse, namely, hyp $= 4\sqrt{3}$. To find a, the length of the side opposite the angle $\alpha = 60°$, we select the sine function. From $\sin\alpha = \text{opp}/\text{hyp}$, we obtain

$$\sin 60° = \frac{a}{4\sqrt{3}} \quad \text{or} \quad a = 4\sqrt{3}\sin 60°.$$

Since $\sin 60° = \sqrt{3}/2$, we have

$$a = 4\sqrt{3}\sin 60° = 4\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 6.$$

Finally, to find the length b of the side adjacent to the 60° angle, we select the cosine function. From $\cos\alpha = \text{adj}/\text{hyp}$, we obtain

$$\cos 60° = \frac{b}{4\sqrt{3}} \quad \text{or} \quad b = 4\sqrt{3}\cos 60°.$$

Because $\cos 60° = \frac{1}{2}$, we find

$$b = 4\sqrt{3}\cos 60° = 4\sqrt{3}\left(\tfrac{1}{2}\right) = 2\sqrt{3}.$$

In Example 3 once we determined a, we could have found b by using either the Pythagorean theorem or the tangent function. In general, there are usually several ways to solve a triangle.

Use of a Calculator If angles other than 30°, 45°, or 60° are involved in a problem, we can obtain approximations of the desired trigonometric function values with a calculator. For the remainder of this chapter, whenever an approximation is used, we will round the final results to the nearest hundredth unless the problem specifies otherwise. To take full advantage of the calculator's accuracy, store the computed values of the trigonometric functions in the calculator for subsequent calculations. If, instead, a rounded version of a displayed value is written down and then later keyed back into the calculator, the accuracy of the final result may be diminished.

EXAMPLE 4 Solving a Right Triangle

Solve the right triangle with legs of length 4 and 5.

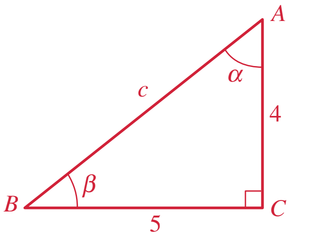


FIGURE 4.10.7 Right triangle in Example 4

Solution After sketching and labeling the triangle as shown in FIGURE 4.10.7, we see that we need to find c, α, and β. From the Pythagorean theorem, the hypotenuse c is given by

$$c = \sqrt{5^2 + 4^2} = \sqrt{41} \approx 6.40.$$

To find β, we use $\tan\beta = \text{opp}/\text{adj}$. (By choosing to work with the given quantities, we avoid error due to previous approximations.) Thus we have

$$\tan\beta = \tfrac{4}{5} = 0.8.$$

From a calculator set in degree mode, we find $\beta \approx 38.66°$. Since $\alpha = 90° - \beta$, we obtain $\alpha \approx 51.34°$.

Exercises 4.10 Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–10, find the values of the six trigonometric functions of the angle θ in the given triangle.

1.

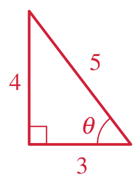


FIGURE 4.10.8 Triangle for Problem 1

2.

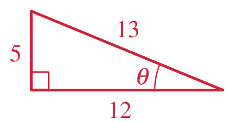


FIGURE 4.10.9 Triangle for Problem 2

3.

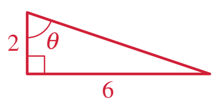


FIGURE 4.10.10 Triangle for Problem 3

4.

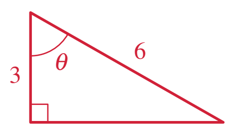


FIGURE 4.10.11 Triangle for Problem 4

5.

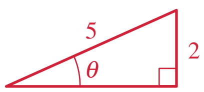


FIGURE 4.10.12 Triangle for Problem 5

6.

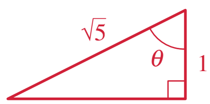


FIGURE 4.10.13 Triangle for Problem 6

7.

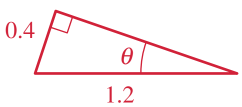


FIGURE 4.10.14 Triangle for Problem 7

8.

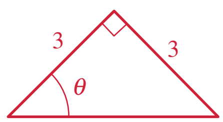


FIGURE 4.10.15 Triangle for Problem 8

9.

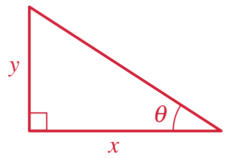


FIGURE 4.10.16 Triangle for Problem 9

10.

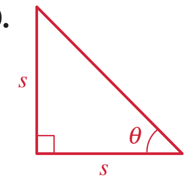


FIGURE 4.10.17 Triangle for Problem 10

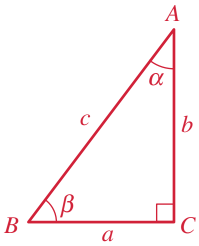


FIGURE 4.10.18 Triangle for Problems 11–22

In Problems 11–22, find the indicated unknowns. Each problem refers to the triangle shown in FIGURE 4.10.18.

11. $a = 4, \beta = 27°; b, c$

12. $c = 10, \beta = 49°; a, b$

13. $b = 8, \beta = 34.33°; a, c$

14. $c = 25, \alpha = 50°; a, b$

15. $b = 1.5, c = 3; \alpha, \beta, a$

16. $a = 5, b = 2; \alpha, \beta, c$

17. $a = 4, b = 10; \alpha, \beta, c$

18. $b = 4, \alpha = 58°; a, c$

19. $a = 9, c = 12; \alpha, \beta, b$

20. $b = 3, c = 6; \alpha, \beta, a$

21. $b = 20, \alpha = 23°; a, c$

22. $a = 11, \alpha = 33.5°; b, c$

In Problems 23 and 24, solve for x in the given triangle.

23.

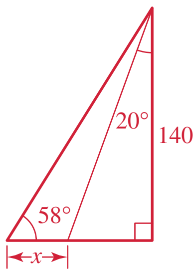


FIGURE 4.10.19 Triangle for Problem 23

24.

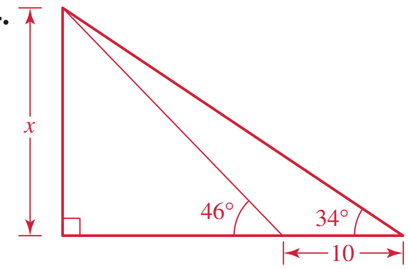


FIGURE 4.10.20 Triangle for Problem 24

In Problems 25 and 26, the cube given in the figure has a side of length s. Find the angle θ between the diagonal AC of its base and the diagonal of the cube AD (Problem 25) and the angle θ between the edge AB and the diagonal of the cube AD (Problem 26).

25.

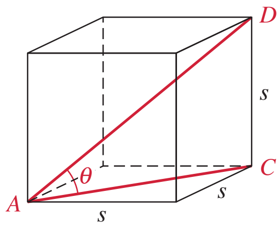


FIGURE 4.10.21 Cube in Problem 25

26.

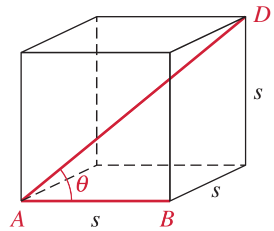


FIGURE 4.10.22 Cube in Problem 26

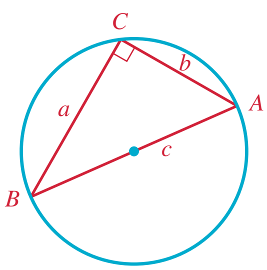


FIGURE 4.10.23 Inscribed right triangle in Problem 27

27. Inscribed Right Triangle If a right triangle is inscribed in a circle, then its hypotenuse is a diameter of the circle. In FIGURE 4.10.23 the blue dot is the center of the circle. Find the area of the red right triangle with sides of length a, b, and c inscribed in a unit circle when the angle at vertex A is 54°.

28. Isosceles Triangle An isosceles triangle is a triangle with two sides of the same length s. As a consequence of this definition, the angles opposite the equal sides have the same measure θ. See FIGURE 4.10.24. Express the area A of an isosceles triangle in terms of s and θ.

29. Equilateral Triangle An equilateral triangle is a triangle with all three sides of the same length s. As a consequence of this definition, the angles opposite the sides have the same measure 60°. See FIGURE 4.10.25. Express the area A of an equilateral triangle as a function of s.

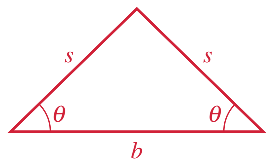


FIGURE 4.10.24 Isosceles triangle in Problem 28

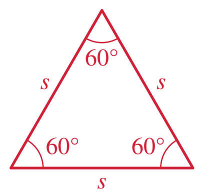


FIGURE 4.10.25 Equilateral triangle in Problem 29

30. Stacked Circles Suppose 10 circles of radius r are stacked within the red triangle shown in FIGURE 4.10.26. Each circle is externally tangent to its neighboring circles and each of the 9 outer circles in the stack is tangent to the red line(s).

(a) Show that the triangle is equilateral.
(b) Express the area A of the equilateral triangle as a function of the radius r.

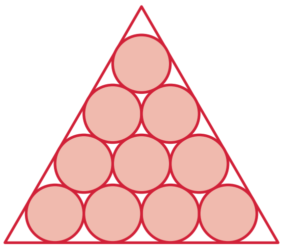


FIGURE 4.10.26 Stacked circles in Problem 30

4.11 Applications of Right Triangles

INTRODUCTION Right triangle trigonometry can be used to solve many practical problems, particularly those involving lengths, heights, and distances.

EXAMPLE 1 Finding the Height of a Tree

A kite is caught in the top branches of a tree. If the 90-ft kite string makes an angle of 22° with the ground, estimate the height of the tree by finding the distance from the kite to the ground.

Solution Let h denote the height of the kite. From FIGURE 4.11.1 we see that

$$\frac{h}{90} = \sin 22^\circ \quad \text{or} \quad h = 90 \sin 22^\circ.$$

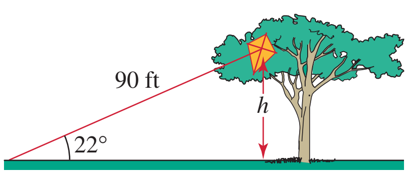


FIGURE 4.11.1 Tree in Example 1

A calculator set in degree mode gives $h \approx 33.71$ ft.

EXAMPLE 2 Length of a Saw Cut

A carpenter cuts the end of a 4-in.-wide board on a 25° bevel from the vertical, starting at a point $1\frac{1}{2}$ in. from the end of the board. Find the lengths of the diagonal cut and the remaining side. See FIGURE 4.11.2.

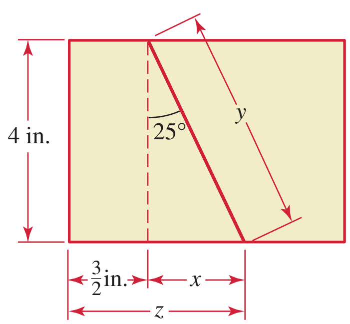


FIGURE 4.11.2 Saw cut in Example 2

Solution Let x, y, and z be the (unknown) dimensions, as labeled in Figure 4.11.2. It follows from the definition of the tangent function that

$$\frac{x}{4} = \tan 25^\circ \qquad \text{so therefore} \qquad x = 4 \tan 25^\circ \approx 1.87 \text{ in.}$$

To find y we observe that

$$\frac{4}{y} = \cos 25° \qquad \text{so} \qquad y = \frac{4}{\cos 25°} \approx 4.41 \text{ in.}$$

Since $z = \frac{3}{2} + x$ and $x \approx 1.87$ in., we see that $z \approx 1.5 + 1.87 \approx 3.37$ in.

Angles of Elevation and Depression The angle between an observer's line of sight to an object and the horizontal is given a special name. As FIGURE 4.11.3 illustrates, if the line of sight is to an object above the horizontal, the angle is called an **angle of elevation**, whereas if the line of sight is to an object below the horizontal, the angle is called an **angle of depression**.

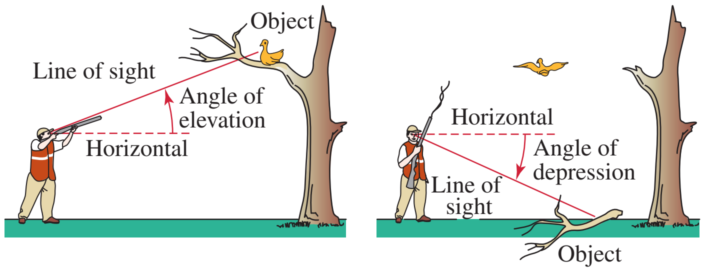


FIGURE 4.11.3 Angles of elevation and depression

EXAMPLE 3 Using Angles of Elevation

A surveyor uses an instrument called a theodolite to measure the angle of elevation between ground level and the top of a mountain. At one point the angle of elevation is measured to be 41°. A half kilometer farther from the base of the mountain, the angle of elevation is measured to be 37°. How high is the mountain?

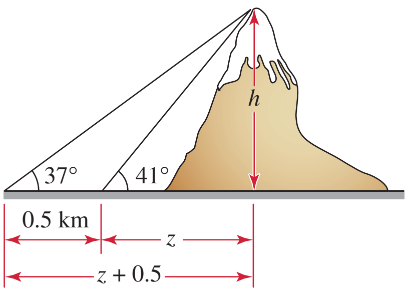


FIGURE 4.11.4 Mountain in Example 3

Solution Let h represent the height of the mountain. FIGURE 4.11.4 shows that there are two right triangles sharing the common side h, so we obtain two equations in two unknowns z and h:

$$\frac{h}{z + 0.5} = \tan 37° \qquad \text{and} \qquad \frac{h}{z} = \tan 41°.$$

We can solve each of these for h, obtaining, respectively,

$$h = (z + 0.5)\tan 37° \qquad \text{and} \qquad h = z\tan 41°.$$

Equating the last two results gives an equation from which we can determine the distance z:

$$(z + 0.5)\tan 37° = z\tan 41°.$$

Solving for z gives us

$$z = \frac{-0.5\tan 37°}{\tan 37° - \tan 41°}.$$

Using $h = z\tan 41°$ we find the height h of the mountain to be

$$h = \frac{-0.5\tan 37°\tan 41°}{\tan 37° - \tan 41°} \approx 2.83 \text{ km.}$$

EXAMPLE 4 Glide Path

Most airplanes approach San Francisco International Airport (SFO) on a straight 3° glide path starting at a point 5.5 mi from the field. A few years ago, the FAA experimented with a computerized two-segment approach where a plane approaches the field on a 6° glide path starting at a point 5.5 mi out and then switches to a 3° glide path 1.5 mi from the point of touchdown. The point of this experimental approach was to reduce the noise of the planes over the outlying residential areas. Compare the height of a plane P' using the standard 3° approach with the height of a plane P using the experimental approach when both planes are 5.5 mi from the airport.

Solution For purposes of illustration, the angles and distances shown in FIGURE 4.11.5 are exaggerated.

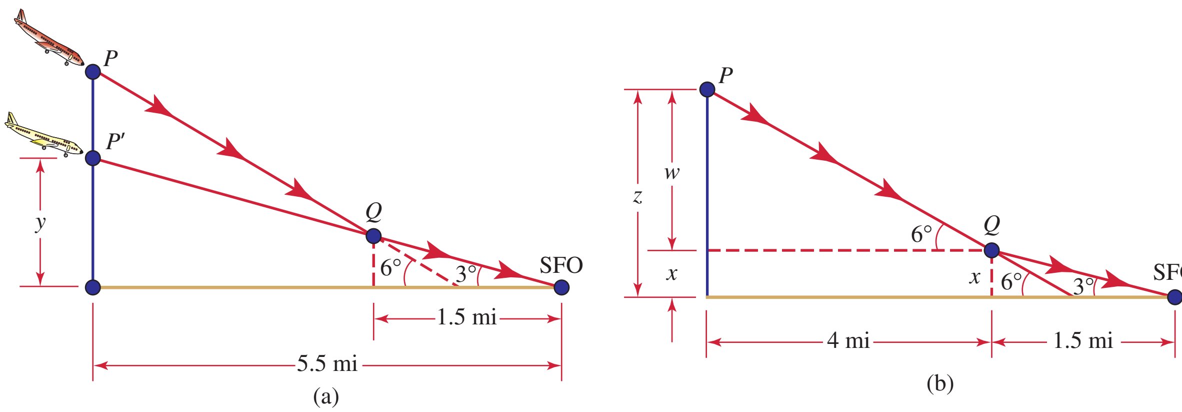


FIGURE 4.11.5 Glide paths in Example 4

First, suppose y is the height of plane P' on the standard approach when it is 5.5 mi out from the airport. As we see in Figure 4.11.5(a),

$$\frac{y}{5.5} = \tan 3° \qquad \text{or} \qquad y = 5.5 \tan 3°.$$

Because distances from the airport are measured in miles, we convert y to feet

$$y = 5.5(5280) \tan 3° \text{ ft} \approx 1522 \text{ ft}.$$

Now, suppose z is the height of plane P on the experimental approach when it is 5.5 mi out from the airport. As shown in Figure 4.11.5(b), $z = x + w$, so we use two right triangles to obtain

$$\frac{x}{1.5} = \tan 3° \qquad \text{or} \qquad x = 1.5 \tan 3°$$

and

$$\frac{w}{4} = \tan 6° \qquad \text{or} \qquad w = 4 \tan 6°.$$

Hence the approximate height of plane P at a point 5.5 mi out from the airport is

$$\begin{aligned} z &= x + w \\ &= 1.5 \tan 3° + 4 \tan 6° \qquad \leftarrow \text{1 mile} = \text{5280 feet} \\ &= 1.5(5280)\tan 3° + 4(5280)\tan 6° \approx 2635 \text{ ft}. \end{aligned}$$

In other words, plane P is approximately 1113 ft higher than plane P'.

Building a Function Section 2.9 was devoted to setting up or constructing functions that were described or expressed in words. As emphasized in that section, this is a task that you will surely face in a course in calculus. Our final example illustrates a recommended procedure of sketching a figure and labeling quantities of interest with appropriate variables.

EXAMPLE 5 Functions That Involve Trigonometry

A plane flying horizontally at an altitude of 2 miles approaches a radar station as shown in FIGURE 4.11.6.

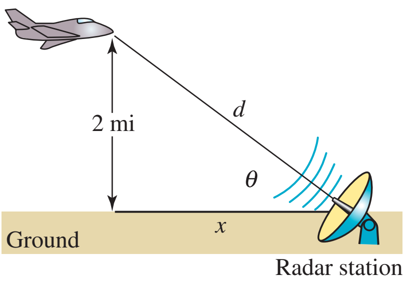


FIGURE 4.11.6 Plane in Example 5

(a) Express the distance d between the plane and the radar station as a function of the angle of elevation θ.

(b) Express the angle of elevation θ of the plane as a function of the horizontal separation x between the plane and the radar station.

Solution As shown in Figure 4.11.6, θ is an acute angle in a right triangle.

(a) We can relate the distance d and the angle θ by $\sin\theta = 2/d$. Solving for d gives

$$d(\theta) = \frac{2}{\sin\theta} \quad \text{or} \quad d(\theta) = 2\csc\theta,$$

where $0 < \theta \leq 90°$.

(b) The horizontal separation x and θ are related by $\tan\theta = 2/x$. We make use of the inverse tangent function to solve for θ:

$$\theta(x) = \tan^{-1}\frac{2}{x},$$

where $0 < x < \infty$.

Exercises 4.11 Answers to selected odd-numbered problems begin on page ANS–18.

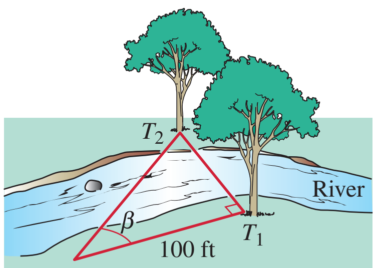


FIGURE 4.11.7 Trees and river in Problem 2

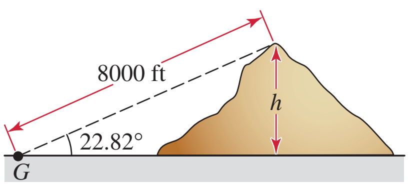


FIGURE 4.11.8 Mountain in Problem 4

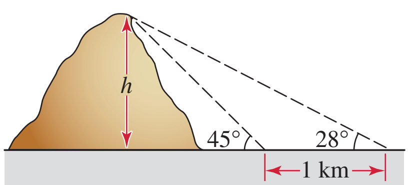


FIGURE 4.11.9 Mountain in Problem 6

1. A building casts a shadow 20 m long. If the angle from the tip of the shadow to a point on top of the building is 69°, how high is the building?
2. Two trees are on opposite sides of a river, as shown in FIGURE 4.11.7. A baseline of 100 ft is measured from tree T_1, and from that position the angle β to T_2 is measured to be 29.7°. If the baseline is perpendicular to the line segment between T_1 and T_2, find the distance between the two trees.
3. A 50-ft tower is located on the edge of a river. The angle of elevation between the opposite bank and the top of the tower is 37°. How wide is the river?
4. A surveyor uses a geodometer to measure the straight-line distance from a point on the ground to a point on top of a mountain. Use the information given in FIGURE 4.11.8 to find the height of the mountain.
5. An observer on the roof of building A measures a 27° angle of depression between the horizontal and the base of building B. The angle of elevation from the same point to the roof of the second building is 41.42°. What is the height of building B if the height of building A is 150 ft? Assume buildings A and B are on the same horizontal plane.
6. Find the height h of a mountain using the information given in FIGURE 4.11.9.
7. The top of a 20-ft ladder is leaning against the edge of the roof of a house. If the angle of inclination of the ladder from the horizontal is 51°, what is the approximate height of the house and how far is the bottom of the ladder from the base of the house?
8. An airplane flying horizontally at an altitude of 25,000 ft approaches a radar station located on a 2000-ft-high hill. At one instant in time, the angle between the radar dish pointed at the plane and the horizontal is 57°. What is the

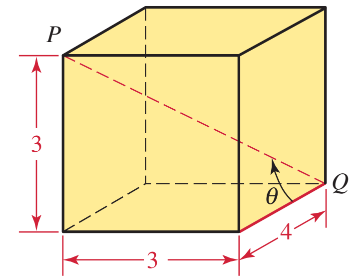


FIGURE 4.11.10 Box in Problem 10

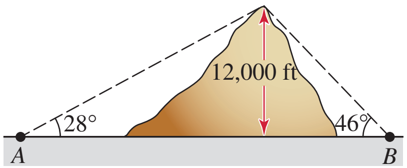


FIGURE 4.11.11 Mountain in Problem 11

straight-line distance in miles between the airplane and the radar station at that particular instant?

9. A 5-mi straight segment of a road climbs a 4000-ft hill. Determine the angle that the road makes with the horizontal.

10. A box has dimensions as shown in FIGURE 4.11.10. Find the length of the diagonal between the corners P and Q. What is the angle θ formed between the diagonal and the bottom edge of the box?

11. Observers in two towns A and B on either side of a 12,000-ft mountain measure the angles of elevation between the ground and the top of the mountain. See FIGURE 4.11.11. Assuming that the towns and the mountaintop lie in the same vertical plane, find the horizontal distance between them.

12. A drawbridge* measures 7.5 m from shore to shore, and when completely open it makes an angle of 43° with the horizontal. See FIGURE 4.11.12(a). When the bridge is closed, the angle of depression from the shore to a point on the surface of the water below the opposite end is 27°. See Figure 4.11.12(b). When the bridge is fully open, what is the distance d between the highest point of the bridge and the water below?

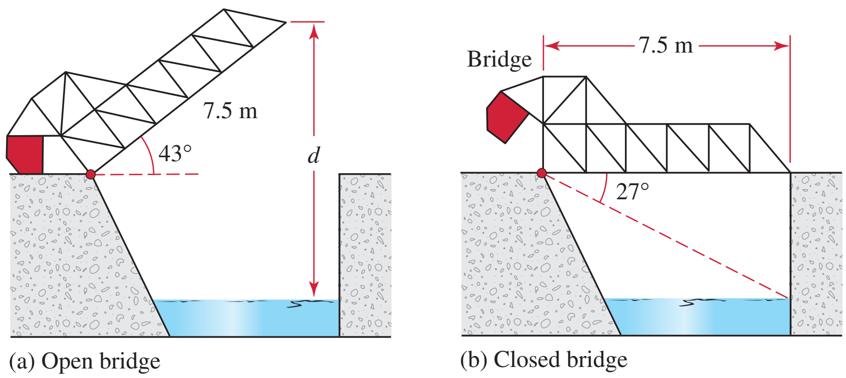


FIGURE 4.11.12 Drawbridge in Problem 12

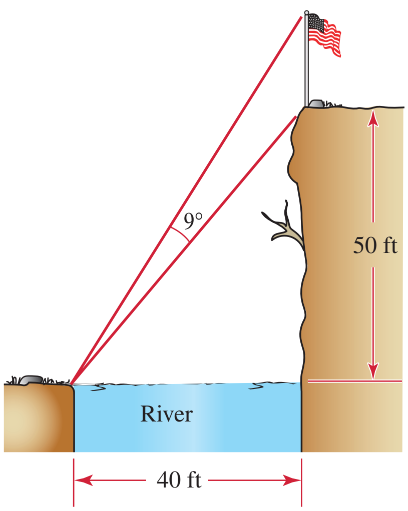
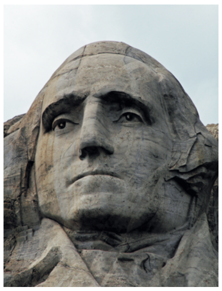


FIGURE 4.11.13 Flagpole in Problem 13

13. A flagpole is located at the edge of a sheer 50-ft cliff at the bank of a river of width 40 ft. See FIGURE 4.11.13. An observer on the opposite side of the river measures an angle of 9° between her line of sight to the top of the flagpole and her line of sight to the top of the cliff. Find the height of the flagpole.

14. From an observation site 1000 ft from the base of Mt. Rushmore the angle of elevation to the top of the sculpted head of George Washington is measured to be 80.05°, whereas the angle of elevation to the bottom of his head is 79.946°. Determine the height of George Washington's head.



Bust of George Washington on Mt. Rushmore

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15. The length of a Boeing 747 airplane is 231 ft. What is the plane's altitude if it subtends an angle of 2° when it is directly above an observer on the ground? See FIGURE 4.11.14.

16. The height of a gnomon (pin) of a sundial is 4 in. If it casts a 6-in. shadow, what is the angle of elevation of the Sun?

Sundial

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*The drawbridge shown in Figure 4.11.12, where the span is continuously balanced by a counterweight, is called a *bascule* bridge.

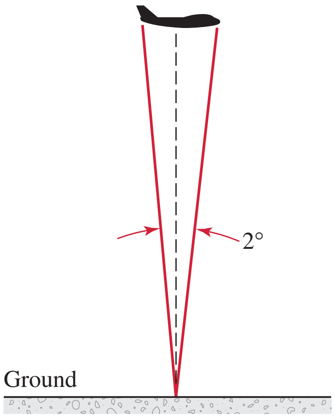


FIGURE 4.11.14 Airplane in Problem 15

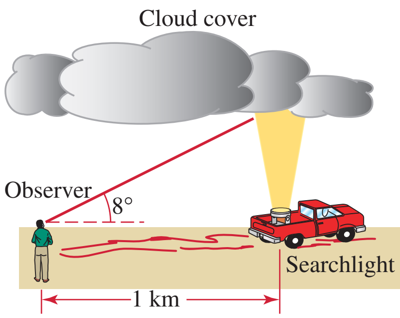


FIGURE 4.11.15 Searchlight in Problem 18

17. Weather radar is capable of measuring both the angle of elevation to the top of a thunderstorm and its range (the horizontal distance to the storm). If the range of a storm is 90 km and the angle of elevation is 4°, can a passenger plane that is able to climb to 10 km fly over the storm?

18. Cloud ceiling is the lowest altitude at which solid cloud is present. The cloud ceiling at airports must be sufficiently high for safe takeoffs and landings. At night the cloud ceiling can be determined by illuminating the base of the clouds with a searchlight pointed vertically upward. If an observer is 1 km from the searchlight and the angle of elevation to the base of the illuminated cloud is 8°, find the cloud ceiling. See FIGURE 4.11.15. (During the day cloud ceilings are generally estimated by sight. However, if an accurate reading is required, a balloon is inflated so that it will rise at a known constant rate. Then it is released and timed until it disappears into the cloud. The cloud ceiling is determined by multiplying the rate by the time of the ascent; trigonometry is not required for this calculation.)

19. On a rescue flight, a U.S. Coast Guard helicopter approaches a container ship at an altitude of 1800 ft as shown in FIGURE 4.11.16. Measured from the front the helicopter, the angle of depression of the ship's stern is 35.3°and the angle of depression of its bow is 26.6°. Approximately how long is the container ship? [*Hint*: Ignore the height of the bow and stern above sea level.]

20. From a room on the top floor of a nearby hotel, the angle of elevation to the top of the Eiffel Tower is 17.5°; from street level in front of the hotel the angle of elevation to the top of the tower is 21.4°. See FIGURE 4.11.17. What is the distance from the hotel to the midpoint M, shown in the figure, of the base of the tower? Approximately how high is the hotel building?

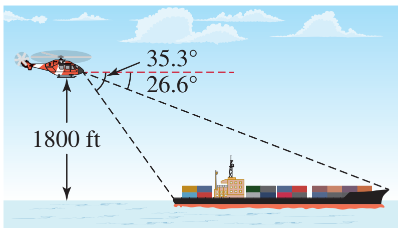


FIGURE 4.11.16 Container ship in Problem 19

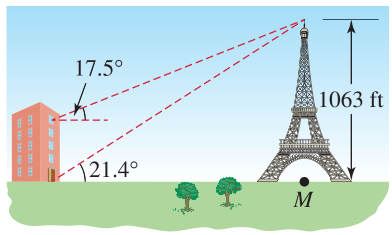


FIGURE 4.11.17 Eiffel Tower in Problem 20

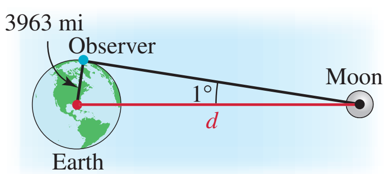


FIGURE 4.11.18 Angle in Problem 21

21. The distance between the Earth and the Moon varies as the Moon revolves around the Earth. At a particular time the **geocentric parallax** angle shown in FIGURE 4.11.18 is measured to be 1°. Calculate to the nearest hundred miles the distance between the center of the Earth and the center of the Moon at this instant. Assume that the radius of the Earth is 3963 miles.

22. The final length of a volcanic lava flow seems to decrease as the elevation of the lava vent from which it originates increases. An empirical study of Mt. Etna gives the final lava flow length L in terms of elevation h by the formula

$$L = 23 - 0.0053h,$$

where L is measured in kilometers and h is measured in meters. Suppose that a Sicilian village at elevation 750 m is on a 10° slope directly below a lava vent at 2500 m. See FIGURE 4.11.19. According to the formula, how close will the lava flow get to the village?

Mt. Etna

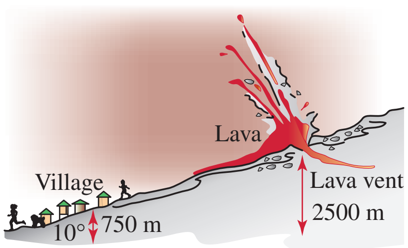


FIGURE 4.11.19 Lava flow in Problem 22

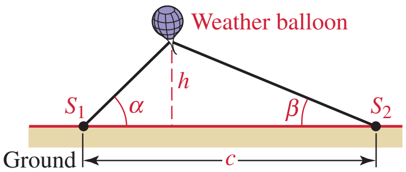


FIGURE 4.11.20 Weather balloon in Problem 23

23. As shown in **FIGURE 4.11.20**, two tracking stations S_1 and S_2 sight a weather balloon between them at elevation angles α and β, respectively. Express the height h of the balloon in terms of α and β, and the distance c between the tracking stations.

24. An entry in a soapbox derby rolls down a hill. Using the information given in **FIGURE 4.11.21**, find the total distance $d_1 + d_2$ that the soapbox travels.

25. Find the height and area of the isosceles trapezoid shown in **FIGURE 4.11.22**.

26. An escalator between the first and second floors of a department store is 58 ft long and makes an angle of 20° with the first floor. See **FIGURE 4.11.23**. Find the vertical distance between the floors.

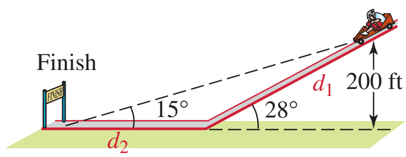


FIGURE 4.11.21 Soapbox in Problem 24

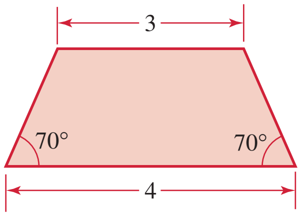


FIGURE 4.11.22 Trapezoid in Problem 25

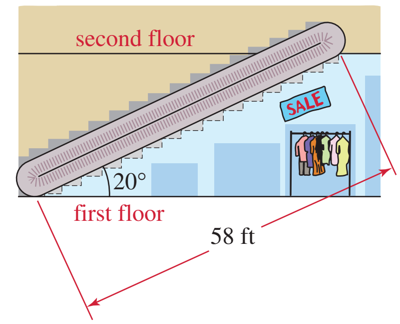


FIGURE 4.11.23 Escalator in Problem 26

27. Recent History According to the online encyclopedia *Wikipedia*, a French helicopter flown by Jean Boulet attained the world's record height of 12,442 m in 1972. What would the angle of elevation to the helicopter have been from a point P on the ground 2000 m from the point directly beneath the helicopter?

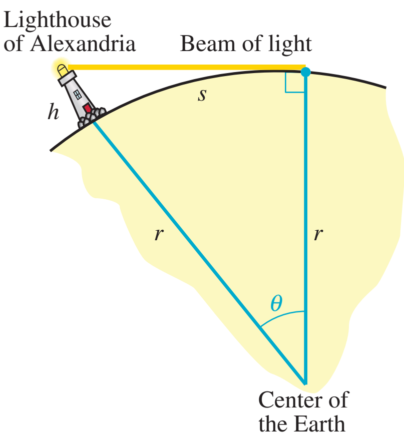


FIGURE 4.11.24 Lighthouse in Problem 28

28. Ancient History In an article from the online encyclopedia *Wikipedia*, the height h of the Lighthouse of Alexandria, one of the Seven Wonders of the Ancient World built between 280 and 247 B.C.E., is estimated to have been between 393 ft and 450 ft. The article goes on to say that there are ancient claims that the light could be seen on the ocean up to 29 miles away. Use the right triangle in **FIGURE 4.11.24** along with the two given heights h to determine the accuracy of the 29 mile claim. Assume that the radius of the Earth is $r = 3963$ mi and s is distance measured in miles on the ocean. [*Hint*: Use 1 ft $= 1/5280$ mi and (7) of Section 4.1.]

Artist's rendering of the Lighthouse of Alexandria

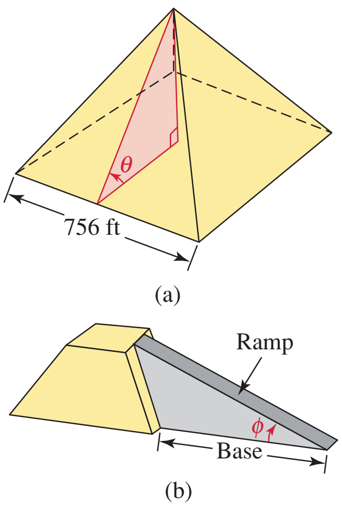


FIGURE 4.11.25 Pyramid in Problem 29

29. Really Ancient History The Great Pyramid of Giza is believed to be the tomb of the Pharaoh Khufu (AKA Cheops) and is the oldest of the Seven Wonders of the Ancient World. When completed around 2560 B.C.E. the pyramid had a square base with length 756 ft on a side and estimated height of 481 ft.

(a) Find the angle of inclination θ shown in **FIGURE 4.11.25(a)** that a side face of the pyramid makes with level ground.

The Great Pyramid of Giza

(b) In one of several theories on how the pyramid was constructed, workers pulled massive stone blocks up an inclined ramp made out of mud bricks and stone rumple. See Figure 4.11.25(b). The height and width of the ramp had to be changed as the height of the pyramid increased. Suppose at the point in time when the unfinished pyramid was 300 ft high, the base of the ramp measured 984 ft. What was the angle of inclination ϕ of the inclined ramp indicated in Figure 4.11.25(b)?

(c) What was the length of the ramp at the 300 ft stage? [*Hint*: Look at Figure 4.11.25(b) in cross section and use part (a).]

30. Medieval History The construction of the campanile, or bell tower, for the Cathedral of Pisa, Italy began in 1173 C.E. After the construction of the second floor one side of its base started sinking into the soft marshy ground. Because it continued to sink long after the completion of its construction in 1372 C.E. the campanile came to be dubbed the Leaning Tower of Pisa. Over the centuries the tower has defied many ingenious attempts to correct its tilt, but after the most recent restoration in 2001 the tower leans at angle 3.99° (corrected from 5.5°) measured from the vertical. This angle is shown in FIGURE 4.11.26 along with the heights, measured from the ground, of the high and low sides of tower belfry. Find the horizontal displacement d of the center P of the roof of the belfry from the vertical. [*Hint*: Keep in mind that the height of the point P measured from the ground is neither of the two heights given in the figure.]

Leaning Tower of Pisa

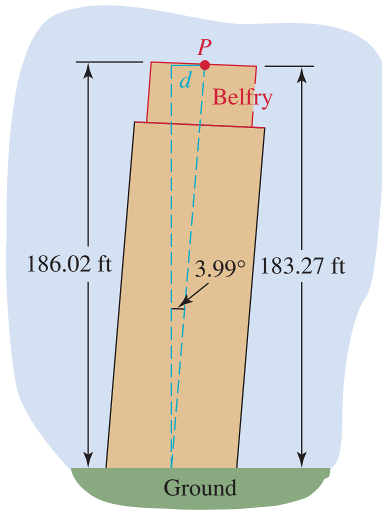


FIGURE 4.11.26 Displacement d of the point P on the belfry roof in Problem 30

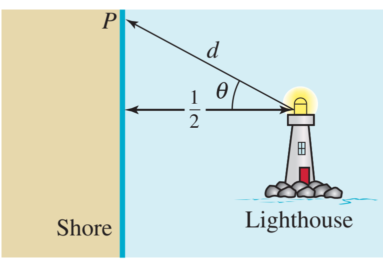


FIGURE 4.11.27 Searchlight in Problem 32

In Problems 31–34, proceed as in Example 5 and translate the words into an appropriate function.

31. A tracking telescope, located 1.25 km from the point of a rocket launch, follows a vertically ascending rocket. Express the height h of the rocket as a function of the angle of elevation θ.

32. A searchlight one-half mile offshore illuminates a point P on the shore. Express the distance d from the searchlight to the point of illumination P as a function of the angle θ shown in FIGURE 4.11.27.

33. A statue is placed on a pedestal as shown in FIGURE 4.11.28. Express the viewing angle θ as a function of the distance x from the pedestal.

34. A woman on an island wishes to reach a point R on a straight shore on the mainland from a point P on the island. The point P is 9 mi from the shore and 15 mi from point R. See FIGURE 4.11.29. If the woman rows a boat at a rate of 3 mi/h to a point Q on the mainland, then walks the rest of the way at a rate of 5 mi/h, express the total time it takes the woman to reach point R as a function of the indicated angle θ. [*Hint*: Distance = rate × time.]

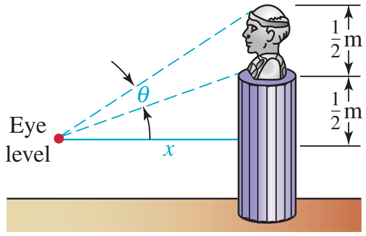


FIGURE 4.11.28 Viewing angle in Problem 33

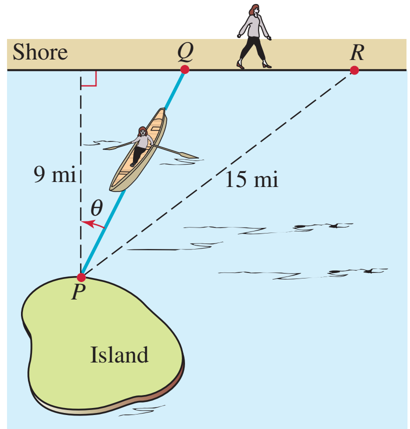


FIGURE 4.11.29 Woman rowing to shore in Problem 34

For Discussion

35. Consider the blue rectangle circumscribed around the red rectangle in FIGURE 4.11.30. With the aid of calculus it can be shown that the area of the blue rectangle is greatest when $\theta = \pi/4$. Find this area in terms of a and b.

36. Home heating oil is often stored in a right circular cylindrical tank of diameter D that rests horizontally. As FIGURE 4.11.31 shows, the depth of the oil can be measured by inserting a dipstick down a vertical diameter. If the dipstick indicates that the depth of the oil is d inches, then show that the volume V of the oil is given by

$$V = \frac{V_0}{\pi}\left[\cos^{-1}\left(1 - \frac{2d}{D}\right) - 2\left(1 - \frac{2d}{D}\right)\sqrt{\left(1 - \frac{d}{D}\right)\frac{d}{D}}\right],$$

where V_0 is the volume of the tank. Although not necessary, assume for simplicity that the tank is less than half full as shown in the figure.

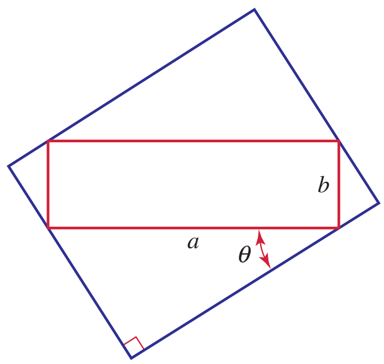


FIGURE 4.11.30 Rectangles in Problem 35

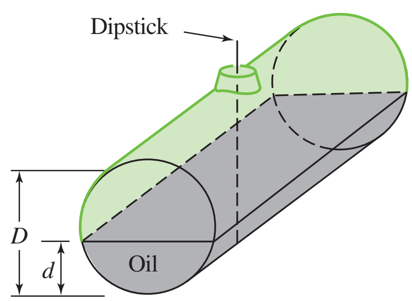


FIGURE 4.11.31 Oil tank in Problem 36

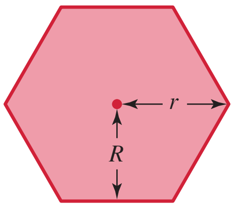


FIGURE 4.11.32 Hexagon in Problem 37

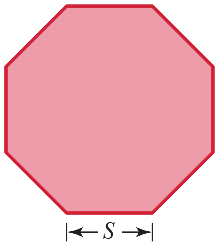


FIGURE 4.11.33 Octagon in Problem 38

37. Regular Polygon Recall, in geometry a polygon is a closed plane figure whose sides are straight-line segments. If all sides of a polygon have equal lengths and if all angles have the same measure, then the polygon is said to be a regular polygon. A regular polygon with three sides is an equilateral triangle; a four-sided regular polygon is a square. The six-sided regular polygon shown FIGURE 4.11.32 is a **hexagon**. In Figure 4.11.32 the perpendicular distance R from the center to a side of the polygon is called the *inradius* or the *apothem*. The distance r from the center to a vertex in Figure 4.11.32 is called the *circumradius* of the polygon.

(a) Use trigonometry to express the area A of a hexagon as a function of the inradius R.

(b) Express the area A of a hexagon as a function of the circumradius r.

38. Express the area A of a eight-sided regular polygon, or **octagon**, shown in FIGURE 4.11.33 as a function of the length S of one side.

39. Using trigonometric functions, generalize the three area formulas in Problems 37 and 38 to any regular polygon having n sides. [*Hint*: You might want to keep track of the number 6 in Problem 37 and the number 8 in Problem 38.]

40. Intersection of Circles–Again The centers of the red and blue circles in FIGURE 4.11.34 are B and C and have coordinates (x_0, y_0) and (x_1, y_1), respectively. Let d denote the distance between the centers.

(a) Suppose each circle has radius r. Use a triangle in Figure 4.11.34(b) to express the area A of the intersection of the circles, the yellow region in the Figure 4.11.34(a), in terms of r and θ.

(b) Show that the answer to Problem 95 in Exercises 4.1 is special case of the area formula in part (a).

(c) Find area of the intersection of the circles

$$(x - 4)^2 + (y - 4)^2 = 9 \quad \text{and} \quad (x - 6)^2 + (y - 8)^2 = 9.$$

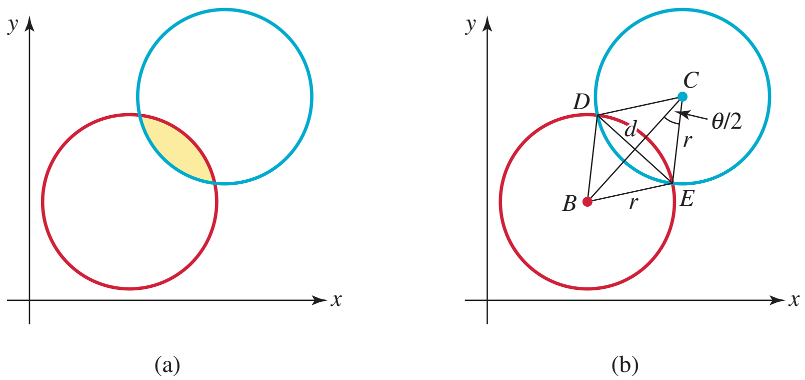


FIGURE 4.11.34 Intersecting circles in Problem 40

4.12 Law of Sines

INTRODUCTION In Section 4.10 we saw how to solve *right* triangles. In this and the next section we consider two techniques for solving an *oblique* triangle, that is, a triangle that has no right angle. An oblique triangle has either three acute angles or two acute angles and an obtuse angle.

Area of an Oblique Triangle Before introducing the principal topic of this section, let us first examine a familiar area problem. From geometry we know that the area A of any triangle is $A = \frac{1}{2}(\text{base} \times \text{height})$. In the case of an oblique triangle we must generally use trigonometry to obtain the height h. As shown in FIGURE 4.12.1(a) we begin

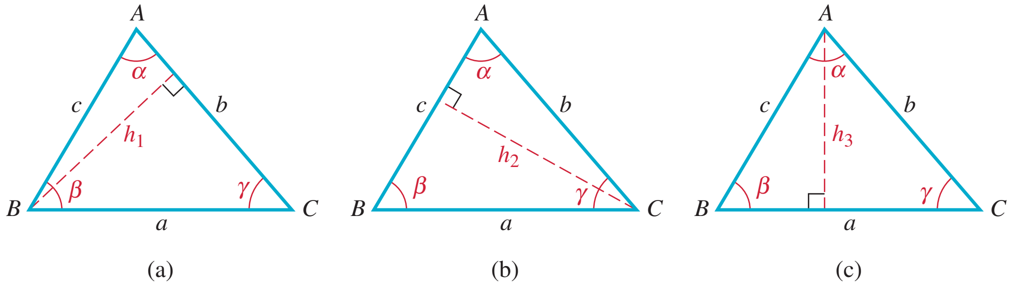


FIGURE 4.12.1 Oblique triangle with three acute angles

by constructing an altitude with length h_1 in the blue triangle from vertex B to side AC. From the right triangle whose hypotenuse has length c we see that

$$\frac{h_1}{c} = \sin\alpha \qquad \text{and so} \qquad h_1 = c\sin\alpha. \tag{1}$$

Next, in Figure 4.12.1(b) let h_2 denote the length of the altitude from vertex C to side AB. Then from the right triangle whose hypotenuse has length a we get

$$\frac{h_2}{a} = \sin\beta \qquad \text{and so} \qquad h_2 = a\sin\beta. \tag{2}$$

Finally, by constructing an altitude from vertex A to side BC we see from Figure 4.12.1(c) that its length h_3 can be expressed in terms γ:

$$\frac{h_3}{b} = \sin\gamma \qquad \text{and so} \qquad h_3 = b\sin\gamma. \tag{3}$$

Using $A = \frac{1}{2}(\text{base} \times \text{height})$ for the three heights h_1, h_2, and h_3 in (1), (2), and (3) along with the respective bases of lengths b, c, and a the area A of the oblique triangle can be expressed in terms of the lengths of two sides and the included angle:

$$A = \frac{1}{2}bc\sin\alpha, \quad A = \frac{1}{2}ac\sin\beta, \quad A = \frac{1}{2}ab\sin\gamma. \tag{4}$$

Although we used an oblique triangle in which all angles were acute, the results in (4) are equally valid for a triangle with an obtuse angle.

The formulas in (4) are certainly useful in their own right in finding area, but coincidentally we have proved a more important result.

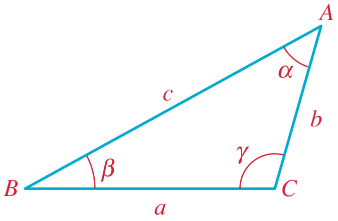


FIGURE 4.12.2 Oblique triangle

Law of Sines Consider the oblique triangle ABC, shown in FIGURE 4.12.2, with angles α, β, and γ, sides BC, AC, and AB with corresponding lengths a, b, and c. If we know the length of one side and two other parts of the triangle, that is, either

- one side and two angles (SAA or ASA), or
- two sides and an angle opposite one of the sides (SSA),

then the remaining three parts can be found using the **Law of Sines**.

THEOREM 4.12.1 The Law of Sines

Suppose angles α, β, and γ, opposite sides of length a, b, and c are as shown in Figure 4.12.2. Then

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c} \tag{5}$$

Because the three formulas in (4) give the same area we see immediately that

$$\frac{1}{2}bc \sin\alpha = \frac{1}{2}ac \sin\beta = \frac{1}{2}ab \sin\gamma.$$

By dividing each term in this double equality by $\frac{1}{2}abc$ we obtain (5).

EXAMPLE 1 Solving an Oblique Triangle (SAA)

Find the remaining parts of the triangle shown in FIGURE 4.12.3.

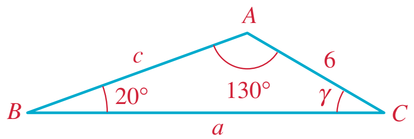


FIGURE 4.12.3 Triangle in Example 1

Solution Let $\beta = 20°$, $\alpha = 130°$, and $b = 6$. Because the sum of the angles in a triangle is 180° we have $\gamma + 20° + 130 = 180°$ and so $\gamma = 180° - 20° - 130° = 30°$. From (5) we then see that

$$\frac{\sin 130°}{a} = \frac{\sin 20°}{6} = \frac{\sin 30°}{c}. \tag{6}$$

We use the first equality in (6) to solve for a:

$$a = 6\frac{\sin 130°}{\sin 20°} \approx 13.44.$$

The second equality in (6) gives c:

$$c = 6\frac{\sin 30°}{\sin 20°} \approx 8.77.$$

EXAMPLE 2 Height of a Building (ASA)

A building is situated on the side of a hill that slopes downward at an angle of 15°. The Sun is uphill from the building at an angle of elevation of 42°. Find the building's height if it casts a shadow 36 ft long.

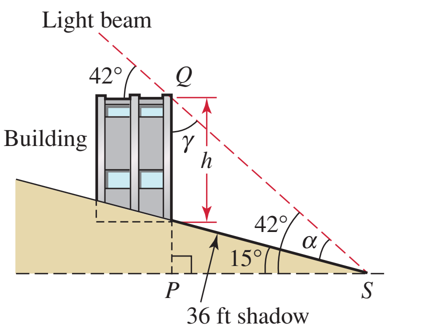


FIGURE 4.12.4 Triangle QPS in Example 2

Solution Denote the height of the building on the downward slope by h and construct a right triangle QPS as shown in FIGURE 4.12.4. Now $\alpha + 15° = 42°$ so that $\alpha = 27°$. Since ΔQPS is a right triangle, $\gamma + 42° = 90°$ gives $\gamma = 90° - 42° = 48°$. From the Law of Sines (5),

$$\frac{\sin 27°}{h} = \frac{\sin 48°}{36} \quad \text{so} \quad h = 36\frac{\sin 27°}{\sin 48°} \approx 21.99 \text{ ft}.$$

In Examples 1 and 2, where we were given *two angles and a side opposite one of these angles*, each triangle had a unique solution. However, this may not always be true for triangles where we know *two sides and an angle opposite one of these sides*. The next example illustrates the latter situation.

EXAMPLE 3 Two Triangles (SSA)

Find the remaining parts of the triangle with $\beta = 50°$, $b = 5$, and $c = 6$.

Solution From the Law of Sines, we have

$$\frac{\sin 50°}{5} = \frac{\sin\gamma}{6} \quad \text{or} \quad \sin\gamma = \frac{6}{5}\sin 50° \approx 0.9193.$$

From a calculator set in degree mode, we obtain $\gamma \approx 66.82°$. At this point it is essential to recall that the sine function is also positive for second quadrant angles. In other

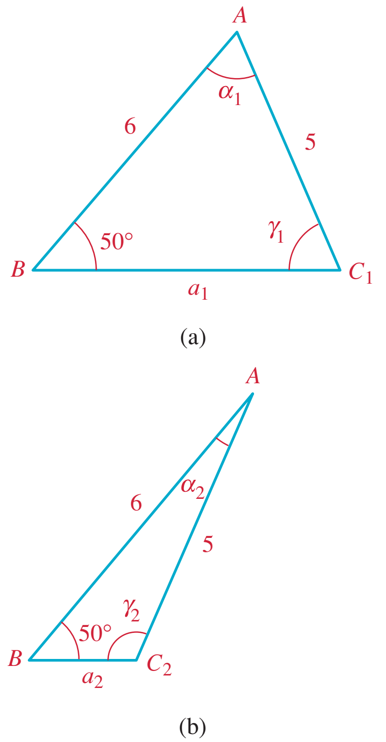


FIGURE 4.12.5 Triangles in Example 3

words, there is another angle satisfying $0° \leq \gamma \leq 180°$ for which $\sin\gamma \approx 0.9193$. Using 66.82° as a reference angle we find the second quadrant angle to be $180° - 66.82° = 113.18°$. Therefore, the two possibilities for γ are $\gamma_1 \approx 66.82°$ and $\gamma_2 \approx 113.18°$. Thus, as shown in FIGURE 4.12.5, there are two possible triangles ABC_1 and ABC_2 satisfying the given three conditions.

To complete the solution of triangle ABC_1 (Figure 4.12.5(a)), we first find $\alpha_1 = 180° - \gamma_1 - \beta$ or $\alpha_1 \approx 63.18°$. To find the side opposite this angle we use

$$\frac{\sin 63.18°}{a_1} = \frac{\sin 50°}{5} \qquad \text{which gives} \qquad a_1 = 5\left(\frac{\sin 63.18°}{\sin 50°}\right)$$

or $a_1 \approx 5.82$.

To complete the solution of triangle ABC_2 (Figure 4.12.5(b)), we find $\alpha_2 = 180° - \gamma_2 - \beta$ or $\alpha_2 \approx 16.82°$. Then from

$$\frac{\sin 16.82°}{a_2} = \frac{\sin 50°}{5} \qquad \text{we find} \qquad a_2 = 5\left(\frac{\sin 16.82°}{\sin 50°}\right)$$

or $a_2 \approx 1.89$.

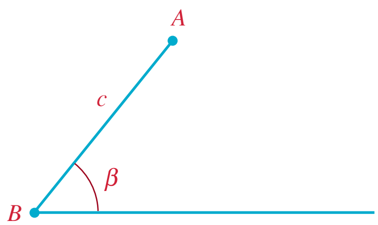
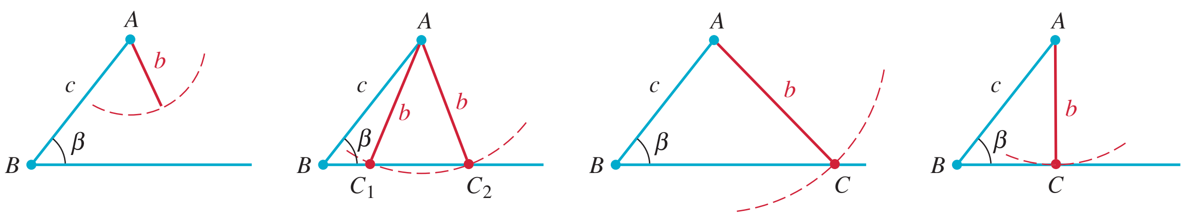


FIGURE 4.12.6 Horizontal base, the angle β, and side AB

Ambiguous Case When solving triangles, the situation where two sides and an angle opposite one of these sides are given (SSA) is called the **ambiguous case**. We have just seen in Example 3 that the given information may determine two different triangles. In the ambiguous case other complications can arise. For instance, suppose that the length of sides AB and AC (that is, c and b, respectively) and the angle β in triangle ABC are specified. As shown in FIGURE 4.12.6, we draw the angle β and mark off side AB with length c to locate the vertices A and B. The third vertex C is located on the base by drawing an arc of a circle of radius b (the length of AC) with center A. As shown in FIGURE 4.12.7, there are four possible outcomes of this construction:

- The arc does not intersect the base and no triangle is formed.
- The arc intersects the base in two distinct points C_1 and C_2 and two triangles are formed (as in Example 3).
- The arc intersects the base in one point and one triangle is formed.
- The arc is tangent to the base and a single right triangle is formed.



(a) No triangle (b) Two triangles (c) Single triangle (d) Right triangle

FIGURE 4.12.7 Solution possibilities for the ambiguous case in the Law of Sines

EXAMPLE 4 Determining the Parts of a Triangle (SSA)

Find the remaining parts of the triangle with $\beta = 40°$, $b = 5$, and $c = 9$.

Solution From the Law of Sines (1), we have

$$\frac{\sin 40°}{5} = \frac{\sin\gamma}{9} \qquad \text{and so} \qquad \sin\gamma = \frac{9}{5}\sin 40° \approx 1.1570.$$

Since the sine of any angle must be between -1 and 1, $\sin\gamma \approx 1.1570$ is impossible. This means the triangle has no solution; the side with length b is not long enough to reach the base. This is the case illustrated in Figure 4.12.7(a).

Exercises 4.12 Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–4, find the area A of the given triangle.

1.

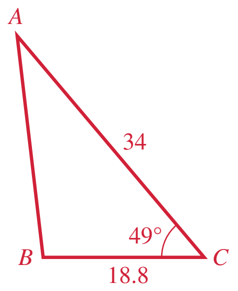


FIGURE 4.12.8 Triangle for Problem 1

2.

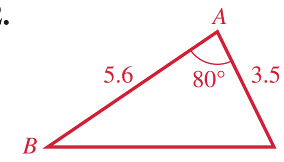


FIGURE 4.12.9 Triangle for Problem 2

3.

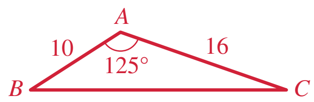


FIGURE 4.12.10 Triangle for Problem 3

4.

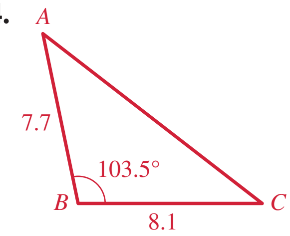


FIGURE 4.12.11 Triangle for Problem 4

In Problems 5–20, refer to Figure 4.12.2.

In Problems 5–20, use the Law of Sines to solve the triangle.

5. $\alpha = 80°, \beta = 20°, b = 7$

6. $\alpha = 60°, \beta = 15°, c = 30$

7. $\beta = 37°, \gamma = 51°, a = 5$

8. $\alpha = 30°, \gamma = 75°, a = 6$

9. $\beta = 72°, b = 12, c = 6$

10. $\alpha = 120°, a = 9, c = 4$

11. $\gamma = 62°, b = 7, c = 4$

12. $\beta = 110°, \gamma = 25°, a = 14$

13. $\gamma = 15°, a = 8, c = 5$

14. $\alpha = 55°, a = 20, c = 18$

15. $\gamma = 150°, b = 7, c = 5$

16. $\alpha = 35°, a = 9, b = 12$

17. $\beta = 30°, a = 10, b = 7$

18. $\alpha = 140°, \gamma = 20°, c = 12$

19. $\alpha = 20°, a = 8, c = 27$

20. $\alpha = 75°, \gamma = 45°, b = 8$

In Problems 21 and 22, solve for x in the given triangle.

21.

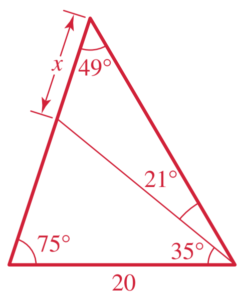


FIGURE 4.12.12 Triangle for Problem 21

22.

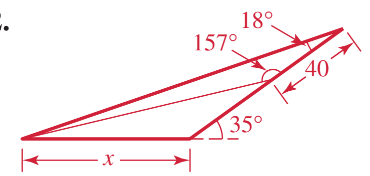


FIGURE 4.12.13 Triangle for Problem 22

Applications

23. Length of a Pool A 10-ft rope that is available to measure the length between two points A and B at opposite ends of a kidney-shaped swimming pool is not long enough. A third point C is found such that the distance from A to C is 10 ft. It is determined that angle ACB is 115° and angle ABC is 35°. Find the distance from A to B. See FIGURE 4.12.14.

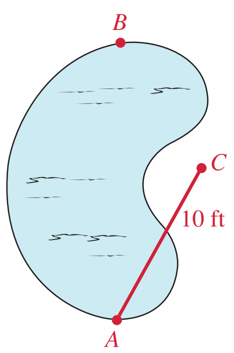


FIGURE 4.12.14 Pool in Problem 23

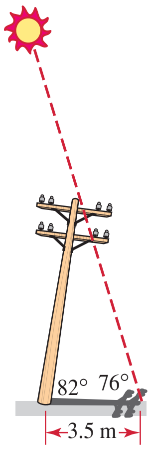


FIGURE 4.12.15 Telephone pole in Problem 25

24. Width of a River Two points A and B lie on opposite sides of a river. Another point C is located on the same side of the river as B at a distance of 230 ft from B. If angle ABC is 105° and angle ACB is 20°, find the distance across the river from A to B.

25. Length of a Telephone Pole A telephone pole makes an angle of 82° with the level ground. As shown in FIGURE 4.12.15, the angle of elevation of the Sun is 76°. Find the length of the telephone pole if its shadow is 3.5 m. (Assume that the tilt of the pole is away from the Sun and in the same plane as the pole and the Sun.)

26. Not on the Level A man 5 ft 9 in. tall stands on a sidewalk that slopes down at a constant angle. A vertical street lamp directly behind him causes his shadow to be 25 ft long. The angle of depression from the top of the man to the tip of his shadow is 31°. Find the angle α, as shown in FIGURE 4.12.16, that the sidewalk makes with the horizontal.

27. How High? If the man in Problem 26 is 20 ft down the sidewalk from the street lamp, find the height of the light above the sidewalk.

28. Plane with an Altitude Angles of elevation to an airplane are measured from the top and the base of a building that is 20 m tall. The angle from the top of the building is 38°, and the angle from the base of the building is 40°. Find the altitude of the airplane.

29. Angle of Drive The distance from the tee to the green on a particular golf hole is 370 yd. A golfer hits his drive and paces its distance off at 210 yd. From the point where the ball lies, he measures an angle of 160° between the tee and the green. Find the angle of his drive off the tee measured from the dashed line from the tee to the green shown in FIGURE 4.12.17.

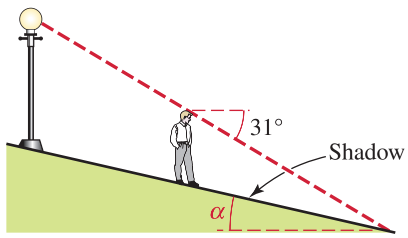


FIGURE 4.12.16 Sloping sidewalk in Problem 26

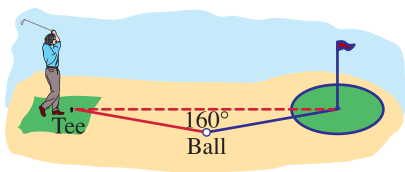


FIGURE 4.12.17 Angle of drive in Problem 29

30. In Problem 29, what is the distance from the ball to the green?

31. Help! One Coast Guard vessel is located 4 nautical miles due south of a second Coast Guard vessel when they receive a distress signal from a sailboat. To offer assistance, the first vessel sails on a bearing of S50°E at 5 knots and the second vessel sails S10°E at 10 knots. Which one of the Coast Guard vessels reaches the sailboat first? [*Hint*: The concept of bearing is reviewed on page 306.]

4.13 Law of Cosines

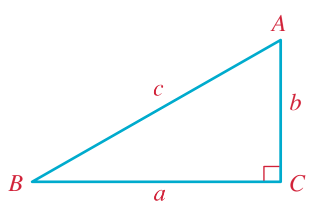


FIGURE 4.13.1 Right triangle

INTRODUCTION In the right triangle shown in FIGURE 4.13.1, the length c of the hypotenuse is related to the lengths a and b of the other two sides by the Pythagorean theorem

$$c^2 = a^2 + b^2. \tag{1}$$

In this section we will see that (1) is just a special case of a general formula that relates the lengths of the sides of *any* triangle.

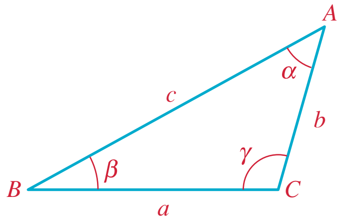


FIGURE 4.13.2 Oblique triangle

Law of Cosines An oblique triangle, such as that in FIGURE 4.13.2, for which we know either

- three sides (SSS), or
- two sides and the included angle (that is, the angle formed by the given sides) (SAS),

cannot be solved directly using the Law of Sines. The **Law of Cosines** that we consider next can be used to solve triangles in these two cases. Like the Law of Sines, (5) of Section 4.12, the Law of Cosines is valid for any oblique triangle, but for convenience we prove the last two equations in (2) using a triangle in which the angles α, β, and γ, are acute.

THEOREM 4.13.1 The Law of Cosines

Suppose angles α, β, and γ, and opposite sides of length a, b, and c are as shown in Figure 4.13.2. Then

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos\alpha \\ b^2 &= a^2 + c^2 - 2ac\cos\beta \\ c^2 &= a^2 + b^2 - 2ab\cos\gamma \end{aligned} \tag{2}$$

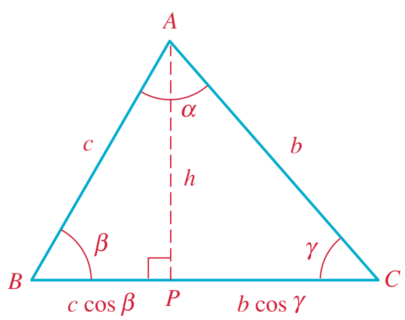


FIGURE 4.13.3 Acute triangle

PROOF: Let P denote the point where the altitude from the vertex A intersects side BC. Then, since both $\triangle BPA$ and $\triangle CPA$ in FIGURE 4.13.3 are right triangles we have from (1),

$$c^2 = h^2 + (c\cos\beta)^2 \tag{3}$$

and

$$b^2 = h^2 + (b\cos\gamma)^2. \tag{4}$$

Now the length of BC is $a = c\cos\beta + b\cos\gamma$ so that

$$c\cos\beta = a - b\cos\gamma. \tag{5}$$

Moreover, from (4),

$$h^2 = b^2 - (b\cos\gamma)^2. \tag{6}$$

Substituting (5) and (6) into (3) and simplifying yields the third equation in (2):

$$\begin{aligned} c^2 &= b^2 - (b\cos\gamma)^2 + (a - b\cos\gamma)^2 \\ &= b^2 - b^2\cos^2\gamma + a^2 - 2ab\cos\gamma + b^2\cos^2\gamma \end{aligned}$$

or

$$c^2 = a^2 + b^2 - 2ab\cos\gamma. \tag{7}$$

Note that equation (7) reduces to the Pythagorean theorem (1) when $\gamma = 90°$.

Similarly, if we use $b\cos\gamma = a - c\cos\beta$ and $h^2 = c^2 - (c\cos\beta)^2$ to eliminate $b\cos\gamma$ and h^2 in (4), we obtain the second equation in (2). ∎

EXAMPLE 1 Solving an Oblique Triangle (SAS)

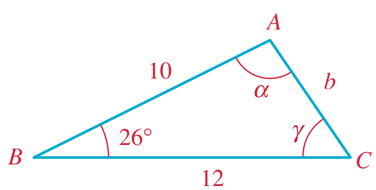


FIGURE 4.13.4 Triangle in Example 1

Find the remaining parts of the triangle shown in FIGURE 4.13.4.

Solution First, if we call the unknown side b and identify $a = 12$, $c = 10$, and $\beta = 26°$, then from the second equation in (2) we can write

$$b^2 = (12)^2 + (10)^2 - 2(12)(10)\cos 26°.$$

Therefore, $b^2 \approx 28.2894$ and so $b \approx 5.32$.

Next, we use the Law of Cosines to determine the remaining angles in the triangle in Figure 4.13.4. If γ is the angle at the vertex C, then the third equation in (2) gives

$$10^2 = 12^2 + (5.32)^2 - 2(12)(5.32)\cos\gamma \qquad \text{or} \qquad \cos\gamma \approx 0.5663.$$

With the aid of a calculator and the inverse cosine we find $\gamma \approx 55.51°$. Note that since the cosine of an angle between 90° and 180° is negative, there is no need to consider two possibilities as we did in Example 3 in Section 4.12. Finally, the angle at the vertex A is $\alpha = 180° - \beta - \gamma$ or $\alpha \approx 98.49°$.

In Example 1, observe that after b is found, we know two sides and an angle opposite one of these sides. Hence we could have used the Law of Sines to find the angle γ.

In the next example we consider the case in which the lengths of the three sides of a triangle are given.

EXAMPLE 2 Determining the Angles in a Triangle (SSS)

Find the angles α, β, and γ in the triangle shown in FIGURE 4.13.5.

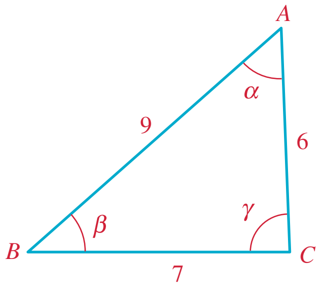


FIGURE 4.13.5 Triangle in Example 2

Solution We use the Law of Cosines to find the angle opposite the longest side:

$$9^2 = 6^2 + 7^2 - 2(6)(7)\cos\gamma \qquad \text{or} \qquad \cos\gamma = \tfrac{1}{21}.$$

A calculator then gives $\gamma \approx 87.27°$. Although we could use the Law of Cosines, we choose to find β by the Law of Sines:

$$\frac{\sin\beta}{6} = \frac{\sin 87.27°}{9} \qquad \text{or} \qquad \sin\beta = \frac{6}{9}\sin 87.27° \approx 0.6659.$$

Since γ is the angle opposite the longest side it is the largest angle in the triangle, so β must be an acute angle. Thus, $\sin\beta \approx 0.6659$ yields $\beta \approx 41.75°$. Finally, from $\alpha = 180° - \beta - \gamma$ we find $\alpha \approx 50.98°$.

Bearing In navigation directions are given using bearings. A **bearing** designates the acute angle that a line makes with the north–south line. For example, FIGURE 4.13.6(a) illustrates a bearing of S40°W, meaning 40 degrees west of south. The bearings in Figures 4.13.6(b) and 4.13.6(c) are N65°E and S80°E, respectively.

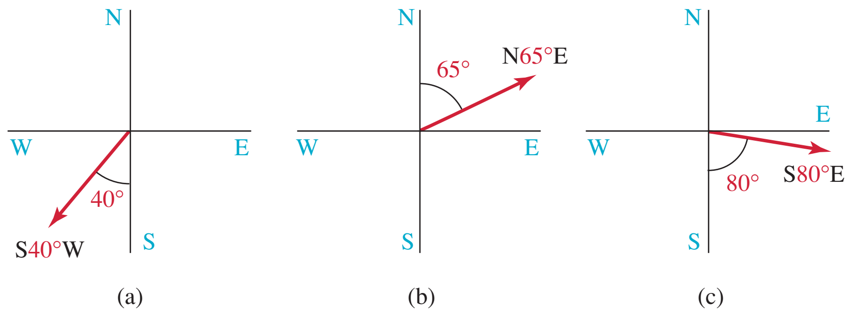


FIGURE 4.13.6 Three examples of bearings

EXAMPLE 3 Bearings of Two Ships (SAS)

Two ships leave a port at 7:00 AM, one traveling at 12 knots (nautical miles per hour) and the other at 10 knots. If the faster ship maintains a bearing of N47°W and the other ship maintains a bearing of S20°W, what is their separation (to the nearest nautical mile) at 11:00 AM that day?

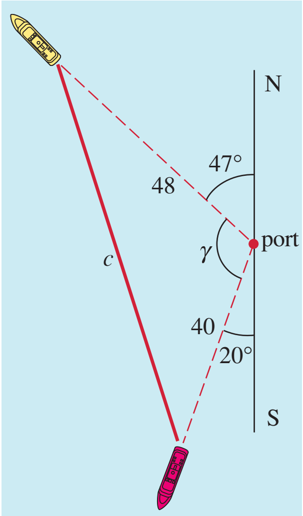


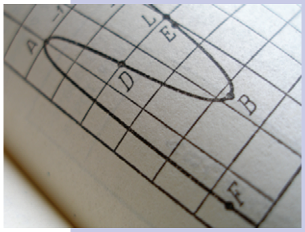
FIGURE 4.13.7 Ships in Example 3

Solution Since the elapsed time is 4 hours, the faster ship has traveled $4 \cdot 12 = 48$ nautical miles from port and the slower ship $4 \cdot 10 = 40$ nautical miles. Using these distances and the given bearings, we can sketch the triangle (valid at 11:00 AM) shown in FIGURE 4.13.7. In the triangle, c denotes the distance separating the ships and γ is the angle opposite that side. Since $47° + \gamma + 20° = 180°$ we find $\gamma = 113°$. Finally, the Law of Cosines

$$c^2 = 48^2 + 40^2 - 2(48)(40)\cos 113°$$

gives $c^2 \approx 5404.41$ or $c \approx 73.51$. Thus the distance between the ships (to the nearest nautical mile) is 74 nautical miles.

NOTES FROM THE CLASSROOM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(*i*) An important first step in solving a triangle is determining which of the three approaches we have discussed to use: right triangle trigonometry, the Law of Sines, or the Law of Cosines. The following table describes the various types of problems and gives the most appropriate approach for each.

Type of Triangle	Information Given	Technique
Right	Two sides or an angle and a side	Basic definitions of sine, cosine, and tangent; the Pythagorean theorem
Oblique	Two angles and a side (SAA or ASA)	Law of Sines
Oblique	Two sides and an angle opposite one of the sides (SSA)	Law of Sines (if the given angle is acute, it is an ambiguous case)
Oblique	Three sides (SSS)	Law of Cosines
Oblique	Two sides and the included angle (SAS)	Law of Cosines

(*ii*) Here are some additional bits of advice for solving triangles.

- Students will frequently use the Law of Sines when a right triangle trigonometric function could have been used. A right triangle approach is the simplest and most efficient.
- In applying the Law of Sines, if you obtain a value greater than 1 for the sine of an angle, there is no solution.
- In the ambiguous case of the Law of Sines, when solving for the first unknown angle, you must consider *both the acute angle found from your calculator and its supplement as possible solutions.* The supplement will be a solution if the sum of the supplement and the angle given in the triangle is less than 180°.
- When three sides are given, check first to see whether the length of the longest side is greater than or equal to the sum of the lengths of the other two sides. If it is, there can be no solution (even though the given information indicates a Law of Cosines approach). This is because the shortest distance between two points is the length of the line segment joining them.

Exercises 4.13 Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–16, refer to Figure 4.13.2.

In Problems 1–16, use the Law of Cosines to solve the triangle.

1. $\gamma = 65°, a = 5, b = 8$
2. $\beta = 48°, a = 7, c = 6$
3. $a = 8, b = 10, c = 7$
4. $\gamma = 31.5°, a = 4, b = 8$
5. $\gamma = 97.33°, a = 3, b = 6$
6. $a = 7, b = 9, c = 4$
7. $a = 11, b = 9.5, c = 8.2$
8. $\alpha = 162°, b = 11, c = 8$
9. $a = 5, b = 7, c = 10$
10. $a = 6, b = 5, c = 7$
11. $a = 3, b = 4, c = 5$
12. $a = 5, b = 12, c = 13$
13. $a = 6, b = 8, c = 12$
14. $\beta = 130°, a = 4, c = 7$
15. $\alpha = 22°, b = 3, c = 9$
16. $\beta = 100°, a = 22.3, b = 16.1$

Applications

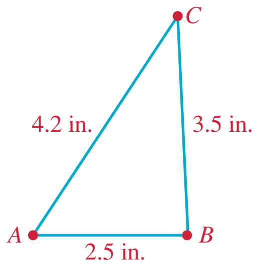

FIGURE 4.13.8 Triangle in Problem 19

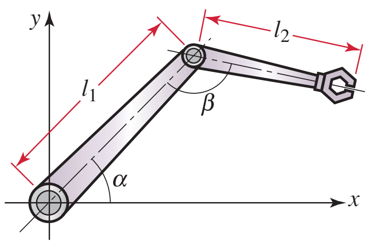

FIGURE 4.13.9 Robotic arm in Problem 21

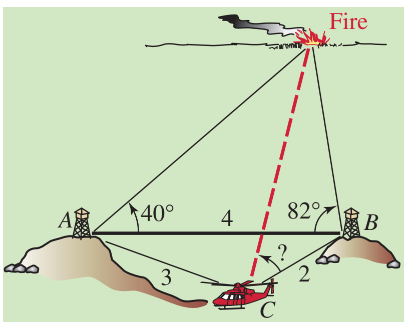

FIGURE 4.13.10 Fire in Problem 22

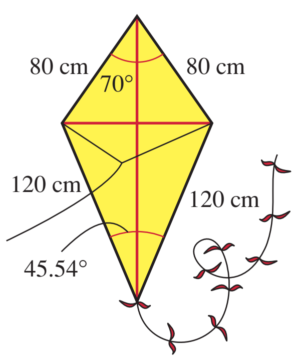

FIGURE 4.13.11 Kite in Problem 23

17. **How Far?** A ship sails due west from a harbor for 22 nautical miles. It then sails S62°W for another 15 nautical miles. How far is the ship from the harbor?
18. **How Far Apart?** Two hikers leave their camp simultaneously, taking bearings of N42°W and S20°E, respectively. If they each average a rate of 5 km/h, how far apart are they after 1 h?
19. **Bearings** On a hiker's map point A is 2.5 in. due west of point B and point C is 3.5 in. from B and 4.2 in. from A, respectively. See FIGURE 4.13.8. Find **(a)** the bearing of A from C, and **(b)** the bearing of B from C.
20. **How Long Will It Take?** Two ships leave port simultaneously, one traveling at 15 knots and the other at 12 knots. They maintain bearings of S42°W and S10°E, respectively. After 3 h the first ship runs aground and the second ship immediately goes to its aid.
    (a) How long will it take the second ship to reach the first ship if it travels at 14 knots?
    (b) What bearing should it take?
21. **A Robotic Arm** A two-dimensional robot arm "knows" where it is by keeping track of a "shoulder" angle α and an "elbow" angle β. As shown in FIGURE 4.13.9, this arm has a fixed point of rotation at the origin. The shoulder angle is measured counterclockwise from the x-axis, and the elbow angle is measured counterclockwise from the upper to the lower arm. Suppose that the upper and lower arms are both of length 2 and that the elbow angle β is prevented from "hyperextending" beyond 180°. Find the angles α and β that will position the robot's hand at the point (1, 2).
22. **Which Way?** Two lookout towers are situated on mountain tops A and B, 4 mi from each other. A helicopter firefighting team is located in a valley at point C, 3 mi from A and 2 mi from B. Using the line between A and B as a reference, a lookout spots a fire at an angle of 40° from tower A and 82° from tower B. See FIGURE 4.13.10. At what angle, measured from CB, should the helicopter fly in order to head directly for the fire?
23. **Making a Kite** For the kite shown in FIGURE 4.13.11, use the Law of Cosines to find the lengths of the two dowels required for the diagonal supports.
24. **Bermuda Triangle** The Bermuda Triangle, also known as the Devil's Triangle, is a triangular patch of the Atlantic Ocean where it is claimed by some to be a place of paranormal activities. This claim is bolstered by the fact that, over the years, many planes and ships have disappeared without a trace within this region. The three vertices of the triangle are generally taken to be at Miami (Florida), San Juan (Puerto Rico), and the islands of Bermuda.

FIGURE 4.13.12 Bermuda Triangle in Problem 24

See FIGURE 4.13.12. The distance from Miami to Bermuda is 1035 mi, the distance from Bermuda to San Juan is 962 mi, and the distance from San Juan to Miami is 1033 mi.

(a) Find the three acute angles in the triangle.

(b) Determine the approximate area of the triangle.

25. Playing Hardball **(a)** A professional baseball diamond is a square 90 feet on a side with a base at each vertex. The rubber (or plate) on the pitcher's mound is 60.5 ft from home base on the line between home base and second base. See FIGURE 4.13.13. What is the distance s between the pitching rubber and first base?

(b) In softball (where the ball is *not* soft), the dimensions of the diamond vary according to age and gender of the players and whether it is fast pitch or slow pitch. What is the distance s for fast-pitch women's college softball where the diamond is 60 ft on a side and the distance from home base to the pitching rubber is 43 ft?

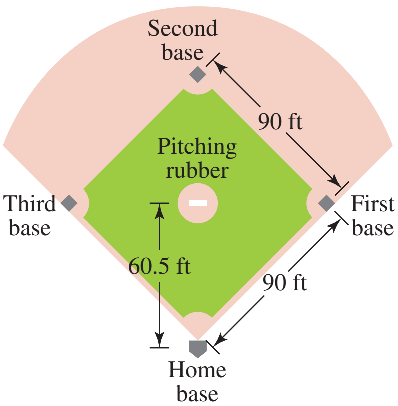

FIGURE 4.13.13 Baseball diamond in Problem 25

26. On the Clock Suppose the lengths of the minute and hour hands of an analog clock are 6 inches and 4.5 inches, respectively. Find the distance d between the tips of the minute and hour hands at 4 PM.

For Discussion

27. Heron's Formula Use the Law of Cosines to derive the formula

$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

for the area of a triangle with sides a, b, c where $s = \frac{1}{2}(a + b + c)$. This formula is named after the Greek mathematician and inventor **Heron of Alexandria** (c. 20–62 C.E.) but should actually be credited to Archimedes.

28. Garden Plot Use Heron's formula in Problem 27 to find the area of a triangular garden plot if the lengths of the three sides are 25, 32, and 41 m, respectively.

29. Corner Lot Find the area of the irregular corner lot shown in FIGURE 4.13.14. [*Hint*: Divide the lot into two triangular lots as shown and then find the area of each triangle. Use Heron's formula in Problem 27 for the area of the acute triangle.]

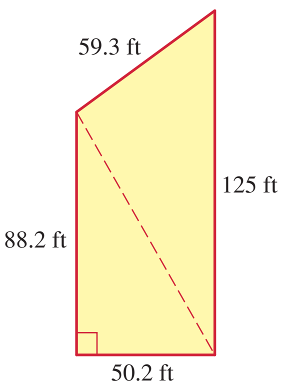

FIGURE 4.13.14 Corner lot in Problem 29

30. Use Heron's formula in Problem 27 to find the area of a triangle with vertices located at $(3, 2)$, $(-3, -6)$, and $(0, 6)$ in a rectangular coordinate system.

31. Blue Man The effort in climbing a flight of stairs depends largely on the flexing angle of the leading knee. A simplified blue stick-figure model of a person walking up a staircase indicates that the maximum flexing of the knee occurs

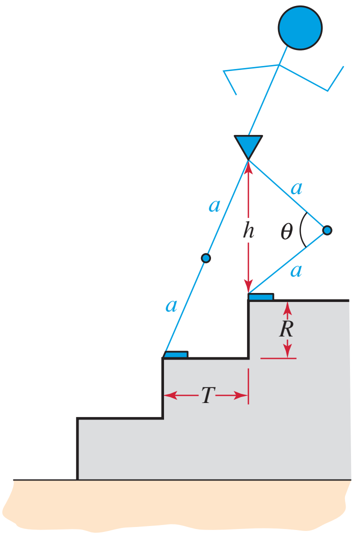

FIGURE 4.13.15 Blue man in Problem 31

when the back leg is straight and the hips are directly over the heel of the front foot. See FIGURE 4.13.15. Show that

$$\cos\theta = \left(\frac{R}{a}\right)\sqrt{4 - \left(\frac{T}{a}\right)^2} + \frac{(T/a)^2 - (R/a)^2}{2} - 1,$$

where θ is the knee joint angle, $2a$ is the length of the leg, R is the rise of a single stair step, and T is the width of a step. [*Hint*: Let h be the vertical distance from hip to heel of the leading leg, as shown in the figure. Set up two equations involving h: one by applying the Pythagorean theorem to the right triangle outlined in color and the other by using the Law of Cosines on the angle θ. Then eliminate h and solve for $\cos\theta$.]

32. For a triangle with sides of lengths a, b, and c and γ is the angle opposite c, we have seen on page 305 that when γ is a right angle the Law of Cosines reduces to the Pythagorean theorem $c^2 = a^2 + b^2$. How is c^2 related to $a^2 + b^2$ when

(a) γ is an acute angle **(b)** γ is an obtuse angle?

33. Intersection of Circles–Finale The centers of the red and blue circles in FIGURE 4.13.16 are B and C and have coordinates (x_0, y_0) and (x_1, y_1), respectively. Let d denote the distance between the centers. In Problem 95 in Exercises 4.1 and in Problem 40 in Exercises 4.11 you were asked to find the area of the intersection of two circles when each circle had the same radius r. Now suppose that the radii of the circles are r_0 and r_1, $r_0 \neq r_1$.

(a) Use a triangle in Figure 4.13.16 to find a general formula for the area of the intersection of the circles in terms of r_0, r_1, θ, and ϕ.

(b) Find area of the intersection of the circles

$$(x - 4)^2 + (y - 5)^2 = 4 \quad \text{and} \quad (x - 6)^2 + (y - 8)^2 = 9.$$

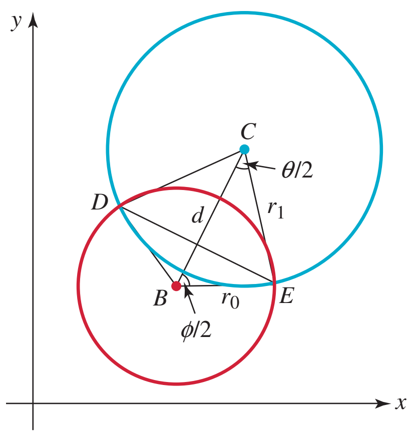

FIGURE 4.13.16 Intersecting circles in Problem 33

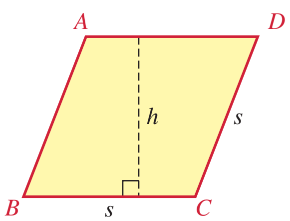

FIGURE 4.13.17 Rhombus in Problem 34

34. Rhombus A rhombus is a parallelogram in which the four sides have the same length s. See FIGURE 4.13.17. Since there is no agreement on whether a square is a rhombus, for purposes of this problem we will exclude it. In a rhombus opposite angles are equal—acute angles at vertices B and D and obtuse angles at A and B. The acute and obtuse angles are supplementary.

(a) Use geometry to show that the area of a rhombus is the product of the base times the height, that is, $A = sh$.

(b) Use trigonometry to show that the area of a rhombus is $A = s^2 \sin\theta$, where θ is any of the four interior angles. Explain why this area formula is valid for any vertex angle in the rhombus.

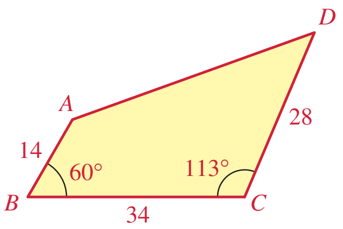

FIGURE 4.13.18 Quadrilateral in Problem 35

(c) Show that the area of a rhombus is $A = \frac{1}{2}d_1d_2$, where d_1 and d_2 are the lengths of the line segments, or diagonals, AC and BD.

(d) Use geometry to show the diagonals connecting opposite vertices bisect the vertex angles.

(e) Use trigonometry to show that the diagonal lines are perpendicular. [*Hint*: See Problem 76 in Exercises 4.7.]

35. Quadrilateral A quadrilateral (also called a quadrangle) is any polygon with four sides. The rhombus discussed in Problem 34 is a quadrilateral. Discuss how to find the area of the quadrilateral given in FIGURE 4.13.18. Carry out your ideas.

4.14 The Limit Concept Revisited

∫Calculus PREVIEW

INTRODUCTION As we saw in Section 2.10, the fundamental motivating problem of differential calculus, *find a tangent line to the graph of the function*, is answered by the concept of a *limit*. In that section we purposely kept the discussion about limits at an intuitive level; our emphasis was on reviewing the appropriate algebra, such as factoring and rationalization, necessary to be able to compute a limit analytically. In the study of the calculus of the trigonometric functions you will, of course, be expected to compute limits involving trigonometric functions. As the examples in this section will illustrate, computation of trigonometric limits entail both algebraic manipulations and knowledge of basic trigonometric identities.

We begin with a fundamental limit result for the sine function.

An Important Trigonometric Limit

To do the calculus of the trigonometric functions, $\sin x$, $\cos x$, $\tan x$, and so on, it is important to realize that the variable x is a real number or an angle x measured in radians. With that in mind, consider the numerical values of $(\sin x)/x$ as x approaches 0 from the right ($x \to 0^+$) given in the table that follows.

$x \to 0^+$	0.1	0.01	0.001	0.0001
$\dfrac{\sin x}{x}$	0.99833416	0.99998333	0.99999983	0.99999999

It is easy to see that the same results given in the table hold as $x \to 0^-$. Because $\sin x$ is an odd function, for $x > 0$ and $-x < 0$ we have $\sin(-x) = -\sin x$ and as a consequence $\dfrac{\sin(-x)}{-x} = \dfrac{\sin x}{x}$. In other words, when the value of x is small in absolute value

$$\frac{\sin x}{x} \approx 1.$$

While numerical calculations such as this do not constitute a proof, they do suggest that $\dfrac{\sin x}{x} \to 1$ as $x \to 0$. Using the limit symbol, we have motivated the following result

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{1}$$

Problem 40 in Exercises 4.14 gives a guided tour through the basic steps of a proof of (1) that is usually presented in calculus.

Important ▸ In this discussion we make the same assumption that we did in Sections 1.5 and 2.10, namely, that all limits under consideration actually exist. Everything that we do, algebraic manipulations, taking limits of products and quotients in the examples in this section is predicated on this assumption.

Other limits of importance are

$$\lim_{x \to a} \sin x = \sin a, \tag{2}$$

$$\lim_{x \to a} \cos x = \cos a. \tag{3}$$

The results (2) and (3) are immediate consequences of the fact that $f(x) = \sin x$ and $g(x) = \cos x$ are continuous functions for all x. As we have seen in Section 4.3 the graphs of $\sin x$ and $\cos x$ are smooth and unbroken. For example, from (2),

$$\lim_{x \to \pi/6} \sin x = \sin\frac{\pi}{6} = \frac{1}{2}$$

and

$$\lim_{x \to 0} \sin x = \sin 0 = 0. \tag{4}$$

Also, from (3),

$$\lim_{x \to 0} \cos x = \cos 0 = 1. \tag{5}$$

The results in (1), (2), and (3) are used often to compute other limits. As in Section 1.5 many of the limits considered in this section are limits of fractional expressions where *both* the numerator and the denominator are approaching 0. Recall, these kinds of limits are said to have the **indeterminate form** 0/0. Note that the limit (1) is of this indeterminate form.

EXAMPLE 1 Using (1)

Find $\lim_{x \to 0} \dfrac{10x - 3\sin x}{x}$.

Solution We rewrite the fractional expression as two fractions with the same denominator x:

$$\begin{aligned}
\lim_{x \to 0} \frac{10x - 3\sin x}{x} &= \lim_{x \to 0} \left[\frac{10x}{x} - \frac{3\sin x}{x}\right] \\
&= \lim_{x \to 0} \frac{10x}{x} - 3\lim_{x \to 0} \frac{\sin x}{x} && \leftarrow \text{cancel the } x \text{ in the first expression} \\
&= \lim_{x \to 0} 10 - 3\lim_{x \to 0} \frac{\sin x}{x} && \leftarrow \text{now use (1)} \\
&= 10 - 3 \cdot 1 \\
&= 7.
\end{aligned}$$

EXAMPLE 2 Using the Double-Angle Formula

Find $\lim_{x \to 0} \dfrac{\sin 2x}{x}$.

Solution To evaluate the given limit, we make use of the double-angle formula $\sin 2x = 2\sin x \cos x$ of Section 4.5 and the results in (1) and (5):

$$\lim_{x \to 0} \frac{\sin 2x}{x} = \lim_{x \to 0} \frac{2\cos x \sin x}{x} = 2\lim_{x \to 0} \cos x \cdot \frac{\sin x}{x} = 2 \cdot \underset{\text{from (5)}}{1} \cdot \underset{\text{from (1)}}{1} = 2.$$

Thus,

$$\lim_{x \to 0} \frac{\sin 2x}{x} = 2. \tag{6}$$

Using a Substitution We are often interested in limits similar to that considered in Example 2. But if we wish to find, say, $\lim_{x \to 0} \frac{\sin 5x}{x}$ the procedure employed in Example 2 breaks down at a practical level since we have not developed a trigonometric identity for $\sin 5x$. There is an alternative procedure that allows us to quickly find $\lim_{x \to 0} \frac{\sin kx}{x}$, where $k \neq 0$ is any real constant, by simply changing the variable by means of a **substitution**. If we let $t = kx$, then $x = t/k$. Notice that as $x \to 0$ then necessarily $t \to 0$. Thus we can write

$$\lim_{x \to 0} \frac{\sin kx}{x} = \lim_{t \to 0} \frac{\sin t}{t/k} = \lim_{t \to 0} \frac{\sin t}{1} \cdot \frac{k}{t} = k \underbrace{\lim_{t \to 0} \frac{\sin t}{t}}_{\text{this limit is 1 from (1)}} = k.$$

Thus we have proved the general result

$$\lim_{x \to 0} \frac{\sin kx}{x} = k. \tag{7}$$

Hence $\lim_{x \to 0} \frac{\sin 5x}{x} = 5$. See Problem 25 in Exercises 4.14.

EXAMPLE 3 Trigonometric Limit

Find $\lim_{x \to 0} \frac{\tan x}{x}$.

Solution Using the definition $\tan x = \sin x/\cos x$ we can write

$$\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \frac{1}{\cos x} \cdot \frac{\sin x}{x} = \lim_{x \to 0} \frac{1}{\cos x} \cdot \frac{\sin x}{x}.$$

From (5) and (1) we know that $\cos x \to 1$ and $(\sin x)/x \to 1$ as $x \to 0$, and so the preceding line becomes

$$\lim_{x \to 0} \frac{\tan x}{x} = \frac{1}{1} \cdot 1 = 1.$$

EXAMPLE 4 Using a Pythagorean Identity

Find $\lim_{x \to 0} \frac{1 - \cos x}{x}$.

Solution To compute this limit we start with a bit of algebraic cleverness by multiplying the numerator and denominator by the conjugate factor of the numerator. Next we use the fundamental Pythagorean identity $\sin^2 x + \cos^2 x = 1$ in the form $1 - \cos^2 x = \sin^2 x$:

$$\begin{aligned}
\lim_{x \to 0} \frac{1 - \cos x}{x} &= \lim_{x \to 0} \frac{1 - \cos x}{x} \cdot \frac{1 + \cos x}{1 + \cos x} \\
&= \lim_{x \to 0} \frac{1 - \cos^2 x}{x(1 + \cos x)} \\
&= \lim_{x \to 0} \frac{\sin^2 x}{x(1 + \cos x)}.
\end{aligned}$$

For the next step we resort back to algebra to rewrite the fractional expression as a product, then use the results in (1), (4), and (5):

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x} &= \lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\ &= \lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\\ &= 1\cdot\frac{0}{2}\\ &= 0.\end{aligned} \tag{8}$$

That is,

$$\lim_{x\to 0}\frac{1-\cos x}{x} = 0.$$

From (8) we obtain a limit result that is used in calculus to find the derivatives of the sine and cosine functions. Since the limit in (8) is equal to 0, we can write

$$\lim_{x\to 0}\frac{1-\cos x}{x} = \lim_{x\to 0}\frac{-(\cos x - 1)}{x} = (-1)\lim_{x\to 0}\frac{\cos x - 1}{x} = 0.$$

Dividing by -1 then gives

$$\lim_{x\to 0}\frac{\cos x - 1}{x} = 0. \tag{9}$$

The Calculus Connection In Section 2.10 we saw that the derivative of a function $y = f(x)$ is the function $f'(x)$ defined by a limit of a difference quotient:

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}. \tag{10}$$

In computing this limit we shrink h to zero, but x is held fixed. Recall too, if a number $x = a$ is in the domains of f and f', then $f(a)$ is the y-coordinate of the point of tangency $(a, f(a))$ and $f'(a)$ is the slope of the tangent line at that point.

Derivatives of $f(x) = \sin x$ and $f(x) = \cos x$ To find the derivative of $f(x) = \sin x$ we use the four-step process illustrated in Example 3 of Section 2.10. In the first step we use from Section 4.5 the sum formula for the sine function:

$$\sin(x_1 + x_2) = \sin x_1\cos x_2 + \cos x_1 \sin x_2. \tag{11}$$

(*i*) With x and h playing the parts of x_1 and x_2, we have from (11):

$$f(x+h) = \sin(x+h) = \sin x\cos h + \cos x\sin h.$$

(*ii*) $$\begin{aligned}f(x+h) - f(x) &= \sin x\cos h + \cos x\sin h - \sin x\\ &= \sin x(\cos h - 1) + \cos x \sin h.\end{aligned}$$

As we see in the next line, we cannot cancel the h's in the difference quotient, but we can rewrite the expression to make use of the limit results in (1) and (9).

(*iii*) $$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{\sin x(\cos h - 1) + \cos x\sin h}{h}\\ &= \sin x\frac{\cos h - 1}{h} + \cos x\frac{\sin h}{h}.\end{aligned}$$

(*iv*) In this line, the symbol h plays the part of the symbol x in (1) and (9):

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \sin x \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h}.$$

From the limit results in (1) and (9), the last line is the same as

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \sin x\cdot 0 + \cos x \cdot 1 = \cos x.$$

In summary:

- The derivative of $f(x) = \sin x$ is $f'(x) = \cos x$. (12)

It is left to you the student to show that:

- The derivative of $f(x) = \cos x$ is $f'(x) = -\sin x$. (13)

See Problems 23 and 24 in Exercises 4.14.

EXAMPLE 5 Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = \sin x$ at $x = 4\pi/3$.

Solution We start by finding the point of tangency. From

$$f\left(\frac{4\pi}{3}\right) = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$$

we see that the point of tangency is $\left(4\pi/3, -\sqrt{3}/2\right)$. The slope of the tangent line at that point is the derivative of $f(x) = \sin x$ evaluated at the x-coordinate. From (12) we know that $f'(x) = \cos x$ and so the slope at $\left(4\pi/3, -\sqrt{3}/2\right)$ is

$$f'\left(\frac{4\pi}{3}\right) = \cos\frac{4\pi}{3} = -\frac{1}{2}.$$

From the point-slope form of a line, an equation of the tangent line is

$$y + \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(x - \frac{4\pi}{3}\right) \quad \text{or} \quad y = -\frac{1}{2}x + \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

See FIGURE 4.14.1.

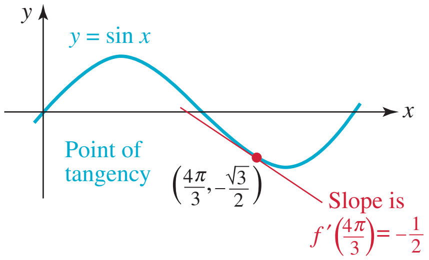

FIGURE 4.14.1 Tangent line in Example 5

Exercises 4.14 Answers to selected odd-numbered problems begin on page ANS-18.

In Problems 1–18, use the results in (1), (2), (3), (7), and (9) to find the indicated limit.

1. $\lim_{x\to 0}\frac{\sin\frac{1}{2}x}{x}$
2. $\lim_{x\to 0}\frac{\sin \pi x}{x}$
3. $\lim_{\theta\to 0}\frac{\sin(-\theta)}{\theta}$
4. $\lim_{t\to 0}\frac{\sin 3t}{4t}$
5. $\lim_{x\to 5\pi/6}\cos x$
6. $\lim_{x\to \pi/4}\sin x$
7. $\lim_{x\to \pi/2}(\cos x + 5\sin x)$
8. $\lim_{x\to \pi/6}\cos x\sin x$
9. $\lim_{x\to 0}\frac{\cos x - 1}{10x}$
10. $\lim_{\theta\to 0}\frac{8(1-\cos\theta)}{\theta}$

11. $\lim_{x\to 0} \dfrac{4x^2 - 2\sin x}{x}$

12. $\lim_{x\to 0} \dfrac{2\sin 4x + 1 - \cos x}{x}$

13. $\lim_{x\to 0} \dfrac{\sin^2 x}{x}$

14. $\lim_{x\to 0} \dfrac{\sin^2 x}{x^2}$

15. $\lim_{x\to \pi/2} \dfrac{\cos x}{\cot x}$

16. $\lim_{x\to 0} \dfrac{\cos x \tan x}{x}$

17. $\lim_{x\to 0} x\cot x$

18. $\lim_{x\to \pi/4} \dfrac{\cos 2x}{\cos x - \sin x}$

In Problems 19–22, proceed as in Example 5 to find an equation of the tangent line to the graph of $f(x) = \sin x$ at the indicated value of x.

19. $x = 0$

20. $x = \pi/2$

21. $x = \pi/6$

22. $x = 2\pi/3$

23. Proceed as on pages 314–315 and find the derivative of $f(x) = \cos x$.

24. Use the result of Problem 23 to find an equation of the tangent line to the graph of $f(x) = \cos x$ at $x = \pi/3$.

25. Use the facts that

$$\lim_{x\to 0} \frac{\cos 5x - 1}{x} = 0 \qquad \text{and} \qquad \lim_{x\to 0} \frac{\sin 5x}{x} = 5$$

to find the derivative of $f(x) = \sin 5x$.

26. Use the result of Problem 25 to find an equation of the tangent line to the graph of $f(x) = \sin 5x$ at $x = \pi$.

Calculator/Computer Problems

In Problems 27 and 28, use a calculator or computer to estimate the given limit by completing each table. Round the entries in each table to eight decimal places.

27. $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$

$x \to 0^+$	0.1	0.01	0.001	0.0001	0.00001
$\dfrac{1 - \cos x}{x^2}$					

Explain why we do not have to consider $x \to 0^-$.

28. $\lim_{x\to 2} \dfrac{x^2 - 4}{\sin(x - 2)}$

$x \to 2^+$	2.1	2.01	2.001	2.0001	2.00001
$\dfrac{x^2 - 4}{\sin(x - 2)}$					

$x \to 2^-$	1.9	1.99	1.999	1.9999	1.99999
$\dfrac{x^2 - 4}{\sin(x - 2)}$					

For Discussion

In Problems 29–36, discuss how to use the result in (1) along with some clever algebra, trigonometry, or a substitution to find the given limit.

29. $\lim_{x\to 0} \dfrac{x}{\sin 3x}$

30. $\lim_{x\to 0} \dfrac{\sin 4x}{\sin 5x}$

31. $\lim_{x\to 0} \dfrac{\sin x^2}{x^2}$

32. $\lim_{x\to \pi} \dfrac{\sin x}{\pi - x}$

33. $\lim_{x\to 0} \dfrac{x^2}{1 - \cos x}$

34. $\lim_{x\to 0} \dfrac{\cos\left(x + \frac{1}{2}\pi\right)}{x}$

35. $\lim_{x\to 0^+} \dfrac{\sin x}{\sqrt{x}}$

36. $\lim_{x\to 1} \dfrac{\sin(x - 1)}{x^2 + 2x - 3}$

37. Using what you have learned in Problems 29 and 36, find the limit

$$\lim_{x\to 2} \frac{x^2 - 4}{\sin(x - 2)}$$

without the aid of the numerical table in Problem 28.

38. **(a)** Use a calculator to complete the following table.

$x \to 0^+$	0.1	0.01	0.001	0.0001	0.00001
$\dfrac{1 - \cos x^2}{x^4}$					

(b) Find the limit $\lim_{x\to 0} \dfrac{1 - \cos x^2}{x^4}$ using the method given in Example 4.

(c) Discuss any differences that you observe between parts **(a)** and **(b)**.

39. **(a)** A regular n-gon is an n-sided polygon inscribed in a circle; the polygon is formed by n equally spaced points on the circle. Suppose the polygon shown in FIGURE 4.14.2 represents a regular n-gon inscribed in a circle of radius r. Use trigonometry to show that the area $A(n)$ of the n-gon is given by

$$A(n) = \frac{n}{2} r^2 \sin\left(\frac{2\pi}{n}\right).$$

See Problems 37–39 in Exercises 4.11.

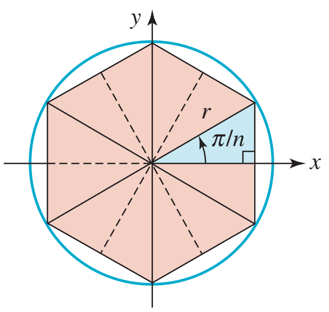

FIGURE 4.14.2 Inscribed n-gon in Problem 39

(b) It stands to reason that the area $A(n)$ approaches the area of the circle as the number of sides of the n-gon increases. Compute A_{100} and A_{1000}.

(c) Let $x = 2\pi/n$ in $A(n)$ and note that as $n \to \infty$ then $x \to 0$. Use (1) of this section to show that $\lim_{n\to\infty} A(n) = \pi r^2$.

40. Consider a circle centered at the origin O with radius 1. As shown in FIGURE 4.14.3(a), let the shaded region OPR be a sector of the circle with central angle t such that $0 < t < \pi/2$. We see from Figures 4.14.3(a)–4.14.3(d) that

$$\text{area of } \triangle OPR < \text{area of sector } OPR < \text{area of } \triangle OQR. \tag{12}$$

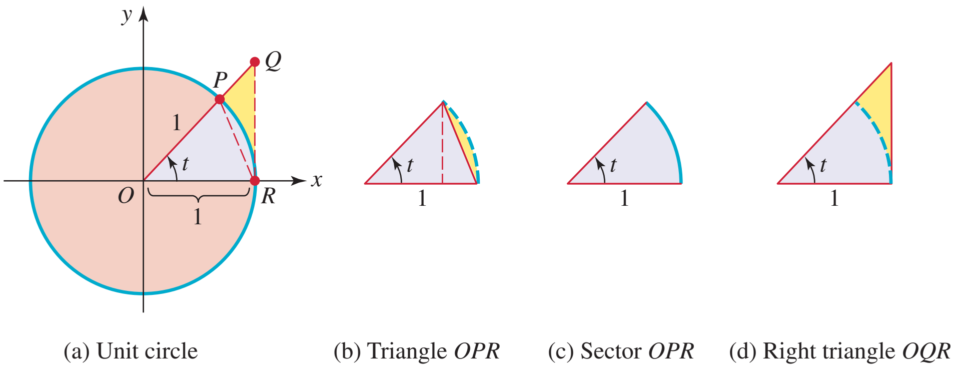

FIGURE 4.14.3 Unit circle in Problem 40

(a) Show that the area of $\triangle OPR$ is $\frac{1}{2}\sin t$ and that the area of $\triangle OQR$ is $\frac{1}{2}\tan t$.
(b) Since the area of a sector of a circle is $\frac{1}{2}r^2\theta$, where r is its radius and θ is measured in radians, it follows that the area of sector OPR is $\frac{1}{2}t$. Use this result, along with the areas in part **(a)**, to show that the inequality in (12) yields

$$\cos t < \frac{\sin t}{t} < 1.$$

(c) Discuss how the preceding inequality proves (1) when we let $t \to 0^+$.

Chapter 4 Review Exercises

Answers to selected odd-numbered problems begin on page ANS-19.

A. Fill in the Blanks

In Problems 1–20, fill in the blanks.

1. $\pi/5$ radians $=$ _______ degrees.
2. 10 degrees $=$ _______ radians.
3. The exact values of the coordinates of the point $P(t)$ on the unit circle corresponding to $t = 5\pi/6$ are _______.
4. The reference angle for $4\pi/3$ radians is _______ radians.
5. $\tan\dfrac{\pi}{3} =$ _______.
6. In standard position, the terminal side of the angle $\dfrac{8\pi}{5}$ radians lies in the _______ quadrant.
7. If $\sin\theta = -\frac{1}{3}$ and θ is in quadrant IV, then $\sec\theta =$ _______.
8. If $\tan t = 2$ and t is in quadrant III, then $\cos t =$ _______.
9. The y-intercept for the graph of the function $y = 2\sec(x + \pi)$ is _______.
10. The values of t in the interval $[0, 2\pi]$ that satisfy $\sin 2t = \frac{1}{2}$ are_______.
11. If $\sin u = \frac{3}{5}$, $0 < u < \pi/2$, and $\cos v = 1/\sqrt{5}$, $3\pi/2 < v < 2\pi$, then $\cos(u + v) =$ _______.
12. If $\cos t = -\frac{2}{3}$, $\pi < t < 3\pi/2$, then $\cos\frac{1}{2}t =$ _______.
13. A sine function with period 4 and amplitude 6 is given by _______.
14. The first vertical asymptote for the graph of $y = \tan\left(x - \dfrac{\pi}{4}\right)$ to the right of the y-axis is _______.

15. $\sin t + \cos t =$ ______ $\sin\left(t + \dfrac{\pi}{4}\right)$.

16. If $\sin t = \frac{1}{6}$, then $\cos\left(t - \dfrac{\pi}{2}\right) =$ ______.

17. The amplitude of $y = -10\cos\dfrac{\pi}{3}x$ is ______.

18. $\cos\left(\dfrac{\pi}{6} - \dfrac{5\pi}{4}\right) =$ ______.

19. The exact value of $\arccos\left(\cos\dfrac{9\pi}{5}\right) =$ ______.

20. The period of the function $y = 2\sin\dfrac{\pi}{3}t$ is ______.

B. True/False

In Problems 1–20, answer true or false.

1. If $\tan t = \frac{3}{4}$, then $\sin t = 3$ and $\cos t = 4$. ______
2. In a right triangle, If $\sin\theta = \frac{11}{61}$, then $\cot\theta = \frac{60}{11}$. ______
3. $\sec(-\pi) = \csc\dfrac{3\pi}{2}$ ______
4. There is no angle t such that $\sec t = \frac{1}{2}$. ______
5. $\sin(2\pi - t) = -\sin t$ ______
6. $1 + \sec^2\theta = \tan^2\theta$ ______
7. $(5, 0)$ is an x-intercept of the graph of $y = 3\sin\pi x$. ______
8. $(2\pi/3, -1/\sqrt{3})$ is a point on the graph of $y = \cot x$. ______
9. The range of the function $y = \csc x$ is $(-\infty, -1] \cup [1, \infty)$. ______
10. The graph of $y = \csc x$ does not intersect the y-axis. ______
11. The line $x = \pi/2$ is a vertical asymptote for the graph of $y = \tan x$. ______
12. If $\tan(x + \pi) = 0.3$, then $\tan x = 0.3$. ______.
13. For the sine function $y = -2\sin x$ we have $-2 \le y \le 2$. ______
14. $\sin 6x = 2\sin 3x\cos 3x$ ______
15. The graph of $y = \sin\left(2x - \dfrac{\pi}{3}\right)$ is the graph of $y = \sin 2x$ shifted $\pi/3$ units to the right. ______
16. Since $\tan(5\pi/4) = 1$, then $\arctan(1) = 5\pi/4$. ______
17. $\arcsin\left(\frac{1}{2}\right) = 30°$ ______
18. $f(x) = \arcsin x$ is not periodic. ______
19. $f(x) = x\sin x$ is 2π periodic. ______
20. $f(x) = \sin(\cos x)$ is an even function. ______

C. Review Exercises

In Problems 1–6, find all t in the interval $[0, 2\pi]$ that satisfy the given equation.

1. $\cos t\sin t - \cos t + \sin t - 1 = 0$
2. $\cos t - \sin t = 0$
3. $4\sin^2 t - 1 = 0$
4. $\sin t = 2\tan t$
5. $\sin t + \cos t = 1$
6. $\tan t - 3\cot t = 2$

In Problems 7–10, solve the triangle satisfying the given conditions.

7. $\alpha = 30°, \beta = 70°, b = 10$
8. $\gamma = 145°, a = 25, c = 20$
9. $\alpha = 51°, b = 20, c = 10$
10. $a = 4, b = 6, c = 3$

In Problems 11–18, find the indicated value without using a calculator.

11. $\cos^{-1}\left(-\frac{1}{2}\right)$
12. $\arcsin(-1)$
13. $\cot\left(\cos^{-1}\frac{3}{4}\right)$
14. $\cos\left(\arcsin\frac{2}{5}\right)$
15. $\sin^{-1}(\sin\pi)$
16. $\cos(\arccos 0.42)$
17. $\sin\left(\arccos\left(\frac{5}{13}\right)\right)$
18. $\arctan(\cos\pi)$

In Problems 19 and 20, write the given expression as an algebraic expression in x.

19. $\sin(\arccos x)$
20. $\sec(\tan^{-1}x)$

In Problems 21–24, the given graph can be interpreted as a rigid/nonrigid transformation of the graph of $y = \sin x$ and of the graph of $y = \cos x$. Find an equation of the graph using the sine function. Then find an equation of the same graph using the cosine function.

21.

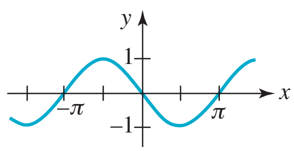

FIGURE 4.R.1 Graph for Problem 21

22.

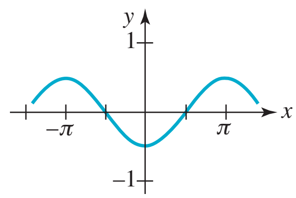

FIGURE 4.R.2 Graph for Problem 22

23.

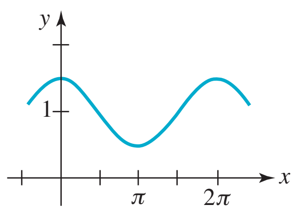

FIGURE 4.R.3 Graph for Problem 23

24.

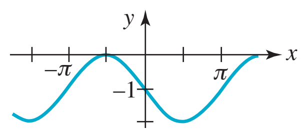

FIGURE 4.R.4 Graph for Problem 24

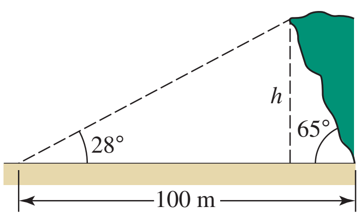

FIGURE 4.R.5 Cliff in Problem 25

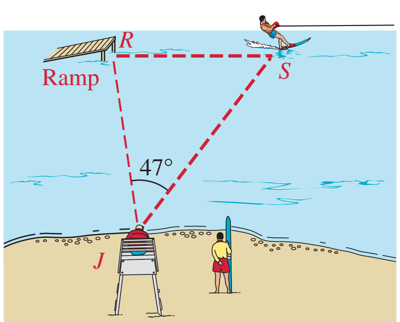

FIGURE 4.R.6 Water-skier in Problem 27

25. A surveyor 100 m from the base of an overhanging cliff measures a 28° angle of elevation from that point to the top of the cliff. See FIGURE 4.R.5. If the cliff makes an angle of 65° with the horizontal ground, determine its height h.

26. A rocket is launched from ground level at an angle of elevation of 43°. If the rocket hits a drone target plane flying at 20,000 ft, find the horizontal distance between the rocket launch site and the point directly beneath the plane. What is the straight-line distance between the rocket launch site and the target plane?

27. A competition water-skier leaves a ramp at point R and lands at S. See FIGURE 4.R.6. A judge at point J measures an $\angle RJS$ as 47°. If the distance from the ramp to the judge is 110 ft, find the length of the jump. Assume that $\angle SRJ$ is 90°.

28. The angle between two sides of a parallelogram is 40°. If the lengths of the sides are 5 and 10 cm, find the lengths of the two diagonals.

29. A weather satellite orbiting the equator of the Earth at a height of $H = 36{,}000$ km spots a thunderstorm to the north at P at an angle of $\theta = 6.5°$ from its vertical. See FIGURE 4.R.7.

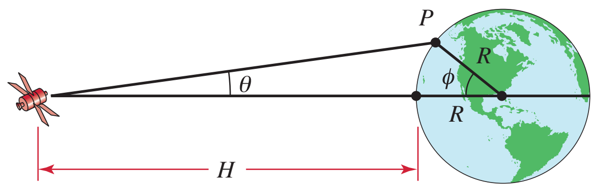

FIGURE 4.R.7 Satellite in Problem 29

(a) Given that the Earth's radius is approximately $R = 6370$ km, find the latitude ϕ of the thunderstorm.

(b) Show that angles θ and ϕ are related by

$$\tan\theta = \frac{R\sin\phi}{H + R(1 - \cos\phi)}.$$

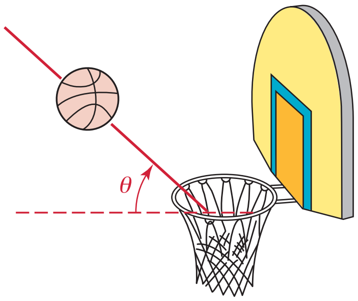

FIGURE 4.R.8 Basketball in Problem 30

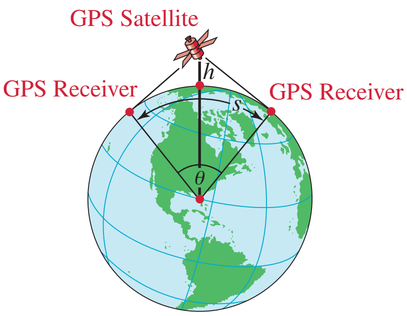

FIGURE 4.R.9 GPS satellite in Problem 31

30. It can be shown that a basketball of diameter d approaching the basket from an angle θ to the horizontal will pass through a hoop of diameter D if $D\sin\theta > d$, where $0° \le \theta \le 90°$. See FIGURE 4.R.8. If the basketball has diameter 24.6 cm and the hoop has diameter 45 cm, what range of approach angles θ will result in a basket?

31. Each of the 24 NAVSTAR Global Positioning System (GPS) satellites orbits the Earth at an altitude of $h = 20{,}200$ km. Using this network of satellites, an inexpensive hand-held GPS receiver can determine its position on the surface of the Earth to within 10 m. Find the greatest distance s (in km) on the surface of the Earth that can be observed from a single GPS satellite. See FIGURE 4.R.9. Take the radius of the Earth to be 6370 km. [*Hint*: Find the central angle θ subtended by s.]

32. An airplane flying horizontally at a speed of 400 miles per hour is climbing at an angle of 6° from the horizontal. When it passes directly over a car traveling 60 miles per hour, it is 2 miles above the car. Assuming that the airplane and the car remain in the same vertical plane, find the angle of elevation from the car to the airplane after 30 minutes.

33. A house measures 45 ft from front to back. The roof measures 32 ft from the front of the house to the peak and 18 ft from the peak to the back of the house. See FIGURE 4.R.10. Find the angles of elevation of the front and back parts of the roof.

34. A regular five-sided polygon is called a regular pentagon. See Figure 4.14.2. Using radian measure, determine the sum of the vertex angles in a regular pentagon.

35. In FIGURE 4.R.11 the blue, green, and red circles are of radii 3, 4, and 6, respectively. The dots represent the centers of the circles. Let d denote the distance between the centers of the blue and red circles. Determine the angle θ shown in the figure that corresponds to $d = 14$.

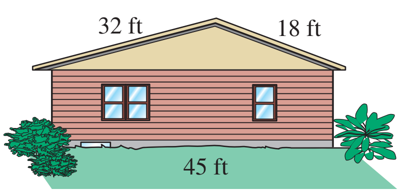

FIGURE 4.R.10 House in Problem 33

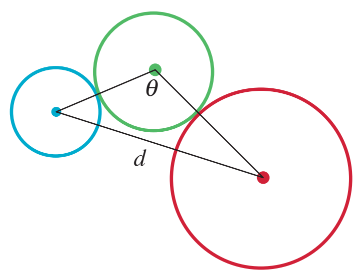

FIGURE 4.R.11 Circles in Problem 35

36. Navigator's Error An airplane is supposed to fly 500 mi due west to a refueling rendezvous point. If a 5°error is made in the heading, how far is the plane from the rendezvous point after flying 400 mi? Through what angle must the airplane turn in order to correct its course at that point?

37. National Historic Landmark Completed in 1902 on a triangular city block, the Flatiron Building in New York City was declared a National Historic Landmark in 1989. See FIGURE 4.R.12. The original 21 story stone clad steel-frame building is considered to be one of the first skyscrapers built in the city. The sides of the building measure 173 ft along Fifth Avenue, 87 ft along East 22nd Street, and 190 ft along Broadway.

(a) Show that the base of the building is approximately a right triangle.

(b) Assuming that the base of the building is a right triangle, find the two acute angles in it.


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Flatiron Building in New York, NY

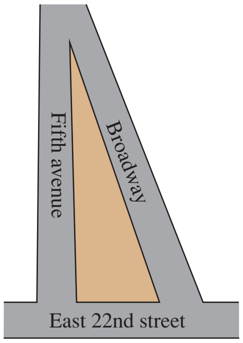

FIGURE 4.R.12 Triangular block in Problem 37

38. Volcanic Cones Viewed from the side, a volcanic cinder cone usually looks like an isosceles trapezoid. See FIGURE 4.R.13. Studies of cinder cones that are less than 50,000 years old indicate that cone height H_{co} and crater width W_{cr} are related to the cone width W_{co} by the equations $H_{co} = 0.18\ W_{co}$ and $W_{cr} = 0.40\ W_{co}$. If $W_{co} = 1.00$, use these equations to determine the base angle ϕ of the trapezoid in Figure 4.R.13.


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Volcanic cinder cones in Haleakala Crater, Maui, Hawaii

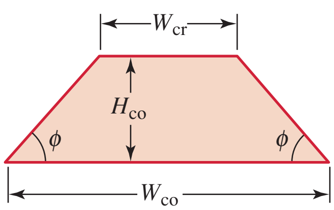

FIGURE 4.R.13 Volcanic cinder cone in Problem 38


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Angels Flight in downtown Los Angeles

39. Angels Flight Claimed to be the world's shortest railway, Angels Flight is a funicular railway consisting of two cars (named Olivet and Sinai) that transports people up and down the steep hill between Hill Street and California Plaza in downtown Los Angeles, CA. The original railway dates back to 1901 and, in its present form, is only 298 ft long. If the angle of elevation of the tracks at its base on Hill Street is 33°, then how high is the hill?

40. Distance Across a Canyon From the floor of a canyon it takes 62 ft of rope to reach the top of one canyon wall and 86 ft to reach the top of the opposite wall.

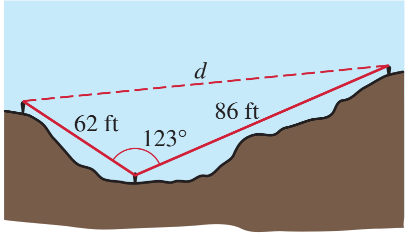

FIGURE 4.R.14 Canyon in Problem 40

See FIGURE 4.R.14. If the two ropes make an angle of 123°, what is the distance d from the top of one canyon wall to the other?

41. How Fast? An observer at a horizontal distance of 1 mile (5280 ft) watches an F-15E Strike Eagle fighter jet go into a vertical climb. See FIGURE 4.R.15. If the angle of elevation of the jet at the observer changes from the initial measurement of 43.44° to 75.21° in 30 seconds, then how fast (in feet per minute) is its rate of climb?

Courtesy of Master Sgt. Lance Cheung, U.S. Air Force

US Air Force F-15E Strike Eagle

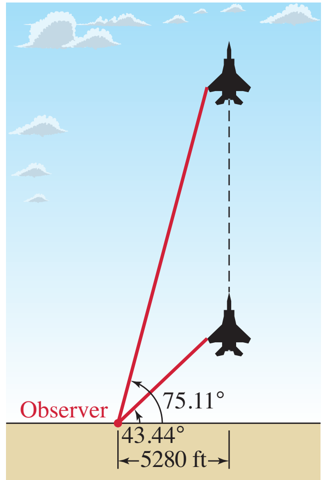

FIGURE 4.R.15 Climbing F-15E in Problem 41

42. Building Height Two buildings were constructed on a inclined lot as shown in FIGURE 4.R.16. The angle of elevation from the right side of the roof of the brown building to the left side of the roof of the gray building is 23°. From the same spot on the roof of the brown building, the angle of depression to the base of the gray building is 48°. Use the additional information in the figure to determine the heights of the facing sides of the buildings relative to the inclined lot.

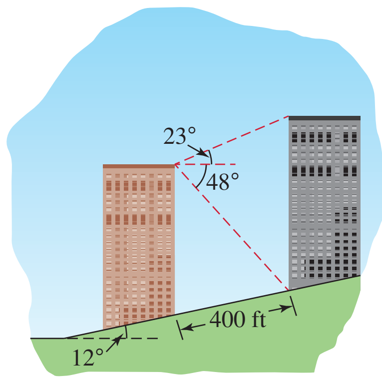

FIGURE 4.R.16 Buildings in Problem 42

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One way of finding the height of a tree: climb it

43. Estimating Tree Height There are many sophisticated instruments, such as range finders and inclinometers (or clinometer), that are invaluable in determining an accurate measurement of the height of an object. A nontechnical method for determining the height of, say, a tree is to climb the tree and then drop the weighted end of a measuring-tape line to the ground. Assuming that you have no measuring devices and that climbing a tree is not practical (or even legal), explain why the following method gives an approximation to the height of a tree:

Find a patch of level ground containing the tree. Guess the distance from the ground to your eyelevel and the length of your walking stride. Find two straight sticks, back away from the tree (*counting your strides*) *holding the sticks at eye level like this* ∠ (*one parallel to the ground and the other adjusted by following the top of the tree*). *Stop when you think the angle between the sticks is* 45°. *The height of the tree is approximately*:

$$(\textit{number of strides}) \times (\textit{stride length}) + \textit{eye level height}.$$

44. Gate of Europe The *Puerta de Europa* towers are twin office buildings in Madrid, Spain. To accommodate a required setback from the wide *Paseo de la Castellana*, the towers were built in 1996 at an angle of 15° from the vertical. In the photo, note the vertical line on the side of each building. The sides of the buildings shown in FIGURE 4.R.17 are congruent parallelograms. Use the information in the figure to find the lengths s_1 and s_2 of the sides of the parallelogram and the distance d between the roofs of the towers.

© Glyn Thomas Photography/Alamy Images

Puerta de Europa towers and the *Paseo de la Castellana* in Madrid, Spain

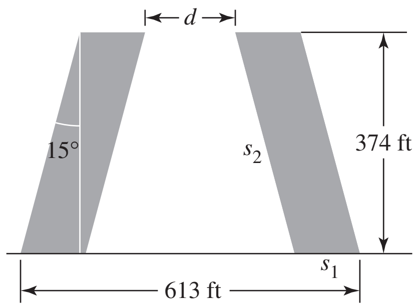

FIGURE 4.R.17 Towers in Problem 44

45. Stairs in Homes In home construction, stairs are usually constructed using two stringers, which are boards that have been notched to accommodate the treads (steps) and the risers. Suppose the stairs consists of 9 risers, the riser height is 7.75 inches, and the tread depth is 10 inches. See FIGURE 4.R.18. Use two different methods to find the approximate length L of a stringer.

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Stairs showing a stringer

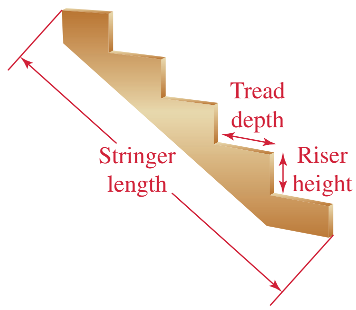

FIGURE 4.R.18 Stringer in Problem 45

46. El Castillo The most prominent feature in the archaeological site of Chichén Itzá is a step pyramid on whose flat top rests a temple to the Mayan feathered-serpent god Kukulkán. The Kukulkán pyramid, more commonly known by the Spanish name *El Castillo*, was built around 900 C.E. and is located in the Mexican state of Yucátan. On each of the four faces of the pyramid there is a protruding stone-block stairway rising to the 79 ft high temple level, although only two of the staircases have been completely restored. The angle of inclination a stairway relative to the ground is 45° whereas the angle of inclination of a face is 53.3°. Prior to 2006, tourists were allowed to climb one of the steep stairways aided by a rope positioned in the middle of the stairway and anchored at the base of the pyramid and at the temple level. Find the approximate length L of the rope. See FIGURE 4.R.19.

Tourists are no longer allowed to climb *El Castillo*

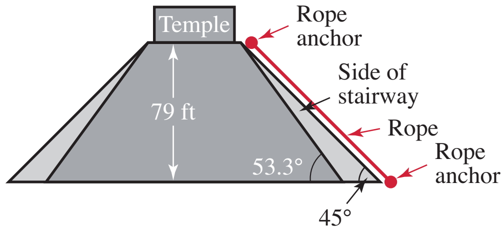

FIGURE 4.R.19 Pyramid in Problem 46

In Problems 47–60, translate the words into an appropriate function.

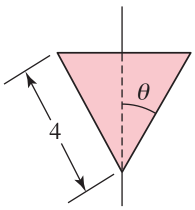

FIGURE 4.R.20 End of water trough in Problem 47

47. A 20-ft-long water trough has ends in the form of isosceles triangles with sides that are 4 ft long. See Figure 2.9.27 in Exercises 2.9. As shown in FIGURE 4.R.20, let θ denote the angle between the vertical and one of the sides of a triangular end. Express the volume of the trough as a function of 2θ.

48. A person driving a car approaches a freeway sign as shown in FIGURE 4.R.21. Let θ be her viewing angle of the sign and let x represent her horizontal distance (measured in feet) to that sign. Express θ as a function of x.

49. As shown in FIGURE 4.R.22, a plank is supported by a sawhorse so that one end rests on the ground and the other end rests against a building. Express the length of the plank as a function of the indicated angle θ.

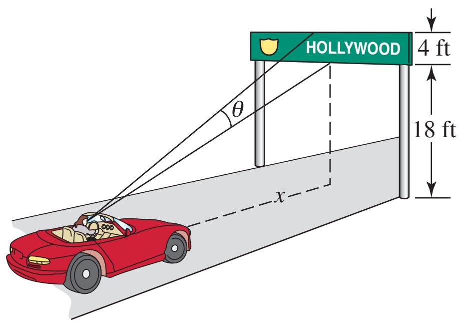

FIGURE 4.R.21 Freeway sign in Problem 48

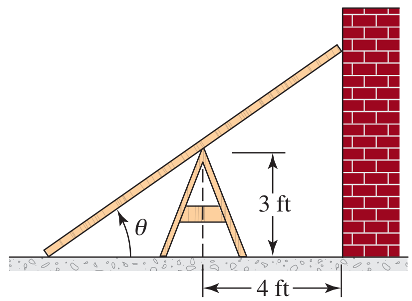

FIGURE 4.R.22 Plank in Problem 49

50. A farmer wishes to enclose a pasture in the form of a right triangle using 2000 ft of fencing on hand. See FIGURE 4.R.23. Show that the area of the pasture as a function of the indicated angle θ is

$$A(\theta) = \frac{1}{2}\cot\theta \cdot \left(\frac{2000}{1 + \cot\theta + \csc\theta}\right)^2.$$

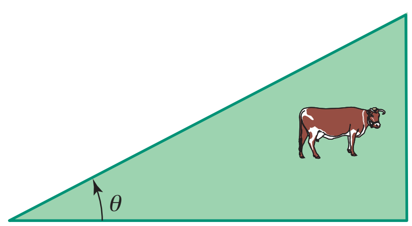

FIGURE 4.R.23 Pasture in Problem 50

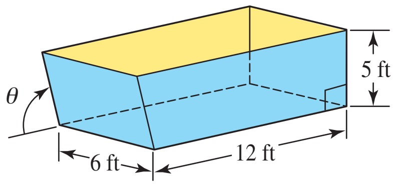

FIGURE 4.R.24 Box in Problem 51

51. Express the volume of the box shown in FIGURE 4.R.24 as a function of the indicated angle θ.

52. A corner of an 8.5 in. $\times$ 11 in. piece of paper is folded over to the other edge of the paper as shown in FIGURE 4.R.25. Express the length L of the crease as a function of the angle θ shown in the figure.

53. A gutter is to be made from a sheet of metal 30 cm wide by turning up the edges of width 10 cm along each side so that the sides make equal angles ϕ with the vertical. See FIGURE 4.R.26. Express the cross-sectional area of the gutter as a function of the angle ϕ.

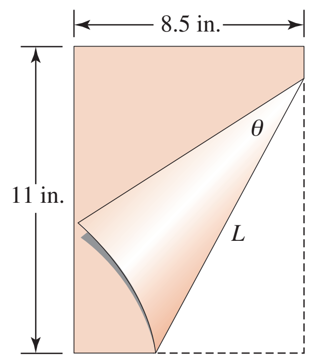

FIGURE 4.R.25 Folded paper in Problem 52

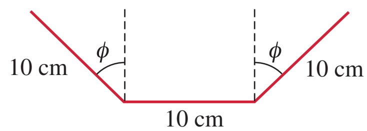

FIGURE 4.R.26 Gutter in Problem 53

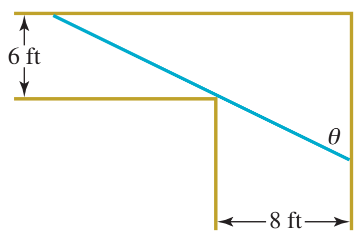

FIGURE 4.R.27 Pipe in Problem 54

54. A metal pipe is to be carried horizontally around a right-angled corner from a hallway 8 feet wide into a hallway that is 6 feet wide. See FIGURE 4.R.27. Express the length L of the pipe as a function of the angle θ shown in the figure.

55. Circle of Latitude A circle of latitude is 2 circle that connects all locations on the Earth that have the same latitude ϕ. A circle of latitude is also referred to as a line of latitude, or a parallel, because if the Earth is represented as a circle in a two-dimensional coordinate system, then the circles of latitude appear as (parallel) horizontal lines. See FIGURE 4.R.28.

(a) If the radius R of the Earth is taken to be 3959 miles, express the radius r of a circle of latitude as a function of ϕ.

(b) Find the radius of the Artic Circle if its latitude is 66°33′44″N.

(c) A line of longitude, or meridian, is one half of a great circle whose center is the center of the Earth. Longitudes are measured east/west from the prime meridian that runs through the Royal Observatory at Greenwich, England. See Figure 4.2.14 in Exercises 4.2. The longitudes of Boston, Massachusetts and Detroit, Michigan are, respectively, 71°3′37″W and 83°2′44″W but the latitude of both cities is approximately the same. What is the distance between Boston and Detroit measured on the circle of latitude at 42°20′N?

(d) Measured on a meridian, what is the distance between two points that differ by one degree of latitude?

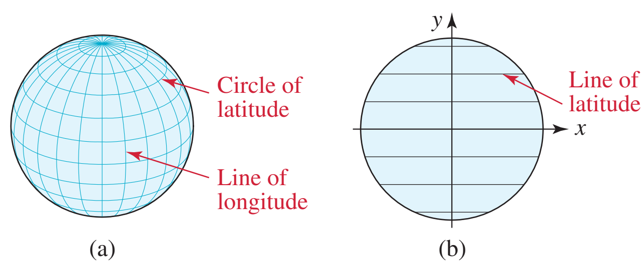

FIGURE 4.R.28 Planet Earth in Problem 55

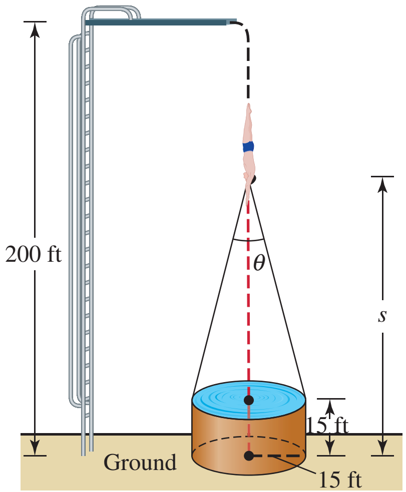

FIGURE 4.R.29 Diver in Problem 57

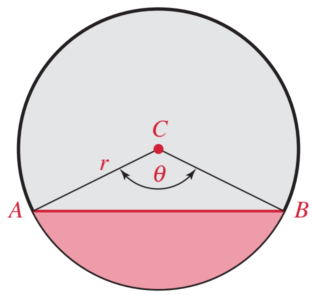

FIGURE 4.R.30 Circular segment in Problem 58

56. More Latitude **(a)** Use Problem 55 to show that the circumference of a circle of latitude as a function of the latitude angle ϕ is given by $C_\phi = C_e \cos \phi$, where C_e is the circumference of the Earth at the equator. Find C_e.
(b) Use part (a) to find the circumference of the Arctic Circle.
(c) Use part (a) to find the distance "around the world" at the latitude 52°45′ N.

57. High Dive A diver jumps from a high platform with an initial downward velocity of 1 ft/s toward the center of a large circular tank of water. See FIGURE 4.R.29. From physics, the height of the diver above ground level is given by $s(t) = -16t^2 - t + 200$, where s is measured in feet and $t \geq 0$ is time in seconds. See (5) in Section 2.4.
(a) Express the angle θ shown in the figure as a function of s.
(b) For the function in part (a), what value does θ approach as $s \to 15$?

58. Circular Segment A circular segment is the region formed between a chord AB of a circle and its associated arc $\widehat{AB}$. This is the light red region in FIGURE 4.R.30. For a circle of fixed radius r, express the area of a circular segment as a function of the central angle θ, where $0 < \theta < \pi$.

59. Equilateral Arch An equilateral arch is obtained by constructing circular arcs on two sides of an equilateral triangle. Let ABC be an equilateral triangle with sides of length s as shown in red in FIGURE 4.R.31. A circular arc $\widehat{CB}$ is drawn from vertex C to vertex B using a circle of radius s centered at A. In like manner, $\widehat{CA}$ is an arc of the circle of radius s centered at B. Equilateral arches were used extensively in Gothic architecture in the design of church windows and doorways. Use the concept of a circular segment discussed in Problem 58 to express the area of an equilateral arch as a function of the length s.

The equilateral arch is found throughout the *Doumo di Milano*, the largest Gothic cathedral in the world.

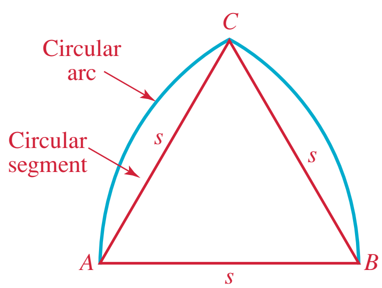

FIGURE 4.R.31 Equilateral arch in Problem 59

60. Express the perimeter of the equilateral arch in Problem 59 as a function of s.