CHAPTER

6

Mathematical Procedures

The multidisciplinary approach to medicine has incorporated a wide variety of mathematical procedures from the fields of physics, chemistry, and engineering. The information presented in this chapter is designed as a self-teaching refresher course to be used as a review of basic mathematical procedures. Some of the more advanced mathematical concepts, including the section on descriptive statistics, should also help the practitioner to interpret data presented in medical journals and scientific articles.

FUNDAMENTAL AXIOMS

Commutative Axiom

$$a + b = b + a$$

$$ab = ba$$

When two or more numbers are added or multiplied together, their order does not affect the result.

Associative Axiom

$$(a + b) + c = a + (b + c)$$

$$(ab)c = a(bc)$$

When three or more numbers are added together, the way they are grouped or associated makes no difference in the result. The same holds true for multiplication.

Distributive Axiom

$$a(b + c) = ab + ac$$

A coefficient (multiplier) of a sum may be distributed as a multiplier of each term.

Order of Precedence

A convention has been established for the order in which numerical operations are performed. This is to prevent confusion when evaluating expressions such as 2×3^2, which could be either 18 or 36. The following rules apply:

1. If the numerical expression *does not* contain fences (such as parentheses), then operations are carried out in the following order:
   a. Raising numbers to powers or extracting roots of numbers.
   b. Multiplication or division.
   c. Addition or subtraction.

Example

$$4 \times 5 + 8 \div 2 + 6^2 - \sqrt{16} + 1 = 20 + 4 + 36 - 4 + 1 = 57$$

2. If the numerical expression *does* contain fences, then follow the procedure in Rule 1, starting with the *innermost* set of parentheses. The sequence is round fences (parentheses), square fences [brackets], double fences {braces}. Once the fences have been eliminated, the expression can be evaluated following Rule 1.

Example

$$\begin{aligned}
&2 + 4 \times \{3 \times 2 - [5 \times 4 + (2 \times 3 - 4 \div 1) - 20] + 12\}\\
&= 2 + 4 \times \{3 \times 2 - [5 \times 4 + (6 - 4) - 20] + 12\}\\
&= 2 + 4 \times \{3 \times 2 - [5 \times 4 + 2 - 20] + 12\}\\
&= 2 + 4 \times \{3 \times 2 - [20 + 2 - 20] + 12\}\\
&= 2 + 4 \times \{3 \times 2 - 2 + 12\}\\
&= 2 + 4 \times \{6 - 2 + 12\}\\
&= 2 + 4 \times 16\\
&2 + 64 = 66
\end{aligned}$$

FRACTIONS

When a number is expressed as a fraction (e.g., $\frac{3}{5}$), the number above the line (3) is called the **numerator** and the number below the line (5) the **denominator**.

Multiplication Property of Fractions

$$\frac{a \times c}{b \times c} = \frac{a}{b} \qquad (c \neq 0)$$

The numerator and denominator of a fraction may be multiplied or divided by the same nonzero number to produce a fraction of equal value.

Example

Simplify (reduce) the fraction $\frac{9}{12}$

Solution

1.	Find the largest integer that will evenly divide both the numerator and denominator.	The largest whole number is 3.
2.	Divide both the numerator and denominator by that number.	$\frac{9}{12} = \left(\frac{9}{3}\right) \div \left(\frac{12}{3}\right) = \frac{3}{4}$

Multiplication of Fractions

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$$

Example

$$\frac{7}{9} \times \frac{3}{4} = ?$$

Solution

1.	Multiply the numerators.	$7 \times 3 = 21$
2.	Multiply the denominators.	$9 \times 4 = 36$
3.	Simplify the resulting fraction if possible.	$\frac{7}{9} \times \frac{3}{4} = \frac{21}{36} = \frac{7}{12}$

Division of Fractions

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

Example	
Find the quotient: $\frac{5}{8} \div \frac{2}{3}$	
Solution	
1. Invert the divisor.	Change $\frac{2}{3}$ to $\frac{3}{2}$
2. Multiply the dividend by the inverted divisor.	$\frac{5}{8} \times \frac{3}{2} = \frac{15}{16}$
3. Simplify if possible.	

Addition and Subtraction of Fractions with the Same Denominator

$$\frac{a}{b} + \frac{c}{b} = \frac{(a + c)}{b}$$

$$\frac{a}{b} - \frac{c}{b} = \frac{(a - c)}{b}$$

Example	
$\frac{5}{32} + \frac{13}{32} - \frac{3}{32} = ?$	
Solution	
1. Combine numerators.	$5 + 13 - 3 = 15$
2. Write the resultant fraction with the new numerator and the same denominator.	$\frac{15}{32}$
3. Simplify if possible.	

Addition and Subtraction of Fractions with Different Denominators

To add or subtract fractions that do not have the same denominator, it is first necessary to express them as fractions having the same denominators. To find a common denominator, find an integer that is evenly divisible by each denominator. The smallest or least common denominator (LCD) is the most convenient.

Example

$$\frac{5}{4} + \frac{7}{18} = ?$$

Solution

1. First find the LCD as follows:	
a. Express each denominator as the product of primes (integers greater than 1 that are evenly divisible by only themselves and 1).	a. $4 = 2 \times 2 = 2^2$ $18 = 2 \times 3 \times 3$ $= 2 \times 3^2$
b. Note the greatest power to which an integer occurs in any denominator.	b. 2^2 is the greatest power of 2 in either denominator, 3^2 is the greatest power of 3 in either denominator.
c. The product of the integers noted in part b is the LCD.	c. $2^2 \times 3^2 = 36$
2. Write fractions as equivalent fractions with denominators equal to the LCD.	$\frac{5}{4} \times \frac{9}{9} = \frac{45}{36}$ $\frac{7}{18} \times \frac{2}{2} = \frac{14}{36}$
3. Combine the numerators and use the LCD as the denominator.	$\frac{45}{36} + \frac{14}{36} = \frac{59}{36}$ $= 1.64$
4. Simplify if possible.	

RATIOS, PROPORTIONS, AND UNIT CONVERSION

The **ratio** of two numbers may be written as follows:

$$a/b = a{:}b$$

Two equivalent ratios form a **proportion**.

$$a/b = c/d$$

$$a{:}b = c{:}d$$

$$a{:}b :: c{:}d$$

Regardless of how the above proportions are expressed, they are read "*a* is to *b* as *c* is to *d*."

Ratios provide a convenient method for converting units. To change the units of a quantity, multiply by ratios whose values are equal to 1 (which does not change the value of the quantity). Select the dimensions of the ratios such that the unit to be changed occurs as a factor of the numerator or as a factor of the denominator. Thus, when the quantity is multiplied by the ratio, the unit is canceled and replaced by an equivalent unit and quantity.

Example		
Convert 2 kilometers/hour to feet/second.		
Solution		
1.	Write an equation expressing the problem.	2 km/hr = x ft/s
2.	Multiply the known quantity by ratios whose value is equal to 1, such that the desired unit remains after canceling pairs of equal dimensions that appear in both the numerator and the denominator.	1 km = 1,000 m 1 m = 3.281 ft 1 hr = 60 min 1 min = 60 s

$$\frac{2\text{ km}}{\text{hr}} \times \frac{1{,}000\text{ m}}{1\text{ km}} \times \frac{3.3\text{ ft}}{1\text{ m}} \times \frac{1\text{ hr}}{60\text{ min}} \times \frac{1\text{ min}}{60\text{ s}} = \frac{6{,}600\text{ ft}}{3{,}600\text{ s}} = \frac{1.8\text{ ft}}{\text{s}}$$

EXPONENTS

When a product is the result of multiplying a factor by itself several times, it is convenient to use a shorthand notation that shows the number used as a factor (a) and the number of factors (n) in the product. In general, such numbers are expressed in the form a^n, where a is the **base** and n the **exponent**. The rules for these numbers are shown in Table 6–7.

Table 6-7 Rules for Exponents

Rule	Example
$a^n \cdot a^m = a^{n+m}$	$x^2 \cdot x^3 = x^5$
$a^n \div a^m = a^{n-m}$	$z^7 \div z^5 = z^2$
$(ab)^n = a^n b^n$	$(2wy)^2 = 4w^2y^2$
$a^0 = 1, (a \neq 0)$	$9x^0 = 9$
$a^1 = a$	$3x^1 = 3x$
$a^{-n} = 1/a^n$	$x^{-2} = 1/x^2$
$a^{1/n} = \sqrt[n]{a}$	$y^{1/3} = \sqrt[3]{y}$
$a^{m/n} = \sqrt[n]{a^m}$	$z^{2/3} = \sqrt[3]{z^2}$
$(a^m)^n = a^{mn}$	$(w^2)^3 = w^6$

SCIENTIFIC NOTATION

A number expressed as a multiple of a power of 10, such as 3.02×10^5, is said to be written in **scientific notation**. Numbers written in this way have two parts: a number between 1 and 10 called the **coefficient**, multiplied by a power of 10 called the **exponent**. This notation has three distinct advantages:

a. It simplifies the expression of very large or very small numbers that would otherwise require many zeros. For example, $681{,}000{,}000 = 6.81 \times 10^8$ and $0.000026 = 2.6 \times 10^{-5}$.
b. Scientific notation clarifies the number of significant figures in a large number. For example, if the radius of the earth is written as 6,378,000 m, it is not clear whether any of the zeros after the 8 is significant. However, when the same number is written as 6.378×10^6 m, it is understood that only the first four digits are significant.
c. Calculations that involve very large or very small numbers are greatly simplified using scientific notation.

Addition and Subtraction

Example	
$6.18 \times 10^3 + 1.9 \times 10^2 - 5.0 \times 10^1$	
Solution	
1. Convert all numbers to the same power of 10 as the number with the highest exponent.	$6.18 \times 10^3 = 6.18 \times 10^3$ $1.9 \times 10^2 = 0.19 \times 10^3$ $5.0 \times 10^1 = 0.05 \times 10^3$
2. Add or subtract the coefficients and retain the same exponent in the answer.	6.18×10^3 $+ 0.19 \times 10^3$ $- 0.05 \times 10^3$ 6.32×10^3

Multiplication and Division

Example	
$(4 \times 10^{23})(2 \times 10^{14})$	
Solution	
1. Multiply (or divide) the coefficients.	$(4 \times 10^{23})(2 \times 10^{14})$ $= 8(10^{23} \times 10^{14})$
2. Combine the powers of 10 using the rules for exponents.	$8(10^{23} \times 10^{14})$ $= 8 \times 10^{23+14}$ $= 8 \times 10^{37}$

Powers and Roots

Example	
$(4 \times 10^5)^2$	
Solution	
1. Raise the coefficient to the indicated power.	$(4 \times 10^5)^2 = (4^2)(10^5)^2$ $= 16(10^5)^2$
2. Multiply the exponent by the indicated power.	$16(10^5)^2 = 16(10^{5 \times 2})$ $= 16 \times 10^{10}$ $= 1.6 \times 10^{11}$

SIGNIFICANT FIGURES

By convention, the number of digits used to express a measured number roughly indicates the error. For example, if a measurement is reported as 35.2 cm, one would assume that the true length was between 35.15 and 35.24 cm (i.e., the error is about 0.05 cm). The last digit (2) in the reported measurement is uncertain, although one can reliably state that it is either 1 or 2. The digit to the right of 2, however, can be any number (5, 6, 7, 8, 9, 0, 1, 2, 3, 4). If the measurement is reported as 35.20 cm, it would indicate that the error is even less (0.005 cm). The number of reliably known digits in a measurement is the number of **significant figures**. Thus, the number 35.2 cm has three significant figures, and the number 35.20 cm has four. The number of significant figures is independent of the decimal point. The numbers 35.2 cm and 0.352 m are the same quantities, both having three significant figures and expressing the same degree of accuracy. The use of significant figures to indicate the accuracy of a result is not as precise as giving the actual error, but is sufficient for most purposes.

Zeros as Significant Figures

Final zeros to the right of the decimal point that are used to indicate accuracy are significant:

179.0	4 significant figures
28.600	5 significant figures
0.30	2 significant figures

For numbers less than one, zeros between the decimal point and the first digit are *not* significant:

0.09	1 significant figure
0.00010	2 significant figures

Zeros between digits are significant:

10.5	3 significant figures
0.8070	4 significant figures
6000.01	6 significant figures

If a number is written with no decimal point, the final zeros may or may not be significant. For instance, the distance between Earth and the sun might be written as 92,900,000 miles, although the accuracy may be only ±5000 miles. This would make only the first zero after the 9 significant. On the other hand, a value of 50 mL measured with a graduated cylinder would be expected to have two significant figures owing to the greater accuracy of the measurement. To avoid ambiguity, numbers are often written as powers of 10 (scientific notation), making all digits significant. Using this convention, 92,900,000 would be written 9.290×10^7, indicating that there are four significant figures.

Calculations Using Significant Figures

The least precise measurement used in a calculation determines the number of significant figures in the answer. Thus, 73.5 + 0.418 = 73.9 rather than 73.918, since the least precise number (73.5) is accurate to only one decimal place. Similarly, 0.394 – 0.3862 = 0.008, with only one significant figure.

For multiplication or division, the rule of thumb is: The product or quotient has the same number of significant figures as the term with the fewest significant figures. As an example, in $28.08 \times 4.6/79.4 = 1.6268$, the term with the fewest significant figures is 4.6. Since this number has at most two significant figures, the result should be rounded off to 1.6.

Rounding Off

The results of mathematical computations are often rounded off to specific numbers of significant figures. This is done so that one does not infer an accuracy in the result that was not present in the measurements. The following rules are universally accepted and will ensure bias-free reporting of results (the number of significant figures desired should be determined first).

1. If the final digits of the number are 1, 2, 3, or 4, they are rounded down (dropped) and the preceding figure is retained unaltered.
2. If the final digits are 6, 7, 8, or 9, they are rounded up (i.e., they are dropped and the preceding figure is increased by one).
3. If the digit to be dropped is a 5, it is rounded down if the preceding figure is even and rounded up if the preceding figure is odd. Thus, 2.45 and 6.15 are rounded off to 2.4 and 6.2, respectively.

FUNCTIONS

A **function** is a particular type of relation between groups of numbers. The uniqueness of a function is that each member of one group is associated with exactly one member of another group. In general, let the variable x stand for the values of one group of numbers and the variable y stand for the values of another group. If each value of x is associated with a unique value of y, then this relation is a function. Specifically, y is said to be a function of x and is denoted $y = f(x)$. With this notation, x is called the *independent variable* and y the *dependent variable*. A function may be represented graphically by using a two-dimensional coordinate (Cartesian) plane formed by two perpendicular axes intersecting each other at a point with coordinates designated as $x = 0$, $y = 0$ (Fig. 6–1). The vertical axis denotes values of y and the horizontal axis values of x. The function is plotted as a series of points whose coordinates are the values of x with their corresponding values of y as determined by the function.

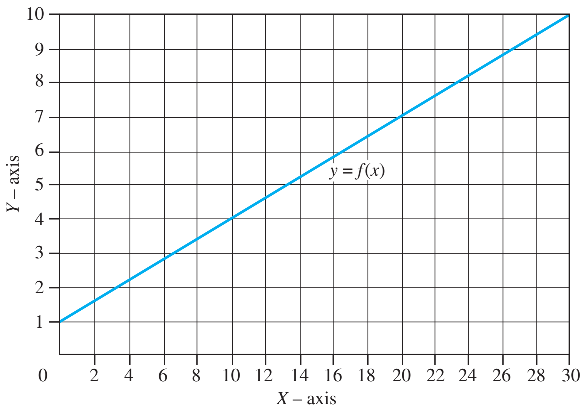

Figure 6-1 Graphic representation of the function y = f(x), where f(x)= 0.3x + 1.

Linear Functions

One of the simplest functions is expressed by the formula

$$y = ax$$

where

y and x are variables
a is a constant

The constant a is sometimes referred to as the *constant of proportionality* and y is said to be *directly proportional* to x (if y is expressed as $y = a/x$, y is said to be inversely proportional to x and the function is no longer linear). The graph of the equation $y = ax$ is a straight line. The constant a is the *slope* of the line.

General Linear Equation

$$y = ax + b$$

where

y = dependent variable
a = slope
x = independent variable
b = y-intercept (the value of y at which the graph of the equation crosses the y-axis)

Solving Linear Equations

To solve a linear equation,

1. Combine similar terms.
2. Use inverse operations to undo remaining additions and subtractions (i.e., add or subtract the same quantities to both sides of the equation). Get all terms with the unknown variable on one side of the equation.
3. If the equation involves fractions, multiply both sides by the least common denominator.
4. If there are multiplications or divisions indicated in the variable term, use inverse operations to find the value of the variable.
5. Check the result by substituting the value into the original equation.

Examples

1. $8 + 10x - 40 = 3x + 7 + 2x + \dfrac{2x}{3}$

$$10x - 32 = 5x + \frac{2x}{3} + 7$$

2. $10x - 32 + 32 = 5x + \dfrac{2x}{3} + 7 + 32$

$$10x = 5x + \frac{2x}{3} + 39$$

$$10x - 5x - \frac{2x}{3} = 39$$

$$5x - \frac{2x}{3} = 39$$

3. $3\left(5x - \dfrac{2x}{3}\right) = 3(39)$

$$15x - 2x = 117$$

4. $13x = 117$

$$\frac{13x}{13} = \frac{117x}{13}$$

$$x = 9$$

5. $8 + 10(9) - 40 = 3(9) + 7 + 2(9) + \dfrac{2(9)}{3}$

$$8 + 90 - 40 = 27 + 7 + 18 + 6$$

$$58 = 58$$

QUADRATIC EQUATIONS

A function of the form $y = ax^2$ is called a **quadratic function**. It is sometimes expressed in the more general form

$$y = ax^2 + bx + c$$

where

a, b, and c are constants

The graph of this equation is a *parabola*. Frequently, it is of interest to know where the parabola intersects the x-axis. The value of y at any point on the x-axis is zero. Therefore, to find the values of x where the graph intersects the x-axis, the quadratic equation is expressed in *standard form:*

$$ax^2 + bx + c = 0$$

with $a \neq 0$.

The solution of any quadratic equation expressed in standard form may be found using the **quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where

a, b, and c are the coefficients in the quadratic equation

Example	
Solve $3x^2 + 7 = 10x$	
Solution	
1. Write the equation in standard form.	$3x^2 - 10x + 7 = 0$
2. Note the coefficients a, b, and c.	$a = 3, b = -10, c = 7$
3. Substitute these values in the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	$x = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(3)(7)}}{2(3)}$
4. Simplify.	$x = \frac{10 \pm \sqrt{100 - 84}}{6}$ $= \frac{10 \pm \sqrt{16}}{6}$ $= 2.33 \text{ or } 1$

LOGARITHMS

The **logarithm** of a number (N) is the exponent (x) to which the base (a) must be raised to produce N. Thus, if $a^x = N$ then $\log_a N = x$ for $a > 0$ and $a \neq 0$. For example, $\log_2 8 = 3$ (read: the log to the base 2 equals 3) because $2^3 = 8$. Logarithms are written as numbers with two parts: an integer, called the characteristic, and a decimal, called the mantissa (e.g., $\log_{10} 86 = 1.9345$).

Common Logarithms

Common logarithms are those that have the base 10. In this book, the base number will be omitted with the assumption that log means $\log_{10}$. Table 6–1 shows the general rules of common logarithms.

x	log x
$10^0 = 1$	$\log 1 = 0$
$10^1 = 10$	$\log 10 = 1$
$10^2 = 100$	$\log 100 = 2$

Table 6-1 Rules of Common Logarithms

Rule	Example
1. $\log ab = \log a + \log b$ $(a > 0, b > 0)$	$x = (746)(384)$ $\log x = \log 746 + \log 384$ $\log 746 = 2.8727$ $\log 384 = 2.5843$ $\log x = 5.4570$ $x = 286{,}400$
2. $\log 1/a = -\log a$ $(a > 0)$	$x = 1/273$ $\log x = -\log 273$ $= -2.4362$ $= \bar{3}.5638$ $x = 0.003663$

Table 6-1 Rules of Common Logarithms (continued)

Rule	Example
3. $\log a/b = \log a - \log b$ $(a > 0, b > 0)$	$x = 478/21$ $\log x = \log 478 - \log 21$ $\log 478 = 2.6794$ $\log 21 = 1.3222$ $\log x = 1.3572$ $x = 22.76$
4. $\log a^n = n \log a$ ($a > 0$, n is a real number)	$x = \sqrt[3]{374}$ $= (374)^{1/3}$ $\log x = 1/3 \log 374$ $\log 374 = 2.5729$ $\log x = 1/3(2.5729)$ $= 0.8576$ $x = 7.204$

The Characteristic

The integer or **characteristic** of the logarithm of a number is determined by the position of the decimal point in the number. The characteristic of a number can easily be found by expressing the number in scientific notation. Once in this form, the exponent is used as the characteristic.

Examples

Number	Characteristic
$3025 = 3.025 \times 10^3$	3
$302.5 = 3.025 \times 10^2$	2
$30.25 = 3.025 \times 10^1$	1
$3.025 = 3.025 \times 10^0$	0
$0.3025 = 3.025 \times 10^{-1}$	-1

Note: The characteristics of logarithms of numbers less than 1 can be written in several ways. Thus, $\log 0.0361 = \log 3.61 \times 10^{-2} = \bar{2}.5575$ (not –2.5575) or 8.5575 – 10. Written as a negative number (as with handheld calculators) $\log 0.0361 = -1.4425$.

The Mantissa

The **mantissa** is the decimal part of the logarithm of a number. The mantissa of a series of digits is the same regardless of the position of the decimal point. Thus, the logarithms of 1.7, 17, and 170 all have the same mantissa, which is 0.230.

Antilogarithms

The number having a given logarithm is called the **antilogarithm** (antilog). The logarithm of 125 is approximately 2.0969. Therefore, the antilog of 2.0969 is $10^{2.0969}$, which is approximately 125.

Antilogs of Negative Logarithms

Using a calculator, negative logarithms can be solved simply by using the 10^x key. For example, the antilog of –2 is $10^{-2} = 0.01$. However, log and antilog tables in reference books are used with positive mantissas only. Therefore, a negative logarithm must be changed to a log with a positive mantissa to find its antilog. This change of form is accomplished by first adding and then subtracting 1, which does not alter the original value of the logarithm.

Example	
Find antilog -1.6415	
Solution	
1. Write the log as a negative characteristic minus the mantissa.	$-1.6415 = -1 - 0.6415$
2. Subtract 1 from the characteristic and add 1 to the mantissa.	$-1 - 0.6415 = -1 - 1 - 0.6415 + 1 = -2 + 0.3585$
3. Express the result as a log having a negative characteristic and a positive mantissa.	$-2 + 0.3585 = \bar{2}.3585$
4. Use the tables to find the antilog.	antilog -1.6415 = antilog $\bar{2}.3585 = 0.02283$

Natural Logarithms

When a logarithmic function must be differentiated or integrated, it is convenient to rewrite the function with the number e as a base. The number e is approximately equal to 2.71828. Logarithms which have the base e are called **natural logarithms** and are denoted by ln (read: "ell-en").

$$\text{If } e^x = N, \text{ then } \log_e N = \ln N = x$$

Any number of the form a^x may be rewritten with e as the base:

$$a^x = e^{x \ln a}$$

Note: e^x is sometimes written as $\exp(x)$.

The same rules apply to natural logarithms that apply to common logarithms. See Table 6–2.

Table 6-2 Rules of Natural Logarithms

1.	$\ln ab = \ln a + \ln b$	$(a > 0, b > 0)$
2.	$\ln 1/a = -\ln a$	$(a > 0)$
3.	$\ln a/b = \ln a - \ln b$	$(a > 0, b > 0)$
4.	$\ln a^x = x \ln a$	$(a > 0, x$ is a real number$)$
5.	$\ln e = 1$	
6.	$\ln e^x = x = e^{\ln x}$	
7.	$a^x = e^{x \ln x}$	$(a > 0)$
8.	$\ln x = (\ln 10)(\log x) = 2.3026(\log x)$	$(x > 0)$

Change of Base

Logarithms to one base can easily be changed to logarithms of another base using the following equation.

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Example

$$\log_{10} x = \frac{\log_e x}{\log_e 10} = \frac{\ln x}{\ln 10}$$

$$\therefore \ \ln x = 2.302585 \log_{10} x$$

TRIGONOMETRY

Systems of Angular Measure

Degree

The degree is defined as 1/360 of a complete revolution.

1 revolution = 360°
1 right angle = 90°
1 degree = 60 minutes (60′)
1 minute = 60 seconds (60″)

Radian

The radian is defined as the angle subtended at the center of a circle by an arc whose length is equal to the radius of the circle. In general, an angle θ in radians is given by

$$\theta = \frac{s}{r}$$

where

s = arc length
r = radius

Relationship Between Degrees and Radians

1 revolution = 2π radians
π = 3.14159 . . .
1 degree = $2\pi/360$ radians = 0.0174.53 radian
30° = $\pi/6$ radians
45° = $\pi/4$ radians
60° = $\pi/3$ radians
90° = $\pi/2$ radians
180° = π radians

Trigonometric Functions

In trigonometry, an angle is considered positive if it is generated by a counterclockwise rotation from standard position, and negative if it is generated by a clockwise rotation (Fig. 6–2). The trigonometric functions of a positive acute angle θ can be defined as ratios of the sides of a right triangle:

$$\text{sine of } \theta = \sin\theta = y/r$$
$$\text{cosine of } \theta = \cos\theta = x/r$$
$$\text{tangent of } \theta = \tan\theta = y/x$$
$$\text{cotangent of } \theta = \cot\theta = x/y$$
$$\text{secant of } \theta = \sec\theta = \mathrm{r}/x$$
$$\text{cosecant of } \theta = \csc\theta = r/y$$

These functions can also be expressed in terms of sine and cosine alone:

$$\tan\theta = \sin\theta/\cos\theta$$
$$\cot\theta = \cos\theta/\sin\theta$$
$$\sec\theta = 1/\cos\theta$$
$$\csc\theta = 1/\sin\theta$$

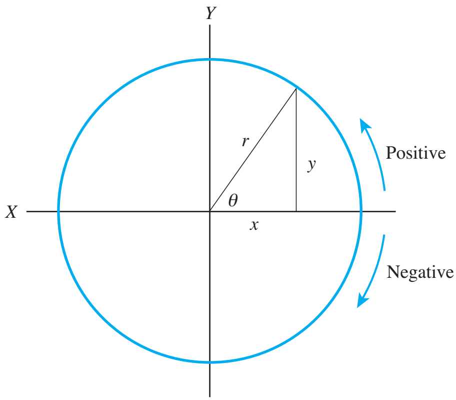

Figure 6–2 An angle is generated by rotating a ray (or half-line) about the origin of a circle. The angle is positive if it is generated by a counterclockwise rotation from the *x*-axis and negative for a clockwise rotation.

Basic Trigonometric Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\sec^2\theta = 1 + \tan^2\theta$$

PROBABILITY

The **probability** of an event A is denoted $p(A)$. It is defined as follows: If an event can occur in p number of ways and can fail to occur in q number of ways, then the probability of the event occurring is $p/(p + q)$. The *odds in favor* of an event occurring are p to q.

Addition Rule

If A and B are any events, then

$$p(A \text{ or } B) = p(A) + p(B) - p(A \text{ and } B)$$

Example

The probability of drawing either a king or a black card from a deck of 52 playing cards is

$$\begin{aligned} p(\text{king or black card}) &= p(\text{king}) + p(\text{black card}) - p(\text{king also black}) \\ &= 4/52 + 26/52 - 2/52 \\ &= 7/13 \end{aligned}$$

Note: If events A and B cannot occur at the same time, they are said to be mutually exclusive, and the addition rule can be simplified to

$$p(A \text{ or } B) = p(A) + p(B)$$

Multiplication Rule

If A and B are any events, then

$$p(A \text{ and } B) = p(A|B) \times p(B)$$

where

$p(A|B)$ = the probability of event A given that event B has occurred