CHAPTER 8

Numerical Methods

In this chapter we look at numerical techniques for solving systems of linear equations. We add four important methods, namely Gausssian elimination, LU decomposition, the Jacobi method, and the Gauss-Seidel method to our library of techniques of solving systems of linear equations. We discuss the merits of the various methods, including their reliability for solving various types of systems. Certain systems of equations can lead to incorrect results unless great care is taken. We discuss ways of recognizing and solving "delicate" systems.

While the determinant approach of Chapter 5 is useful for finding eigenvalues and eigenvectors of small matrices and for developing the theory, it is not practical for finding eigenvalues and eigenvectors of large matrices that occur in applications. We introduce a numerical technique for finding such eigenvalues and eigenvectors.

Applications discussed include an analysis of networks—ways of describing connectivities of networks and the accessibility of their vertices. Such measures are used to compare connectivities of cities and regions, and to plan where roads should be built.

*8.1 Gaussian Elimination

There are many elimination methods in addition to the method of Gauss-Jordan elimination for solving systems of linear equations. In this section we introduce another elimination method called **Gaussian elimination**. Different methods are suitable for different occasions. It is important to choose the best method for the purpose in mind. We shall discuss the relative merits of Gauss-Jordan elimination and Gaussian elimination. The merits and drawbacks of other methods will be discussed later.

The method of Gaussian elimination involves an *echelon form* of the augmented matrix of the system of equations. An echelon form satisfies the first three of the conditions of the reduced echelon form.

DEFINITION A matrix is in **echelon form** if

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each row is 1. This element is called a **leading 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row. (This implies that all the elements below a leading 1 are zero.)

The following matrices are all in echelon form.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & -6 & 4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 4 & 6 & 2 & 5 & 2 \\ 0 & 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 6 \end{bmatrix}$$

The difference between a reduced echelon form and an echelon form is that the elements above and below a leading 1 are zero in a reduced echelon form, while only the elements below the leading 1 need be zero in an echelon form.

The Gaussian elimination algorithm is as follows.

Gaussian Elimination

1. Write down the augmented matrix of the system of linear equations.
2. Find an echelon form of the augmented matrix using elementary row operations. This is done by creating leading 1's, then zeros below each leading 1, column by column, starting with the first column.
3. Write down the system of equations corresponding to the echelon form.
4. Use back substitution to arrive at the solution.

We illustrate the method with the following example.

EXAMPLE 1 Solve the following system of linear equations using the method of Gaussian elimination.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ -x_1 - 2x_2 - 2x_3 + x_4 &= 2 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= 4 \end{aligned}$$

SOLUTION

Starting with the augmented matrix, create zeros below the pivot in the first column.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \underset{\substack{R2 + R1 \\ R3 + (-2)R1}}{\approx} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 2 & 8 & 6 \end{bmatrix}$$

At this stage we create a zero only *below* the pivot.

$$\underset{R3 + (-2)R2}{\approx} \begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 2 & 4 \end{bmatrix} \underset{(1/2)\,R3}{\approx} \underbrace{\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}}_{\text{echelon form}}$$

We have arrived at the echelon form.
The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ x_3 + 3x_4 &= 1 \\ x_4 &= 2 \end{aligned}$$

Observe that the effect of performing the row operations in this manner to arrive at an echelon form is to eliminate variables from equations. This is called **forward elimination**. This system is now solved by **back substitution**. (The terms **forward pass** and **backward pass** are also used.) The value of x_4 is substituted into the second equation to give x_3. x_3 and x_4 are then substituted into the first equation to get x_1. We get

$$\begin{aligned} x_3 + 3(2) &= 1, \\ x_3 &= -5 \end{aligned}$$

Substituting $x_4 = 2$ and $x_3 = -5$ into the first equation,

$$\begin{aligned} x_1 + 2x_2 + 3(-5) + 2(2) &= -1, \\ x_1 + 2x_2 &= 10, \\ x_1 &= -2x_2 + 10 \end{aligned}$$

Let $x_2 = r$. The system has many solutions. The solutions are

$$x_1 = -2r + 10, \quad x_2 = r, \quad x_3 = -5, \quad x_4 = 2$$

The forward elimination of variables in this method was performed using matrices and elementary row operations. The back substitution can also be performed using matrices. The final matrix is then the reduced echelon form of the system. This way of performing the back substitution can be implemented on a computer. We illustrate the method for the system of equations of the previous example.

EXAMPLE 2 Solve the following system of linear equations using the method of Gaussian elimination, performing back substitution using matrices.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= -1 \\ -x_1 - 2x_2 - 2x_3 + x_4 &= 2 \\ 2x_1 + 4x_2 + 8x_3 + 12x_4 &= 4 \end{aligned}$$

SOLUTION

We arrive at the echelon form as in the previous example.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ -1 & -2 & -2 & 1 & 2 \\ 2 & 4 & 8 & 12 & 4 \end{bmatrix} \approx \cdots \approx \underbrace{\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}}_{\text{echelon form}}$$

This marks the end of the forward elimination of variables from equations. We now commence the back substitution using matrices.

$$\begin{bmatrix} 1 & 2 & 3 & 2 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{R1 + (-2)R3 \\ R2 + (-3)R3}}{\approx} \begin{bmatrix} 1 & 2 & 3 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(Create zeros above the leading 1 in row 3. This is equivalent to substituting for x_4 from Equation 3 into Equations 1 and 2.)

$$\underset{R1 + (-3)R2}{\approx} \begin{bmatrix} 1 & 2 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

(Create a zero above the leading 1 in row 2. This is equivalent to substituting for x_3 from Equation 2 into Equation 1.)

This matrix is the reduced echelon form of the original augmented matrix. The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 \qquad\qquad &= 10 \\ x_3 \qquad &= -5 \\ x_4 &= 2 \end{aligned}$$

Let $x_2 = r$. We get the same solution as previously,

$$x_1 = -2r + 10, \quad x_2 = r, \quad x_3 = -5, \quad x_4 = 2$$

Comparison of Gauss-Jordan and Gaussian Elimination

The method of Gaussian elimination is in general more efficient than Gauss-Jordan elimination in that it involves fewer operations of addition and multiplication. It is during the back substitution that Gaussian elimination picks up this advantage. We now illustrate how Gaussian elimination saves two operations over Gauss-Jordan elimination in the preceding example. Consider the final transformation that brings the matrix to reduced echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & -5 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \underset{R1 + (-3)R2}{\approx} \begin{bmatrix} 1 & 2 & 0 & 0 & 10 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

The aim of this transformation is to create a 0 in location $(1, 3)$. Note that changing the 3 in location $(1, 3)$ to 0 need not in practice involve any arithmetic operations, as one (or the computer) knows in advance that the element is to be zero. The 0 in the $(1, 4)$ location remains unchanged; no arithmetic operations need be performed on it. The row operation **R**1 $+ (-3)$**R**2 in fact uses only two arithmetic operations—one of multiplication and one of addition—in changing the -5 in the $(1, 5)$ location to 10:

$$-5 \underset{\substack{\nearrow \\ \text{addition}}}{+} (-3)\underset{\substack{\nwarrow \\ \text{multiplication}}}{(-5)} = 10$$

On the other hand, when the zero is created in the $(1, 3)$ location during Gauss-Jordan elimination, the $(1, 4)$ element and the $(1, 5)$ element are both changed, each involving two operations, one addition and the other multiplication. These two operations for the change in the $(1, 4)$ element are two additional operations involved in Gauss-Jordan elimination.

In larger systems of equations, many more operations are saved in Gaussian elimination during back substitution. The reduction in the number of operations not only saves time on a computer but also increases the accuracy of the final answer. With each arithmetic operation there is a possibility of round-off error on a computer. With large systems, the method of Gauss-Jordan elimination involves approximately 50% more arithmetic operations than does Gaussian elimination (see the following table).

Count of Operations for an $n \times n$ System with Unique Solution

	Number of Multiplications	Number of Additions
Gauss-Jordan elimination	$\frac{n^3}{2} + \frac{n^2}{2} \approx \frac{n^3}{2}$ (for large n)	$\frac{n^3}{2} - \frac{n}{2} \approx \frac{n^3}{2}$
Gaussian elimination	$\frac{n^3}{3} + n^2 - \frac{n}{3} \approx \frac{n^3}{3}$	$\frac{n^3}{3} + \frac{n^2}{2} - \frac{5n}{6} \approx \frac{n^3}{3}$

Gauss-Jordan elimination, on the other hand, has the advantage of being more straightforward for hand computations. It is easier for solving small systems and it is the method that we use in this course when we solve systems of linear equations by hand.

We complete this section with a discussion of the formulas for the total number of operations involved in solving a system of n linear equations in n variables that has a unique solution, using both Gauss-Jordan elimination and Gaussian elimination. We group multiplications and divisions together as multiplications and additions and subtractions together as additions.

If there are many equations, with n large, then the term with the highest power of n dominates the other terms in the preceding formulas. The total number of operations with Gauss-Jordan elimination is approximately n^3 while the total in Gaussian elimination is approximately $\frac{2n^3}{3}$. Gaussian elimination is, thus, approximately 50% more efficient than Gauss-Jordan elimination.

We now derive the above formulas for Gauss-Jordan elimination, leaving it for the reader to arrive at the formulas for Gaussian elimination in the exercises that follow. Let us denote general elements in the matrices by $*$. Assume that there are no row interchanges. Note that when an element is known to become a 1 or a zero there are no arithmetic operations involved; substitution is used. Thus, for example, there are no operations involved in the location where a leading one is created. We get, starting with the $n \times (n + 1)$ augmented matrix of the system,

$$
\begin{array}{c}
n \\ \text{rows}
\end{array}
\left\{ \overbrace{\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}}^{n\,+\,1 \text{ columns}} \right.
\underset{n \text{ mults}}{\approx}
\begin{bmatrix} 1 & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ * & * & \cdots & * \end{bmatrix}
\underset{\substack{n \text{ mults} \\ n \text{ adds} \\ (\text{per row})}}{\approx}
\begin{bmatrix} 1 & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}
$$

$$
\underset{n\,-\,1 \text{ mults}}{\approx}
\begin{bmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}
\underset{\substack{n\,-\,1 \text{ mults} \\ n\,-\,1 \text{ adds} \\ (\text{per row})}}{\approx}
\begin{bmatrix} 1 & 0 & \cdots & * \\ 0 & 1 & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & * \end{bmatrix}
$$

$$
\underset{1 \text{ mult}}{\approx \cdots \approx}
\begin{bmatrix} 0 & 1 & * & * \\ 0 & 1 & * & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & * \end{bmatrix}
\underset{\substack{1 \text{ mult} \\ 1 \text{ add} \\ (\text{per row})}}{\approx}
\begin{bmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & * \end{bmatrix}
$$

We now add up these operations. Remember that there are $(n - 1)$ rows involved every time zeros are created "per row" in the above manner. The following formula is used for the sum of the first n integers

$$[n + (n - 1) + \cdots + 1] = \frac{n(n + 1)}{2}$$

Total number of multiplications

$$= \underbrace{[n + (n+1) + \cdots + 1]}_{\text{to create leading 1's}} + \underbrace{(n-1)[n + (n-1) + \cdots + 1]}_{\text{to create zeros}}$$

$$= n[n + (n-1) + \cdots + 1] = n\left[\frac{n(n+1)}{2}\right] = \frac{n^3}{2} + \frac{n^2}{2}$$

Total number of additions

$$= (n-1)[n + (n+1) + \cdots + 1] = (n-1)\left[\frac{n(n+1)}{2}\right] = \frac{n^3}{2} - \frac{n}{2}$$

EXERCISE SET 8.1

Echelon Form

1. Determine whether or not each of the following matrices is in echelon form.

(a) $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 5 & 6 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 7 & 6 & 2 \\ 0 & 0 & 1 & 8 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. Determine whether or not each of the following matrices is in echelon form.

(a) $\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & 4 & 6 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 4 & 2 & 3 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & -1 & 4 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Gaussian Elimination

In Exercises 3–6, solve the systems of equations using Gaussian elimination.

(a) Perform the back substitution using equations.

(b) Perform the back substitution using matrices.

3.
$$\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 - x_2 + x_3 &= 2 \\ x_1 + 2x_2 + 3x_3 &= 14 \end{aligned}$$

4.
$$\begin{aligned} x_1 - x_2 - x_3 &= 2 \\ x_1 - x_2 + x_3 &= 2 \\ 3x_1 - 2x_2 + x_3 &= 5 \end{aligned}$$

5.
$$\begin{aligned} x_1 - x_2 + 2x_3 &= 3 \\ 2x_1 - 2x_2 + 5x_3 &= 4 \\ x_1 + 2x_2 - x_3 &= -3 \\ 2x_2 + 2x_3 &= 1 \end{aligned}$$

6.
$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 - 2x_5 &= 1 \\ 2x_1 - x_2 - x_3 + 3x_4 - x_5 &= 3 \\ -x_1 - x_2 + 5x_3 \qquad\quad - 4x_5 &= -3 \end{aligned}$$

Miscellaneous Results

7. Consider a system of four equations in five variables. In general, how many arithmetic operations will be saved by using Gaussian elimination rather than Gauss-Jordan elimination to solve the system? Where are these operations saved?

8. Compare Gaussian elimination to Gauss-Jordan elimination for a system of four equations in six variables. Determine where operations are saved during Gaussian elimination.

9. Can a matrix have more than one echelon form? (*Hint*: Consider a 2×3 matrix and arrive at an echelon form using two distinct sequences of row operations.)

10. Consider a system of n linear equations in n variables that has a unique solution. Show that Gaussian elimination involves $n^3/3 + n^2/2 + n/6$ multiplications and $n^3/3 - n/3$ additions to arrive at an echelon form, and then a further $n^2/2 - n/2$ multiplications and $n^2/2 - n/2$ additions to arrive at the reduced echelon form. Thus a total of $n^3/3 + n^2 - n/3$ multiplications and $n^3/3 + n^2/2 - 5n/6$ additions are involved in solving the system of equations using Gaussian elimination. (The formula for the sum of squares is $n^2 + (n-1)^2 + \cdots + 1 = [n(n+1)(2n+1)]/6$.)

11. Construct a table that gives the number of multiplications and additions in both Gauss-Jordan elimination and Gaussian elimination for systems of n equations in n variables having unique solutions, for $n = 2$ to 10.