City Hall in London, England, stands on the left side of the River Thames, in the More London development by Tower Bridge. The building, designed by British architect Norman Foster, was opened in 2002. Its bulbous shape reduces surface area and thus improves its energy efficiency.

CHAPTER

1

Linear Equations

Mathematics is, of course, a discipline in its own right. It is, however, more than that—it is a tool used in many other fields. Linear algebra is a branch of mathematics that plays a central role in modern mathematics, and also is of importance to engineers and physical, social, and behavioral scientists. In this course the reader will learn mathematics, will learn to think mathematically, and will be instructed in the art of applying mathematics. The course is a blend of theory, numerical techniques, and interesting applications.

When mathematics is used to solve a problem it often becomes necessary to find a solution to a so-called system of linear equations. Historically, linear algebra developed from studying methods for solving such equations. This chapter introduces methods for solving systems of linear equations. We shall discuss two applications of systems of linear equations. We shall determine currents through electrical networks and analyze traffic flows through road networks.

1.1 Matrices and Systems of Linear Equations

An equation in the variables x and y that can be written in the form $ax + by = c$, where a, b, and c are real constants (a and b not both zero), is called a *linear equation*. The graph of such an equation is a straight line in the xy plane. Consider the system of two linear equations,

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 4 \end{aligned}$$

A pair of values of x and y that satisfies both equations is called a **solution**. It can be seen by substitution that $x = 3$, $y = 2$ is a solution to this system. A solution to such a system will be a point at which the graphs of the two equations intersect. The following examples, Figures 1.1, 1.2, and 1.3, illustrate that three possibilities can arise for such systems of equations. There can be a unique solution, no solution, or many solutions. We use the point/slope form $y = mx + b$, where m is the slope and b is the y-intercept, to graph these lines.

Unique solution

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 4 \end{aligned}$$

Write as $y = -x + 5$ and $y = 2x - 4$. The lines have slopes -1 and 2, and y-intercepts 5 and -4. They intersect at a point, the solution. There is a unique solution, $x = 3$, $y = 2$.

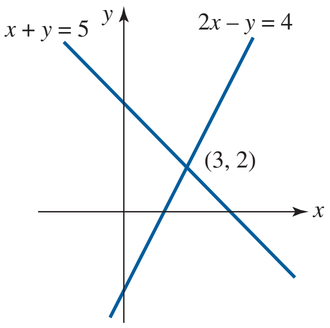


Figure 1.1

No solution

$$\begin{aligned} -2x + y &= 3 \\ -4x + 2y &= 2 \end{aligned}$$

Write as $y = 2x + 3$ and $y = 2x + 1$. The lines have slope 2, and y-intercepts 3 and 1. They are parallel. There is no point of intersection. No solution.

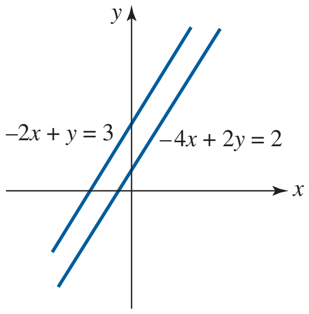


Figure 1.2

Many solutions

$$\begin{aligned} 4x - 2y &= 6 \\ 6x - 3y &= 9 \end{aligned}$$

Each equation can be written as $y = 2x - 3$. The graph of each equation is a line with slope 2 and y-intercept -3. Any point on the line is a solution. Many solutions

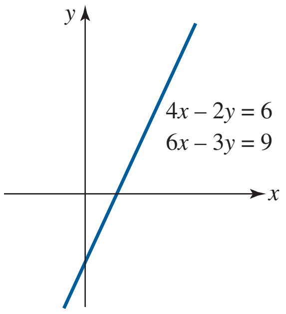


Figure 1.3

Our aim in this chapter is to analyze larger systems of linear equations. A **linear equation** *in n variables* $x_1, x_2, x_3, \ldots, x_n$ is one that can be written in the form

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n = b$$

where the coefficients $a_1, a_2, \ldots, a_n$ and b are constants. The following is an example of a system of three linear equations.

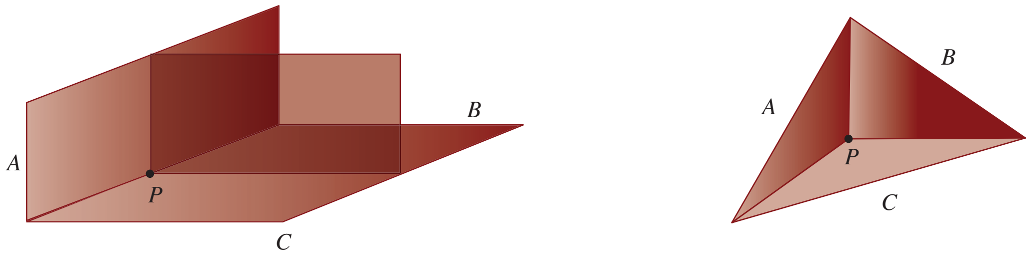
$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

It can be seen on substitution that $x_1 = -1$, $x_2 = 1$, $x_3 = 2$ is a solution to this system. (We arrive at this solution in Example 1 of this section.)

A linear equation in three variables corresponds to a plane in three-dimensional space. Solutions to a system of three such equations will be points that lie on all three planes. As for systems of two equations there can be a unique solution, no solution, or many solutions. We illustrate some of the various possibilities in Figure 1.4.

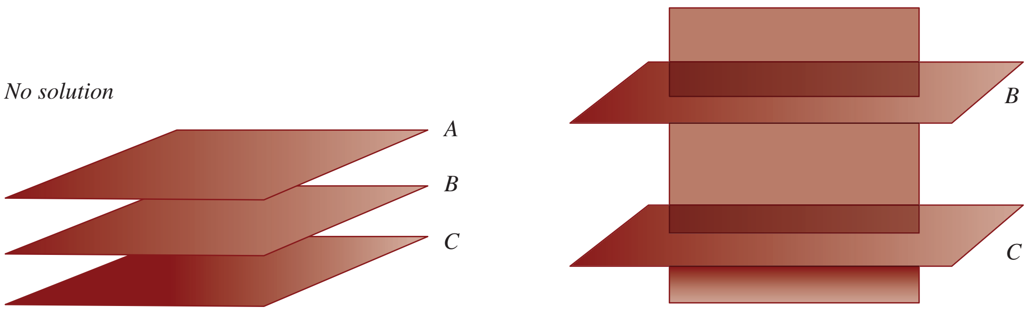
As the number of variables increases, a geometrical interpretation of such a system of equations becomes increasingly complex. Each equation will represent a space embedded in a larger space. Solutions will be points that lie on all the embedded spaces. While a general geometrical way of thinking about a problem is often useful, we rely on algebraic methods for arriving at and interpreting the solution. We introduce a method for solving systems

Unique solution



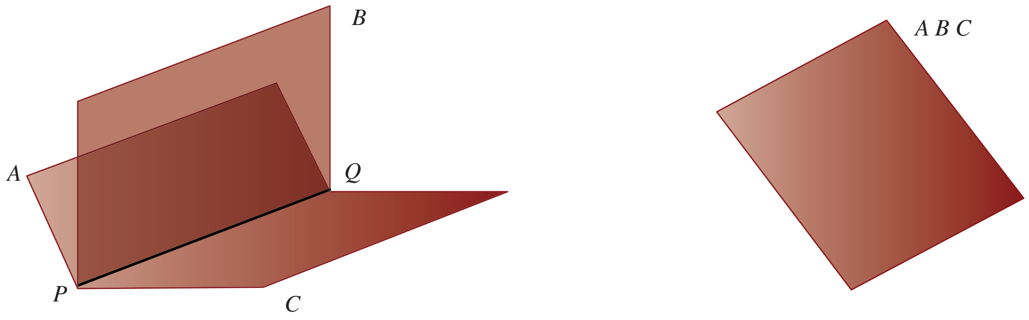
Three planes *A*, *B*, and *C* intersect at a single point *P*.
P corresponds to a unique solution.

No solution



Planes *A*, *B*, and *C* have no points in common.
There is no solution.

Many solutions



Three planes *A*, *B*, and *C* intersect in a line *PQ*. Any point on the line is a solution.

Three equations represent the same plane. Any point on the plane is a solution.

Figure 1.4

of linear equations called **Gauss-Jordan elimination**.[1] This method involves systematically eliminating variables from equations. In this section we shall see how this method applies to systems of equations that have a unique solution. In the following section we shall extend the method to more general systems of linear equations.

We shall use rectangular arrays of numbers called matrices to describe systems of linear equations. At this time we introduce the necessary terminology.

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[1] Carl Friedrich Gauss (1777–1855) was one of the greatest mathematical scientists ever. Among his discoveries was a way to calculate the orbits of asteroids. He taught for forty-seven years at the University of Göttingen, Germany. He made contributions to many areas of mathematics, including number theory, probability, and statistics. Gauss has been described as "not really a physicist in the sense of searching for new phenomena, but rather a mathematician who attempted to formulate in exact mathematical terms the experimental results of others." Gauss had a turbulent personal life, suffering financial and political problems because of revolutions in Germany.

Wilhelm Jordan (1842–1899) taught geodesy at the Technical College of Karlsruhe, Germany. His most important work was a handbook on geodesy that contained his research on systems of equations. Jordan was recognized as being a master teacher and an excellent writer.

DEFINITION A **matrix** is a rectangular array of numbers. The numbers in the array are called the *elements* of the matrix.

Matrices are usually denoted by capital letters. Examples of matrices in standard notation are

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 1 \\ 0 & 5 \\ -8 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 5 & 6 \\ 0 & -2 & 5 \\ 8 & 9 & 12 \end{bmatrix}$$

Rows and Columns Matrices consist of rows and columns. Rows are labeled from the top of the matrix, columns from the left. The following matrix has two rows and three columns.

$$\begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix}$$

The rows are:

$$\underset{\text{row 1}}{\begin{bmatrix} 2 & 3 & -4 \end{bmatrix}}, \qquad \underset{\text{row 2}}{\begin{bmatrix} 7 & 5 & -1 \end{bmatrix}}$$

The columns are:

$$\underset{\text{column 1}}{\begin{bmatrix} 2 \\ 7 \end{bmatrix}}, \qquad \underset{\text{column 2}}{\begin{bmatrix} 3 \\ 5 \end{bmatrix}}, \qquad \underset{\text{column 3}}{\begin{bmatrix} -4 \\ -1 \end{bmatrix}}$$

Submatrix A submatrix of a given matrix is an array obtained by deleting certain rows and columns of the matrix. For example, consider the following matrix A. The matrices P, Q, and R are submatrices of A.

$$A = \underset{\text{matrix } A}{\begin{bmatrix} 1 & 7 & 4 \\ 2 & 3 & 0 \\ 5 & 1 & -2 \end{bmatrix}} \qquad P = \begin{bmatrix} 1 & 7 \\ 2 & 3 \\ 5 & 1 \end{bmatrix} \quad Q = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} \quad R = \begin{bmatrix} 1 & 4 \\ 5 & -2 \end{bmatrix}$$

submatrices of A

Size and Type The size of a matrix is described by specifying the number of rows and columns in the matrix. For example, a matrix having two rows and three columns is said to be a 2×3 matrix; the first number indicates the number of rows, the second indicates the number of columns. When the number of rows is equal to the number of columns, the matrix is said to be a **square matrix**. A matrix consisting of one row is called a **row matrix**. A matrix consisting of one column is a **column matrix**. The following matrices are of the stated sizes and types.

$$\underset{2 \times 3 \text{ matrix}}{\begin{bmatrix} 1 & 0 & 3 \\ -2 & 4 & 5 \end{bmatrix}} \qquad \underset{\substack{3 \times 3 \text{ matrix} \\ \text{a square matrix}}}{\begin{bmatrix} 2 & 5 & 7 \\ -9 & 0 & 1 \\ -3 & 5 & 8 \end{bmatrix}} \qquad \underset{\substack{1 \times 4 \text{ matrix} \\ \text{a row matrix}}}{\begin{bmatrix} 4 & -3 & 8 & 5 \end{bmatrix}} \qquad \underset{\substack{3 \times 1 \text{ matrix} \\ \text{a column matrix}}}{\begin{bmatrix} 8 \\ 3 \\ 2 \end{bmatrix}}$$

Location The location of an element in a matrix is described by giving the row and column in which the element lies. For example, consider the following matrix.

$$\begin{bmatrix} 2 & 3 & -4 \\ 7 & 5 & -1 \end{bmatrix}$$

The element 7 is in row 2, column 1. We say that it is in location (2, 1).

The element in location (1, 3) is -4. Note that the convention is to give the row in which the element lies, followed by the column.

Identity Matrices An identity matrix is a square matrix with 1s in the *diagonal* locations (1, 1), (2, 2), (3, 3), etc., and zeros elsewhere. We write I_n for the $n \times n$ identity matrix. The following matrices are identity matrices.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We are now ready to continue the discussion of systems of linear equations. We use matrices to describe systems of linear equations. There are two important matrices associated with every system of linear equations. The coefficients of the variables form a matrix called the **matrix of coefficients** of the system. The coefficients, together with the constant terms, form a matrix called the **augmented matrix** of the system. For example, the matrix of coefficients and the augmented matrix of the following system of linear equations are as shown:

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned} \qquad \underset{\text{matrix of coefficients}}{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 1 & -1 & -2 \end{bmatrix}} \qquad \underset{\text{augmented matrix}}{\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}}$$

Observe that the matrix of coefficients is a submatrix of the augmented matrix. The augmented matrix completely describes the system.

Transformations called **elementary transformations** can be used to change a system of linear equations into another system of linear equations that has the same solution. These transformations are used to solve systems of linear equations by eliminating variables. In practice it is simpler to work in terms of matrices using analogous transformations called **elementary row operations**. It is not necessary to write down the variables x_1, x_2, x_3, at each stage. Systems of linear equations are in fact described and manipulated on computers in terms of such matrices. These transformations are as follows.

Elementary Transformations	**Elementary Row Operations**
1. Interchange two equations.	1. Interchange two rows of a matrix.
2. Multiply both sides of an equation by a nonzero constant.	2. Multiply the elements of a row by a nonzero constant.
3. Add a multiple of one equation to another equation.	3. Add a multiple of the elements of one row to the corresponding elements of another row.

Systems of equations that are related through elementary transformations are called **equivalent systems**. Matrices that are related through elementary row operations are called **row equivalent matrices**. The symbol $\approx$ is used to indicate equivalence in both cases.

Elementary transformations preserve solutions since the order of the equations does not affect the solution, multiplying an equation throughout by a nonzero constant does not change the truth of the equality, and adding equal quantities to both sides of an equality results in an equality.

The method of Gauss-Jordan elimination uses elementary transformations to eliminate variables in a systematic manner, until we arrive at a system that gives the solution. We illustrate Gauss-Jordan elimination using equations and the analogous matrix implementation of the method side by side in the following example. The reader should note the way in which the variables are eliminated in the equations in the left column. At the same time observe how this is accomplished in terms of matrices in the right column by creating zeros in certain locations. We shall henceforth be using the matrix approach.

EXAMPLE 1 Solve the system of linear equations

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

SOLUTION

Equation Method
Initial System

$$\begin{aligned} x_1 + x_2 + x_3 &= 2 \\ 2x_1 + 3x_2 + x_3 &= 3 \\ x_1 - x_2 - 2x_3 &= -6 \end{aligned}$$

Eliminate x_1 from 2nd and 3rd equations.

$$\begin{array}{cr} \approx & x_1 + x_2 + x_3 = 2 \\ \text{Eq2} + (-2)\text{Eq1} & x_2 - x_3 = -1 \\ \text{Eq3} + (-1)\text{Eq1} & -2x_2 - 3x_3 = -8 \end{array}$$

Eliminate x_2 from 1st and 3rd equations.

$$\begin{array}{cr} \approx & x_1 + 2x_3 = 3 \\ \text{Eq1} + (-1)\text{Eq2} & x_2 - x_3 = -1 \\ \text{Eq3} + (2)\text{Eq2} & -5x_3 = -10 \end{array}$$

Make coefficient of x_3 in 3rd equation 1 (i.e., solve for x_3).

$$\begin{array}{cr} & x_1 + 2x_3 = 3 \\ \approx & x_2 - x_3 = -1 \\ (-1/5)\text{Eq3} & x_3 = 2 \end{array}$$

Eliminate x_3 from 1st and 2nd equations.

$$\begin{array}{cr} \approx & x_1 = -1 \\ \text{Eq1} + (-2)\text{Eq3} & x_2 = 1 \\ \text{Eq2} + \text{Eq3} & x_3 = 2 \end{array}$$

The solution is $x_1 = -1$, $x_2 = 1$, $x_3 = 2$.

Analogous Matrix Method
Augmented Matrix

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 3 \\ 1 & -1 & -2 & -6 \end{bmatrix}$$

Create zeros in column 1.

$$\begin{array}{c} \approx \\ \text{R2} + (-2)\text{R1} \\ \text{R3} + (-1)\text{R1} \end{array} \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & -1 \\ 0 & -2 & -3 & -8 \end{bmatrix}$$

Create appropriate zeros in column 2.

$$\begin{array}{c} \approx \\ \text{R1} + (-1)\text{R2} \\ \text{R3} + (2)\text{R2} \end{array} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & -5 & -10 \end{bmatrix}$$

Make the (3, 3) element 1 (called "normalizing" the element).

$$\begin{array}{c} \approx \\ (-1/5)\text{R3} \end{array} \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Create zeros in column 3.

$$\begin{array}{c} \approx \\ \text{R1} + (-2)\text{R3} \\ \text{R2} + \text{R3} \end{array} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Matrix corresponds to the system.

$$\begin{aligned} x_1 \qquad\qquad &= -1 \\ x_2 \qquad &= 1 \\ x_3 &= 2 \end{aligned}$$

Solution is $x_1 = -1$, $x_2 = 1$, $x_3 = 2$.

Geometrically, each of the three original equations in this example represents a plane in three-dimensional space. The fact that there is a unique solution means that these three planes intersect at a single point. The solution $(-1, 1, 2)$ gives the coordinates of this point where the three planes intersect. We now give another example to reinforce the method.

EXAMPLE 2 Solve the following system of linear equations.

$$\begin{aligned} x_1 - 2x_2 + 4x_3 &= 12 \\ 2x_1 - x_2 + 5x_3 &= 18 \\ -x_1 + 3x_2 - 3x_3 &= -8 \end{aligned}$$

SOLUTION

Start with the augmented matrix and use the first row to create zeros in the first column. (This corresponds to using the first equation to eliminate x_1 from the second and third equations.)

$$\begin{bmatrix} 1 & -2 & 4 & 12 \\ 2 & -1 & 5 & 18 \\ -1 & 3 & -3 & -8 \end{bmatrix} \underset{\substack{R2 + (-2)R1 \\ R3 + R1}}{\approx} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & 3 & -3 & -6 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

Next multiply row 2 by $\frac{1}{3}$ to make the (2, 2) element 1. (This corresponds to making the coefficient of x_2 in the second equation 1.)

$$\underset{(\frac{1}{3})R2}{\approx} \begin{bmatrix} 1 & -2 & 4 & 12 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & 1 & 4 \end{bmatrix}$$

Create zeros in the second column as follows. (This corresponds to using the second equation to eliminate x_2 from the first and third equations.)

$$\underset{\substack{R1 + (2)R2 \\ R3 + (-1)R2}}{\approx} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 2 & 6 \end{bmatrix}$$

Multiply row 3 by $\frac{1}{2}$. (This corresponds to making the coefficient of x_3 in the third equation 1.)

$$\underset{(\frac{1}{2})R3}{\approx} \begin{bmatrix} 1 & 0 & 2 & 8 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Finally, create zeros in the third column. (This corresponds to using the third equation to eliminate x_3 from the first and second equations.)

$$\underset{\substack{R1 + (-2)R3 \\ R2 + R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This matrix corresponds to the system

$$\begin{aligned} x_1 \qquad\qquad &= 2 \\ x_2 \qquad &= 1 \\ x_3 &= 3 \end{aligned}$$

The solution is $x_1 = 2$, $x_2 = 1$, $x_3 = 3$.

■

This Gauss-Jordan method of solving a system of linear equations using matrices involves creating 1s and 0s in certain locations of matrices. These numbers are created in a systematic manner, column by column. The following example illustrates that it may be necessary to interchange two rows at some stage in order to proceed in the preceding manner.

EXAMPLE 3 Solve the system

$$\begin{aligned} 4x_1 + 8x_2 - 12x_3 &= 44 \\ 3x_1 + 6x_2 - 8x_3 &= 32 \\ -2x_1 - x_2 \qquad\quad &= -7 \end{aligned}$$

SOLUTION

We start with the augmented matrix and proceed as follows. (Note the use of zero in the augmented matrix as the coefficient of the missing variable x_3 in the third equation.)

$$\begin{bmatrix} 4 & 8 & -12 & 44 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{bmatrix} \underset{(\frac{1}{4})R1}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 3 & 6 & -8 & 32 \\ -2 & -1 & 0 & -7 \end{bmatrix}$$

$$\underset{\substack{R2 + (-3)R1 \\ R3 + (2)R1}}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & 0 & 1 & -1 \\ 0 & 3 & -6 & 15 \end{bmatrix}$$

At this stage we need a nonzero element in the location (2, 2) in order to continue. To achieve this we interchange the second row with the third row (a *later* row) and then proceed.

$$\underset{R2 \leftrightarrow R3}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & 3 & -6 & 15 \\ 0 & 0 & 1 & -1 \end{bmatrix} \underset{(\frac{1}{3})R2}{\approx} \begin{bmatrix} 1 & 2 & -3 & 11 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\underset{R1 + (-2)R2}{\approx} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix} \underset{\substack{R1 + (-1)R3 \\ R2 + (2)R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The solution is $x_1 = 2$, $x_2 = 3$, $x_3 = -1$.

■

Summary

We now summarize the method of Gauss-Jordan elimination for solving a system of n linear equations in n variables that has a unique solution. The augmented matrix is made up of a matrix of coefficients A and a column matrix of constant terms B. Let us write $[A:B]$ for this matrix. Use row operations to gradually transform this matrix, column by column, into a matrix $[I_n:X]$, where I_n is the identity $n \times n$ matrix.

$$[A:B] \approx \cdots \approx [I_n:X]$$

This final matrix $[I_n:X]$ is called the **reduced echelon form** of the original augmented matrix. The matrix of coefficients of the final system of equations is I_n and X is the column matrix of constant terms. This implies that the elements of X are the unique solution. Observe that as $[A:B]$ is being transformed to $[I_n:X]$, A is being changed to I_n. Thus:

> If A is the matrix of coefficients of a system of n equations in n variables that has a unique solution, then it is row equivalent to I_n.

If $[A:B]$ cannot be transformed in this manner into a matrix of the form $[I_n:X]$, the system of equations does not have a unique solution. More will be said about such systems in the next section.

Many Systems

Certain applications involve solving a number of systems of linear equations, all having the same square matrix of coefficients A. Let the systems be

$$[A:B_1], [A:B_2], \ldots, [A:B_k]$$

The constant terms $B_1, B_2, \ldots, B_k$, might for example be test data, and one wants to know the solutions that would lead to these results. The situation often dictates that the solutions be unique. One could of course go through the method of Gauss-Jordan elimination for each system, solving each system independently. This procedure would lead to the reduced echelon forms

$$[I_n:X_1], [I_n:X_2], \ldots, [I_n:X_k]$$

and the solutions would be $X_1, X_2, \ldots, X_k$. However, the same reduction of A to I_n would be repeated for each system; this involves a great deal of unnecessary duplication. The systems can be represented by one large augmented matrix $[A:B_1\ B_2 \cdots B_k]$, and the Gauss-Jordan method can be applied to this one matrix. We would get

$$[A:B_1\ B_2 \cdots B_k] \approx \cdots \approx [I_n:X_1\ X_2 \cdots X_k]$$

leading to the solutions $X_1, X_2, \ldots, X_k$.

EXAMPLE 4 Solve the following three systems of linear equations, all of which have the same matrix of coefficients.

$$\begin{aligned} x_1 - x_2 + 3x_3 &= b_1 \\ 2x_1 - x_2 + 4x_3 &= b_2 \\ -x_1 + 2x_2 - 4x_3 &= b_3 \end{aligned} \quad \text{for} \quad \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 11 \\ -11 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ -4 \end{bmatrix} \quad \text{in turn.}$$

SOLUTION

Construct the large augmented matrix that describes all three systems and determine the reduced echelon form as follows.

$$\begin{aligned}
\begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 2 & -1 & 4 & 11 & 1 & 3 \\ -1 & 2 & -4 & -11 & 2 & -4 \end{bmatrix}
&\underset{\substack{R2+(-2)R1\\ R3+R1}}{\approx}
\begin{bmatrix} 1 & -1 & 3 & 8 & 0 & 3 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 1 & -1 & -3 & 2 & -1 \end{bmatrix} \\
&\underset{\substack{R1+R2\\ R3+(-1)R2}}{\approx}
\begin{bmatrix} 1 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & -5 & 1 & -3 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix} \\
&\underset{\substack{R1+(-1)R3\\ R2+2R3}}{\approx}
\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 2 & 1 & 2 \end{bmatrix}
\end{aligned}$$

The solutions to the three systems of equations are given by the last three columns of the reduced echelon form. They are

$$\begin{aligned}
x_1 &= 1, & x_2 &= -1, & x_3 &= 2 \\
x_1 &= 0, & x_2 &= 3, & x_3 &= 1 \\
x_1 &= -2, & x_2 &= 1, & x_3 &= 2
\end{aligned}$$

■

In this section we have limited our discussion to systems of n linear equations in n variables that have a unique solution. In the following section we shall extend the method of Gauss-Jordan elimination to accommodate other systems that have a unique solution, and also to include systems that have many solutions or no solutions.

EXERCISE SET 1.1

Matrices

1. Give the sizes of the following matrices.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 4 & 5 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 9 \\ -6 & 4 \\ -3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 4 & 5 \end{bmatrix}$ **(d)** $\begin{bmatrix} -7 \\ 4 \\ 3 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & 9 & -8 & 7 \\ 4 & 2 & 5 & 7 & 2 \\ 4 & -6 & 4 & 0 & 0 \end{bmatrix}$

(f) $\begin{bmatrix} 2 & -3 & 4 & 7 \end{bmatrix}$

2. Give the (1, 1), (2, 2), (3, 3), (1, 5), (2, 4), (3, 2) elements of the following matrix.

$$\begin{bmatrix} 1 & 2 & 3 & 0 & -1 \\ -2 & 4 & -5 & 3 & 6 \\ 5 & 8 & 9 & 2 & 3 \end{bmatrix}$$

3. Give the (2, 3), (3, 2), (4, 1), (1, 3), (4, 4), (3, 1) elements of the following matrix.

$$\begin{bmatrix} 1 & 2 & 7 & 0 \\ -1 & 2 & 4 & 5 \\ 3 & 5 & 0 & -1 \\ 6 & 9 & 0 & 2 \end{bmatrix}$$

4. Write down the identity matrix I_4.

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[*] Answers to exercises marked in red are provided in the back of the book.

Matrices and Systems of Equations

5. Determine the matrix of coefficients and augmented matrix of each of the following systems of equations.

(a)
$$\begin{aligned} x_1 + 3x_2 &= 7 \\ 2x_1 - 5x_2 &= -3 \end{aligned}$$

(b)
$$\begin{aligned} 5x_1 + 2x_2 - 4x_3 &= 8 \\ x_1 + 3x_2 + 6x_3 &= 4 \\ 4x_1 + 6x_2 - 9x_3 &= 7 \end{aligned}$$

(c)
$$\begin{aligned} -x_1 + 3x_2 - 5x_3 &= -3 \\ 2x_1 - 2x_2 + 4x_3 &= 8 \\ x_1 + 3x_2 \qquad &= 6 \end{aligned}$$

(d)
$$\begin{aligned} 5x_1 + 4x_2 &= 9 \\ 2x_1 - 8x_2 &= -4 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

(e)
$$\begin{aligned} 5x_1 + 2x_2 - 4x_3 &= 8 \\ 4x_2 + 3x_3 &= 0 \\ x_1 \qquad - x_3 &= 7 \end{aligned}$$

(f)
$$\begin{aligned} -x_1 + 3x_2 - 9x_3 &= -4 \\ x_1 \qquad - 4x_3 &= 11 \\ x_1 + 8x_2 \qquad &= 1 \end{aligned}$$

(g)
$$\begin{aligned} x_1 \qquad\qquad &= -3 \\ x_2 \qquad &= 12 \\ x_3 &= 8 \end{aligned}$$

(h)
$$\begin{aligned} -4x_1 + 2x_2 - 9x_3 + x_4 &= -1 \\ x_1 + 6x_2 - 8x_3 - 7x_4 &= 15 \\ -x_2 + 3x_3 - 5x_4 &= 0 \end{aligned}$$

6. Interpret the following matrices as augmented matrices of systems of equations. Write down each system of equations.

(a) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ **(b)** $\begin{bmatrix} 7 & 9 & 8 \\ 6 & 4 & -3 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 9 & -3 \\ 5 & 0 & 2 \end{bmatrix}$ **(d)** $\begin{bmatrix} 8 & 7 & 5 & -1 \\ 4 & 6 & 2 & 4 \\ 9 & 3 & 7 & 6 \end{bmatrix}$

(e) $\begin{bmatrix} 2 & -3 & 6 & 4 \\ 7 & -5 & -2 & 3 \\ 0 & 2 & 4 & 0 \end{bmatrix}$ **(f)** $\begin{bmatrix} 0 & -2 & 4 \\ 5 & 7 & -3 \\ 6 & 0 & 8 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 8 \\ 0 & 0 & 1 & 4 \end{bmatrix}$ **(h)** $\begin{bmatrix} 1 & 2 & -1 & 6 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 1 & -2 \end{bmatrix}$

Elementary Row Operations

7. In the following exercises you are given a matrix followed by an elementary row operation. Determine each resulting matrix.

(a) $\begin{bmatrix} 2 & 6 & -4 & 0 \\ 1 & 2 & -3 & 6 \\ 8 & 3 & 2 & 5 \end{bmatrix} \underset{(\frac{1}{2})R1}{\approx}$

(b) $\begin{bmatrix} 0 & -8 & 4 & 3 \\ 2 & 7 & 5 & 1 \\ 3 & -5 & 8 & 9 \end{bmatrix} \underset{R1 \leftrightarrow R2}{\approx}$

(c) $\begin{bmatrix} 1 & 2 & 3 & -1 \\ -1 & 1 & 7 & 1 \\ 2 & -4 & 5 & -3 \end{bmatrix} \underset{\substack{R2 + R1 \\ R3 + (-2)R1}}{\approx}$

(d) $\begin{bmatrix} 1 & 2 & 3 & -4 \\ 0 & 1 & 2 & 1 \\ 0 & -4 & 3 & -5 \end{bmatrix} \underset{\substack{R1 + (-2)R2 \\ R3 + (4)R2}}{\approx}$

(e) $\begin{bmatrix} 1 & 0 & 4 & -3 \\ 0 & 1 & -3 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix} \underset{\substack{R1 + (-4)R3 \\ R2 + (3)R3}}{\approx}$

(f) $\begin{bmatrix} 1 & 0 & 2 & 7 \\ 0 & 1 & 5 & -3 \\ 0 & 0 & -2 & 8 \end{bmatrix} \underset{(-\frac{1}{2})R3}{\approx}$

8. Interpret each of the following row operations as a stage in arriving at the reduced echelon form of a matrix. Why have the indicated operations been selected? What particular aims do they accomplish in terms of the systems of linear equations that are described by the matrices?

(a)
$$\begin{bmatrix} 1 & -4 & 3 & 5 \\ -2 & 1 & 7 & 5 \\ 4 & 0 & -3 & 6 \end{bmatrix} \underset{\substack{R2 + (2)R1 \\ R3 + (-4)R1}}{\approx} \begin{bmatrix} 1 & -4 & 3 & 5 \\ 0 & -7 & 13 & 15 \\ 0 & 16 & -15 & -14 \end{bmatrix}$$

(b) $\begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 3 & 9 & -6 \\ 0 & 4 & 7 & -8 \end{bmatrix} \underset{(\frac{1}{3})R2}{\approx} \begin{bmatrix} 1 & 2 & -4 & 7 \\ 0 & 1 & 3 & -2 \\ 0 & 4 & 7 & -8 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & -4 & 5 \\ 0 & 0 & -2 & 6 \\ 0 & 1 & 3 & -8 \end{bmatrix} \underset{R2 \leftrightarrow R3}{\approx} \begin{bmatrix} 1 & 3 & -4 & 5 \\ 0 & 1 & 3 & -8 \\ 0 & 0 & -2 & 6 \end{bmatrix}$

(d)
$$\begin{bmatrix} 1 & 2 & 5 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & -3 & 1 & -2 \end{bmatrix} \underset{\substack{R1 + (-2)R2 \\ R3 + (3)R2}}{\approx} \begin{bmatrix} 1 & 0 & 1 & 6 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 7 & -11 \end{bmatrix}$$

9. Interpret each of the following row operations as a stage in arriving at the reduced echelon form of a matrix. Why have these operations been selected?

(a) $$\begin{bmatrix} 1 & 0 & 2 & 6 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \underset{\substack{\text{R1} + (-2)\text{R3} \\ \text{R2} + \text{R3}}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(b) $$\begin{bmatrix} 0 & 2 & 4 & -1 \\ 4 & 3 & 2 & -8 \\ 5 & -7 & 1 & 2 \end{bmatrix} \underset{\text{R1} \leftrightarrow \text{R2}}{\approx} \begin{bmatrix} 4 & 3 & 2 & -8 \\ 0 & 2 & 4 & -1 \\ 5 & -7 & 1 & 2 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & -2 & 6 \end{bmatrix} \underset{(-\frac{1}{2})\text{R3}}{\approx} \begin{bmatrix} 1 & 0 & 3 & 7 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & -3 \end{bmatrix} \underset{\substack{\text{R1} + (2)\text{R3} \\ \text{R2} + (-3)\text{R3}}}{\approx} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & -3 \end{bmatrix}$$

Solving Systems of Linear Equations

10. The following systems of equations all have unique solutions. Solve these systems using the method of Gauss-Jordan elimination with matrices.

(a) $$\begin{aligned} x_1 - 2x_2 &= -8 \\ 2x_1 - 3x_2 &= -11 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 + 2x_2 &= 4 \\ 3x_1 + 2x_2 &= 3 \end{aligned}$$

(c) $$\begin{aligned} x_1 \phantom{{}+2x_2} + x_3 &= 3 \\ 2x_2 - 2x_3 &= -4 \\ x_2 - 2x_3 &= 5 \end{aligned}$$

(d) $$\begin{aligned} x_1 + x_2 + 3x_3 &= 6 \\ x_1 + 2x_2 + 4x_3 &= 9 \\ 2x_1 + x_2 + 6x_3 &= 11 \end{aligned}$$

(e) $$\begin{aligned} x_1 - x_2 + 3x_3 &= 3 \\ 2x_1 - x_2 + 2x_3 &= 2 \\ 3x_1 + x_2 - 2x_3 &= 3 \end{aligned}$$

(f) $$\begin{aligned} -x_1 + x_2 - x_3 &= -2 \\ 3x_1 + x_2 + x_3 &= 10 \\ 4x_1 + 2x_2 + 3x_3 &= 14 \end{aligned}$$

11. The following systems of equations all have unique solutions. Solve these systems using the method of Gauss-Jordan elimination with matrices.

(a) $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 14 \\ 2x_1 + 5x_2 + 8x_3 &= 36 \\ x_1 - x_2 &= -4 \end{aligned}$$

(b) $$\begin{aligned} x_1 - x_2 - x_3 &= -1 \\ -2x_1 + 6x_2 + 10x_3 &= 14 \\ 2x_1 + x_2 + 6x_3 &= 9 \end{aligned}$$

(c) $$\begin{aligned} 2x_1 + 2x_2 - 4x_3 &= 14 \\ 3x_1 + x_2 + x_3 &= 8 \\ 2x_1 - x_2 + 2x_3 &= -1 \end{aligned}$$

(d) $$\begin{aligned} 2x_2 + 4x_3 &= 8 \\ 2x_1 + 2x_2 &= 6 \\ x_1 + x_2 + x_3 &= 5 \end{aligned}$$

(e) $$\begin{aligned} x_1 - x_3 &= 3 \\ -x_1 + 2x_3 &= -8 \\ 3x_1 + x_2 - x_3 &= 0 \end{aligned}$$

12. The following systems of equations all have unique solutions. Solve these systems using the method of Gauss-Jordan elimination with matrices.

(a) $$\begin{aligned} \tfrac{3}{2}x_1 + 3x_3 &= 15 \\ -x_1 + 7x_2 - 9x_3 &= -45 \\ 2x_1 + 5x_3 &= 22 \end{aligned}$$

(b) $$\begin{aligned} -3x_1 - 6x_2 - 15x_3 &= -3 \\ 2x_1 + 3x_2 + 9x_3 &= 1 \\ -4x_1 - 7x_2 - 17x_3 &= -4 \end{aligned}$$

(c) $$\begin{aligned} 3x_1 + 6x_2 - 3x_4 &= 3 \\ x_1 + 3x_2 - x_3 - 4x_4 &= -12 \\ x_1 - x_2 + x_3 + 2x_4 &= 8 \\ 2x_1 + 3x_2 &= 8 \end{aligned}$$

(d) $$\begin{aligned} x_1 + 2x_2 + 2x_3 + 5x_4 &= 11 \\ 2x_1 + 4x_2 + 2x_3 + 8x_4 &= 14 \\ x_1 + 3x_2 + 4x_3 + 8x_4 &= 19 \\ x_1 - x_2 + x_3 &= 2 \end{aligned}$$

(e) $$\begin{aligned} x_1 + x_2 + 2x_3 + 6x_4 &= 11 \\ 2x_1 + 3x_2 + 6x_3 + 19x_4 &= 36 \\ 3x_2 + 4x_3 + 15x_4 &= 28 \\ x_1 - x_2 - x_3 - 6x_4 &= -12 \end{aligned}$$

13. The following exercises involve many systems of linear equations with unique solutions that have the same matrix of coefficients. Solve the systems by applying the method of Gauss-Jordan elimination to a large augmented matrix that describes many systems.

(a)
$$\begin{aligned} x_1 + 2x_2 &= b_1 \\ 3x_1 + 5x_2 &= b_2 \end{aligned}$$
for $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ 9 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix}$ in turn.

(b)
$$\begin{aligned} x_1 + x_2 &= b_1 \\ 2x_1 + 3x_2 &= b_2 \end{aligned}$$
for $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 13 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in turn.

(c)
$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= b_1 \\ x_1 - x_2 + 2x_3 &= b_2 \\ 2x_1 - 3x_2 + 6x_3 &= b_3 \end{aligned}$$
for $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 14 \end{bmatrix}, \begin{bmatrix} -5 \\ -3 \\ -8 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 9 \end{bmatrix}$ in turn.

(d)
$$\begin{aligned} x_1 + 2x_2 - x_3 &= b_1 \\ -x_1 - x_2 + x_3 &= b_2 \\ 3x_1 + 7x_2 - x_3 &= b_3 \end{aligned}$$
for $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 6 \\ -4 \\ 18 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix}$ in turn.

1.2 Gauss-Jordan Elimination

In the previous section we used the method of Gauss-Jordan elimination to solve systems of n equations in n variables that had a unique solution. We shall now discuss the method in its more general setting, where the number of equations can differ from the number of variables and where there can be a unique solution, many solutions, or no solutions. Our approach again will be to start from the augmented matrix of the given system and to perform a sequence of elementary row operations that will result in a simpler matrix (the reduced echelon form), which leads directly to the solution.

We now give the general definition of reduced echelon form. The reader will observe that the reduced echelon forms discussed in the previous section all conform to this definition.

DEFINITION A matrix is in **reduced echelon form** if:

1. Any rows consisting entirely of zeros are grouped at the bottom of the matrix.
2. The first nonzero element of each other row is 1. This element is called a **leading 1**.
3. The leading 1 of each row after the first is positioned to the right of the leading 1 of the previous row.
4. All other elements in a column that contains a leading 1 are zero.

The following matrices are all in reduced echelon form.

$$\begin{bmatrix} 1 & 0 & 8 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{bmatrix} \quad \begin{bmatrix} 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 & 0 & 0 & 8 \\ 0 & 1 & 7 & 0 & 0 & 9 \\ 0 & 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The following matrices are not in reduced echelon form for the reasons stated.

$$\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Row of zeros not at bottom of matrix

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

First nonzero element in row 2 is not 1

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 3 \end{bmatrix}$$

Leading 1 in row 3 not to the right of leading 1 in row 2

$$\begin{bmatrix} 1 & 7 & 0 & 8 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Nonzero element above leading 1 in row 2

There are usually many sequences of row operations that can be used to transform a given matrix to reduced echelon form—they all, however, lead to the same reduced echelon form. We say that *the reduced echelon form of a matrix is unique*. The method of Gauss-Jordan elimination is an important systematic way (called an algorithm) for arriving at the reduced echelon form. It can be programmed on a computer. We now summarize the method, then give examples of its implementation.

Gauss-Jordan Elimination

1. Write down the augmented matrix of the system of linear equations.
2. Derive the reduced echelon form of the augmented matrix using elementary row operations. This is done by creating leading 1s, then zeros above and below each leading 1, column by column, starting with the first column.
3. Write down the system of equations corresponding to the reduced echelon form. This system gives the solution.

We stress the importance of mastering this algorithm. Not only is getting the correct solution important, the method of arriving at the solution is important. We shall, for example, be interested in the efficiency of this algorithm (the number of additions and multiplications used) and comparing it with other algorithms that can be used to solve systems of linear equations.

EXAMPLE 1 Use the method of Gauss-Jordan elimination to find the reduced echelon form of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 & -2 & 2 \\ 3 & 3 & -3 & 9 & 12 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

SOLUTION

Step 1 Interchange rows, if necessary, to bring a nonzero element to the top of the first nonzero column. This nonzero element is called a **pivot**.

$$\underset{R1 \leftrightarrow R2}{\approx} \begin{bmatrix} \overset{\swarrow \text{pivot}}{\textcircled{3}} & 3 & -3 & 9 & 12 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Step 2 Create a 1 in the pivot location by multiplying the pivot row by $\frac{1}{\text{pivot}}$.

$$\underset{(\frac{1}{3})R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 4 & 4 & -2 & 11 & 12 \end{bmatrix}$$

Step 3 Create zeros elsewhere in the pivot column by adding suitable multiples of the pivot row to all other rows of the matrix.

$$\underset{R3 + (-4)R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

Step 4 Cover the pivot row and all rows above it. Repeat Steps 1 and 2 for the remaining submatrix. Repeat step 3 for the whole matrix. Continue thus until the reduced echelon form is reached.

$$\begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 2 & -2 & 2 \\ 0 & 0 & \underset{\uparrow}{2} & -1 & -4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & \textcircled{2} & -2 & 2 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix}$$

first nonzero column
of the submatrix.

pivot

$$\underset{(\frac{1}{2})R2}{\approx} \begin{bmatrix} 1 & 1 & -1 & 3 & 4 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 2 & -1 & -4 \end{bmatrix} \quad \underset{\substack{R1 + R2 \\ R3 + (-2)R2}}{\approx} \begin{bmatrix} 1 & 1 & 0 & 2 & 5 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & \underset{\uparrow}{\textcircled{1}} & -6 \end{bmatrix}$$

pivot

$$\underset{\substack{R1 + (-2)R3 \\ R2 + R3}}{\approx} \begin{bmatrix} 1 & 1 & 0 & 0 & 17 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & -6 \end{bmatrix}$$

This matrix is the reduced echelon form of the given matrix.

We now illustrate how this method is used to solve various systems of equations. The following example illustrates how to solve a system of linear equations that has many solutions. The reduced echelon form is derived. It then becomes necessary to interpret the reduced echelon form, expressing the many solutions in a clear manner.

EXAMPLE 2 Solve, if possible, the system of equations

$$\begin{aligned} 3x_1 - 3x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned}$$

SOLUTION

Start with the augmented matrix and follow the Gauss-Jordan algorithm. Pivots and leading ones are circled.

$$\begin{bmatrix} \textcircled{3} & -3 & 3 & 9 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix} \underset{(\frac{1}{3})R1}{\approx} \begin{bmatrix} \textcircled{1} & -1 & 1 & 3 \\ 2 & -1 & 4 & 7 \\ 3 & -5 & -1 & 7 \end{bmatrix}$$

$$\underset{\substack{R2 + (-2)R1\\ R3 + (-3)R1}}{\approx}\begin{bmatrix} 1 & -1 & 1 & 3\\ 0 & \textcircled{1} & 2 & 1\\ 0 & -2 & -4 & -2 \end{bmatrix} \underset{\substack{R1 + R2\\ R3 + (2)R2}}{\approx}\begin{bmatrix} 1 & 0 & 3 & 4\\ 0 & 1 & 2 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We have arrived at the reduced echelon form. The corresponding system of equations is

$$\begin{aligned} x_1 + \quad 3x_3 &= 4\\ x_2 + 2x_3 &= 1 \end{aligned}$$

There are many values of x_1, x_2, and x_3 that satisfy these equations. This is a system of equations that has many solutions. x_1 is called the **leading variable** of the first equation and x_2 is the leading variable of the second equation. To express these many solutions, we write the leading variables in each equation in terms of the remaining variables. We get

$$\begin{aligned} x_1 &= -3x_3 + 4\\ x_2 &= -2x_3 + 1 \end{aligned}$$

Let us assign the arbitrary value r to x_3. The **general solution** to the system is

$$x_1 = -3r + 4,\ x_2 = -2r + 1,\ x_3 = r$$

As r ranges over the set of real numbers we get many solutions. r is called a **parameter**. We can get specific solutions by giving r different values. For example,

$$\begin{aligned} r &= 1 \quad \text{gives} \quad x_1 = 1,\ x_2 = -1,\ x_3 = 1\\ r &= -2 \quad \text{gives} \quad x_1 = 10,\ x_2 = 5,\ x_3 = -2 \end{aligned}$$

EXAMPLE 3 This example illustrates that the general solution can involve a number of parameters. Solve the system of equations

$$\begin{aligned} x_1 + 2x_2 - x_3 + 3x_4 &= 4\\ 2x_1 + 4x_2 - 2x_3 + 7x_4 &= 10\\ -x_1 - 2x_2 + x_3 - 4x_4 &= -6 \end{aligned}$$

SOLUTION

On applying the Gauss-Jordan algorithm we get

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 4\\ 2 & 4 & -2 & 7 & 10\\ -1 & -2 & 1 & -4 & -6 \end{bmatrix} \underset{\substack{R2 + (-2)R1\\ R3 + R1}}{\approx}\begin{bmatrix} 1 & 2 & -1 & 3 & 4\\ 0 & 0 & 0 & 1 & 2\\ 0 & 0 & 0 & -1 & -2 \end{bmatrix}$$

$$\underset{\substack{R1 + (-3)R2\\ R3 + R2}}{\approx}\begin{bmatrix} 1 & 2 & -1 & 0 & -2\\ 0 & 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We have arrived at the reduced echelon form. The corresponding system of equations is

$$\begin{aligned} x_1 + 2x_2 - x_3 \qquad &= -2 \\ x_4 &= 2 \end{aligned}$$

Expressing the leading variables in terms of the remaining variables we get

$$x_1 = -2x_2 + x_3 - 2,\ x_4 = 2$$

Let us assign the arbitrary values r to x_2 and s to x_3. The general solution is

$$x_1 = -2r + s - 2,\ x_2 = r,\ x_3 = s,\ x_4 = 2$$

Specific solutions can be obtained by giving r and s various values.

EXAMPLE 4 This example illustrates a system that has no solution. Let us try to solve the system

$$\begin{aligned} x_1 + x_2 + 5x_3 &= 3 \\ x_2 + 3x_3 &= -1 \\ x_1 + 2x_2 + 8x_3 &= 3 \end{aligned}$$

SOLUTION

Starting with the augmented matrix we get

$$\begin{bmatrix}1&1&5&3\\0&1&3&-1\\1&2&8&3\end{bmatrix} \underset{\text{R3} + (-1)\text{R1}}{\approx} \begin{bmatrix}1&1&5&3\\0&1&3&-1\\0&1&3&0\end{bmatrix}$$

$$\underset{\substack{\text{R1} + (-1)\text{R2}\\ \text{R3} + (-1)\text{R2}}}{\approx} \begin{bmatrix}1&0&2&4\\0&1&3&-1\\0&0&0&1\end{bmatrix} \underset{\substack{\text{R1} + (-4)\text{R3}\\ \text{R2} + \text{R3}}}{\approx} \begin{bmatrix}1&0&2&0\\0&1&3&0\\0&0&0&1\end{bmatrix}$$

The last row of this reduced echelon form gives the equation

$$0x_1 + 0x_2 + 0x_3 = 1$$

This equation cannot be satisfied for any values of x_1, x_2, and x_3. Thus the system has no solution. (This information was in fact available from the next-to-last matrix.)

Homogeneous Systems of Linear Equations

A system of linear equations is said to be **homogeneous** if all the constant terms are zeros. As we proceed in the course we shall find that homogeneous systems of linear equations have many interesting properties and play a key role in our discussions.

The following system is a homogeneous system of linear equations.

$$\begin{aligned} x_1 + 2x_2 - 5x_3 &= 0 \\ -2x_1 - 3x_2 + 6x_3 &= 0 \end{aligned}$$

Observe that $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, is a solution to this system. It is apparent that this result can be extended as follows to any homogeneous system of equations.

THEOREM 1.1

A homogeneous system of linear equations in n variables always has the solution $x_1 = 0, x_2 = 0, \ldots, x_n = 0$. This solution is called the **trivial solution**.

Let us see if the preceding homogeneous system has any other solutions. We solve the system using Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & 2 & -5 & 0 \\ -2 & -3 & 6 & 0 \end{bmatrix} \underset{\text{R2} + (2)\text{R1}}{\approx} \begin{bmatrix} 1 & 2 & -5 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$
$$\underset{\text{R1} + (-2)\text{R2}}{\approx} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & -4 & 0 \end{bmatrix}$$

This reduced echelon form gives the system

$$\begin{aligned} x_1 \qquad + 3x_3 &= 0 \\ x_2 - 4x_3 &= 0 \end{aligned}$$

Expressing the leading variables in terms of the remaining variable we get

$$\begin{aligned} x_1 &= -3x_3 \\ x_2 &= 4x_3 \end{aligned}$$

Letting $x_3 = r$ we see that the system has many solutions,

$$x_1 = -3r,\ x_2 = 4r,\ x_3 = r$$

Observe that the solution $x_1 = 0, x_2 = 0, x_3 = 0$ is obtained by letting $r = 0$.

In a similar manner, the augmented matrix of any homogeneous system of linear equations that has more variables than equations will have a reduced echelon form that has more nonzero columns than rows with the last column being zero. The corresponding system of equations, and thus the original system, will have many solutions, one of which is the trivial solution as in the last example. We summarize this important observation in the following theorem.

THEOREM 1.2

A homogeneous system of linear equations that has more variables than equations has many solutions. One of these solutions is the trivial solution.

In the first two sections of this chapter we introduced the method of Gauss-Jordan elimination for solving systems of linear equations. As we proceed in the course we shall introduce other methods and compare the merits of the methods. There is another popular elimination method for solving systems of linear equations, for example, called **Gaussian elimination**. We introduce that method in Section 8.1.[2] The following discussion reveals some of the numerical concerns when solving systems of equations.

Numerical Considerations

In practice, systems of linear equations are solved on computers. Numbers are represented on computers in the form $\pm 0.\, a_1 \ldots a_n \times 10^r$, where $a_1, \ldots, a_n$ are integers between 0

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[2] Gaussian elimination can in fact be used in place of Gauss-Jordan elimination as the standard method for this course if so desired.

and 9 and r is an integer (positive or negative). Such a number is called a **floating-point number**. The quantity $a_1, \ldots, a_n$ is called the **mantissa**, and r is the **exponent**. For example, the number 125.6 is written in floating-point form as 0.1256×10^3. An arithmetic operation of multiplication, division, addition, or subtraction on floating-point numbers is called a **floating point operation**, or **flop**.

Computers can handle only a limited number of integers in the mantissa of a number. The mantissa is rounded to a certain number of places during each operation and consequently errors called **round-off errors** occur in methods such as Gauss-Jordan elimination. The fewer flops that are performed during computation the faster and more accurate the result will be. (Ways of minimizing these errors are discussed in Chapter 8.) To compute the reduced echelon form of a system of n equations in n variables, the method of Gauss-Jordan elimination requires $\frac{1}{2}n^3 + \frac{1}{2}n^2$ multiplications and $\frac{1}{2}n^3 - \frac{1}{2}n$ additions (Section 8.1). The number of multiplications required to solve a system of, say, ten equations in ten variables $(n = 10)$ is 550, and the number of additions is 495. The total number of flops is the sum of these, namely 1045. Algorithms are usually measured and compared using such data.

EXERCISE SET 1.2

Reduced Echelon Form of a Matrix

1. Determine whether the following matrices are in reduced echelon form. If a matrix is not in reduced echelon form give a reason.

(a) $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 7 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 5 & 6 \\ 0 & 1 & 3 & -7 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 4 & 0 & 5 \\ 0 & 0 & 2 & 9 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **(f)** $\begin{bmatrix} 1 & 5 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 9 \end{bmatrix}$ **(h)** $\begin{bmatrix} 1 & 0 & 0 & 3 & 2 \\ 0 & 2 & 0 & 6 & 1 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. Determine whether the following matrices are in reduced echelon form. If a matrix is not in reduced echelon form give a reason.

(a) $\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 0 & 1 & 8 \\ 0 & 1 & 4 & 9 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 2 & 0 & 0 & 4 \\ 0 & 0 & 1 & 0 & 6 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 5 & 0 & 2 & 0 \\ 0 & 0 & 1 & 9 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 4 & 2 & 6 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 2 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 7 \\ 0 & 0 & 0 & 1 & 3 \end{bmatrix}$ **(f)** $\begin{bmatrix} 1 & 0 & 4 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(g) $\begin{bmatrix} 1 & 0 & 0 & 5 & 3 \\ 0 & 0 & 1 & 0 & 3 \\ 0 & 1 & 2 & 3 & 7 \end{bmatrix}$ **(h)** $\begin{bmatrix} 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \\ 0 & 1 & 0 & 0 & 3 \end{bmatrix}$

(i) $\begin{bmatrix} 1 & 5 & -3 & 0 & 7 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

3. Each of the following matrices is the reduced echelon form of the augmented matrix of a system of linear equations. Give the solution (if it exists) to each system of equations.

(a) $\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -3 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 3 & 0 & 6 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 0 & 0 & 5 & 3 \\ 0 & 1 & 0 & 6 & -2 \\ 0 & 0 & 1 & 2 & -4 \end{bmatrix}$

(f) $\begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 5 \end{bmatrix}$

4. Each of the following matrices is the reduced echelon form of the augmented matrix of a system of linear equations. Give the solution (if it exists) to each system of equations.

(a) $$\begin{bmatrix} 1 & 0 & 2 & 4 & 1 \\ 0 & 1 & -3 & 5 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(b) $$\begin{bmatrix} 1 & -3 & 2 & 0 & 4 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) $$\begin{bmatrix} 1 & -2 & 0 & 3 & 0 & 4 \\ 0 & 0 & 1 & 2 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 8 \end{bmatrix}$$

(d) $$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 6 \\ 0 & 1 & 5 & 0 & 4 & 7 \\ 0 & 0 & 0 & 1 & 9 & -3 \end{bmatrix}$$

Solving Systems of Linear Equations

5. Solve (if possible) each of the following systems of three equations in three variables using the method of Gauss-Jordan elimination.

(a) $$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 1 \\ 2x_1 + 8x_2 + 11x_3 &= 7 \\ x_1 + 6x_2 + 7x_3 &= 3 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 + 4x_3 &= 15 \\ 2x_1 + 4x_2 + 9x_3 &= 33 \\ x_1 + 3x_2 + 5x_3 &= 20 \end{aligned}$$

(c) $$\begin{aligned} x_1 + x_2 + x_3 &= 7 \\ 2x_1 + 3x_2 + x_3 &= 18 \\ -x_1 + x_2 - 3x_3 &= 1 \end{aligned}$$

(d) $$\begin{aligned} x_1 + 4x_2 + x_3 &= 2 \\ x_1 + 2x_2 - x_3 &= 0 \\ 2x_1 + 6x_2 \phantom{{}+x_3} &= 3 \end{aligned}$$

(e) $$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ 2x_1 - x_2 + 4x_3 &= 7 \\ 3x_1 - 5x_2 - x_3 &= 7 \end{aligned}$$

(f) $$\begin{aligned} 3x_1 - 3x_2 + 9x_3 &= 24 \\ 2x_1 - 2x_2 + 7x_3 &= 17 \\ -x_1 + 2x_2 - 4x_3 &= -11 \end{aligned}$$

6. Solve (if possible) each of the following systems of three equations in three variables using the method of Gauss-Jordan elimination.

(a) $$\begin{aligned} 3x_1 + 6x_2 - 3x_3 &= 6 \\ -2x_1 - 4x_2 - 3x_3 &= -1 \\ 3x_1 + 6x_2 - 2x_3 &= 10 \end{aligned}$$

(b) $$\begin{aligned} x_1 + 2x_2 + x_3 &= 7 \\ x_1 + 2x_2 + 2x_3 &= 11 \\ 2x_1 + 4x_2 + 3x_3 &= 18 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ 2x_1 + 4x_2 - 2x_3 &= 6 \\ 3x_1 + 6x_2 + 2x_3 &= -1 \end{aligned}$$

(d) $$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 8 \\ 3x_1 + 7x_2 + 9x_3 &= 26 \\ 2x_1 \phantom{{}+7x_2} + 6x_3 &= 11 \end{aligned}$$

(e) $$\begin{aligned} x_2 + 2x_3 &= 5 \\ x_1 + 2x_2 + 5x_3 &= 13 \\ x_1 \phantom{{}+2x_2} + 2x_3 &= 4 \end{aligned}$$

(f) $$\begin{aligned} x_1 + 2x_2 + 8x_3 &= 7 \\ 2x_1 + 4x_2 + 16x_3 &= 14 \\ x_2 + 3x_3 &= 4 \end{aligned}$$

7. Solve (if possible) each of the following systems of equations using the method of Gauss-Jordan elimination.

(a) $$\begin{aligned} x_1 + x_2 - 3x_3 &= 10 \\ -3x_1 - 2x_2 + 4x_3 &= -24 \end{aligned}$$

(b) $$\begin{aligned} 2x_1 - 6x_2 - 14x_3 &= 38 \\ -3x_1 + 7x_2 + 15x_3 &= -37 \end{aligned}$$

(c) $$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 0 \\ x_1 + 2x_2 \phantom{{}-x_3} + x_4 &= 4 \\ -x_1 - 2x_2 + 2x_3 + 4x_4 &= 5 \end{aligned}$$

(d) $$\begin{aligned} x_1 + 2x_2 \phantom{{}+3x_3} + 4x_4 &= 0 \\ -2x_1 - 4x_2 + 3x_3 - 2x_4 &= 0 \end{aligned}$$

(A homogeneous system)

(e) $$\begin{aligned} x_2 - 3x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 + 4x_4 &= 0 \\ -2x_1 - 2x_2 + 2x_3 - 8x_4 &= 0 \end{aligned}$$

(A homogeneous system)

8. Solve (if possible) each of the following systems of equations using the method of Gauss-Jordan elimination.

(a)
$$\begin{aligned} x_1 + x_2 + x_3 - x_4 &= -3 \\ 2x_1 + 3x_2 + x_3 - 5x_4 &= -9 \\ x_1 + 3x_2 - x_3 - 6x_4 &= -7 \\ -x_1 - x_2 - x_3 \phantom{{}- 6x_4} &= 1 \end{aligned}$$

(b)
$$\begin{aligned} x_2 + 2x_3 &= 7 \\ x_1 - 2x_2 - 6x_3 &= -18 \\ -x_1 - x_2 - 2x_3 &= -5 \\ 2x_1 - 5x_2 - 15x_3 &= -46 \end{aligned}$$

(c)
$$\begin{aligned} 2x_1 - 4x_2 + 16x_3 - 14x_4 &= 10 \\ -x_1 + 5x_2 - 17x_3 + 19x_4 &= -2 \\ x_1 - 3x_2 + 11x_3 - 11x_4 &= 4 \\ 3x_1 - 4x_2 + 18x_3 - 13x_4 &= 17 \end{aligned}$$

(d)
$$\begin{aligned} x_1 - x_2 + 2x_3 \phantom{{}- 4x_4} &= 7 \\ 2x_1 - 2x_2 + 2x_3 - 4x_4 &= 12 \\ -x_1 + x_2 - x_3 + 2x_4 &= -4 \\ -3x_1 + x_2 - 8x_3 - 10x_4 &= -29 \end{aligned}$$

(e)
$$\begin{aligned} x_1 + 6x_2 - x_3 - 4x_4 &= 0 \\ -2x_1 - 12x_2 + 5x_3 + 17x_4 &= 0 \\ 3x_1 + 18x_2 - x_3 - 6x_4 &= 0 \end{aligned}$$

(A homogeneous system)

(f)
$$\begin{aligned} 4x_1 + 8x_2 - 12x_3 &= 28 \\ -x_1 - 2x_2 + 3x_3 &= -7 \\ 2x_1 + 4x_2 - 8x_3 &= 16 \\ -3x_1 - 6x_2 + 9x_3 &= -21 \end{aligned}$$

(g)
$$\begin{aligned} x_1 + x_2 &= 2 \\ 2x_1 + 3x_2 &= 3 \\ x_1 + 3x_2 &= 0 \\ x_1 + 2x_2 &= 1 \end{aligned}$$

Understanding Systems of Linear Equations

9. Construct examples of the following:

(a) A system of linear equations with more variables than equations, having no solution.

(b) A system of linear equations with more equations than variables, having a unique solution.

10. The reduced echelon forms of the matrices of systems of two equations in two variables, and the types of solutions they represent can be classified as follows. (• corresponds to possible nonzero elements.)

$$\begin{bmatrix} 1 & 0 & \bullet \\ 0 & 1 & \bullet \end{bmatrix} \quad \begin{bmatrix} 1 & \bullet & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & \bullet & \bullet \\ 0 & 0 & 0 \end{bmatrix}$$

unique solution   no solutions   many solutions

Classify in a similar manner the reduced echelon forms of the matrices, and the types of solutions they represent, of

(a) systems of three equations in two variables,

(b) systems of three equations in three variables.

11. Consider the homogeneous system of linear equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

(a) Show that if $x = x_0$, $y = y_0$ is a solution, then $x = kx_0$, $y = ky_0$, is also a solution, for any value of the constant k.

(b) Show that if $x = x_0$, $y = y_0$, and $x = x_1$, $y = y_1$, are any two solutions, then $x = x_0 + x_1$, $y = y_0 + y_1$, is also a solution.

12. Show that $x = 0$, $y = 0$ is a solution to the homogeneous system of linear equations

$$\begin{aligned} ax + by &= 0 \\ cx + dy &= 0 \end{aligned}$$

Prove that this is the only solution if and only if $ad - bc \neq 0$.

13. Consider two systems of linear equations having augmented matrices $[A : B_1]$ and $[A : B_2]$, where the matrix of coefficients of both systems is the same 3×3 matrix A.

(a) Is it possible for $[A : B_1]$ to have a unique solution and $[A : B_2]$ to have many solutions?

(b) Is it possible for $[A : B_1]$ to have a unique solution and $[A : B_2]$ to have no solutions?

(c) Is it possible for $[A : B_1]$ to have many solutions and $[A : B_2]$ to have no solutions?

14. Solve the following systems of linear equations by applying the method of Gauss-Jordan elimination to a large augmented matrix that represents two systems with the same matrix of coefficients.

(a)
$$\begin{aligned} x_1 + x_2 + 5x_3 &= b_1 \\ x_1 + 2x_2 + 8x_3 &= b_2 \\ 2x_1 + 4x_2 + 16x_3 &= b_3 \end{aligned}$$

$$\text{for } \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}, \text{ in turn.}$$

(b) $$\begin{aligned} x_1 + 2x_2 + 4x_3 &= b_1 \\ x_1 + x_2 + 2x_3 &= b_2 \\ 2x_1 + 3x_2 + 6x_3 &= b_3 \end{aligned}$$

for $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 5 \\ 13 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \\ 11 \end{bmatrix}$, in turn.

15. Write down a 3 × 3 matrix at random. Find its reduced echelon form. The reduced echelon form is probably the identity matrix I_3! Explain this. [*Hint*: Think about the geometry.]

16. If a 3 × 4 matrix is written down at random, what type of reduced echelon form is it likely to have and why?

17. Computers can only carry a finite number of digits. This causes errors called round-off errors to occur when numbers are truncated. Because of this phenomenon computers can give incorrect results. Much research goes into developing algorithms that minimize such round-off errors. (Readers who are interested in these algorithms should read Section 8.3.) A computer is used to determine the reduced echelon form of an augmented matrix of a system of linear equations. Which of the following is most likely to happen, and why?

(a) The computer gives a solution to the system, when in fact a solution does not exist.

(b) The computer gives that a solution does not exist, when in fact a solution does exist.

*1.3 Curve Fitting, Electrical Networks, and Traffic Flow

Systems of linear equations are used in such diverse fields as electrical engineering, economics, and traffic analysis. We now discuss applications in some of these fields.

Curve Fitting

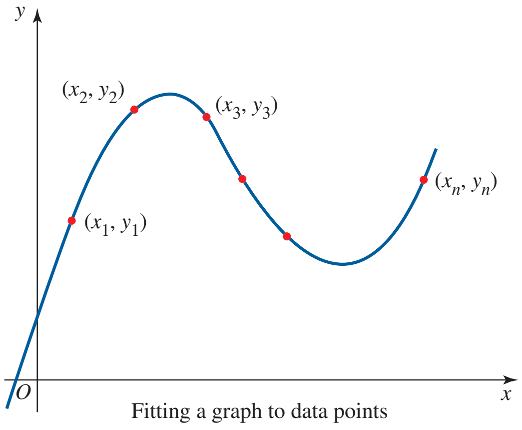
The following problem occurs in many different branches of science. A set of data points

$$(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$$

is given and it is necessary to find a polynomial whose graph passes through the points. The points are often measurements in an experiment. The x-coordinates are called **base points**. It can be shown that if the base points are all distinct, then a unique polynomial of degree $n - 1$ (or less)

$$y = a_0 + a_1 x + \cdots + a_{n-2}x^{n-2} + a_{n-1}x^{n-1}$$

can be **fitted** to the points. See Figure 1.5.



Fitting a graph to data points

Figure 1.5

The coefficients $a_0, a_1, \ldots, a_{n-2}, a_{n-1}$ of the appropriate polynomial can be found by substituting the points into the polynomial equation and then solving a system of linear equations. (It is usual to write the polynomial in terms of ascending powers of x for the pur-

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[*] Sections and chapters marked with an asterisk are optional. The instructor can use these sections to build around the core material to give the course the desired flavor.

pose of finding these coefficients. The columns of the matrix of coefficients of the system of equations then often follow a pattern. More will be said about this later.)

We now illustrate the procedure by fitting a polynomial of degree two, a parabola, to a set of three such data points.

EXAMPLE 1 Determine the equation of the polynomial of degree two whose graph passes through the points (1, 6), (2, 3), (3, 2).

SOLUTION

Observe that in this example we are given *three* points and we want to find a polynomial of degree *two* (one less than the number of data points). Let the polynomial be

$$y = a_0 + a_1x + a_2x^2$$

We are given three points and shall use these three sets of information to determine the three unknowns a_0, a_1, and a_2. Substituting

$$x = 1, y = 6;\ x = 2, y = 3;\ x = 3, y = 2$$

in turn into the polynomial leads to the following system of three linear equations in a_0, a_1, and a_2.

$$\begin{aligned} a_0 + a_1 + a_2 &= 6 \\ a_0 + 2a_1 + 4a_2 &= 3 \\ a_0 + 3a_1 + 9a_2 &= 2 \end{aligned}$$

Solve this system for a_2, a_1, and a_0 using Gauss-Jordan elimination.

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 4 & 3 \\ 1 & 3 & 9 & 2 \end{bmatrix} \begin{array}{c} \approx \\ R2 + (-1)R1 \\ R3 + (-1)R1 \end{array} \begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & -3 \\ 0 & 2 & 8 & -4 \end{bmatrix}$$

$$\begin{array}{c} \approx \\ R1 + (-1)R2 \\ R3 + (-2)R2 \end{array} \begin{bmatrix} 1 & 0 & -2 & 9 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 2 & 2 \end{bmatrix} \begin{array}{c} \approx \\ (\frac{1}{2})R3 \end{array} \begin{bmatrix} 1 & 0 & -2 & 9 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\begin{array}{c} \approx \\ R1 + (2)R3 \\ R2 + (-3)R3 \end{array} \begin{bmatrix} 1 & 0 & 0 & 11 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We get $a_0 = 11$, $a_1 = -6$, $a_2 = 1$. The parabola that passes through these points is $y = 11 - 6x + x^2$. See Figure 1.6.

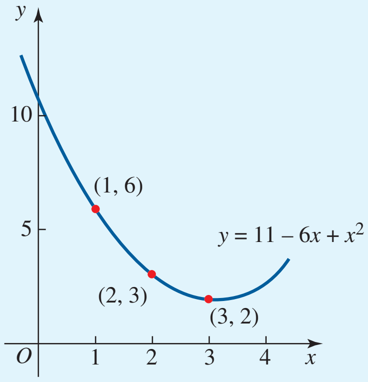

Figure 1.6

Electrical Network Analysis

Systems of linear equations are used to determine the currents through various branches of electrical networks. The following two laws, which are based on experimental verification in the laboratory, lead to the equations.

Kirchhoff's Laws*

1. **Junctions**: All the current flowing into a junction must flow out of it.
2. **Paths**: The sum of the *IR* terms (*I* denotes current, *R* resistance) in any direction around a closed path is equal to the total voltage in the path in that direction.

EXAMPLE 2 Consider the electrical network of Figure 1.7. Let us determine the currents through each branch of this network.

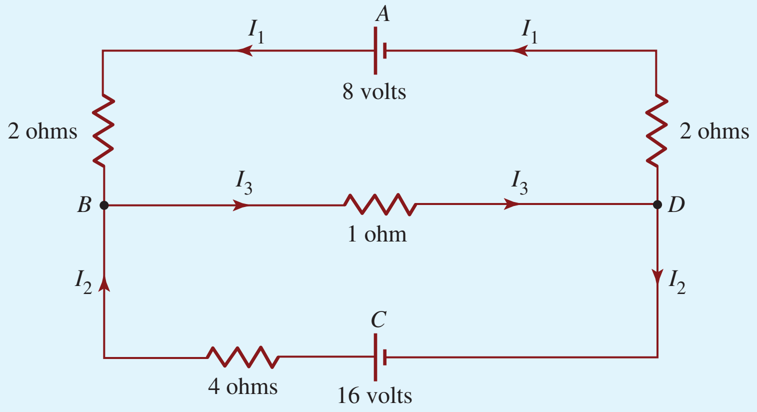

Figure 1.7

SOLUTION

The batteries, (denoted |ı) are 8 volts and 16 volts. The following convention is used in electrical engineering to indicate the terminal of the battery out of which the current flows: ↻. The resistances (denoted ∿∿) are one 1-ohm, one 4-ohm, and two 2-ohm. The current entering each battery will be the same as that leaving it.

Let the currents in the various branches of the above circuit be I_1, I_2, and I_3. Kirchhoff's laws refer to junctions and closed paths. There are two junctions in this circuit, namely the points *B* and *D*. There are three closed paths, namely *ABDA*, *CBDC*, and *ABCDA*. Apply the laws to the junctions and paths.

Junctions:

$$\text{Junction } B,\ I_1 + I_2 = I_3$$
$$\text{Junction } D,\ I_3 = I_1 + I_2$$

These two equations result in a single linear equation

$$I_1 + I_2 - I_3 = 0$$

Paths:

$$\text{Path } ABDA,\quad 2I_1 + 1I_3 + 2I_1 = 8$$
$$\text{Path } CBDC,\quad 4I_2 + 1I_3 = 16$$

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*Gustav Rubert Kirchhoff (1824–1887) was educated at the University of Königsberg and did most of his teaching at the University of Heidelberg. His most important contributions were in the discovery and analysis of the laws of electromagnetic radiation. Kirchhoff was an excellent teacher and writer whose books influenced teaching in German universities. Kirchhoff was described as "not easily drawn out but of a cheerful and obliging disposition."

It is not necessary to look further at path *ABCDA*. We now have a system of three linear equations in three unknowns, I_1, I_2, and I_3. Path *ABCDA* in fact leads to an equation that is a combination of the last two equations; there is no new information.

The problem thus reduces to solving the following system of three linear equations in three variables.

$$\begin{aligned} I_1 + I_2 - I_3 &= 0 \\ 4I_1 \qquad + I_3 &= 8 \\ 4I_2 + I_3 &= 16 \end{aligned}$$

Using the method of Gauss-Jordan elimination, we get

$$\begin{bmatrix} 1 & 1 & -1 & 0 \\ 4 & 0 & 1 & 8 \\ 0 & 4 & 1 & 16 \end{bmatrix} \underset{R2 + (-4)R1}{\approx} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & -4 & 5 & 8 \\ 0 & 4 & 1 & 16 \end{bmatrix}$$

$$\underset{(-\frac{1}{4})R2}{\approx} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -\frac{5}{4} & -2 \\ 0 & 4 & 1 & 16 \end{bmatrix} \underset{\substack{R1 + (-1)R2 \\ R3 + (-4)R2}}{\approx} \begin{bmatrix} 1 & 0 & \frac{1}{4} & 2 \\ 0 & 1 & -\frac{5}{4} & -2 \\ 0 & 0 & 6 & 24 \end{bmatrix}$$

$$\underset{(\frac{1}{6})R3}{\approx} \begin{bmatrix} 1 & 0 & \frac{1}{4} & 2 \\ 0 & 1 & -\frac{5}{4} & -2 \\ 0 & 0 & 1 & 4 \end{bmatrix} \underset{\substack{R1 + (-\frac{1}{4})R3 \\ R2 + (\frac{5}{4})R3}}{\approx} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{bmatrix}$$

The currents are $I_1 = 1$, $I_2 = 3$, $I_3 = 4$. The units are amps. The solution is unique, as is to be expected in this physical situation.

EXAMPLE 3 Determine the currents through the various branches of the electrical network in Figure 1.8. This example illustrates how one has to be conscious of direction in applying law 2 for closed paths.

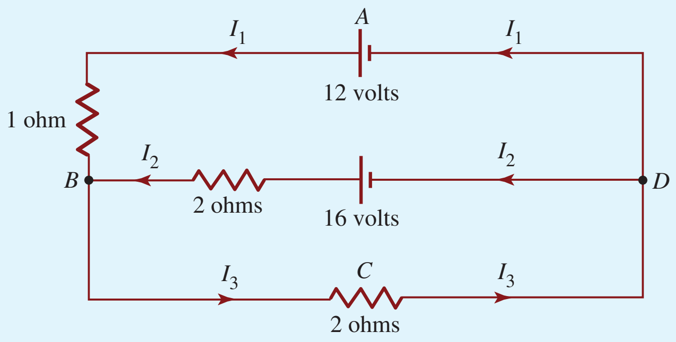


Figure 1.8

SOLUTION

Junctions:

$$\begin{aligned} &\text{Junction } B,\ I_1 + I_2 = I_3 \\ &\text{Junction } D,\ I_3 = I_1 + I_2 \end{aligned}$$

giving $I_1 + I_2 - I_3 = 0$.

Paths:

$$\begin{aligned} &\text{Path } ABCDA,\ 1I_1 + 2I_3 = 12 \\ &\text{Path } ABDA,\ 1I_1 + 2(-I_2) = 12 + (-16) \end{aligned}$$

Observe that we have selected the direction *ABDA* around this last path. The current along the branch *BD* in this direction is $-I_2$, and the voltage is -16. We now have three equations in the three variables I_1, I_2, and I_3.

$$\begin{array}{rcrcrcr} I_1 & + & I_2 & - & I_3 & = & 0 \\ I_1 & & & + & 2I_3 & = & 12 \\ I_1 & - & 2I_2 & & & = & -4 \end{array}$$

Solving these equations, we get $I_1 = 2$, $I_2 = 3$, $I_3 = 5$ amps.

In practice, electrical networks can involve many resistances and circuits; determining currents through branches involves solving large systems of equations on a computer.

Traffic Flow

Network analysis, as we saw in the previous discussion, plays an important role in electrical engineering. In recent years, the concepts and tools of network analysis have been found to be useful in many other fields, such as information theory and the study of transportation systems. The following analysis of traffic flow that was mentioned in the introduction illustrates how systems of linear equations with many solutions can arise in practice.

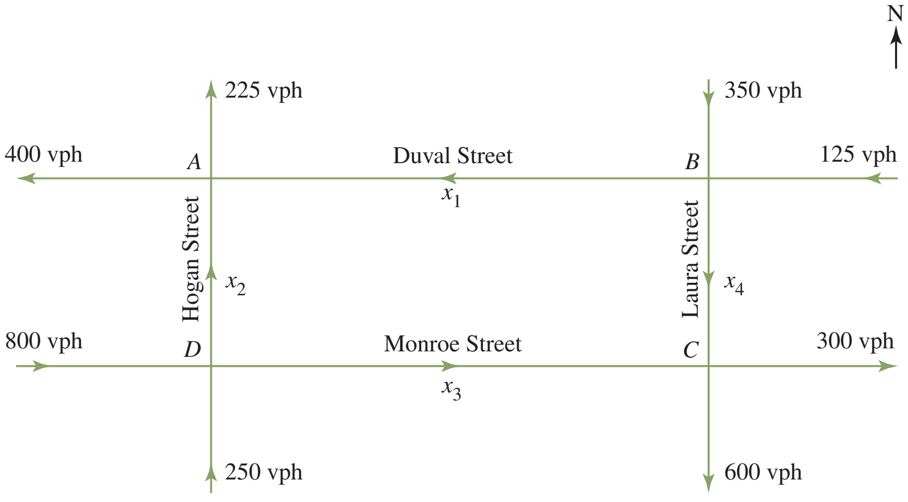
Consider the typical road network of Figure 1.9. It represents an area of downtown Jacksonville, Florida. The streets are all one-way with the arrows indicating the direction of traffic flow. The traffic is measured in vehicles per hour (vph). The figures in and out of the network given here are based on midweek peak traffic hours, 7 A.M. to 9 A.M. and 4 P.M. to 6 P.M. Let us construct a mathematical model that can be used to analyze the flow $x_1, \ldots, x_4$ within the network.

Assume that the following traffic law applies.

All traffic entering an intersection must leave that intersection.

This conservation of flow constraint (compare it to the first of Kirchhoff's laws for electrical networks) leads to a system of linear equations. These are, by intersection:

A: Traffic in $= x_1 + x_2$. Traffic out $= 400 + 225$. Thus $x_1 + x_2 = 625$.
B: Traffic in $= 350 + 125$. Traffic out $= x_1 + x_4$. Thus $x_1 + x_4 = 475$.
C: Traffic in $= x_3 + x_4$. Traffic out $= 600 + 300$. Thus $x_3 + x_4 = 900$.
D: Traffic in $= 800 + 250$. Traffic out $= x_2 + x_3$. Thus $x_2 + x_3 = 1050$.



Downtown Jacksonville, FL

Figure 1.9

The constraints on the traffic are described by the following system of linear equations.

$$\begin{aligned} x_1 + x_2 &= 625 \\ x_1 + x_4 &= 475 \\ x_3 + x_4 &= 900 \\ x_2 + x_3 &= 1050 \end{aligned}$$

The method of Gauss-Jordan elimination is used to solve this system of equations. The augmented matrix and reduced echelon form of the preceding system are as follows:

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 625 \\ 1 & 0 & 0 & 1 & 475 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 1 & 1 & 0 & 1050 \end{bmatrix} \approx \cdots \approx \begin{bmatrix} 1 & 0 & 0 & 1 & 475 \\ 0 & 1 & 0 & -1 & 150 \\ 0 & 0 & 1 & 1 & 900 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The system of equations that corresponds to this reduced echelon form is

$$\begin{aligned} x_1 + x_4 &= 475 \\ x_2 - x_4 &= 150 \\ x_3 + x_4 &= 900 \end{aligned}$$

Expressing each leading variable in terms of the remaining variable, we get

$$\begin{aligned} x_1 &= -x_4 + 475 \\ x_2 &= x_4 + 150 \\ x_3 &= -x_4 + 900 \end{aligned}$$

As was perhaps to be expected the system of equations has many solutions—there are many traffic flows possible. One does have a certain amount of choice at intersections. Let us now use this mathematical model to arrive at information. Suppose it becomes necessary to perform road work on the stretch *DC* of Monroe Street. It is desirable to have as small a flow x_3 as possible along this stretch of road. The flows can be controlled along various branches by means of traffic lights. What is the minimum value of x_3 along *DC* that would not lead to traffic congestion? We use the preceding system of equations to answer this question.

All traffic flows must be nonnegative (a negative flow would be interpreted as traffic moving in the wrong direction on a one-way street). The third equation tells us that x_3 will be a minimum when x_4 is as large as possible, as long as it does not go above 900. The largest value x_4 can be without causing negative values of x_1 or x_2 is 475. Thus the smallest value of x_3 is $-475 + 900$, or 425. Any road works on Monroe should allow for at least 425 vph.

In practice, networks are much vaster than the one discussed here, leading to larger systems of linear equations that are handled on computers. Various values of variables can be fed in and different scenarios created.

EXERCISE SET 1.3

Curve Fitting

In Exercises 1–5, determine the equations of the polynomials of degree two whose graphs pass through the given points.

1. $(1, 2), (2, 2), (3, 4)$

2. $(1, 14), (2, 22), (3, 32)$

3. $(1, 5), (2, 7), (3, 9)$

4. $(1, 8), (3, 26), (5, 60)$. What is the value of y when $x = 2$?

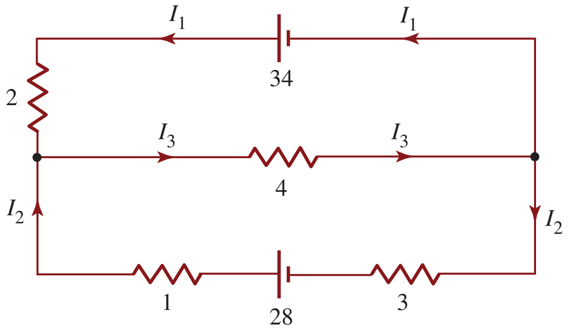
5. $(-1, -1), (0, 1), (1, -3)$. What is the value of y when $x = 3$?

6. Find the equation of the polynomial of degree three whose graph passes through the points $(1, -3), (2, -1), (3, 9), (4, 33)$.

Electrical Networks

In Exercises 7–14, determine the currents in the various branches of the electrical networks. The units of current are amps and the units of resistance are ohms. (*Hint*: In Exercise 14 it is difficult to decide the direction of the current along AB. Make a guess. A negative result for the current means that your guess was the wrong one—the current is in the opposite direction. However, the magnitude will be correct. There is no need to rework the problem.)

7.



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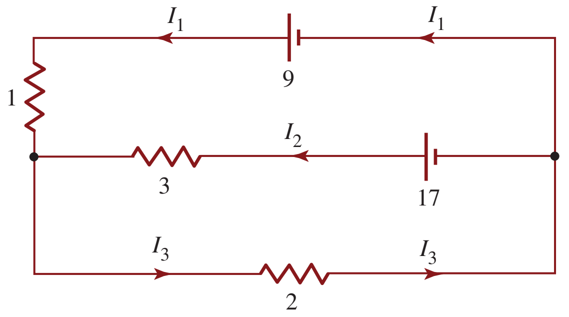

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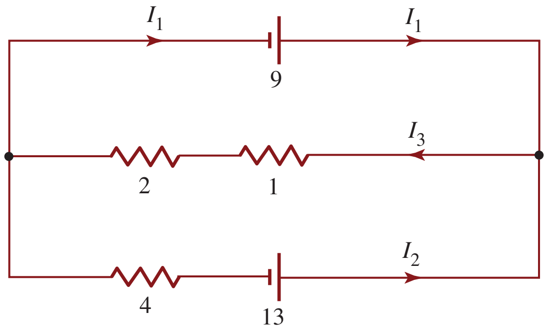

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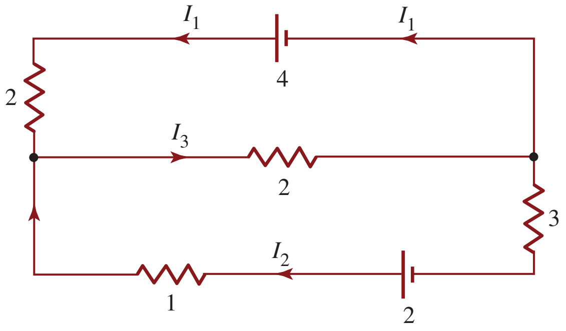

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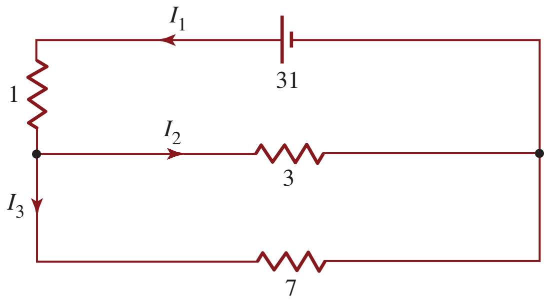

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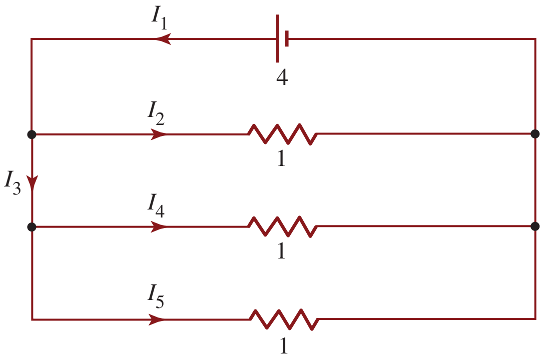

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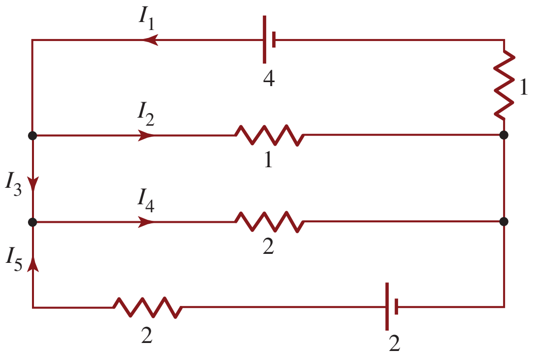

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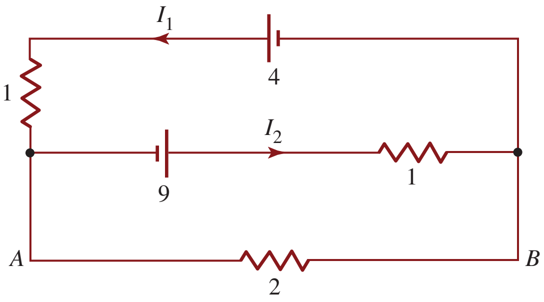

15. Determine the currents through the various branches of the electrical network in Figure 1.10.

(a) when battery C is 9 volts

(b) when battery C is 23 volts

Note how the current through the branch AB is reversed in (b). What would the voltage of C have to be for no current to pass through AB?

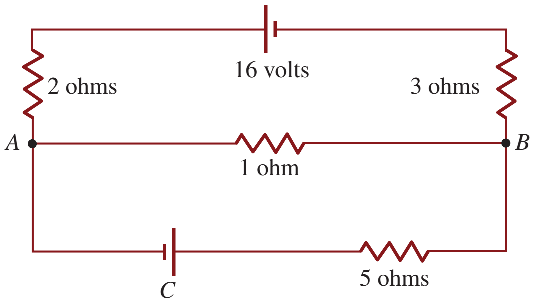

Figure 1.10

Traffic Flow

16. Construct a system of linear equations that describes the traffic flow in the road network of Figure 1.11. All streets are one-way streets in the directions indicated. The units are vehicles per hour. Give two distinct possible flows of traffic. What is the minimum possible flow that can be expected along branch AB?

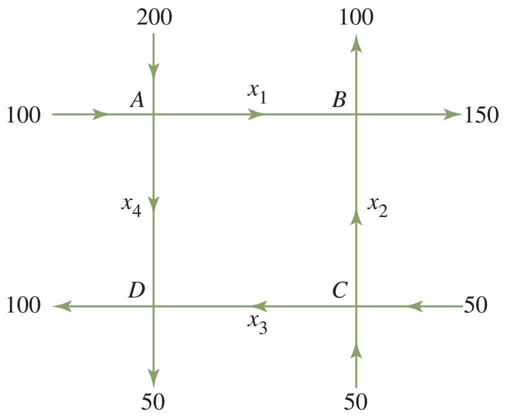

Figure 1.11

17. Figure 1.12 represents the traffic entering and leaving a "roundabout" road junction. Such junctions are very common in Europe. Construct a system of equations that describes the flow of traffic along the various branches. What is the minimum flow possible along the branch BC? What are the other flows at that time? (Units are vehicles per hour.)

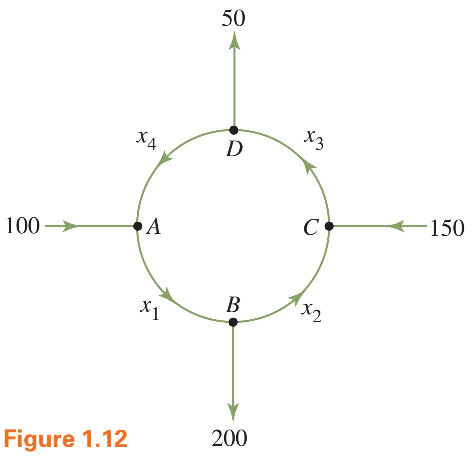

Figure 1.12

18. Figure 1.13 represents the traffic entering and leaving another type of roundabout road junction in Continental Europe. Such roundabouts ensure the continuous smooth flow of traffic at road junctions. Construct linear equations that describe the flow of traffic along the various branches. Use these equations to determine the minimum flow possible along x_1. What are the other flows at that time? (It is not necessary to compute the reduced echelon form. Use the fact that traffic flow cannot be negative.)

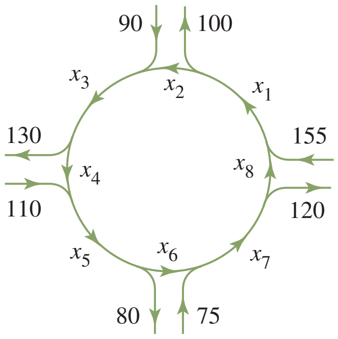

Figure 1.13

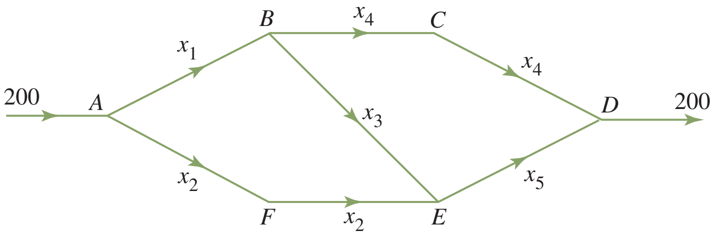

Figure 1.14

19. Figure 1.14 describes a flow of traffic, the units being vehicles per hour.

(a) Construct a system of linear equations that describes this flow.

(b) The total time it takes the vehicles to travel any stretch of road is proportional to the traffic along that stretch. For example, the total time it takes x_1 vehicles to traverse AB is kx_1 minutes. Assuming that the constant is the same for all sections of road, the total time for all these 200 vehicles to be in this network is $kx_1 + 2kx_2 + kx_3 + 2kx_4 + kx_5$. What is this total time if $k = 4$? Give the average time for each car.

20. There will be many polynomials of degree 2 that pass through the points (1, 2) and (3, 4). The situation can be described by a system of two linear equations in three variables that has many solutions. Find an equation (involving a parameter) that represents this family of polynomials. Determine the polynomials that open up and those that open down.

21. There will be many polynomials of degree 3 that pass through the points (1, 2), (3, 4), and (4, 8). The situation can be described by a system of three linear equations in four variables that has many solutions. Find an equation (involving a parameter) that represents this family of polynomials. Determine the unique polynomial of degree 3 passing through these points for which the coefficient of x^3 is 1.

CHAPTER 1 REVIEW EXERCISES*

1. Give the sizes of the following matrices.

(a) $\begin{bmatrix} 4 & 3 & -2 \\ 1 & 5 & 7 \end{bmatrix}$ **(b)** $\begin{bmatrix} 0 & 2 \\ 4 & 6 \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 3 & 2 & 7 \end{bmatrix}$ **(d)** $\begin{bmatrix} -2 \\ 3 \\ 6 \end{bmatrix}$

(e) $\begin{bmatrix} 8 & 5 & 3 & -7 & 5 & 9 \\ -2 & 3 & 5 & 7 & 0 & 2 \\ 4 & -3 & 5 & 1 & 2 & 3 \\ 0 & -8 & -1 & 5 & 3 & 8 \end{bmatrix}$

2. Give the (1, 3), (2, 1), (3, 3), (2, 5), (3, 6) elements of the following matrix.

$$\begin{bmatrix} 3 & 2 & 0 & 7 & 8 & 4 \\ 6 & 7 & 4 & 2 & 1 & 0 \\ 0 & 2 & 5 & 7 & 8 & 9 \end{bmatrix}$$

3. Write down the identity matrix I_5.

4. Determine the matrix of coefficients and the augmented matrix of each of the following systems of equations.

(a) $\begin{aligned} x_1 + 2x_2 &= 6 \\ 4x_1 - 3x_2 &= -1 \end{aligned}$

(b) $\begin{aligned} 2x_1 + x_2 - 4x_3 &= 1 \\ x_1 - 2x_2 + 8x_3 &= 0 \\ 3x_1 + 5x_2 - 7x_3 &= -3 \end{aligned}$

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*Answers to all review exercises are provided in the back of the book.

(c)
$$\begin{aligned} -x_1 + 2x_2 - 7x_3 &= -2 \\ 3x_1 - x_2 + 5x_3 &= 3 \\ 4x_1 + 3x_2 &= 5 \end{aligned}$$

(d)
$$\begin{aligned} x_1 &= 1 \\ x_2 &= 5 \\ x_3 &= -3 \end{aligned}$$

(e)
$$\begin{aligned} -2x_1 + 3x_2 - 8x_3 + 5x_4 &= -2 \\ x_1 + 5x_2 - 6x_4 &= 0 \\ -x_2 + 2x_3 + 3x_4 &= 5 \end{aligned}$$

5. Interpret the following matrices as augmented matrices of systems of equations. Write down each system of equations.

(a) $\begin{bmatrix} 4 & 2 & 0 \\ -3 & 7 & 8 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 9 & -3 \\ 0 & 3 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 0 & -3 & 6 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 4 & -1 & 7 \\ 0 & 1 & 3 & 8 \\ 0 & 0 & 1 & -5 \end{bmatrix}$

6. Determine whether the following matrices are in reduced echelon form. If a matrix is not in reduced echelon form give a reason.

(a) $\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 7 \end{bmatrix}$ **(b)** $\begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & 0 & 6 \\ 0 & 1 & 0 & -7 \\ 0 & 0 & 1 & 9 \end{bmatrix}$ **(d)** $\begin{bmatrix} 1 & 3 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 & 7 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 8 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & -7 \\ 0 & 0 & 1 & 0 & 3 \end{bmatrix}$

7. The following systems of equations have unique solutions. Solve these systems using the method of Gauss-Jordan elimination with matrices.

(a)
$$\begin{aligned} 2x_1 + 4x_2 &= 2 \\ 3x_1 + 7x_2 &= 2 \end{aligned}$$

(b)
$$\begin{aligned} x_1 - 2x_2 - 6x_3 &= -17 \\ 2x_1 - 6x_2 - 16x_3 &= -46 \\ x_1 + 2x_2 - x_3 &= -5 \end{aligned}$$

(c)
$$\begin{aligned} x_2 + 2x_3 + 6x_4 &= 21 \\ x_1 - x_2 + x_3 + 5x_4 &= 12 \\ x_1 - x_2 - x_3 - 4x_4 &= -9 \\ 3x_1 - 2x_2 - 6x_4 &= -4 \end{aligned}$$

8. Solve (if possible) the following systems of equations using the method of Gauss-Jordan elimination.

(a)
$$\begin{aligned} x_1 - x_2 + x_3 &= 3 \\ -2x_1 + 3x_2 + x_3 &= -8 \\ 4x_1 - 2x_2 + 10x_3 &= 10 \end{aligned}$$

(b)
$$\begin{aligned} x_1 + 3x_2 + 6x_3 - 2x_4 &= -7 \\ -2x_1 - 5x_2 - 10x_3 + 3x_4 &= 10 \\ x_1 + 2x_2 + 4x_3 &= 0 \\ x_2 + 2x_3 - 3x_4 &= -10 \end{aligned}$$

9. Let A be an $n \times n$ matrix in reduced echelon form. Show that if $A \neq I_n$, then A has a row consisting entirely of zeros.

10. Let A and B be row equivalent matrices. Show that A and B have the same reduced echelon form.

11. Determine the equation of the polynomial of degree two whose graph passes through the points (1, 3), (2, 6), (3, 13).

12. Determine the currents through the branches of the network in Figure 1.15.

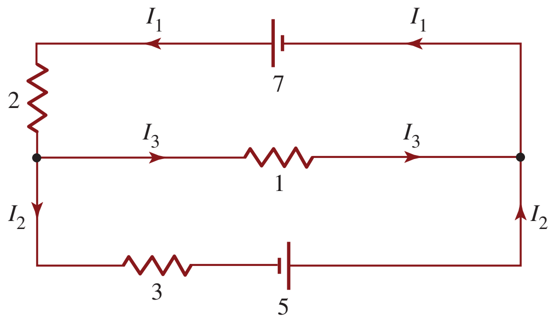

Figure 1.15

13. Consider the traffic entering and leaving a roundabout road junction in Britain, shown in Figure 1.16. (Observe that the traffic goes around this roundabout in the opposite direction to the one on the Continent in Exercise 18, Section 1.3. They drive on the left of the road in Britain.) Construct a system of linear equations that describe the flow of traffic along the various branches. Determine the minimum flow possible along x_8. What are the other flows at that time?

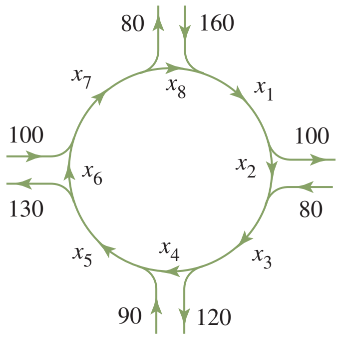

Figure 1.16