Chapter

6

Integration on Curves

In this chapter, we will continue with the discussion of integration. We will be concerned with the total accumulation of a function defined on a curve in the plane or in space. In particular, we will investigate the integral of a vector field along a curve.

In Section 6.1, we will introduce the idea of the total accumulation of a function on a curve, which will lead to the concept of the path integral of a function along a curve. Following the program we developed in Chapter 5, we will first approximate the total accumulation by a Riemann sum and then define the integral to be a limit of Riemann sums. Once again, we will partition the region of integration, which is now a curve, sample the values of the function in each cell of the partition, scale the sampled values, and sum them to produce our Riemann sum. We express the partition and the Riemann sum in terms of a parametrization of the curve. An important question will be to understand how the integral depends on the choice of parametrization.

In Section 6.2, we will introduce a new concept, the line integral of a vector field along a curve, which is the total accumulation of the component of the vector field in the direction of the curve. This will be motivated by a discussion of the concept of work in physics, and we will apply this to a number of examples. In particular, we will explore the relationship between work and energy and define the concepts of kinetic and potential energy.

In Section 6.3, we will focus on the line integral of a vector field around a closed curve. This will lead to a discussion of Green's theorem, a surprising result that connects the line integral of a vector field around a closed curve to a double integral of derivatives of the vector field over the region enclosed by the curve. We will apply these results in a discussion of the flux of a vector field across a curve, where we revisit diffusion in the plane. The section ends with a proof of Green's theorem.

A Collaborative Exercise—A Charged Wire

In Chapter 5, we used Riemann sums to approximate the total accumulation of a function defined over a region in the plane or in space. The process we used to create the Riemann sum was to *partition* the region, *sample* the function in each cell of the partition, *scale* our sampled function value by multiplying it by the size (area or volume) of the cell, and then *sum* these sampled values over all cells in our partition of the region. We saw that we could improve our approximation by *refining* our partition, that is, shrinking the size of the cells and using more sampled values.

Now we will turn our attention to the problem of approximating the total accumulation of a function defined on a path in the plane or in space.

We begin with an example from physics. Let us consider the charge on an insulated copper wire in the presence of a stationary (not changing in time) electric field. Since the wire is insulated, the electric charge cannot leave the wire. However, since copper is a conductor, the charge will move freely in the wire in response to the electric field until it reaches an equilibrium distribution. The charge will generally be unevenly distributed in the wire and can be represented by a *charge density* function ρ that is defined on the wire. The total charge of the wire is the total accumulation of the charge density function over the length of the wire. In this discussion, we will try to develop a method to approximate the total charge of the wire.

1. Assume that a charged wire in space is in the presence of a stationary electric field. Further, assume we can measure the equilibrium charge density ρ at any point (x, y, z) on the wire and that the charge is measured in charge per unit length.
   a. Following the process we used in Chapter 5 (and outlined above), describe how you would set up a Riemann sum to approximate the total charge in the wire.
   b. What would you use for the *scaling factor*?
   c. Write out your Riemann sum in terms of ρ and points (x, y, z) on the wire.
2. Assume that the wire is represented by the image of a parametrization $\alpha : [a, b] \subset \mathbb{R} \to \mathbb{R}^3$. Thus points on the wire are of the form $\alpha(t) = (x(t), y(t), z(t))$ for $a \leq t \leq b$. Here we want to express the Riemann sum from 1c in terms of t.
   a. How would you represent your partition of the wire into N cells in terms of the parameter t?
   b. How would you represent your sampling scheme in terms of the parameter t?
   c. What would the charge density be at the sampled points?
   d. Can you represent the scaling factor for the i^{th} cell in terms of t? If not, can you use the parametrization to approximate the scaling factor?
   e. Using your answers to a through d, express your Riemann sum from 1c in terms of t.
3. Based on our previous work with Riemann sums, what would you expect to do to calculate an exact value for the total charge on the wire?

6.1 Path Integrals

In this section, we will turn our attention to the problem of determining the total accumulation of a function defined on a curve in the plane or in space. Before we begin, let us consider several examples of physical phenomena that can be modeled by functions on curves.

Example 6.1 **Total Accumulation on a Curve**

A. Total Charge. As in the collaborative exercise, consider an electric charge in an insulated wire. In the presence of an electric field, the electric charge in the wire will be unevenly distributed in the wire. This can be represented by a ***charge density*** function ρ that is defined on the wire. The total charge of the wire is the total accumulation of the charge density function over the length of the wire.

B. Total Mass. Suppose we have a function that represents a density distribution along a curve, for example, a wire coated by ice of varying thickness. The total accumulation of the density function along the curve is the ***total mass***.

C. Arc Length. If a function takes a constant value 1 everywhere on a curve, the total accumulation of the function is the ***arc length*** of the curve.

D. Potential Energy. Consider a system that consists of a uniformly charged wire and a charged particle located at a point $\mathbf{x}$ not on the wire. The potential energy function for the system can be expressed as the total accumulation of the potential energy function for the particle in space and a charged particle on the wire, where the accumulation is taken over the particles in the wire.

As we did in Chapter 5, we will begin by constructing a Riemann sum to approximate the total accumulation of a function defined on a curve. In particular, we will express the Riemann sum in terms of the coordinate of a parametrization of the curve. That is, we will partition the domain of the parametrization, which will then give rise to a partition of the curve that is the image of the parametrization. Then we will sample the function on the curve, scale the sampled values, and sum the scaled values. In this case, since the domain is a subset of $\mathbb{R}$, the scaling factor will be the length of a cell rather than the area of a cell. The result will be a Riemann sum for a function of one variable. Thus, we will be able to use the fundamental theorem of calculus to evaluate the integral. Initially, we will work with curves in the plane; the generalization to curves in space is immediate.

Let us begin with a function f defined on a curve $\mathcal{C}$ in the plane. We will assume that we have been given or have constructed a parametrization $\alpha : [a, b] \to \mathbb{R}^2$ of $\mathcal{C}$. If $\alpha(t) = (x(t), y(t))$,

$$\mathcal{C} = \{(x(t), y(t)) : a \leq t \leq b\} \subset \mathbb{R}^2.$$

We want to construct a Riemann sum to approximate the total accumulation of f on $\mathcal{C}$, the image of α. In order to partition $\mathcal{C}$, we start with a partition P of $[a, b]$, the domain of α, into N subintervals. We choose $N + 1$ points t_i, $i = 0, \ldots, N$, so that

$$a = t_0 < t_1 < t_2 < \ldots < t_{N-1} < t_N = b.$$

The i^{th} subinterval of the partition, $[t_{i-1}, t_i]$, has length $\Delta t_i = t_i - t_{i-1}$. The image of the partition P of $[a, b]$ gives rise to a partition of $\mathcal{C}$ into N cells. The i^{th} cell of this partition is the image of the i^{th} subinterval of P,

$$\{(x(t), y(t)) : t_{i-1} \leq t \leq t_i\}.$$

(See Figure 6.1.) In order to sample f in the i^{th} cell of $\mathcal{C}$, choose a point t_i^* in the i^{th} subinterval of the partition P of $[a, b]$, that is, $t_{i-1} \leq t_i^* \leq t_i$. We sample f at the point $\alpha(t_i^*)$, so that the sampled value of f in the i^{th} cell is $f(\alpha(t_i^*))$.

Following the procedure we developed for constructing Riemann sums in Chapter 5, we scale the sampled value $f(\alpha(t_i^*))$ by the length Δs_i of the i^{th} cell of $\mathcal{C}$. Thus the Riemann sum for the partition P is

$$R(f, \alpha, P) = \sum_{i=1}^{N} f(\alpha(t_i^*))\Delta s_i.$$

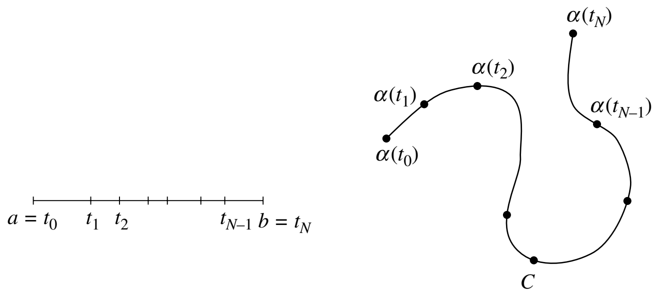

Figure 6.1 A partition of the interval $[a, b]$ and the corresponding partition of the curve $\mathcal{C}$ that is the image of α.

In order to obtain an exact value for the total accumulation of f on α, we evaluate the limit of $R(f, \alpha, P)$ as the mesh of P approaches 0, where the mesh of P, mesh(P), is the largest length Δs_i in the partition. Once again, if this limit exists and is independent of the choices of partitions and sampling schemes, we define the limit to be an integral of f. Since this construction depends on the choice of parametrization α, we will say that the integral is over α rather than $\mathcal{C}$. Note that the analogous construction applies to functions f of three variables and parametrizations α of curves in space. If we assume f is continuous, and α is continuously differentiable, that is, α is differentiable and α' is continuous, then one can prove this limit exists and is independent of choices. We summarize this in the following proposition:

Proposition 6.1 Let $\alpha : [a, b] \to \mathbb{R}^2$ or $\mathbb{R}^3$ be a continuously differentiable function, and let f be a continuous function defined on the image of α. Then

$$\lim_{\text{mesh}(P) \to 0} R(f, \alpha, P)$$

exists and is independent of the choices of partitions and sampling schemes. We call this limit the ***path integral*** of f on α, and we denote it by $\int_\alpha f\ ds$. Thus,

$$\int_\alpha f\ ds = \lim_{\text{mesh}(P) \to 0} R(f, \alpha, P). \quad \blacklozenge$$

Before we develop a method for evaluating path integrals, let us consider an example of a calculation of a Riemann sum and a path integral for the potential energy of a charged particle and a uniformly charged wire. (See Example 6.1D.)

Example 6.2 **Potential Energy of a Uniformly Charged Wire.** Consider a uniformly charged wire of charge density C in the shape of a circle of radius 1 and a particle of charge q that is located at a point not on the wire. If the wire is fixed in space, potential energy depends only on the position of the charged particle. Intuitively, the potential energy of the system is obtained by summing the potential energy of pairs of points consisting of the point in space and a point in the wire. The potential energy of two charged particles with charges q_1 and q_2 is

$$\frac{q_1 q_2}{4\pi\epsilon_0} \frac{1}{r},$$

where r is the distance between the two particles and ϵ_0 is a constant.

To simplify matters, we will identify the wire with the circle parametrized by $\alpha(t) = (\cos t, \sin t, 0)$, $0 \le t \le 2\pi$, and will locate the particle of charge q at $(0, 1, 1)$. We will partition the domain $[0, 2\pi]$ into N subintervals, and we will let Δt_i be the length of the i^{th} subinterval and t_i^* be a point in the i^{th} subinterval of this partition. The partition of

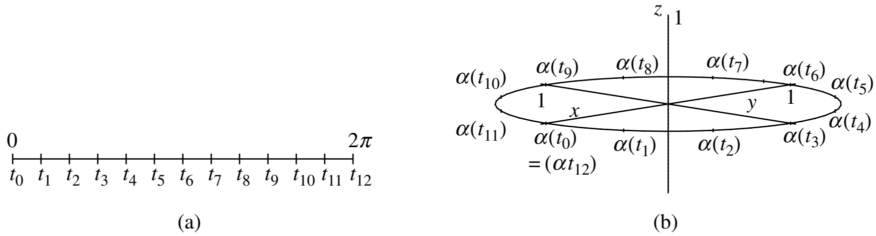

Figure 6.2 (a) A partition of the domain $[0, 2\pi]$ of Example 6.2 into $N = 12$ cells. (b) The image under α of the partition of $[0, 2\pi]$ is a partition of the unit circle into $N = 12$ cells.

the wire is the image of this partition under α. (See Figure 6.2.) For this parametrization, the length of the i^{th} cell in the partition of the wire is also Δt_i. (Why?)

Since the length of a cell is small, the charge in the cell acts like a point charge of magnitude $C\Delta t_i$. Thus the potential energy due to the i^{th} cell is approximately

$$\begin{aligned}\frac{qC\Delta t_i}{4\pi\epsilon_0}\frac{1}{r} &= \frac{qC\Delta t_i}{4\pi\epsilon_0}\frac{1}{\|(0,1,1)-\alpha(t_i^*)\|}\\ &= \frac{qC}{4\pi\epsilon_0}\frac{\Delta t_i}{\sqrt{\cos(t_i^*)^2+(1-\sin(t_i^*))^2+1}}.\end{aligned}$$

The potential energy can be approximated by the Riemann sum

$$\frac{Cq}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{\Delta t_i}{\sqrt{\cos(t_i^*)^2+(1-\sin(t_i^*))^2+1}}.$$

We must now evaluate the limit of this Riemann sum as the mesh of the partition approaches 0, that is, as Δt_i approaches 0. The limit is equal to

$$\frac{Cq}{4\pi\epsilon_0}\int_0^{2\pi}\frac{1}{\sqrt{\cos^2 t+(1-\sin t)^2+1}}\,dt = \frac{Cq}{4\pi\epsilon_0}\int_0^{2\pi}\frac{1}{\sqrt{3-2\sin t}}\,dt.$$

This defines the potential energy U of the system when the charged particle is located at $(0, 1, 1)$.

The Path Integral

Returning to the general case, let us assume that f is continuous and that α is continuously differentiable on $[a, b]$. In order to evaluate the path integral of f on α we must equate it with a definite integral in one variable.

We begin by approximating the Riemann sum

$$R(f, \alpha, P) = \sum_{i=1}^{N} f(\alpha(t_i^*))\Delta s_i$$

with a Riemann sum for a function of t on $[a, b]$. Since we cannot, in general, compute the length Δs_i of a cell explicitly, we will approximate it by the length of the straight line segment between the endpoints of the cell,

$$\Delta s_i \approx \|\alpha(t_i) - \alpha(t_{i-1})\|.$$

However, this is not yet in the form that we want. Since α is differentiable, we can approximate $\alpha'(t)$ by the difference quotient $\frac{1}{\Delta t}(\alpha(t+\Delta t) - \alpha(t))$ for small values of Δt. It follows that

$$\alpha(t + \Delta t) - \alpha(t) \approx \alpha'(t)\Delta t.$$

Since $t_i = t_{i-1} + \Delta t_i$ where Δt_i is positive, we have

$$\begin{aligned} \|\alpha(t_i) - \alpha(t_{i-1})\| &= \|\alpha(t_{i-1} + \Delta t_i) - \alpha(t_{i-1})\| \\ &\approx \|\alpha'(t_{i-1})\Delta t_i\| \\ &= \|\alpha'(t_{i-1})\|\Delta t_i. \end{aligned}$$

Returning to our Riemann sum, if we choose $t_i^* = t_{i-1}$, then $\Delta s_i \approx \|\alpha'(t_i^*)\|\Delta t_i$ and

$$R(f, \alpha, P) \approx \sum_{i=1}^{N} f(\alpha(t_i^*))\|\alpha'(t_i^*)\|\Delta t_i.$$

This sum is a Riemann sum for the function of t given by $f(\alpha(t))\|\alpha'(t)\|$. Since f, α, and α' are continuous, the limit of Riemann sums of this form exists and is equal to

$$\int_a^b f(\alpha(t))\|\alpha'(t)\|\, dt.$$

It is possible to make this intuitive argument rigorous, though we will not do so here. We summarize this discussion in the following theorem. Notice that the criteria for the existence of the integral of f on α depend on both f and α.

Theorem 6.1 Let $\alpha : [a, b] \to \mathbb{R}^2$ be a continuously differentiable function, and let f be a continuous function defined on the image of α. Then

$$\int_{\alpha} f \, ds = \int_a^b f(\alpha(t)) \|\alpha'(t)\| \, dt. \; \blacklozenge$$

If α is a parametrization of a curve in space, the substance of the argument remains the same and the Riemann sums and limit take the same form. We will use the same notation for the integral of a function on a space curve.

We use this result in the following example.

Example 6.3 **Path Integrals**

A. Total Mass on a Curve. Let $\alpha(t) = (t, t^3)$, $0 \le t \le 2$, be a parametrization of the curve $y = x^3$ between $(0, 0)$ and $(2, 8)$, and let $\delta(x, y) = y$ be a function that represents a density distribution on the curve. The total mass along α is the total accumulation of δ. Thus the total mass is

$$\begin{aligned} \int_{\alpha} \delta \, ds &= \int_0^2 \delta(\alpha(t)) \|\alpha'(t)\| \, dt \\ &= \int_0^2 t^3 \sqrt{1 + 9t^4} \, dt \\ &= \frac{1}{54} \left(1 + 9t^4\right)^{3/2} \Big|_0^2 \\ &= \frac{(145)^{3/2} - 1}{54}. \end{aligned}$$

B. Total Accumulation on a Helix. Let $f(x, y, z) = x^2 + y^2 + z^2$, and let $\alpha(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$, a parametrization of a spiral helix of radius 1 and height 2π. The total accumulation of f on α is

$$\begin{aligned} \int_{\alpha} f \, ds &= \int_0^{2\pi} f(\alpha(t)) \|\alpha'(t)\| \, dt \\ &= \int_0^{2\pi} f(\cos t, \sin t, t) \sqrt{2} \, dt \\ &= \int_0^{2\pi} \left(1 + t^2\right) \sqrt{2} \, dt \\ &= \sqrt{2} \left[t + t^3/3\right]_0^{2\pi} \\ &= \sqrt{2} \left(2\pi + \frac{8}{3}\pi^3\right). \end{aligned}$$

If the function f is the constant function equal to 1, $f(x, y) = 1$ for all x and y, then the Riemann sum for f over α is

$$R(f, \alpha, P) = \sum_{i=1}^{N} \Delta s_i \approx \sum_{i=1}^{N} \|\alpha(t_i) - \alpha(t_{i-1})\|.$$

Each term is an approximation of the length of an individual cell of the partition of the image of α. The limit of these Riemann sums is defined to be the arc length of the parametrization α. Thus we have the following definition.

Definition 6.1 Let $\alpha : [a, b] \to \mathbb{R}^2$ or $\mathbb{R}^3$ be a continuously differentiable function. We define the ***arc length*** of α to be the path integral $\int_\alpha 1\, ds = \int_a^b \|\alpha'(t)\|\, dt$. ♦

If $\alpha : [a, b] \to \mathbb{R}^2$ or $\mathbb{R}^3$ is continuously differentiable and f is a continuous function defined on the image of α, then we define the ***average value*** of f on α to be the total accumulation of f on α divided by the arc length α. In the following example, we use this to compute the average value of the function $f(x, y) = x$ on a parametrization α.

Example 6.4 **Arc Length and Average Value.** Let $\alpha(t) = (t, t^2)$, $0 \le t \le 2$, which parametrizes an arc $\mathcal{C}$ of the parabola given by $y = x^2$, and let $f(x, y) = x$. To find the average value of f on $\mathcal{C}$, we first compute the arc length of α.

$$\begin{aligned} \int_\alpha ds &= \int_0^2 \|\alpha'(t)\|\, dt \\ &= \int_0^2 \|(1, 2t)\|\, dt \\ &= \int_0^2 \sqrt{1 + 4t^2}\, dt \\ &= \left[\frac{t}{2}\sqrt{1 + 4t^2} + \frac{1}{4}\ln(2t + \sqrt{1 + 4t^2})\right]_0^2 \\ &= \left[\sqrt{17} + \frac{1}{4}\ln(4 + \sqrt{17})\right]. \end{aligned}$$

The total accumulation of $f(x, y) = x$ over α is

$$\begin{aligned} \int_\alpha f\, ds &= \int_0^2 t\sqrt{1 + 4t^2}\, dt \\ &= \left[\frac{2}{3 \cdot 8}(1 + 4t^2)^{3/2}\right]_0^2 \\ &= \frac{1}{12}(17^{3/2} - 1). \end{aligned}$$

Thus the average value of f on α is the quotient of these two values, which is approximately 1.239.

Independence of Parametrization

Since our calculations depend on a parametrization of a curve, it would appear that a different choice of parametrization might yield a different value for the integral. Fortunately, this is not the case, since, as we will now demonstrate, the value of a path integral does not depend on the choice of the parametrization of the curve.

First, we must consider the relationship between different parametrizations of the same curve. Let us begin with a differentiable parametrization $\alpha = \alpha(u)$ of a curve $\mathcal{C}$, $c \leq u \leq d$, that is one-to-one. This ensures that α traces $\mathcal{C}$ just once. If $u = h(t)$ is a differentiable function of one variable, $h : [a, b] \to [c, d]$, with $h(a) = c$, $h(b) = d$, and $h'(t) > 0$ on $[a, b]$, then the composition $\beta(t) = (\alpha \circ h)(t) = \alpha(h(t))$ is also a differentiable parametrization of the curve $\mathcal{C}$ that traces $\mathcal{C}$ just once. We claim that the total accumulation of a function f over α and over β are the same. That is, we have the following proposition, whose proof is left as an exercise. (See Exercise 8.)

Proposition 6.2 Under the above hypotheses on the parametrizations β and α of $\mathcal{C}$,

$$\int_\beta f\ ds = \int_\alpha f\ ds. \quad \blacklozenge$$

Notice that since $\beta'(t) = \alpha'(h(t))h'(t)$ and $h'(t) > 0$, β' is a positive multiple of α', so that β and α parametrize $\mathcal{C}$ in the same direction. If instead h had satisfied $h(a) = d$, $h(b) = c$, and $h'(t) < 0$, β' would be a negative multiple of α', and β and α would parametrize $\mathcal{C}$ in opposite directions. The above proposition also holds in this case.

Finally, if α and β are one-to-one continuously differentiable parametrizations of the same curve $\mathcal{C}$, then it can be shown that there is a differentiable function h such that $\beta = \alpha \circ h$. Combining this fact with the proposition, it follows that the total accumulation of a function on the image of a parametrization depends on the image curve $\mathcal{C}$ and not on the choice of parametrization, as long as the parametrization is one-to-one. In particular, if $\mathcal{C}$ is a curve and α is a one-to-one parametrization of $\mathcal{C}$, then we can define the arc length of the curve $\mathcal{C}$ to be the arc length of α. Thus we have the following definitions of the path integral of a function on a curve and the arc length of a curve.

Definition 6.2 Let $\mathcal{C}$ be a curve in the plane or in space, and let f be a continuous function whose domain contains $\mathcal{C}$.

We define the ***path integral*** of f on $\mathcal{C}$ to be the path integral of f on α, where α is ***any*** one-to-one continuously differentiable parametrization of $\mathcal{C}$. We denote the path integral of f on $\mathcal{C}$ by $\int_C f\ ds$.

We define the ***arc length*** of $\mathcal{C}$ to be the arc length of α, where α is ***any*** one-to-one continuously differentiable parametrization of $\mathcal{C}$. Thus the arc length of $\mathcal{C}$ is $\int_C ds = \int_\alpha ds$. ♦

In the following example of total accumulation on a curve, we are given $\mathcal{C}$ and must find a convenient parametrization in order to compute the path integral.

Example 6.5

Total Accumulation on a Curve. Let us compute the total accumulation of $f(x, y) = xy$ on the portion of the curve $x^2 + 4y^2 = 4$ in the first quadrant. First we must parametrize the curve. Let us use the parametrization $\alpha(t) = (2\cos t, \sin t)$, $0 \le t \le \pi/2$. Then the total accumulation of f on $\mathcal{C}$ can be computed as follows.

$$\begin{aligned}
\int_C f\ ds &= \int_0^{\pi/2} f(\alpha(t))\|\alpha'(t))\|\,dt \\
&= \int_0^{\pi/2} 2\cos t \sin t\sqrt{4\sin^2 t + \cos^2 t}\ dt \\
&= \int_0^{\pi/2} 2\cos t \sin t\sqrt{3\sin^2 t + 1}\,dt \\
&= \left[(2/9)\left(3\sin^2 t + 1\right)^{3/2}\right]_0^{\pi/2} \\
&= 14/9.
\end{aligned}$$

Summary

In this section, we introduced the concept of the ***path integral*** of a function, which extends integration to functions defined on curves in the plane or in space.

We began by constructing a Riemann sum for the total accumulation of a function f on the curve $\mathcal{C}$. We first ***partitioned the domain of a parametrization*** $\alpha : [a, b] \to \mathbb{R}^2$ of $\mathcal{C}$. This gave rise to a partition of $\mathcal{C}$, which we used to construct a Riemann sum for the total accumulation of f. By taking the limit of Riemann sums constructed in this manner, we obtained an exact value for the total accumulation of f on α. This value is called the ***path integral*** of f over α, which we denoted by $\int_\alpha f\ ds$.

The integral exists when f and α are continuously differentiable. Further,

$$\int_\alpha f\ ds = \int_a^b f(\alpha(t))\|\alpha'(t)\|\, dt.$$

Using this form, it is possible to show that the total accumulation of f over a parametrization of $\mathcal{C}$ does not depend on the choice of the parametrization α. Thus $\int_{\mathcal{C}} f\ ds = \int_\alpha f\ ds$ for any one-to-one parametrization α of $\mathcal{C}$.

Section 6.1 Exercises

1. A Riemann Sum Calculation. Figure 6.3 is a contour plot of a trail up Stinson Mountain in New Hampshire from the parking lot, located at approximately 1500 ft above sea level, to the peak, which is approximately 2850 ft above sea level. The contours are at 40-ft intervals.

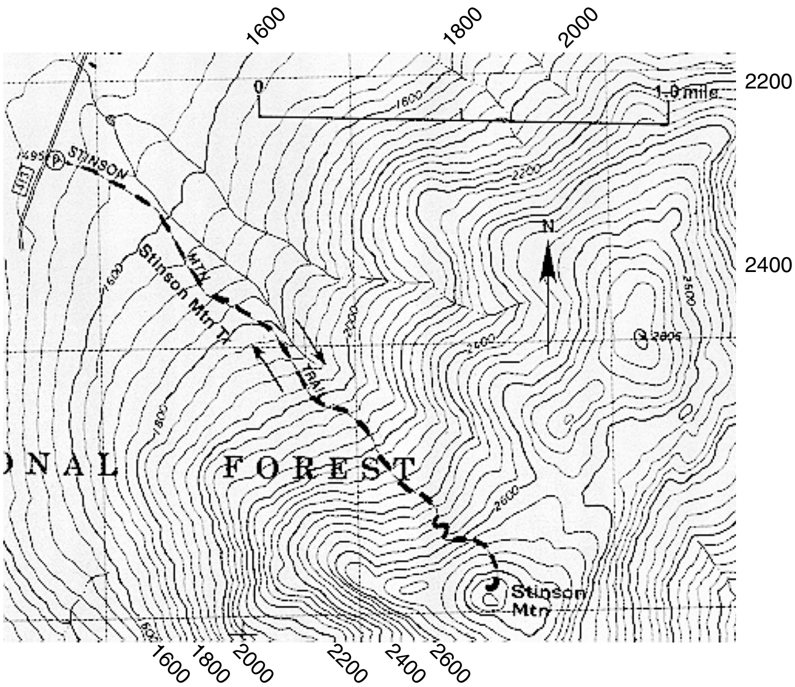

Figure 6.3 The contour plot of Stinson Mountain, New Hampshire, for Exercise 1. Map courtesy of the U.S. Geological Survey.

(a) Explain how you would use the contour map to construct a Riemann sum to approximate the length of this trail. (*Hint:* In order to do this, we must partition the path and then approximate Δs_i, the length of each cell in the partition. Keep in mind that the contour plot is a two-dimensional representation of the mountain, and thus the path drawn on the contour map is a two-dimensional representation of a path in space.)

(b) Use a Riemann sum with $N = 4$ cells to approximate the length of this trail. Carefully describe the endpoints of the cells in your partition of the trail, give the approximating value of Δs_i for each cell, and compute the sum that will approximate the length.

(c) We know we can improve the approximation by increasing the number of cells in the partition. Use a Riemann sum with $N = 8$ cells to approximate the length of this trail. Carefully describe the endpoints of the cells in your partition of the trail, give the approximating value of Δs_i for each cell, and compute the sum that will approximate the length.

(d) Why should your approximation in (c) be better than your approximation in (b)?

2. Arc Length. Compute the arc length of the following parametrizations.

(a) $\alpha(t) = (t, t^{3/2})$ for $1 \leq t \leq 4$.
(b) $\alpha(t) = (\cos(3t), \sin(3t), t)$ for $0 \leq t \leq \pi$.
(c) $\alpha(t) = (3t^2, 2t^3, \frac{3}{4}t^4)$ for $0 \leq t \leq 2$.

3. Path Integrals over $\mathcal{C}$. For each of the following functions f and curves $\mathcal{C}$, (i) construct a parametrization α of $\mathcal{C}$ and (ii) set up and evaluate an integral to compute the total accumulation of f on $\mathcal{C}$.

(a) $\mathcal{C}$ is the line segment from $(1, -1, 2)$ to $(3, 0, 1)$ and $f(x, y, z) = e^{x+y+z}$.
(b) $\mathcal{C}$ is the circle of radius 2 in the xy-plane centered at the origin and $f(x, y) = x^2 y$.

4. Total Mass. For each of the following parametrizations α and density functions δ, compute the total mass, $\int_\alpha \delta \, ds$.

(a) $\alpha(t) = (t, \ln(t))$ for $1 \leq t \leq 3$ with density function $\delta(x, y) = x^2$.
(b) $\alpha(t) = (t, t^2, t^2)$ for $0 \leq t \leq 2$ with density function $\delta(x, y, z) = x$.

5. Average Value. Compute the average value of $f(x, y, z) = x^2 + y^2 + z^2$ over each of the following parametrizations.

(a) $\alpha(t) = (\cos(3t), \sin(3t), t)$ for $0 \leq t \leq \pi$. (See Exercise 2(b).)
(b) $\alpha(t) = (3t^2, 2t^3, \frac{3}{4}t^4)$ for $0 \leq t \leq 2$. (See Exercise 2(c).)

6. Piecewise Defined Curves. Let $\mathcal{C}$ be a curve in the plane or space, where $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2 \cup \ldots \cup \mathcal{C}_n$ with $\mathcal{C}_i$ parametrized by $\alpha_i = \alpha_i(t)$, $a_i \leq t \leq b_i$. Assume that each α_i is continuously differentiable. Then we define the integral of f over $\mathcal{C}$ by

$$\int_C f \, ds = \int_{C_1} f \, ds + \int_{C_2} f \, ds + \cdots + \int_{C_n} f \, ds.$$

That is, to compute the total accumulation of f, identify the pieces C_i of the curve C, parametrize each one separately, evaluate the integral of f over each piece, and then sum the results to obtain the total accumulation of f over C.

(a) Let $f(x, y) = xy$. Compute the total accumulation of f over the curve that is the boundary of the square with vertices at $(0, 0)$, $(2, 0)$, $(2, 2)$, and $(0, 2)$.
(b) Let $f(x, y) = x^2 + y^2$. Compute the total accumulation of f over the curve that is the boundary of the upper half of the disk of radius 2 centered at the origin.
(c) Let $f(x, y, z) = (x + y)e^z$. Compute the total accumulation of f over the curve that is the boundary of the triangle with vertices $(1, 0, 0)$, $(0, 0, 1)$, and $(0, 1, 0)$.

7. **Properties of Path Integrals.** Let $\alpha : [a, b] \to \mathbb{R}^2$ or $\mathbb{R}^3$ be a continuously differentiable function and let f and g be continuous functions defined on the image of α. Use the definition of the path integral to justify the following equations.

(a) If c is a constant, $\int_\alpha cf\ ds = c\int_\alpha f\ ds$.
(b) $\int_\alpha (f + g)\ ds = \int_\alpha f\ ds + \int_\alpha g\ ds$.

8. **Independence of Parametrization I.** Let $\alpha(u)$ be a differentiable parametrization of a curve $\mathcal{C}$, $c \le u \le d$, that is one-to-one and has nonzero derivative on (c, d). Suppose $h(t)$ is a differentiable function of one variable, $h : [a, b] \to [c, d]$.

(a) Assume $h(a) = c$, $h(b) = d$, and $h'(t) > 0$ on $[a, b]$. Let $\beta(t) = \alpha(h(t))$; then β and α parametrize $\mathcal{C}$ in the same direction. Show that

$$\int_\beta f\ ds = \int_\alpha f\ ds.$$

(*Hint:* Evaluate $\int_\alpha f\ ds$ using the substitution $u = h(t)$.)
(b) Assume $h(a) = d$, $h(b) = c$, and $h'(t) < 0$ on $[a, b]$. Let $\beta(t) = \alpha(h(t))$; then β and α parametrize $\mathcal{C}$ in opposite directions. Show that

$$\int_\beta f\ ds = \int_\alpha f\ ds.$$

9. **Independence of Parametrization II.** In the previous exercise, we proved that the path integral of f on a curve $\mathcal{C}$ is independent of the choice of parametrization of $\mathcal{C}$ as long as the parametrizations are one-to-one. Explain why it is necessary that the parametrizations be one-to-one.

10. **Potential Energy.** In Example 6.2, suppose that the particle of charge q is located at the point $(0, 0, z)$ on the z-axis. Show that the potential energy is given by $\frac{Cq}{4\pi\epsilon_0}\int_0^{2\pi}\frac{1}{\sqrt{1+z^2}}\ dt$.

11. **Potential Energy of a Charged Segment.** Consider a system that consists of a charged straight wire of length 2 and a charged particle not on the wire. Assume that the wire is modeled by the line segment $\{(x, 0, 0) : -1 \le x \le 1\}$ and has constant charge density C and that the particle of charge q is located at $\mathbf{x} = (x, y, z)$.

(a) Set up a Riemann sum to approximate the potential energy function $U(\mathbf{x})$ of this system.
(b) If you consider the limit of these Riemann sums as the mesh of the partition approaches 0, what is the resulting integral?
(c) Evaluate the integral from part (b) when $(x, y, z) = (0, 0, 1)$.
(d) Evaluate the integral from part (b) when $(x, y, z) = (x_0, y_0, z_0)$.

6.2 Line Integrals

In this section, we want to explore an important use of path integration in physics, the definition and calculation of the work done by a force on an object. Later, we will relate work to kinetic and potential energy. This definition will also give rise to the more general notion of a line integral of a vector field over the image of a parametrization. In order to simplify our presentation, we will state our results for the plane, but keep in mind that they also apply in space.

We begin by recalling the definition of the work done by a constant force on a particle moving in a straight line in the plane. (See Section 1.3.)

Definition 6.3 Let $\mathbf{F}$ be a constant force field; that is, $\mathbf{F}(x, y) = \mathbf{F}_0$ for all (x, y). If $\mathbf{F}$ moves a particle a distance d_0 in the direction $\mathbf{u}$, where $\|\mathbf{u}\| = 1$, then the ***work done by* $\mathbf{F}$ *on the particle*** is the product of the component of $\mathbf{F}$ in the direction of motion with the distance that the particle moves, that is,

$$W = (\mathbf{F} \cdot \mathbf{u})d_0. \quad \blacklozenge$$

Now let us suppose that an object moves through a continuous nonconstant force field

$$\mathbf{F}(x, y) = (u(x, y), v(x, y))$$

from a point P to a point Q. Suppose also that the motion of the object is described by a parametrization $\alpha(t) = (x(t), y(t))$, $a \le t \le b$, with $\alpha(a) = P$, $\alpha(b) = Q$, and $\alpha'(t) \ne \mathbf{0}$. Since $\mathbf{F}$ is not constant and α may not be linear, the above definition does not apply to this situation. However, since α is differentiable and $\mathbf{F}$ is continuous, over small segments of the image, the motion is approximately linear and the force is approximately constant. Thus we can use the above definition to approximate the work done by the force on small segments of the image. Summing these approximations over the image, we will arrive at an approximation to the total work done by $\mathbf{F}$ as the particle moves according to α.

We can carry out such an approximation by following the procedure we established in Section 6.1. We partition the interval $[a, b]$ into N subintervals $[t_{i-1}, t_i]$ of length

$\Delta t_i = t_i - t_{i-1}$ and select a point t_i^* in the i^{th} subinterval. We will approximate the force acting on the object as it moves from $\alpha(t_{i-1})$ to $\alpha(t_i)$ by $\mathbf{F}(\alpha(t_i^*))$, and we will approximate the direction of the motion of the particle by the unit vector

$$\mathbf{T}(\alpha(t_i^*)) = \alpha'(t_i^*)/\|\alpha'(t_i^*)\|.$$

(See Figure 6.4.) Note that since $\alpha'(t) \neq 0$, this expression is well defined. Thus the component of the force in the direction of motion on the i^{th} cell is approximated by $\mathbf{F}(\alpha(t_i^*)) \cdot \mathbf{T}(\alpha(t_i^*))$.

The work done by $\mathbf{F}$ in moving the particle along the i^{th} cell of the partition is approximated by the product of $\mathbf{F}(\alpha(t_i^*)) \cdot \mathbf{T}(\alpha(t_i^*))$ with the distance Δs_i that the particle moves in going from $\alpha(t_{i-1})$ to $\alpha(t_i)$. Thus the approximation of the work for a particular cell is

$$(\mathbf{F}(\alpha(t_i^*)) \cdot \mathbf{T}(\alpha(t_i^*)))\,\Delta s_i.$$

The sum of these values gives an approximation to the total work done by the force as the particle moves according to α. It is a Riemann sum for the total accumulation of the function $\mathbf{F} \cdot \mathbf{T}$ over α:

$$R(\mathbf{F} \cdot \mathbf{T}, \alpha, P) = \sum_{i=1}^{N} (\mathbf{F}(\alpha(t_i^*)) \cdot \mathbf{T}(\alpha(t_i^*)))\,\Delta s_i.$$

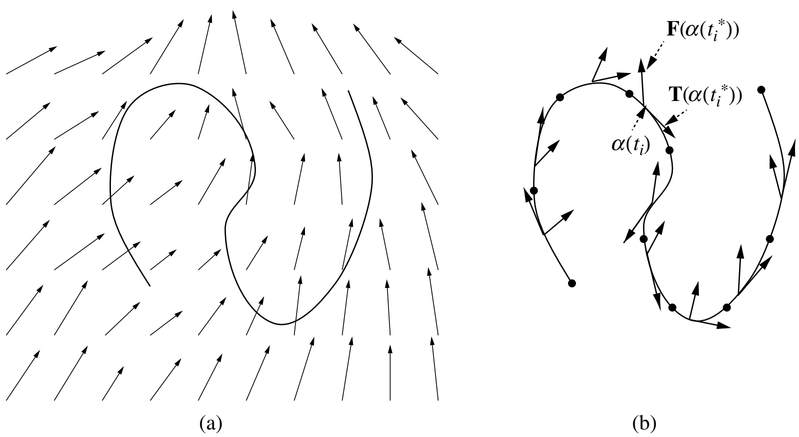

Figure 6.4 (a) A curve in a vector field. (b) The vectors $\mathbf{F}(\alpha(t_i^*))$ and $\mathbf{T}(\alpha(t_i^*))$ at the points $\alpha(t_i^*)$ on the image of α.

Since $\mathbf{F}$, α, and $\mathbf{T}$ are continuous, $\mathbf{F}(\alpha(\mathbf{t})) \cdot \mathbf{T}(\alpha(t))$ is also continuous. Consequently, the $\lim_{\text{mesh}(P)\to 0} R(\mathbf{F}\cdot\mathbf{T}, \alpha, P)$ exists and is equal to the path integral of the function $\mathbf{F}\cdot\mathbf{T}$ on α, $\int_\alpha \mathbf{F}\cdot\mathbf{T}\, ds$. The path integral is equal to

$$\int_\alpha \mathbf{F}\cdot\mathbf{T}\, ds = \int_a^b (\mathbf{F}(\alpha(t)) \cdot \mathbf{T}(\alpha(t)))\, \|\alpha'(t)\|\, dt.$$

This discussion leads to the following general definition of work:

Definition 6.4 If $\mathbf{F} = \mathbf{F}(x, y)$ is a force field, then the ***work*** done by $\mathbf{F}$ on a particle moving according to α, W_α, is the path integral over α of the component of $\mathbf{F}$ in the direction of α. That is,

$$W_\alpha = \int_\alpha \mathbf{F}\cdot\mathbf{T}\, ds,$$

where $\mathbf{T}$ is the unit tangent vector to α. ◆

More generally, suppose $\mathbf{F}$ is a vector field defined on a domain in $\mathbb{R}^2$ that contains the image of the parametrization α.

Definition 6.5 The ***line integral*** of $\mathbf{F}$ over the image of α is $\int_\alpha \mathbf{F}\cdot\mathbf{T}\, ds$, where $\mathbf{T}$ is the unit tangent vector to α. ◆

Thus the line integral of $\mathbf{F}$ over α is defined to be the path integral of $\mathbf{F}\cdot\mathbf{T}$ over α. This integral can then be evaluated as a one-variable integral in terms of the parameter t. Since $\mathbf{T}(\alpha(t)) = \alpha'(t)/\|\alpha'(t)\|$, we can simplify the resulting definite integral:

$$\begin{aligned}\int_\alpha \mathbf{F}\cdot\mathbf{T}\, ds &= \int_a^b \mathbf{F}(\alpha(t)) \cdot \mathbf{T}(\alpha(t))\|\alpha'(t)\|\, dt \\ &= \int_a^b \mathbf{F}(\alpha(t)) \cdot \frac{\alpha'(t)}{\|\alpha'(t)\|}\|\alpha'(t)\|\, dt.\end{aligned}$$

Canceling the $\|\alpha'\|$ terms, we have

$$\int_\alpha \mathbf{F}\cdot\mathbf{T} = \int_a^b \mathbf{F}(\alpha(t)) \cdot \alpha'(t)\ dt,$$

which is considerably easier to evaluate than the original path integral formulation. The following example illustrates the use of this form for a line integral.

Example 6.6

A Line Integral. Let $\mathbf{F}(x, y) = (-y, x)$, and let $\alpha(t) = (t^2, t^3)$, $0 \leq t \leq 1$. The line integral of $\mathbf{F}$ over α is given by

$$\begin{aligned}\int_\alpha \mathbf{F} \cdot \mathbf{T}\, ds &= \int_0^1 \mathbf{F}(\alpha(t)) \cdot \alpha'(t)\, dt \\ &= \int_0^1 (-t^3, t^2) \cdot (2t, 3t^2)\, dt \\ &= \int_0^1 (-2t^4 + 3t^4)\, dt \\ &= \int_0^1 t^4\, dt \\ &= \left[\frac{1}{5} t^5\right]_0^1 = 1/5.\end{aligned}$$

Now let us consider an example of work involving a charged particle moving through an electric field. The force exerted by an electric field $\mathbf{E}$ on a particle of charge q is $q\mathbf{E}$. If the particle moves through $\mathbf{E}$ according to α, the work done by the force on the particle is

$$W_\alpha = \int_\alpha q\mathbf{E} \cdot \mathbf{T}\, ds = \int_a^b q\mathbf{E}(\alpha(t)) \cdot \alpha'(t)\, dt.$$

Here we consider the particular case of an electric field generated by a charged particle.

Example 6.7

Work Done by an Electric Field. A particle of charge q_0 located at the origin gives rise to an electric field

$$\mathbf{E}(\mathbf{x}) = \frac{q_0}{4\pi\epsilon_0} \frac{\mathbf{x}}{\|\mathbf{x}\|^3},$$

where ϵ_0 is a constant. Thus the force exerted by the field on a particle of charge q located at $\mathbf{x}$ is

$$\mathbf{F}(\mathbf{x}) = q\mathbf{E}(\mathbf{x}) = \frac{qq_0}{4\pi\epsilon_0} \frac{\mathbf{x}}{\|\mathbf{x}\|^3}.$$

If a particle of charge q moves through $\mathbf{E}$ according to $\alpha : [a, b] \to \mathbb{R}^2$, then the work done by $\mathbf{F}$ on the particle in motion is given by

$$W_\alpha = \int_a^b q\mathbf{E}(\alpha(t)) \cdot \alpha'(t)\, dt = \int_a^b \left(\frac{qq_0}{4\pi\epsilon_0} \frac{\alpha(t)}{\|\alpha(t)\|^3} \right) \cdot \alpha'(t)\, dt.$$

For a particle located at $\alpha(t)$, $\|\alpha(t)\| = \sqrt{\alpha(t) \cdot \alpha(t)}$. We can rewrite the integrand in terms of the dot product and use the fact that $(\alpha(t) \cdot \alpha(t))' = 2\alpha(t) \cdot \alpha'(t)$ to evaluate the integral:

$$\begin{aligned} \frac{qq_0}{4\pi\epsilon_0} \int_a^b \frac{\alpha(t) \cdot \alpha'(t)}{(\alpha(t) \cdot \alpha(t))^{3/2}}\, dt &= \frac{qq_0}{4\pi\epsilon_0} \left. \frac{-1}{(\alpha(t) \cdot \alpha(t))^{1/2}} \right|_a^b \\ &= \frac{qq_0}{4\pi\epsilon_0} \left. \frac{-1}{\|\alpha(t)\|} \right|_a^b \\ &= \frac{-qq_0}{4\pi\epsilon_0} \left(\frac{1}{\|\alpha(b)\|} - \frac{1}{\|\alpha(a)\|} \right). \end{aligned}$$

Notice that this quantity depends on the endpoints of α and not on any of the intermediate positions of the moving particle. This is an important property of electric fields that we will return to later in this section. For now, let us note that the quantity $-W_\alpha/q$ is called the ***electric potential difference*** along α. It depends on the electric field and not on the charge of the moving particle.

The Orientation of a Curve

In Section 6.1, we stated that if α and β are parametrizations of the same curve $\mathcal{C}$, then the path integral of a function f over α is equal to the path integral of f over β. That is, the total accumulation of a function on a curve does not depend on which parametrization we use to parametrize the curve, as long as the parametrization traces the curve just once. There is a corresponding result for line integrals, but it requires the added assumption that α and β trace $\mathcal{C}$ in the same direction. Intuitively, this makes sense because the amount of work done moving a particle through a force field should depend on whether the particle moves "with the force" or "against the force." Let us see how this works.

Suppose that $\alpha : [a, b] \to \mathbb{R}^2$, with $\alpha = \alpha(u)$, and $\beta : [c, d] \to \mathbb{R}^2$, with $\beta = \beta(t)$, are continuously differentiable parametrizations of the same curve $\mathcal{C}$ that are related by composition with a differentiable function h of one variable. That is, $\beta = \alpha \circ h$, where $u = h(t)$ so that $\beta(t) = \alpha(h(t)) = \alpha(u)$. Thus $\beta'(t) = \alpha'(h(t))h'(t)$. Since we want α and β to trace $\mathcal{C}$ in the same direction, we will also require that $h : [a, b] \to [c, d]$ with $h(a) = c$

and $h(b) = d$. With these hypotheses, let us compute the line integral of a vector field $\mathbf{F}$ along α using the substitution $u = h(t)$:

$$\begin{aligned}\int_\alpha \mathbf{F}\cdot\mathbf{T}\,ds &= \int_a^b \mathbf{F}(\alpha(u))\cdot\alpha'(u)\,du\\ &= \int_c^d \mathbf{F}(\alpha(h(t)))\cdot\alpha'(h(t))h'(t)\,dt\\ &= \int_c^d \mathbf{F}(\beta(t))\cdot\beta'(t)\,dt\\ &= \int_\beta \mathbf{F}\cdot\mathbf{T}\,ds.\end{aligned}$$

It follows that the line integral of $\mathbf{F}$ over a curve $\mathcal{C}$ does not depend on the parametrization, as long as the parametrizations trace the curve in the same direction.

Now let us see what happens if the parametrizations trace $\mathcal{C}$ in opposite directions. Thus we assume that $h(a) = d$ and $h(b) = c$. Repeating the above calculation, we have

$$\begin{aligned}\int_\alpha \mathbf{F}\cdot\mathbf{T}\,ds &= \int_a^b \mathbf{F}(\alpha(u))\cdot\alpha'(u)\,du\\ &= \int_d^c \mathbf{F}(\alpha(h(t)))\cdot\alpha'(h(t))h'(t)\,dt\\ &= \int_d^c \mathbf{F}(\beta(t))\cdot\beta'(t)\,dt.\end{aligned}$$

Since $d > c$, the next step is to reverse the order of integration, which changes the sign of the integral.

$$\begin{aligned}\int_d^c \mathbf{F}(\beta(t))\cdot\beta'(t)\,dt &= -\int_c^d \mathbf{F}(\beta(t))\cdot\beta'(t)\,dt\\ &= -\int_\beta \mathbf{F}\cdot\mathbf{T}\,ds.\end{aligned}$$

It follows that if α and β trace a curve $\mathcal{C}$ in opposite directions, the line integrals of $\mathbf{F}$ over α and β have the same absolute value but opposite signs.

Before we formally state these results, we would like to introduce the concept of orientation.

Definition 6.6 Let α and β be parametrizations of a curve $\mathcal{C}$.

1. If α and β trace $\mathcal{C}$ in the same direction, we say that α and β parametrize $\mathcal{C}$ with the ***same orientation***. If α and β trace $\mathcal{C}$ in opposite directions, we say that α and β parametrize $\mathcal{C}$ with ***opposite orientations***.

2. We say that $\mathcal{C}$ is ***oriented*** if we have chosen a direction for a parametrization of $\mathcal{C}$. Thus a given curve has two possible orientations. We say that a parametrization ***agrees with the orientation of the curve*** if its direction agrees with the direction we have chosen for the curve. If $\mathcal{C}$ is oriented, we will denote the ***opposite*** or ***reverse orientation*** of $\mathcal{C}$ by $-\mathcal{C}$. ♦

Using this new terminology, we summarize our earlier calculations.

Proposition 6.3 Let α and β be parametrizations of $\mathcal{C}$. If α and β parametrize C with the same orientation, then

$$\int_\beta \mathbf{F}\cdot\mathbf{T}\,ds = \int_\alpha \mathbf{F}\cdot\mathbf{T}\,ds.$$

If α and β parametrize C with opposite orientations, then

$$\int_\beta \mathbf{F}\cdot\mathbf{T}\,ds = -\int_\alpha \mathbf{F}\cdot\mathbf{T}\,ds. \quad ♦$$

Since the line integral of $\mathbf{F}$ over a parametrization of a curve $\mathcal{C}$ depends only on the orientation of the parametrization, we can define the line integral of a vector field over a curve independent of the choice of parametrization as long as we specify the orientation of the curve.

Definition 6.7 Let $\mathcal{C}$ be an oriented curve in the domain of the vector field $\mathbf{F}$. Then we define the ***line integral*** of $\mathbf{F}$ over $\mathcal{C}$ to be the line integral of $\mathbf{F}$ over α, where α is any parametrization of C that agrees with the orientation of C. We denote the line integral of $\mathbf{F}$ over the oriented curve C by

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

Thus, we have

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_\alpha \mathbf{F}\cdot\mathbf{T}\,ds. \quad ♦$$

Notice that combining the notation of Definitions 6.6 and 6.7, we have

$$\int_{-C} \mathbf{F}\cdot\mathbf{T}\,ds = -\int_C \mathbf{F}\cdot\mathbf{T}\,ds.$$

Returning to the concept of work, this result implies that if $\mathbf{F}$ is a force field, the work done on a particle moving along a curve $\mathcal{C}$ in a particular direction is independent of the way in which the particle moves along the curve. The work done on a particle moving along $\mathcal{C}$ in the opposite direction is the opposite of this value.

Independence of Path

In Example 6.7, we carried out the calculation of the work done by an electric field on a charged particle. We showed that for the electric field $\mathbf{E}$ due to a particle of charge q_0 located at the origin, the work done by the field on a particle of charge q moving along α is given by

$$W_\alpha = \frac{-qq_0}{4\pi\epsilon_0}\left(\frac{1}{\|\alpha(b)\|} - \frac{1}{\|\alpha(a)\|}\right).$$

We made the point that this expression depends only on the endpoints of α, not on any of the intermediate points. To clarify what is going on, let us define a function f by

$$f(\mathbf{x}) = -\frac{qq_0}{4\pi\epsilon_0\|\mathbf{x}\|},$$

so that

$$W_\alpha = f(\alpha(b)) - f(\alpha(a)).$$

A calculation shows that the gradient of f is the force field $q\mathbf{E}$, that is, $\nabla f = q\mathbf{E}$. (See Exercise 3.) Thus, in this case, we have expressed the integral of a gradient vector field of a function as the difference between the values of the function at the endpoints. This leads us to consider the general question of computing the line integral of a gradient vector field along a curve.

Let f be a differentiable function, let $\mathbf{F}$ be its gradient vector field, and let $\alpha : [a, b] \to \mathbb{R}^2$ be a differentiable parametrization of a curve $\mathcal{C}$ contained in the domain of f. We can use the chain rule in the form $\frac{d}{dt}f(\alpha(t)) = \nabla f(\alpha(t)) \cdot \alpha'(t)$ and the fundamental theorem of calculus to evaluate the line integral of $\mathbf{F} = \nabla f$ over α:

$$\begin{aligned}\int_\alpha \mathbf{F}\cdot\mathbf{T}\,ds &= \int_a^b \mathbf{F}(\alpha(t))\cdot\alpha'(t)\,dt\\ &= \int_a^b \nabla f(\alpha(t))\cdot\alpha'(t)\,dt\\ &= \int_a^b \frac{d}{dt}\left(f(\alpha(t))\right)\,dt\\ &= f(\alpha(t))\Big|_{t=a}^{t=b}\\ &= f(\alpha(b)) - f(\alpha(a)).\end{aligned}$$

We see that the line integral of a gradient vector field depends only on the endpoints of the parametrization. That is, if $\alpha : [a, b] \to \mathbb{R}^2$ and $\beta : [c, d] \to \mathbb{R}^2$ are parametrizations with the same initial point and the same endpoint, $\alpha(a) = \beta(c)$ and $\alpha(b) = \beta(d)$, then

$$\begin{aligned} \int_\alpha \nabla f \cdot \mathbf{T}\, ds &= f(\alpha(b)) - f(\alpha(a)) \\ &= f(\beta(d)) - f(\beta(c)) \\ &= \int_\beta \nabla f \cdot \mathbf{T}\, ds. \end{aligned}$$

Alternatively, if α is a parametrization of any path from P to Q,

$$\int_\alpha \nabla f \cdot \mathbf{T}\, ds = f(Q) - f(P).$$

In particular, the work done in moving an object through a force field ∇f from one point to another is independent of the choice of path for the motion of the object, as we saw for the electric force field of Example 6.7.

This fact about gradient vector fields leads us to make the following definition.

Definition 6.8 A vector field $\mathbf{F} = \mathbf{F}(\mathbf{x})$ is said to have the ***path independence*** property if $\int_\alpha \mathbf{F} \cdot \mathbf{T}\, ds$ depends only on the endpoints of α; that is, if $\alpha : [a, b] \to \mathbb{R}^2$ and $\beta : [c, d] \to \mathbb{R}^2$ satisfy $\alpha(a) = \beta(c)$ and $\alpha(b) = \beta(d)$, then

$$\int_\alpha \mathbf{F} \cdot \mathbf{T}\, ds = \int_\beta \mathbf{F} \cdot \mathbf{T}\, ds. \quad \blacklozenge$$

Since every gradient vector field has the path independence property, it is natural to ask if this property is sufficient to imply that a vector field is a gradient vector field. That is, if $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ has the path independence property, is there a differentiable function f so that $\nabla f = \mathbf{F}$? In fact, there is such a function, and the path independence property is exactly the condition that is required to construct the function. The construction of f is given in the proof of the following theorem.

Theorem 6.2 Let $\mathbf{F}(\mathbf{x}, \mathbf{y}) = (\mathbf{u}(\mathbf{x}, \mathbf{y}), \mathbf{v}(\mathbf{x}, \mathbf{y}))$ be a continuous vector field that satisfies the path independence property, and let $\mathbf{x}_0$ be a fixed point in the domain of $\mathbf{F}$. Let f be the function defined by

$$f(\mathbf{x}) = \int_\alpha \mathbf{F} \cdot \mathbf{T}\, ds,$$

where α is a parametrization of a curve with initial point $\mathbf{x}_0$ and final point $\mathbf{x}$. Then f is a function whose gradient is $\mathbf{F}$, $\nabla f = \mathbf{F}$. $\blacklozenge$

Proof: In order to construct f, we must choose a fixed point $\mathbf{x}_0 = (x_0, y_0)$. Given any other point $\mathbf{x} = (x, y)$, we define f by

$$f(\mathbf{x}) = \int_\alpha \mathbf{F} \cdot \mathbf{T}\, ds,$$

where $\alpha : [a, b] \to \mathbb{R}^2$ is a parametrization that satisfies $\alpha(a) = \mathbf{x}_0$ and $\alpha(b) = \mathbf{x}$. Since $\mathbf{F}$ has the path independence property, $f(\mathbf{x})$ is uniquely defined independent of the choice of α. It remains to show that $\nabla f = \mathbf{F}$, that is,

$$\left(\frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y)\right) = (u(x, y), v(x, y)).$$

To verify this, we will use the limit definition of the partial derivative and make use of the path independence property.

First, let us show that $\frac{\partial f}{\partial x}(x, y) = u(x, y)$. Using the definition of f,

$$\begin{aligned}\frac{\partial f}{\partial x}(x, y) &= \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h} \\ &= \lim_{h \to 0} \frac{\int_{\alpha_h} \mathbf{F} \cdot \mathbf{T}\, ds - \int_{\alpha_0} \mathbf{F} \cdot \mathbf{T}\, ds}{h}.\end{aligned}$$

Here α_0 denotes a parametrization with initial point $\mathbf{x}_0$ and final point $\mathbf{x}$, and α_h denotes a parametrization with initial point $\mathbf{x}_0$ and final point $(x + h, y)$. Since $\mathbf{F}$ has the path independence property, we can choose α_h in such a way as to simplify the calculation of the partial derivative. In particular, we will choose α_h so that its image is the union of the image of α_0 and the horizontal line segment from $\mathbf{x} = (x, y)$ to $(x + h, y)$. (See Figure 6.5.) We will parametrize the segment of the image from $\mathbf{x}_0$ to $\mathbf{x}$ by α_0 and the

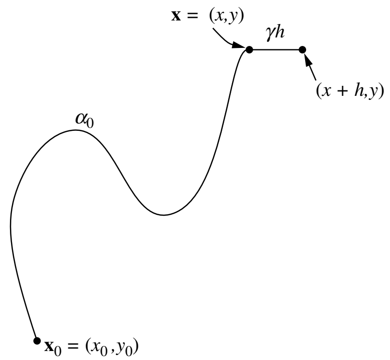

Figure 6.5 The image of the curve α_h consists of the union of the image of α_0 and the image of γ_h, which is the line segment from $\mathbf{x} = (x, y)$ to $(x + h, y)$.

horizontal segment by γ_h, where $\gamma_h(t) = (x + t, y)$, with $0 \le t \le h$ if h is positive and $h \le t \le 0$ if h is negative. Thus

$$\int_{\alpha_h} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\alpha_0} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{\gamma_h} \mathbf{F} \cdot \mathbf{T}\, ds,$$

and

$$\int_{\alpha_h} \mathbf{F} \cdot \mathbf{T}\, ds - \int_{\alpha_0} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\gamma_h} \mathbf{F} \cdot \mathbf{T}\, ds.$$

Using this, we are able to simplify our limit,

$$\frac{\partial f}{\partial x}(x, y) = \lim_{h \to 0} \frac{\int_{\gamma_h} \mathbf{F} \cdot \mathbf{T}\, ds}{h}.$$

Since $\gamma_h'(t) = (1, 0)$, the unit tangent vector of γ_h is also equal to $(1, 0)$, and

$$\mathbf{F} \cdot \mathbf{T} = (u(x, y), v(x, y)) \cdot (1, 0) = u(x, y).$$

Therefore, the limit is equal to

$$\lim_{h \to 0} \frac{\int_0^h u(x + t, y)\, dt}{h}.$$

Notice that the quotient inside the limit is equal to the average value of u on the line segment from (x, y) to $(x + h, y)$. Since u is a continuous function, the limit of its average value on the line segment as the length of the segment approaches 0 is equal to the value of the function at the endpoint, that is,

$$\lim_{h \to 0} \frac{\int_0^h u(x + t, y)\, dt}{h} = u(x, y).$$

Thus we have shown that for f defined as above,

$$\frac{\partial f}{\partial x}(x, y) = u(x, y).$$

In order to compute the partial derivative of f with respect to y, we must replace γ_h in this argument by a curve that parametrizes the vertical line segment from $Q = (x, y)$ to $(x, y + h)$. Then a similar argument shows that

$$\frac{\partial f}{\partial y}(x, y) = v(x, y).$$

We conclude that every vector field satisfying the path independence property is a gradient vector field. ▮

The function f of the theorem depends on the choice of $\mathbf{x}_0$ as well as the vector field $\mathbf{F}$. Suppose that $\tilde{\mathbf{x}}_0 \neq \mathbf{x}_0$ and that we use $\tilde{\mathbf{x}}_0$ as in the theorem to define a function $\tilde{f}$. Then $\tilde{f}$ also satisfies $\nabla \tilde{f} = \mathbf{F}$. In Exercise 4 we show that f and $\tilde{f}$ differ by a constant, that is, there is a constant C so that $\tilde{f}(\mathbf{x}) = f(\mathbf{x}) + C$ for all $\mathbf{x}$. Since this construction yields $f(\mathbf{x}_0) = 0$, the different choices of the basepoint are equivalent to fixing a point where f has the value 0. We will use this fact in the discussion that follows.

Application: Work, Energy, and Conservative Forces

We are now in a position to explore the relationship between work and energy, which plays a fundamental role in physics. Let us suppose that $\mathbf{F}$ is a force field and that a particle of mass m moves through the field according to α. Here we assume that no other forces act on the object, so that the motion of the object is due solely to this force. From Newton's second law, we know that the force acting on the particle at an instant in time is equal to the product of the mass of the particle and its acceleration, $\mathbf{F} = m\alpha''$. Let us substitute this into the integral expression for work.

$$\begin{aligned} W_\alpha &= \int_\alpha \mathbf{F} \cdot \mathbf{T}\, ds \\ &= \int_a^b m\alpha''(t) \cdot \alpha'(t)\, dt \\ &= \left[\frac{m}{2}\alpha'(t) \cdot \alpha'(t)\right]_a^b \\ &= \frac{m}{2}\alpha'(b) \cdot \alpha'(b) - \frac{m}{2}\alpha'(a) \cdot \alpha'(a). \end{aligned}$$

Note that we used the fundamental theorem of calculus and Exercise 3(c) of Section 2.2 to evaluate the integral.

The quantity $K(t) = \frac{m}{2}\alpha'(t) \cdot \alpha'(t)$ is called the ***kinetic energy*** of the particle. In physics, the velocity α' of an object is denoted by $\mathbf{v}$, so that $K = \frac{m}{2}\mathbf{v} \cdot \mathbf{v}$. This calculation says that the amount of work done by the force in moving the object is equal to the difference between its kinetic energy at the initial point of its motion and at the final point of its motion, $W = K(b) - K(a)$. This is known as the ***work-energy*** theorem for a particle. (See Exercise 11.)

Certain forces have the property that if an object moves subject to the force through a closed loop, then the kinetic energy of the object upon return to its initial position is the same as it was initially. For example, neglecting air resistance, gravity has this property. If an object is launched vertically from the surface of the earth with velocity v_0, it will return to earth with velocity $-v_0$. Consequently, the work done by gravity on the object is zero, $W = \frac{m}{2}v_0^2 - \frac{m}{2}(-v_0)^2 = 0$. Forces with this property are called ***conservative*** forces. In addition to gravity, the electric force of Example 6.7 is a conservative force. If a force is conservative, it follows that it has the path independence property. (See Exercise 8.) Thus from the theorem, there is a function f with the property that $\nabla f = \mathbf{F}$.

The function $U = -f$, which satisfies $-\nabla U = \mathbf{F}$, is called the ***potential energy*** of the system. Along any path α from Q_1 to Q_2, $W_\alpha = -(U(Q_2) - U(Q_1))$. Since W depends on Q_1 and Q_2, and not on α, we will use the notation W_{Q_1,Q_2} to denote the work done by a conservative field in moving a particle from Q_1 to Q_2. Because the basepoint P that is used to define U will satisfy $U(P) = 0$, the choice of P is determined by the choice of a convenient location for the 0 value of the potential energy. After choosing a basepoint, $U(Q) = -W_{P,Q}$.

Since for a conservative system work is equal to the change in kinetic energy *and* the change in potential energy, we have for any Q_1 and Q_2,

$$K(Q_2) - K(Q_1) = -(U(Q_2) - U(Q_1)).$$

This implies that

$$K(Q_2) + U(Q_2) = K(Q_1) + U(Q_1).$$

This holds for any Q_1 and Q_2, so it follows that the sum $K(Q) + U(Q)$ must be a constant. This constant is called the ***total energy*** of the system. If a conservative force acts on a particle in motion, changing its kinetic energy, since the total energy is constant, there is a corresponding change in the potential energy that compensates for the change in the kinetic energy. (In Section 3.5 we assumed that $E = K + U$ is constant and then proved that $\nabla U = -\mathbf{F}$.)

An interesting application of this discussion is to determine the escape velocity for the earth, which we do in the following example.

Example 6.8 Escape Velocity of the Earth

A. Assume that the center of the earth is located at the origin of a three-dimensional coordinate system and that an object of mass m is located at $\mathbf{x}$. Then the gravitational force acting on the object is

$$\mathbf{F} = -GM_e m \frac{\mathbf{x}}{\|\mathbf{x}\|^3},$$

where G is the gravitational constant and M_e is the mass of the earth. A calculation similar to the one in Example 6.7 shows that the work done by gravity in moving the object from $\mathbf{x}_1$ to $\mathbf{x}_2$, which we denote by $W_{\mathbf{x}_1,\mathbf{x}_2}$, is

$$W_{\mathbf{x}_1,\mathbf{x}_2} = GM_e m \left(\frac{1}{\|\mathbf{x}_2\|} - \frac{1}{\|\mathbf{x}_1\|} \right),$$

so that $\mathbf{F}$ has the path independence property, which confirms that gravity is a conservative force. As above, the ***gravitational potential energy*** U is defined by $U(\mathbf{x}) = -W_{\mathbf{x}_0,\mathbf{x}}$, where $\mathbf{x}_0$ is a basepoint chosen so that $U(\mathbf{x}_0) = 0$. If our focus is on the behavior of gravity near the surface of the earth, it is convenient to choose $\mathbf{x}_0$ to be a point on the ground. However, if we are concerned with a larger distance scale, it is convenient to have the potential energy at ∞ be equal to 0. To do this, we define the potential energy to be the limit of $-W_{\mathbf{x}_0,\mathbf{x}}$ as $\|\mathbf{x}_0\| \to \infty$. Thus

$$U(\mathbf{x}) = \lim_{\|\mathbf{x}_0\| \to \infty} -W_{\mathbf{x}_0,\mathbf{x}} = -\frac{GM_e m}{\|\mathbf{x}\|}.$$

Note that U satisfies $-\nabla U = \mathbf{F}$. Intuitively, the potential energy of the object at $\mathbf{x}$ is the negative of the amount of work required to bring the object from ∞ to $\mathbf{x}$.

B. The potential energy of an object on the surface of the earth is $-GM_e m/R_e$, where R_e is the radius of the earth. The amount of work required to move the object to infinity is the negative of the amount of work required to bring the object from infinity, so it is also equal $-GM_e m/R_e$. Neglecting air resistance, we can use the work-energy theorem to find the initial velocity necessary to propel the object from the surface of the earth to a point at infinity with zero velocity. Since the final velocity is zero, the initial velocity $\mathbf{v}_0$ must satisfy $-GM_e m/R_e = -m\mathbf{v}_0 \cdot \mathbf{v}_0/2$, so that the speed $\|\mathbf{v}_0\|$ must satisfy $\|\mathbf{v}_0\| = \sqrt{2GM_e/R_e} \approx 7$ mi/s. (Note that $G = 6.67 \times 10^{-11}$ N-m/kg^2, $M_e \approx 6.0 \times 10^{24}$ kg, and $R_e \approx 6.37 \times 10^6$ m.) This speed is known as the ***escape velocity*** of the earth.

Summary

In this section, we introduced the ***line integral*** of a vector field **F** over a curve in the plane or space. The construction was motivated by the effort to extend the concept of ***work*** in physics to the work done on a particle by a continuous nonconstant vector field.

If α is a parametrization of the curve and **T** is the unit tangent vector to α, the ***line integral*** of **F** along α is the path integral of $\mathbf{F}\cdot\mathbf{T}$ along α. Thus,

$$\int_\alpha \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b (\mathbf{F}(\alpha(t))\cdot\mathbf{T}(\alpha(t)))\,\|\alpha'(t)\|\,dt.$$

The integral on the right-hand side can be simplified to obtain the formula

$$\int_\alpha \mathbf{F}\cdot\mathbf{T}\,ds = \int_a^b \mathbf{F}(\alpha(t))\cdot\alpha'(t)\,dt.$$

If α is the path of a particle moving through the force field **F**, this integral defines the ***work done by*** **F** on the particle as it moves along α.

We defined the ***orientation*** of a path and showed that the line integral of **F** over two parametrizations of the same curve is the same if the parametrizations trace the curve with the same orientation.

Building on these ideas, we showed that if **F** is a gradient field, $\mathbf{F} = \nabla f$, for a continuously differentiable function f, then the line integral of **F** along a curve depends only on the values of f at the endpoints of the curve. We called this property ***path independence***, and we proved the converse. ***Every vector field with the path independence property is a gradient vector field.*** We used these constructions to develop several ideas from physics, including the ***work-energy theorem***, ***conservative forces***, ***potential energy***, and ***total energy***.

Section 6.2 Exercises

1. Line Integrals. For each of the following force fields **F**, compute the line integral of **F** over the parametrization α.

(a) $\mathbf{F}(x,y) = (-y,x)$ and $\alpha(t) = (e^t, e^{-t})$, $0 \le t \le 1$.
(b) $\mathbf{F}(x,y) = (x,y)$ and $\alpha(t) = (\cos^3 t, \sin^3 t)$, $0 \le t \le 2\pi$.
(c) $\mathbf{F}(x,y,z) = (x+y,y,y)$ and $\alpha(t) = (t,t,t^2)$, $0 \le t \le 3$.

2. Line Integrals over Curves. For each of the following vector fields $\mathbf{F}$, evaluate the line integral of $\mathbf{F}$ along the given curve.

(a) $\mathbf{F}(x, y) = (x, y)$ along the circle of radius 2 centered at the origin traversed clockwise.
(b) $\mathbf{F}(x, y) = (y^2, x^2)$ along the parabola $y = x^2$ from $(0, 0)$ to $(2, 4)$.
(c) $\mathbf{F}(x, y) = (x^2, xy)$ along the perimeter of the unit square from $(0, 0)$ to $(1, 0)$ to $(1, 1)$ to $(0, 1)$ and back to $(0, 0)$.

3. A Gradient Calculation. Show that the gradient of $f(\mathbf{x}) = -\frac{qq_0}{4\pi\epsilon_0}\frac{1}{\|\mathbf{x}\|}$ is equal to the force field $\mathbf{F} = q\mathbf{E}$, where $\mathbf{E}$ is the electric field of Example 6.7, $\mathbf{E}(\mathbf{x}) = \frac{q_0}{4\pi\epsilon_0}\frac{\mathbf{x}}{\|\mathbf{x}\|^3}$.

4. Path Independence. Suppose that the vector field $\mathbf{F}$ has the path independence property. Let $\mathbf{x}_0$ be a fixed point and define the function f by

$$f(\mathbf{x}) = \int_\alpha \mathbf{F} \cdot \mathbf{T}\, ds,$$

where α is a parametrization of a curve with initial point $\mathbf{x}_0$ and final point $\mathbf{x}$.

(a) Show that if $\mathcal{C}$ is any curve in the domain of $\mathbf{F}$ from a point $\mathbf{x}_1$ to a point $\mathbf{x}_2$, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = f(\mathbf{x}_2) - f(\mathbf{x}_1).$$

(*Hint:* Since $\mathbf{F}$ has the path independence property, the integral on the left-hand side can be evaluated along a path C that goes from $\mathbf{x}_1$ to $\mathbf{x}_0$ to $\mathbf{x}_2$.)

(b) Suppose that $\tilde{\mathbf{x}}_0 \neq \mathbf{x}_0$ is another point in the domain of $\mathbf{F}$ and that $\tilde{f}$ is defined by

$$\tilde{f}(\mathbf{x}) = \int_\alpha \mathbf{F} \cdot \mathbf{T}\, ds,$$

where α is a parametrization of a curve with initial point $\tilde{\mathbf{x}}_0$ and final point $\mathbf{x}$. Show that f and $\tilde{f}$ differ by

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$$

where $\mathcal{C}$ is a curve from $\mathbf{x}_0$ to $\tilde{\mathbf{x}}_0$. (*Hint:* To define $f(\mathbf{x})$, use a path from $\mathbf{x}_0$ to $\tilde{\mathbf{x}}_0$ to $\mathbf{x}$.

(c) Conclude that f and $\tilde{f}$ differ by a constant.

5. Mixed Partials and Path Independence. Let $\mathbf{F}$ be a continuous and differentiable vector field on a domain $\mathcal{D}$ that satisfies the path independence property.

(a) If $\mathbf{F}(x, y) = (F_1(x, y), F_2(x, y))$, show that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

(b) If $\mathbf{F}(x,y,z) = (F_1(x,y,z), F_2(x,y,z), F_3(x,y,z))$, show that $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$, and $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$.

As a consequence of this exercise, if $\mathbf{F}$ is a vector field in the plane and $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$, then $\mathbf{F}$ does not satisfy the path independence property. Similarly, if $\mathbf{F}$ is a vector field in space and $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$, $\frac{\partial F_1}{\partial z} \neq \frac{\partial F_3}{\partial x}$, or $\frac{\partial F_2}{\partial z} \neq \frac{\partial F_3}{\partial y}$, then $\mathbf{F}$ does not satisfy the path independence property.

6. **Potential Functions.** A function f with the property that $\nabla f = \mathbf{F}$ is called a ***potential function*** for $\mathbf{F}(x,y) = (u(x,y), v(x,y))$. Since this means that $\frac{\partial f}{\partial x}(x,y) = u(x,y)$ and $\frac{\partial f}{\partial y}(x,y) = v(x,y)$, an ad hoc method for finding f is to evaluate and compare the indefinite integrals $\int u(x,y)\,dx$ and $\int v(x,y)\,dy$. However, since the partial derivative with respect to x of a term involving only y is zero, for example, $\frac{\partial \sin y}{\partial x} = 0$, the constant of integration for $\int u(x,y)\,dx$ may depend on y. Similarly, the constant of integration for $\int v(x,y)\,dy$ may depend on x. Call these C_y and C_x, respectively. By comparing the antiderivatives, we can find all the terms of f. For example, if $F(x,y) = (1+y, x+y^2)$, then $\int (1+y)\,dx = x + xy + C_y$ and $\int (x+y^2)\,dy = xy + y^3/3 + C_x$. Equating these two expressions, we see that the term xy appears in both expressions and that $C_y = y^3/3$ and $C_x = x$. We conclude that $f(x,y) = x + xy + y^3/3$ is a potential function for $\mathbf{F}$.

For each of the following vector fields, use Exercise 5 to determine if $\mathbf{F}$ is not path independent. For each vector field with equal mixed partial derivatives, use this technique to find a potential function for $\mathbf{F}$.

(a) $\mathbf{F}(x,y) = (-y, x)$, $\mathcal{D} = \mathbb{R}^2$.
(b) $\mathbf{F}(x,y) = (3x^2 + y, e^y + x)$, $\mathcal{D} = \mathbb{R}^2$.
(c) $\mathbf{F}(x,y,z) = (y^2, 2xy, z)$, $\mathcal{D} = \mathbb{R}^3$.
(d) $\mathbf{F}(x,y,z) = (z^2, y^2 z, 2xz)$, $\mathcal{D} = \mathbb{R}^3$.

7. **Path Independent Vector Fields.** Each of the following vector fields $\mathbf{F}$ satisfies the path independence property. Evaluate $\int_C \mathbf{F}\cdot\mathbf{T}\,ds$ by finding a potential function f with $\nabla f = \mathbf{F}$ and using f to evaluate the integral.

(a) $\mathbf{F}(x,y) = (x,y)$ and C is the upper half of a circle of radius 2 centered at the origin traversed counterclockwise.
(b) $\mathbf{F}(x,y) = (\cos x \cos y, -\sin x \sin y)$ and C is the path along $y = x^3$ from $(-1,1)$ to $(1,1)$.
(c) $\mathbf{F}(x,y,z) = (x + yz, y + xz, z + xy)$ and C is the path along the helix parametrized by $\alpha(t) = (\cos t, \sin t, t)$ from $(1,0,0)$ to $(0,1,\pi/2)$.

8. **Integrals around Closed Curves.** We know that every vector field with the path independence property is a conservative vector field. Here we show the converse: that every conservative vector field has the path independence property. Suppose that $\mathbf{F}$ is a conservative vector field, so that $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = 0$ for every closed curve $\mathcal{C}$. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be oriented curves from the P to Q in the domain of $\mathbf{F}$.

(a) Let $-\mathcal{C}_2$ denote $\mathcal{C}_2$ with the opposite orientation. Then $\mathcal{C} = \mathcal{C}_1 \cup (-\mathcal{C}_2)$ is a closed curve with initial point and endpoint equal to P. Show that $\int_{\mathcal{C}_1} \mathbf{F}\cdot\mathbf{T}\,ds = -\int_{-\mathcal{C}_2} \mathbf{F}\cdot\mathbf{T}\,ds$.

(b) Use part (a) to show that $\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathbf{T}\, ds$. Since this argument holds for any P, Q, $\mathcal{C}_1$, and $\mathcal{C}_2$, this shows that $\mathbf{F}$ has the path independence property.

9. Sum of Conservative Forces. If forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on an object, then the total force $\mathbf{F}$ acting on the object is $\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2$. Show that if $\mathbf{F}_1$ and $\mathbf{F}_2$ are conservative forces, then $\mathbf{F}$ is a conservative force.

10. Friction. Assume that the force of friction acts in a direction opposite to the direction of motion with magnitude proportional to the speed. Thus if the path of a particle is parametrized by $\alpha = \alpha(t)$, the force due to friction at $\alpha(t)$ is $\mathbf{F}(\alpha(t)) = -k\alpha'(t)$ where k is a constant. Compute $\int_\alpha \mathbf{F} \cdot \mathbf{T}\; ds$ and show that $\mathbf{F}$ is not a conservative force.

11. Work-Energy Theorem. In this exercise, we want to investigate the consequences of the work-energy theorem.

(a) Suppose that a force acts on a particle and the motion of the particle is described by a parametrization α with constant speed, that is, $\|\alpha'(t)\|$ is constant. How much work was done by the field in moving the particle?

(b) Suppose that a force acts on a particle so that the initial speed of the particle is equal to the final speed of the particle. How much work was done by the field in moving the particle?

12. Work and Conservative Fields. If a force does a quantity of work W on a particle, it is said that the particle does work on whatever produced the force in the amount $-W$. It follows that in a conservative field, when a particle moves through a closed loop, the ***particle does no work*** on the field and that the particle's ability to do work is ***conserved***. This interpretation is the source of the name "conservative" field. Use the work-energy theorem to explain the following statements:

(a) The kinetic energy of a particle decreases by an amount equal to the amount of work that a particle does.

(b) The kinetic energy of a body in motion is equal to the work it can do in being brought to rest.

13. Inverse Square Fields. An ***inverse square force field*** $\mathbf{F}$ takes the form

$$\mathbf{F}(\mathbf{x}) = \frac{K}{r^2}\mathbf{u},$$

where $r = \|\mathbf{x}\|$ and $\mathbf{u} = \mathbf{x}/\|\mathbf{x}\|$. For example, the electric force field generated by a point charge and a gravitational field generated by an object are inverse square force fields.

(a) Show that $\mathbf{F}$ is conservative (for all $\mathbf{x} \neq \mathbf{0}$) by finding a potential function f with $\nabla f = \mathbf{F}$.

(b) Show that the work done in moving a particle from $\mathbf{x}$ to a point infinitely far from the origin is K/r.

14. Gravity. Since gravity is a conservative force, the total energy of an object subject only to gravity is constant. This applies, for example, to a satellite orbiting the earth above the earth's atmosphere. Thus if α parametrizes the orbit of a satellite, the kinetic energy and the potential energy of the satellite satisfy

$$K(\alpha(t)) + U(\alpha(t)) = \frac{m}{2}\mathbf{v}(t)\cdot\mathbf{v}(t) - \frac{GM_e m}{\|\alpha(t)\|} = E,$$

where E is constant. Assuming that the orbit is a closed curve, where in the orbit will the speed of the satellite be greatest and where will it be least? Explain your answer.

6.3 Integration over Closed Curves

In this section, we will continue to study line integrals focusing on the integral of a vector field over a simple closed curve in the plane. The primary result will be Green's[1] theorem, which relates the line integral of a vector field over a simple closed curve to a double integral over the region enclosed by the curve. Green's theorem applies to vector fields $\mathbf{F}$ whose coordinate functions are continuously differentiable. The generalization of Green's theorem to closed curves in space, Stokes' theorem, will be considered in the next chapter. We will introduce and discuss Green's theorem first, reserving the proof for the end of the section.

Intuitively, a curve is closed if it begins and ends at the same point. We say the curve is a simple closed curve if it does not intersect itself at any other points. More precisely, we have the following definition.

Definition 6.9 A curve is called ***closed*** if it has a parametrization $\alpha : [a, b] \to \mathbb{R}^2$ with $\alpha(a) = \alpha(b)$. A closed curve is called a ***simple closed curve*** if for any distinct points $t_1, t_2 \in [a, b)$, $\alpha(t_1) \neq \alpha(t_2)$. ♦

A simple closed curve $\mathcal{C}$ in $\mathbb{R}^2$ divides the plane into two regions, the ***interior*** of $\mathcal{C}$ and the ***exterior*** of $\mathcal{C}$. The interior of $\mathcal{C}$ is the region contained inside $\mathcal{C}$, and the exterior of $\mathcal{C}$ is the region outside $\mathcal{C}$. Each of these regions is an open set in the plane. The curve $\mathcal{C}$ is the boundary of these regions. (See Section 3.3 for the definition of open sets and boundary.) We will consider curves that can be expressed as the finite union of curves can be parametrized by differentiable functions. We call these ***piecewise differentiable*** curves.

In Section 6.2, we showed that a continuous vector field $\mathbf{F}$ has the path independence property, or is conservative, on its domain, if and only if it is a gradient vector field. If a vector field has the path independence property, it follows that its line integral around a

[1] G. Green (1793–1841) was a self-taught British mathematician and physicist known for creating the first mathematical theory of electricity and magnetism.

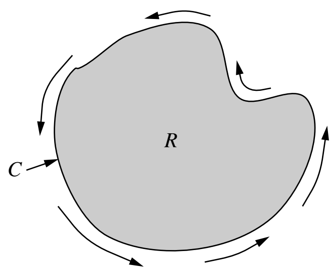

Figure 6.6 A region $\mathcal{R}$ with boundary a simple closed curve $\mathcal{C}$ oriented positively.

closed curve is zero. Conversely, in Exercise 8 of Section 6.2, we showed that if the line integral of a vector field around every closed curve is zero, then the vector field has the path independence property. This required that the interior of $\mathcal{C}$ was contained in the domain of $\mathbf{F}$. Combining these two results, a vector field $\mathbf{F}$ is a gradient vector field if and only if the line integral of $\mathbf{F}$ over every closed curve whose interior is contained in the domain of $\mathbf{F}$ is zero.

Green's theorem extends this result in that it makes a claim about the line integral of any continuously differentiable vector field $\mathbf{F}$ around a simple closed curve. It expresses the line integral of $\mathbf{F}$ around a simple closed curve $\mathcal{C}$ as a double integral of the partial derivatives of the coordinate functions of $\mathbf{F}$ over the interior of $\mathcal{C}$. To state Green's theorem, we must orient $\mathcal{C}$. The conventional orientation of $\mathcal{C}$ orients it so that the interior remains on the left as the curve is traced in the direction of the orientation. For simple closed curves in the plane, this orients the boundary curve in a ***counterclockwise*** direction around the interior $\mathcal{R}$. We will call this the ***positive orientation*** of the boundary. (See Figure 6.6.) We can now state Green's theorem. The proof appears at the end of the section.

Theorem 6.3 **Green's Theorem.** Let $\mathcal{R}$ be a region in the plane whose boundary $\mathcal{C}$ is a piecewise differentiable, simple closed curve that is positively oriented with respect to $\mathcal{R}$. Let $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ be a vector field that is defined and continuously differentiable on an open set containing $\mathcal{C}$ and $\mathcal{R}$. Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA_{x,y}. \quad \blacklozenge$$

In the special case when $\mathbf{F} = \nabla f$ is a gradient vector field, $u = \frac{\partial f}{\partial x}$ and $v = \frac{\partial f}{\partial y}$. Because the mixed partial derivatives of f are equal, it follows immediately that the integrand of the double integral in the theorem is zero.

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} - \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = 0.$$

Thus Green's theorem gives us an alternate way to understand the fact that the line integral of a gradient vector field around a closed curve is zero.

In addition, there is an interpretation of Green's theorem when $\mathbf{F}$ is the velocity field of a fluid flow. In this case, the line integral of $\mathbf{F}$ around a closed curve $\mathcal{C}$ gives the total accumulation of the component of the flow in the direction of $\mathcal{C}$. This quantity is called the ***circulation*** of the fluid flow around $\mathcal{C}$. If the velocity field $\mathbf{F}$ is continuous and differentiable on $\mathcal{R}$ and $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ on $\mathcal{R}$, then the circulation around any closed curve $\mathcal{C} \subset \mathcal{R}$ is 0. In this context, we say the fluid flow is ***irrotational.*** To check if a flow is irrotational, we need only check that $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$.

Since Green's theorem equates a line integral with a double integral, it can be used in two ways as a calculational tool. For example, if we are interested in the line integral of a vector field around a closed curve, it might be the case that it is simpler to evaluate the corresponding double integral over the interior of the closed curve rather than the line integral. Conversely, we might prefer to evaluate the line integral over $\mathcal{C}$ in place of the double integral over the interior of $\mathcal{C}$. The following example illustrates both types of calculations.

Example 6.9 **Green's Theorem Calculations**

A. Let $\mathcal{R}$ be the region in the first quadrant bounded by the x-axis, the y-axis, the line $y = x + 2$, and the line $y = 2x - 4$. Let $\mathbf{F}(x, y) = (xy, x^2 - y^2)$. (See Figure 6.7.) The boundary $\mathcal{C}$ of this region consists of four straight line segments. In order to compute the line integral of $\mathbf{F}$ around $\mathcal{C}$ in a counterclockwise direction, we would parametrize each of these line segments with the appropriate orientation and then compute the corresponding line integrals. Using Green's theorem, we can replace this calculation by the calculation of the double integral over the region $\mathcal{R}$ of

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2x - x = x.$$

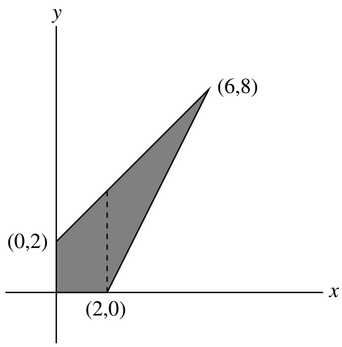

Figure 6.7 The region in Example 6.9A.

Thus we have

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds &= \int\int_{\mathcal{R}} x\,dA_{x,y}\\ &= \int_0^2\int_0^{x+2} x\,dy\,dx + \int_2^6\int_{2x-4}^{x+2} x\,dy\,dx\\ &= 20/3 + 80/3 = 100/3.\end{aligned}$$

B. If $\mathcal{C}$ is a simple closed curve enclosing a region $\mathcal{R}$, and $\mathbf{F}(x,y) = \frac{1}{2}(-y,x)$, then using Green's theorem, we have

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \int\int_{\mathcal{R}} 1\,dA_{x,y} = \text{Area}(\mathcal{R}).$$

Thus we can use the line integral, $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds$, to compute the area of the region enclosed by $\mathcal{C}$. For example, suppose $\mathcal{R}$ is the interior of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The ellipse can be parametrized by $\alpha(t) = (a\cos t, b\sin t)$, $t\in[0,2\pi]$. The area enclosed by the ellipse can be computed by evaluating the iterated integral

$$\int_{-a}^{a}\int_{-\frac{b}{a}\sqrt{a^2-x^2}}^{\frac{b}{a}\sqrt{a^2-x^2}} 1\,dA.$$

Using Green's theorem with $\mathbf{F} = \frac{1}{2}(-y,x)$, we know the area enclosed by the ellipse is

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{2\pi} \frac{1}{2}(-b\sin t, a\cos t)\cdot(-a\sin t, b\cos t)\,dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.$$

Green's theorem can be extended to regions whose boundary consists of a collection of simple closed curves. A region of this type is obtained by removing the interiors of a collection of simple closed curves $\mathcal{C}_2, \mathcal{C}_3, ..., \mathcal{C}_n$ from the interior of a simple closed curve $\mathcal{C}_1$. It is important that these curves be oriented correctly. Each component of the boundary of $\mathcal{R}$ must be oriented so that when it is traversed in the direction of the orientation, the region stays on the left. Thus the curve $\mathcal{C}_1$ must be traced counterclockwise and the curves $\mathcal{C}_2, \mathcal{C}_3, \ldots, \mathcal{C}_n$ must be traced clockwise. (See Figure 6.8(a).)

We are now in a position to state the generalization of Green's theorem and to sketch a proof of the result.

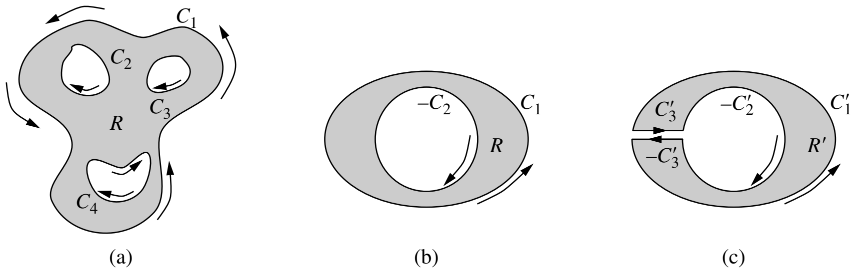

Figure 6.8 (a) A region that is the interior of a simple closed curve with a number of holes. (b) and (c) The region between the ellipse $x^2 + 2y^2 = 4$ and the circle $x^2 + y^2 = 1$ used in the proof of Green's theorem for more general regions. (See Theorem 6.4 and Example 6.10.)

Theorem 6.4 **Green's Theorem for More General Regions.** Let $\mathcal{R}$ be a region in $\mathbb{R}^2$ whose boundary $\mathcal{C}$ consists of the union of a finite number of piecewise differentiable, simple closed curves that are positively oriented with respect to $\mathcal{R}$. Let $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ be a vector field that is defined and continuously differentiable on an open set containing $\mathcal{C}$ and $\mathcal{R}$. Then

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA_{x,y}. \quad \blacklozenge$$

Proof Sketch: We consider a special case to demonstrate the general situation. Suppose $\mathcal{C}_1$ is the ellipse $x^2 + 2y^2 = 4$ oriented counterclockwise and $\mathcal{C}_2$ is the circle $x^2 + y^2 = 1$, also oriented counterclockwise. Let $\mathcal{R}$ be the region between them. (See Figure 6.8(b).) We want to replace $\mathcal{R}$ by a region $\mathcal{R}'$ arbitrarily close to $\mathcal{R}$ whose boundary is a simple closed curve. To do so, we insert a curve $\mathcal{C}_3$ connecting $\mathcal{C}_1$ to $\mathcal{C}_2$ as shown in Figure 6.8(c). Parametrize $\mathcal{C}_3$ from $\mathcal{C}_1$ to $\mathcal{C}_2$, so that $-\mathcal{C}_3$ is traced from $\mathcal{C}_2$ to $\mathcal{C}_1$. Separating $\mathcal{C}_3$ and $-\mathcal{C}_3$ slightly, we obtain a region $\mathcal{R}'$, whose boundary $\mathcal{C}_1' \cup \mathcal{C}_3' \cup -\mathcal{C}_2' \cup -\mathcal{C}_3'$ is a simple closed curve oriented counterclockwise, or with the region on the left. (The prime notation indicates the modification that results from separating $\mathcal{C}_3$ and $-\mathcal{C}_3$.) Thus we may apply Green's theorem to $\mathcal{R}'$ and its boundary:

$$\int\int_{\mathcal{R}'} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA_{x,y} = \int_{\mathcal{C}_1'} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{\mathcal{C}_3'} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{-\mathcal{C}_2'} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{-\mathcal{C}_3'} \mathbf{F} \cdot \mathbf{T}\, ds.$$

Intuitively, if we allow the separation between $\mathcal{C}_3'$ and $-\mathcal{C}_3'$ to approach 0, the integrals over $\mathcal{C}_3'$ and $-\mathcal{C}_3'$ are opposites of each other, and we may remove the primes from the remaining terms. Separating these steps, we have

$$\int\int_{\mathcal{R}'} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA_{x,y} = \int_{\mathcal{C}_1'} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{\mathcal{C}_3'} \mathbf{F} \cdot \mathbf{T}\, ds - \int_{\mathcal{C}_2'} \mathbf{F} \cdot \mathbf{T}\, ds - \int_{\mathcal{C}_3'} \mathbf{F} \cdot \mathbf{T}\, ds,$$

and then

$$\int\int_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA_{x,y} = \int_{\mathcal{C}_1} \mathbf{F}\cdot\mathbf{T}\,ds - \int_{\mathcal{C}_2} \mathbf{F}\cdot\mathbf{T}\,ds.$$

This is the generalized version of Green's theorem for $\mathcal{R}$.

To extend this approach to regions of the type shown in Figure 6.8(a), we can proceed inductively, increasing the number of boundary components, or we can introduce sufficiently many segments to produce a region whose boundary is a simple closed curve as we did above. ∎

The extension of Green's theorem can be used to simplify the evaluation of line integrals around closed curves, because it allows us to replace more complicated line integrals by less complicated line integrals. We demonstrate this for the circle and ellipse used in the proof sketch.

Example 6.10 **Green's Theorem on a Punctured Region.** Let $\mathbf{F}(x,y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$. The vector field $\mathbf{F}$ is continuous, differentiable, and satisfies $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ at every point except the origin. Thus if $\mathcal{C}$ is a simple closed curve that does not enclose the origin, so that $\mathbf{F}$ is defined and differentiable everywhere on the interior of $\mathcal{C}$, then Green's theorem tells us that $\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = 0$. However, it does not apply directly to curves that circle the origin. Nevertheless, here we see how it can be of use.

A. Suppose $\mathcal{C}_2$ is the circle $x^2+y^2=1$ oriented counterclockwise. We cannot apply Green's theorem to evaluate the line integral of $\mathbf{F}$ around $\mathcal{C}_2$, but we can evaluate the line integral directly. If we parametrize the unit circle by $\alpha(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$, we have

$$\begin{aligned}\int_{\mathcal{C}_2} \mathbf{F}\cdot\mathbf{T}\,ds &= \int_0^{2\pi} \mathbf{F}(\alpha(t))\cdot\alpha'(t)\,dt \\ &= \int_0^{2\pi} (-\sin t, \cos t)\cdot(-\sin t, \cos t)\,dt = 2\pi.\end{aligned}$$

B. Now suppose $\mathcal{C}_1$ is the ellipse $x^2+2y^2=4$ oriented counterclockwise. Again, we cannot apply Green's theorem directly to evaluate the line integral of $\mathbf{F}$ around $\mathcal{C}_1$, because the origin is also contained in the interior of $\mathcal{C}_1$. If we parametrize $\mathcal{C}_1$ and try to evaluate the line integral directly, we find that the integral is difficult to evaluate symbolically. Instead we can use the generalized version of Green's theorem to evaluate the integral. Let $\mathcal{R}$ be the region between $\mathcal{C}_1$ and $\mathcal{C}_2$. (See Figure 6.8(b).) In this case, $\mathbf{F}$ is defined on $\mathcal{R}$ and $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$ on $\mathcal{R}$. Thus, applying the generalized form of Green's theorem, we have

$$0 = \int\int_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA_{x,y} = \int_{\mathcal{C}_1} \mathbf{F}\cdot\mathbf{T}\,ds - \int_{\mathcal{C}_2} \mathbf{F}\cdot\mathbf{T}\,ds.$$

It follows that

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathbf{T}\, ds = 2\pi.$$

We can see from the structure of the argument that we could replace $\mathcal{C}_1$ by any simple closed curve that circles the origin once and is oriented counterclockwise.

The Flux of a Vector Field

There is an important application of Green's theorem to fluid flows in the plane when the velocity field $\mathbf{F}$ of the flow does not vary in time. If $\mathcal{C}$ is a simple closed curve in the plane, we are interested in the amount of fluid that crosses $\mathcal{C}$ in a unit of time. This quantity is called the ***total flux*** of $\mathbf{F}$ across $\mathcal{C}$. This is equal to the total accumulation of the component of $\mathbf{F}$ in a direction normal to $\mathcal{C}$. Initially, we will express the total flux as a path integral, and then we will see how to rewrite it as a line integral.

First, let us establish our notation. Let $\alpha(t) = (x(t), y(t))$, $t \in [a, b]$, be a continuously differentiable parametrization of $\mathcal{C}$ with a nonzero derivative, so that $\|\alpha'(t)\| \neq 0$. The unit tangent vector to α is given by

$$\mathbf{T}(t) = \frac{\alpha'(t)}{||\alpha'(t)||} = \frac{1}{||\alpha'(t)||}(x'(t), y'(t)).$$

The unit vector $\mathbf{N}$ given by

$$\mathbf{N}(t) = \frac{1}{||\alpha'(t)||}(y'(t), -x'(t))$$

is orthogonal to $\mathbf{T}$. It can be shown that $\mathbf{N}$ always points to the same side of $\mathcal{C}$. Thus we can use $\mathbf{N}$ to specify a direction from one side of $\mathcal{C}$ to the other everywhere on $\mathcal{C}$ in a consistent manner. (See Exercise 11.) At each point of $\mathcal{C}$, the quantity $\mathbf{F} \cdot \mathbf{N}$ is the component of the velocity of the flow in a direction normal to or across $\mathcal{C}$. The ***total flux*** of $\mathbf{F}$ across $\mathcal{C}$ in the direction $\mathbf{N}$ is defined to be the path integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{N}\, ds.$$

Keep in mind that although this is the integral of the dot product of vectors, it is *not* a line integral because $\mathbf{T}$ is not one of the vectors. However, after a brief calculation, we will be able to express the total flux as a line integral.

Let $\mathbf{F}(x, y) = (u(x, y), v(x, y))$. Then the total flux of $\mathbf{F}$ across $\mathcal{C}$ in the direction $\mathbf{N}$ is

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{N}\, ds &= \int_a^b (u(\alpha(t)), v(\alpha(t))) \cdot \frac{1}{||\alpha'(t)||}(y'(t), -x'(t))||\alpha'(t)||\, dt \\ &= \int_a^b -v(\alpha(t))x'(t) + u(\alpha(t))y'(t)\, dt \\ &= \int_a^b (-v(\alpha(t)), u(\alpha(t))) \cdot \alpha'(t)\, dt.\end{aligned}$$

This last integral is the line integral of the vector field $(-v, u)$ along $\mathcal{C}$. Thus

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{N}\, ds = \int_{\mathcal{C}} (-v, u) \cdot \mathbf{T}\, ds.$$

If $\mathcal{C}$ is a simple closed curve with interior $\mathcal{R}$, we can use Green's theorem to express the total flux across $\mathcal{C}$ as a double integral. If α parametrizes $\mathcal{C}$ in a counterclockwise manner, then the vector $\mathbf{N}$ given above points out of $\mathcal{R}$, so that the total flux of $\mathbf{F}$ across $\mathcal{C}$ in the direction $\mathbf{N}$ measures the amount of fluid flowing out of $\mathcal{R}$ in a unit of time. Applying Green's theorem, we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{N}\, ds = \int_{\mathcal{C}} (-v, u) \cdot \mathbf{T}\, ds = \int\int_{\mathcal{R}} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)\, dA_{x,y}.$$

This result is a two-dimensional version of the divergence theorem, which we will encounter in Section 7.4. Since the double integral represents the total flux of $\mathbf{F}$ out of $\mathcal{R}$ and the integrand is defined at every point of $\mathcal{R}$, it makes sense on an intuitive level to interpret the integrand

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$$

as a measure of the flux of the flow at a point. Thus we will call this quantity the ***pointwise*** or ***infinitesimal flux*** of $\mathbf{F}$. Following through on this idea, if the integrand or infinitesimal flux is always positive, then the total flux is also positive. We will give a precise definition of infinitesimal flux in Section 7.3.

Application: Diffusion

This result can also be applied to a time-dependent diffusion process to measure the flux of the solute out of a region. The velocity of the fluid is assumed to be zero so that the movement of the solute is due solely to diffusion. This is an important consideration because diffusion is a microscopic process that is due to the random motion of molecules, whereas a fluid flow is a macroscopic process, so that the effects of fluid velocity are more significant than those of diffusion. The movement of the solute is represented by the flux vector. We will assume that the diffusion process satisfies ***Fick's*[2] *law***, that is, the flux vector is a multiple of the gradient vector of the concentration of the solute. The following example computes the flux of a source centered at the origin.

Example 6.11

Total Flux of a Diffusion Process. If an amount M of solute is released instantaneously from the origin in the plane at time $t = 0$, the concentration of the solute at time t at (x, y) is given by

$$c(x, y, t) = \frac{M}{4\pi\delta t} e^{-(x^2+y^2)/(4\delta t)},$$

where the constant δ is the diffusivity of the solute in the solvent. According to Fick's law, the flux vector $\mathbf{J}$ of the diffusion process is given by

$$\begin{aligned} \mathbf{J}(x, y, t) &= -\delta \nabla c(x, y, t) \\ &= -\delta\left(\frac{\partial c}{\partial x}(x, y, t), \frac{\partial c}{\partial y}(x, y, t)\right). \end{aligned}$$

Also, a calculation shows that c satisfies the equation

$$\frac{\partial c}{\partial t}(x, y, t) = \delta \left(\frac{\partial^2 c}{\partial x^2}(x, y, t) + \frac{\partial^2 c}{\partial y^2}(x, y, t) \right).$$

This equation is called the ***diffusion equation***. The expression on the right-hand side is usually written $\Delta c(x, y, t)$ where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. This combination of second derivatives of c is called the ***Laplacian***[3] derivative of c.

Let us compute the flux of the concentration through a circle $\mathcal{C}$ of radius 1 centered at the origin in the outward direction. Here we assume that $\mathcal{C}$ is positively oriented relative to its interior $\mathcal{R}$, the unit disk centered at the origin. Thus, we want to compute

$$\int_{\mathcal{C}} \mathbf{J} \cdot \mathbf{N}\, ds = \int_{\mathcal{C}} -\delta(\nabla c \cdot \mathbf{N})\, ds = \int_{\mathcal{C}} -\delta\left(-\frac{\partial c}{\partial y}, \frac{\partial c}{\partial x}\right) \cdot \mathbf{T}\, ds.$$

Applying Green's theorem to the right-most integral, we see that

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{J} \cdot \mathbf{N}\, ds &= \int\int_{\mathcal{R}} -\delta \left(\frac{\partial^2 c}{\partial x^2} + \frac{\partial^2 c}{\partial y^2} \right) dA_{x,y} \\ &= -\int\int_{\mathcal{R}} \frac{\partial c}{\partial t}(x, y, t)\, dA_{x,y}, \end{aligned}$$

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[2] A. E. Fick (1829–1901) was a German physiologist interested in diffusion of gas through a membrane in physiology and physics.

[3] Pierre-Simon Laplace, marquis de Laplace (1749–1827), was a French mathematician and astronomer known for his groundbreaking work in analysis and applied mathematics.

where we have used the fact that c satisfies the diffusion equation to make the last substitution. Thus the total flux of the diffusion process is equal to the negative of the double integral of the time derivative of the concentration.

For the particular c we started with,

$$\frac{\partial c}{\partial t}(x, y, t) = \frac{M}{4\pi\delta}\left(\frac{-1}{t^2} + \frac{x^2+y^2}{4\delta t^3}\right) e^{-(x^2+y^2)/(4\delta t)}.$$

Substituting this expression into the double integral for the total flux, a calculation shows that the flux at time t is equal to

$$\frac{M}{4\delta t^2} e^{-1/(4\delta t)}.$$

We will explore this function in Exercise 13.

In the course of this example, we have shown that if $c = c(x, y, t)$ satisfies the diffusion equation, then the flux of c across $\mathcal{C}$ at time t is the negative total accumulation of $\frac{\partial c}{\partial t}$ on the interior of $\mathcal{C}$,

$$\int_{\mathcal{C}} \mathbf{J} \cdot \mathbf{N}\, ds = -\int\int_{\mathcal{R}} \frac{\partial c}{\partial t}(x, y, t)\, dA_{x,y}.$$

Notice the importance of the minus sign. Assuming that there is no addition or removal of substance within $\mathcal{R}$, if the integral of $\frac{\partial c}{\partial t}$ is positive, then the total amount of substance in $\mathcal{R}$ is increasing, so that there must be a net influx of substance into $\mathcal{R}$. It follows that the total flux of $\mathbf{J}$ ***out*** of $\mathcal{R}$ must be ***negative***. Conversely, if the integral of $\frac{\partial c}{\partial t}$ is negative, then the total amount of substance in $\mathcal{R}$ is decreasing, so that there must be a net efflux of substance out of $\mathcal{R}$ and the total flux of $\mathbf{J}$ out of $\mathcal{R}$ must be positive.

If $\mathbf{F}$ represents the velocity field of a flow or the flux field of a diffusion process, then the total flux across $\mathcal{C}$ represents the total amount of substance that leaves $\mathcal{R}$ in one unit of time. If no substance is added or lost at a point inside $\mathcal{R}$, the total flux is also the total change in the amount of substance in $\mathcal{R}$ per unit time. In this case, we might interpret the integrand of the double integral as the change in the amount of substance per unit area per unit time, so that its total accumulation is the total change in the amount of substance in $\mathcal{R}$ per unit time. We will give a more careful justification of this interpretation in Section 7.3, when we consider the generalization of this result to vector fields in space.

The Proof of Green's Theorem

Let us begin by recalling the hypothesis of Green's theorem: $\mathcal{R}$ is a region in the plane whose boundary $\mathcal{C}$ is a piecewise differentiable, simple closed curve that is positively oriented with respect to $\mathcal{R}$, and $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ is a vector field that is defined and differentiable on an open set containing $\mathcal{C}$ and $\mathcal{R}$. We want to show that

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA.$$

The proof of Green's theorem involves a calculation of both sides of this expression to show they are equal. We will carry out this calculation in three steps: First we will show that the result holds for a particularly simple type of region that is bounded by the graphs of functions of x or functions of y, then we will indicate how to partition a general region into subregions of this type, and lastly we will show that Green's theorem holds on the general region if it holds on the subregions.

Step 1

There are two special cases for $\mathcal{R}$ that we want to consider. The simpler of the two cases, when $\mathcal{R}$ is a rectangle, is considered in Exercise 4. We will consider the more involved case here. Assume that $\mathcal{R}$ is a region that is bounded by a horizontal line, a vertical line, and a curve that can be written both in the form $y = g(x)$ and $x = h(y)$. (See Figure 6.9.)

The region $\mathcal{R}$ can be described in the form $a \le x \le b$ and $c \le y \le g(x)$ or in the form $c \le y \le d$ and $a \le x \le h(y)$. Using the additivity of the double integral, the double integral of $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ over $\mathcal{R}$ can be expressed as the difference of the double integral of $\frac{\partial v}{\partial x}$

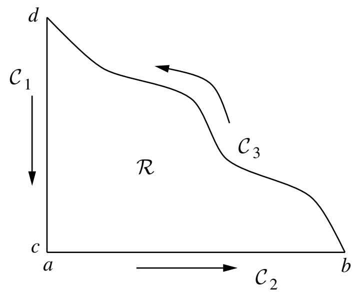

Figure 6.9 Region bounded by two lines and a curve that can be written both in the form $y = g(x)$ and $x = h(y)$.

over the region $\mathcal{R}$ and the double integral of $\frac{\partial u}{\partial y}$ over $\mathcal{R}$. We will use the two descriptions of $\mathcal{R}$ to express each of these double integrals as an iterated integral.

$$\begin{aligned}\int\int_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dA_{x,y} &= \int\int_{\mathcal{R}} \frac{\partial v}{\partial x} dA_{x,y} - \int\int_{\mathcal{R}} \frac{\partial u}{\partial y} dA_{x,y} \\ &= \int_c^d \int_a^{h(y)} \frac{\partial v}{\partial x}(x,y)\, dx\, dy - \int_a^b \int_c^{g(x)} \frac{\partial u}{\partial y}(x,y)\, dy\, dx \\ &= \int_c^d (v(h(y),y) - v(a,y))\, dy - \int_a^b (u(x,g(x)) - u(x,c))\, dx.\end{aligned}$$

The boundary of the region $\mathcal{R}$ consists of three pieces: $\mathcal{C}_1$, the line segment $x = a,\ c \leq y \leq d$; $\mathcal{C}_2$, the line segment $y = c,\ a \leq x \leq b$; and $\mathcal{C}_3$, the portion of the curve $x = h(y),\ c \leq y \leq d$. These pieces can be parametrized by $\alpha_1(y) = (a, y)$, $c \leq y \leq d$, $\alpha_2(x) = (x, c)$, $a \leq x \leq b$, and $\alpha_3(y) = (h(y), y)$, $c \leq y \leq d$, respectively. The boundary $\mathcal{C}$ is oriented counterclockwise by $-\alpha_1$, α_2, and α_3, where $-\alpha_1$ means that we parametrize $\mathcal{C}_1$ from (a, d) to (a, c). (See Figure 6.9.) Then,

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\, ds &= \int_{-\alpha_1} \mathbf{F}\cdot\mathbf{T}\, ds + \int_{\alpha_2} \mathbf{F}\cdot\mathbf{T}\, ds + \int_{\alpha_3} \mathbf{F}\cdot\mathbf{T}\, ds \\ &= \int_d^c (u(a,y), v(a,y))\cdot(0,1)\, dy + \int_a^b (u(x,c), v(x,c))\cdot(1,0)\, dx \\ &\quad + \int_c^d (u(h(y),y), v(h(y),y))\cdot(h'(y),1)\, dy \\ &= \int_d^c v(a,y)\, dy + \int_a^b u(x,c)\, dx + \int_c^d u(h(y),y)h'(y)\, dy + \int_c^d v(h(y),y)\, dy.\end{aligned}$$

If we substitute $x = h(y)$ into the third integral above and use the fact that for all points on $\mathcal{C}_3$, $y = g(x)$, we have

$$\int_c^d u(h(y),y)h'(y)\, dy = \int_b^a u(x,g(x))\, dx = -\int_a^b u(x,g(x))\, dx.$$

Combining this with our first calculation, we have

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\, ds &= \int_c^d (v(h(y),y) - v(a,y))\ dy + \int_a^b (u(x,c) - u(x,g(x)))\ dx \\ &= \int\int_{\mathcal{R}} \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} dA_{x,y},\end{aligned}$$

which proves Green's theorem for three-sided regions of the specified type.

Step 2

Now let us suppose that $\mathcal{R}$ is the interior of a simple closed curve $\mathcal{C}$. We would like to partition $\mathcal{R}$ into rectangles and region of the form given in Step 1. Figure 6.10 indicates how to partition such a region. Intuitively, we fill sufficiently much of the interior of $\mathcal{R}$ with nonoverlapping rectangles, so that we can fill the remaining portion of $\mathcal{R}$ with three-sided regions of the type specified in Step 1, where the boundary of any three-sided region is made up of a vertical and a horizontal line segment and a portion of $\mathcal{C}$. By using sufficiently many rectangles and choosing sufficiently small three-sided regions, it is possible to guarantee that the third side of each triangular region can be written both as the graph of a function of x and as the graph of a function of y. From Step 1 and Exercise 4, we have that Green's theorem holds for each of these subregions.

Step 3

Now let us assume that the interior $\mathcal{R}$ of a simple closed curve $\mathcal{C}$ has been partitioned into regions $\mathcal{R}_1, \ldots, \mathcal{R}_m$ and that Green's theorem holds on each of these regions. Since $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \ldots \cup \mathcal{R}_m$, the double integral of a function over $\mathcal{R}$ is the sum of the double integrals of the function over each $\mathcal{R}_i$. Thus we can calculate the right-hand side of the expression in Green's theorem by calculating

$$\int\int_{\mathcal{R}_i} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA_{x,y}$$

for each i and summing the results.

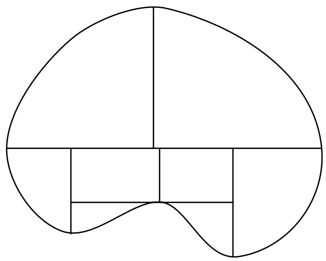

Figure 6.10 A region partitioned into regions for which Green's theorem can be verified directly.

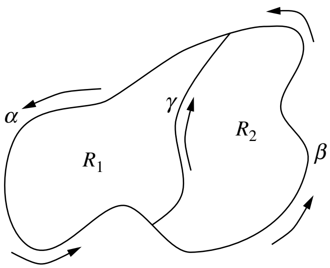

Figure 6.11 A region $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ with positively oriented boundaries. The boundary $\mathcal{C}_1$ of $\mathcal{R}_1$ is parametrized by α and γ. The boundary $\mathcal{C}_2$ of $\mathcal{R}_2$ is parametrized by β and $-\gamma$.

After we have partitioned $\mathcal{R}$, let $\mathcal{C}_i$ be the boundary of $\mathcal{R}_i$ oriented counterclockwise. In order to compute the left-hand side of the expression in Green's theorem, we must relate the line integral of $\mathbf{F}$ over $\mathcal{C}$ to the line integrals of $\mathbf{F}$ over the $\mathcal{C}_i$. We claim that

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \int_{\mathcal{C}_1} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{\mathcal{C}_2} \mathbf{F}\cdot\mathbf{T}\,ds + \cdots + \int_{\mathcal{C}_m} \mathbf{F}\cdot\mathbf{T}\,ds.$$

We will demonstrate this for the case where $\mathcal{R}$ is partitioned into two regions. (See Figure 6.11.) If $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ with boundaries $\mathcal{C}_1$ and $\mathcal{C}_2$, then we want to show that

$$\int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{T}\,ds = \int_{\mathcal{C}_1} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{\mathcal{C}_2} \mathbf{F}\cdot\mathbf{T}\,ds.$$

Let $\tilde{\mathcal{C}}$ denote the common portion of $\mathcal{C}_1$ and $\mathcal{C}_2$. Let γ be a parametrization of $\tilde{\mathcal{C}}$ oriented so that it agrees with the orientation of $\mathcal{C}_1$ and let $-\gamma$ denote the parametrization in the opposite direction so that $-\gamma$ agrees with the orientation of $\mathcal{C}_2$. Let α parametrize the portion of $\mathcal{C}_1$ not including $\tilde{\mathcal{C}}$ and let β parametrize the portion of $\mathcal{C}_2$ not including $\tilde{\mathcal{C}}$. We will assume that α and β have the orientation inherited from $\mathcal{C}$. Notice that these orientations agree with the counterclockwise orientations of $\mathcal{C}_1$ and $\mathcal{C}_2$.

Then we have

$$\int_{\mathcal{C}_1} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{\mathcal{C}_2} \mathbf{F}\cdot\mathbf{T}\,ds = \int_{\alpha} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{\gamma} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{-\gamma} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{\beta} \mathbf{F}\cdot\mathbf{T}\,ds.$$

From Section 6.2, we know that

$$\int_{-\gamma} \mathbf{F}\cdot\mathbf{T}\,ds = -\int_{\gamma} \mathbf{F}\cdot\mathbf{T}\,ds.$$

Thus

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\alpha} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{\beta} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds,$$

which verifies the claim when $\mathcal{R}$ is the union of two regions. If $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \cdots \cup \mathcal{R}_m$, apply the preceding result to $\mathcal{R}_1 \cup \mathcal{R}_2$, then $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$, and so on. Since we have shown that the interior of every simple closed curve can be partitioned into regions for which Green's theorem holds, this completes the proof of Green's theorem. ∎

The proof of Green's theorem for more general regions follows from Green's theorem by partitioning a region $\mathcal{R}$ into a union of regions bounded by simple closed curves. Since Green's theorem applies to a region bounded by a simple closed curve, it applies to each of the regions making up $\mathcal{R}$. Then, by applying the conclusions of Step 3 of the proof of Green's theorem, we can show that Green's theorem applies to $\mathcal{R}$.

Summary

In this section, we stated and proved ***Green's theorem***, which relates the line integral of a vector field $\mathbf{F}$ around a closed curve $\mathcal{C}$ to the double integral of partial derivatives of $\mathbf{F}$ over $\mathcal{R}$, the interior of $\mathcal{C}$.

If $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ and $\mathcal{C}$ is ***positively oriented***, Green's theorem states

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \int\int_{\mathcal{R}} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) dA_{x,y}.$$

By positive orientation, we mean that when $\mathcal{C}$ is traced in a ***counterclockwise*** manner, $\mathcal{R}$ is to the ***left*** of $\mathcal{C}$. For the theorem to hold, it is sufficient that $\mathcal{C}$ be ***piecewise differentiable***. The theorem can also be applied to a region $\mathcal{R}$ whose boundary is a union of simple closed curves.

We presented two physical applications of Green's theorem involving fluids. First, we showed how Green's theorem could be used to relate the ***total flux*** of a fluid through a closed curve to the integral of the ***infinitesimal flux*** over the interior of the curve. Second, we showed how this result could be applied to a ***diffusion process*** that is governed by the ***diffusion equation***. Here the total flux is equal to the double integral of the rate change of the concentration over the interior.

Section 6.3 Exercises

1. Positive Orientation. For each of the following regions with boundary, give a parametrization of the boundary of the region that positively orients the boundary.

(a) The unit square, $[0, 1] \times [0, 1]$.

(b) The region bounded by the ellipse $x^2/4 + y^2 = 1$.
(c) The region in the upper half-plane bounded by $y = |x|$ and $x^2 + y^2 = 4$.
(d) The annulus $1 \leq x^2 + y^2 \leq 4$.

2. **Green's Theorem Calculations I.** Using Green's theorem, set up and evaluate a double integral that is equal to $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds$, for each of the following vector fields $\mathbf{F}$ and curves $\mathcal{C}$.

   (a) $\mathbf{F}(x, y) = (x^2, xy)$ and $\mathcal{C}$ is the triangle with vertices $(0, 0)$, $(2, 0)$, and $(2, 3)$.
   (b) $\mathbf{F}(x, y) = (-y^3, x^3)$ and $\mathcal{C}$ is the boundary of the region in the upper half-plane bounded by $y = |x|$ and $x^2 + y^2 = 4$.
   (c) $\mathbf{F}(x, y) = (x + y, \cos x)$ and $\mathcal{C}$ is the square with vertices $(0, 0)$, $(4, 0)$, $(4, 4)$, and $(0, 4)$.

3. **Green's Theorem Calculations II.** Using Green's theorem, set up and evaluate a line integral to compute the area of each of the following regions. (See Example 6.9B. Be sure to parametrize the boundary in a counterclockwise manner.)

   (a) The region enclosed by the cycloid $\alpha(t) = (2t - 2\sin t, 2 - 2\cos t)$ for $0 \leq t \leq 2\pi$ and the x-axis between $x = 0$ and $x = 4\pi$.
   (b) The region that is the intersection of the interiors of the circles $x^2 + y^2 = 1$ and $x^2 + (y - 1)^2 = 1$.

4. **Proof of Green's Theorem.** Complete the proof of Green's theorem by verifying Green's theorem for a rectangle $\mathcal{R} = [a, b] \times [c, d]$. Parametrize the boundary $\mathcal{C}$ of $\mathcal{R}$ counterclockwise and show that the line integral of $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ around $\mathcal{C}$ is equal to the integral of $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$ over $\mathcal{R}$.

5. **Green's Theorem and Area.** In Example 6.9B, we showed that if $\mathbf{F}(x, y) = \frac{1}{2}(-y, x)$, then by applying Green's theorem

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = \text{Area}(\mathcal{R}),$$

   where $\mathcal{R}$ is the interior of the simple closed curve $\mathcal{C}$.

   (a) Find another vector field $\mathbf{G}$ with the property that $\int_{\mathcal{C}} \mathbf{G} \cdot \mathbf{T}\, ds = \text{Area}(\mathcal{R})$.
   (b) What is the line integral of $\mathbf{F} - \mathbf{G}$ around $\mathcal{C}$?
   (c) Based on (b), what can we say about a vector field $\mathbf{G}$ that satisfies $\int_{\mathcal{C}} \mathbf{G} \cdot \mathbf{T}\, ds = \text{Area}(\mathcal{R})$?

6. **Verifying Green's Theorem.** Verify Green's theorem for each of the following regions $\mathcal{R}$ and vector fields $\mathbf{F}$.

   (a) $\mathcal{R} = \{(x, y) : 1 \leq x^2 + y^2 \leq 4\}$ and the vector field $\mathbf{F}(x, y) = (xy^2, -yx^2)$.
   (b) $\mathcal{R} = \{(x, y) : 0 \leq x \leq 1, 1 - x \leq y \leq 2 - x\}$ and the vector field $\mathbf{F}(x, y) = (x + y, -yx)$.

7. Generalized Green's Theorem. Consider the region $a^2 \leq x^2 + y^2 \leq b^2$. Prove the general case of Green's theorem for this region by partitioning the annulus into regions for which the first version of Green's theorem applies.

8. Total Flux. For each of the following vector fields $\mathbf{F}$ and regions with boundary, compute the total flux of the vector field across the boundary of the region.

(a) The region is a unit disk centered at the origin and $\mathbf{F}(x, y) = (e^y + x^3, y^3)$.
(b) The region is a unit square centered at the origin and $\mathbf{F}(x, y) = (x^2 - y^3, x^3 + y^2)$.
(c) The region is the annulus $1 \leq x^2 + y^2 \leq 4$ and $\mathbf{F}(x, y) = (yx^2, xy^2)$.

9. The Plane Minus the Origin. (See Example 6.10.) Let $\mathbf{F}(x, y) = (\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2})$.

(a) Let $\mathcal{C}$ be an oriented simple closed curve that encloses the origin. Show that the line integral of $\mathbf{F}$ around $\mathcal{C}$ is 2π if $\mathcal{C}$ is oriented counterclockwise and -2π if $\mathcal{C}$ is oriented clockwise.
(b) Suppose that $\mathcal{C}$ is a closed curve that wraps twice around the origin in a counterclockwise manner. Show that the line integral of $\mathbf{F}$ around $\mathcal{C}$ is 4π.

10. A Punctured Region. Let $\mathbf{F} = (u, v)$ be a continuously differentiable vector field that is defined everywhere except at a single point (x_0, y_0) in $\mathbb{R}^2$. Suppose also that $\frac{\partial v}{\partial x}(x, y) - \frac{\partial u}{\partial y}(x, y) = 0$ for all $(x, y) \neq (x_0, y_0)$.

(a) Show that if $\mathcal{C}$ is any simple closed curve that does not enclose the point (x_0, y_0), then $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{T}\, ds = 0$.
(b) Show that if $\mathcal{C}_1$ and $\mathcal{C}_2$ are simple closed curves oriented counterclockwise, which enclose the point (x_0, y_0), then $\int_{\mathcal{C}_1} \mathbf{F} \cdot \mathbf{T}\, ds = \int_{\mathcal{C}_2} \mathbf{F} \cdot \mathbf{T}\, ds$.

11. Normal Field to a Curve. Let α be a continuously differentiable parametrization of a curve from P to Q, $\alpha(t) = (x(t), y(t))$ with $\|\alpha(t)\| \neq 0$. (Recall that α is continuously differentiable if the coordinate functions of α are differentiable and have continuous derivatives.) Let the vector field $\mathbf{N} = \mathbf{N}(t)$ at $\alpha(t)$ be defined by

$$\mathbf{N}(t) = \frac{1}{||\alpha'(t)||}(y'(t), -x'(t)).$$

(a) Show that $\mathbf{N}$ is normal to the image of α.
(b) Construct an intuitive argument to justify the claim that $\mathbf{N}$ always points to the same side of the image of α.
(c) If $\mathcal{C}$ is a simple closed curve in the plane parametrized in a counterclockwise direction by $\alpha(t) = (x(t), y(t))$, show that $\mathbf{N}$ points out of the interior of $\mathcal{C}$.

12. Irrotational Flows. Let $\mathbf{F}$ be a differentiable vector field that represents the velocity field of a fluid flow. Discuss the relationship between the property of path independence for $\mathbf{F}$ and the property of $\mathbf{F}$ being irrotational.

13. **Total Diffusive Flux.** In Example 6.11, we outlined the calculation of the total flux through the unit circle of the diffusion process whose concentration is given by

$$c(x, y, t) = \frac{M}{4\pi\delta t} e^{-(x^2+y^2)/(4\delta t)}.$$

Here we fill in the details of that calculation and explore the resulting expression for the total flux.

(a) Verify the calculation of $\frac{\partial c}{\partial t}$.
(b) Use polar coordinates to show that

$$\int\int_{\mathcal{R}} -\frac{\partial c}{\partial t}(x, y, t)\, dA_{x,y} = \frac{M}{4\delta t^2} e^{-1/(4\delta t)},$$

where $\mathcal{R}$ is the disk of radius 1 centered at the origin.
(c) Setting $M = 1$ and $\delta = 1$, describe the behavior of the total flux $\frac{M}{4\delta t^2} e^{-1/(4\delta t)}$ as a function of time t. What does this tell us about the diffusion process as a function of time? Explain.

6.4 End of Chapter Exercises

1. **Arc Length and Average Value.** Let $\alpha(t) = (e^t \cos t, e^t \sin t, 2)$ with $0 \leq t \leq 1$.

(a) Compute the arc length of α.
(b) Compute the average value of $f(x, y, z) = 2x^2 + 2y^2 + 2z^2$ along α.

2. **Total Mass.** For each of the following parametrizations α and density functions δ, compute the total mass along the parametrization.

(a) $\alpha(t) = (\cos(2t), \sin(2t), t^2)$ for $0 \leq t \leq 2\pi$ with density function $\delta(x, y, z) = \sqrt{z}$.
(b) $\alpha(t) = (t \cos t, t \sin t, t)$ for $0 \leq t \leq \pi/2$ with density function $\delta(x, y, z) = z$.

3. **Path Integrals over $\mathcal{C}$.** For each of the following functions f and curves $\mathcal{C}$, (i) construct a parametrization α of $\mathcal{C}$ and (ii) set up and evaluate an integral to compute the total accumulation of f on $\mathcal{C}$.

(a) $\mathcal{C}$ is the semicircle of radius 1/2 with $x \geq 0$ in the xy-plane centered at the origin and $f(x, y) = xy^3$.
(b) $\mathcal{C}$ is the portion of the parabola $y = x^2 - 1$ from $(1, 0)$ to $(2, 3)$ and $f(x, y) = xy$.

4. **Potential Energy of a Charged Ellipse.** Consider a system that consists of a charged wire in the shape of an ellipse with axes of length $2a$ and $2b$ and a charged particle not on

the wire. Assume that the wire is modeled by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the xy-plane and has constant charge density C and that the particle of charge q is located at $\mathbf{x} = (x, y, z)$.

(a) Set up a Riemann sum to approximate the potential energy function $U = U(\mathbf{x})$ of this system.
(b) If you consider the limit of these Riemann sums as the mesh of the partition approaches zero, what is the resulting integral?

5. Total Accumulation by Computer. Use a computer algebra system to compute the following:

(a) The length of the ellipse parametrized by $\alpha(t) = (\cos t, 2\sin t)$.
(b) The length of the curve $y = x^3$ from $(0, 0)$ to $(2, 8)$.
(c) The average value of $f(x, y, z) = z$ on the curve parametrized by $\alpha(t) = (\cos t, \sin t, t^2)$, $0 \leq t \leq 2\pi$.

6. Line Integrals. For each of the following vector fields $\mathbf{F}$, compute the line integral of $\mathbf{F}$ over the parametrization α.

(a) $\mathbf{F}(x, y, z) = (x, y, z)$ and $\alpha(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 1$.
(b) $\mathbf{F}(x, y, z) = (y, z, x)$ and $\alpha(t) = (t, t^2, t^3)$, $0 \leq t \leq 1$.

7. Line Integrals over Curves. For each of the following vector fields $\mathbf{F}$, evaluate the line integral of $\mathbf{F}$ along the given curve. (*Hint:* Green's theorem might be helpful.)

(a) $\mathbf{F}(x, y, z) = (e^x, e^{-y}, e^z)$ along the straight line from $(-1, 1, 1)$ to $(1, 0, 1)$.
(b) $\mathbf{F}(x, y, z) = (yz, xz, xy)$ along the perimeter of the triangle from the vertex $(1, 0, 0)$ to $(0, 0, 1)$ to $(0, 1, 0)$ and back to $(1, 0, 0)$.

8. Potential Functions. Determine whether or not the following vector fields satisfy the path independence property on the given domain by finding a potential function f with $\mathbf{F} = \nabla f$ or by using Exercise 5 of Section 6.2.

(a) $\mathbf{F}(x, y) = (2xy, x^2)$, $\mathcal{D} = \mathbb{R}^2$.
(b) $\mathbf{F}(x, y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$, $\mathcal{D}$ is the annulus $1 \leq x^2 + y^2 \leq 4$.
(c) $\mathbf{F}(x, y, z) = (3x^2 - yz, 3y^2 - xz, -xy)$, $\mathcal{D} = \mathbb{R}^3$.

9. Positive Orientation. For each of the following regions with boundary, give a parametrization of the boundary of the region that positively orients the boundary.

(a) The triangular region with vertices $(0, 0)$, $(1, 1)$, and $(-1, 1)$.
(b) The unit disk centered at $(0, 1)$.
(c) The region outside the circle of radius $1/2$ centered at the origin and inside the square $[-1, 1] \times [-1, 1]$.

10. **Green's Theorem Calculations I.** Using Green's theorem, set up and evaluate a double integral to compute the line integral of $\mathbf{F}$ around the given curves.

    (a) $\mathbf{F}(x, y) = (-y, yx)$ and $\mathcal{C}$ is the quadrilateral with vertices $(0, 0)$, $(4, 0)$, $(4, 2)$, and $(2, 4)$.
    (b) $\mathbf{F}(x, y) = (x - y, x + y)$ and $\mathcal{C}$ is the curve consisting of the interval from $(-1, 0)$ to $(1, 0)$ on the x-axis and the graph of $y = 1 - x^2$ from -1 to 1.

11. **Green's Theorem Calculations II.** Using Green's theorem, set up and evaluate a line integral to compute the area of each of the following regions. (See Example 6.9B.)

    (a) The region enclosed by the hypocycloid $x^{2/3} + y^{2/3} = 1$.
    (b) The region inside the right half-plane between the circle of radius 2 centered at the origin and the ellipse $y^2/4 + x^2 = 1$.

12. **Total Flux.** For each of the following vector fields $\mathbf{F}$ and regions with boundary, compute the total flux of the vector field across the boundary of the region.

    (a) The region is a triangle with vertices $(0, 0)$, $(3, 0)$, and $(0, 3)$ and $\mathbf{F}(x, y) = (y - 1, x - 1)$.
    (b) The region is bounded by the ellipse $x^2 + 4y^2 = 4$ and $\mathbf{F}(x, y) = (\sin y - x, \cos x + y)$.
    (c) The region is bounded by the graphs of $y = 4 - x^2$ and the lower half of the semicircle of radius 2 centered at the origin and $\mathbf{F}(x, y) = (2x - y, x + 2y)$.

13. **Line Integral of Acceleration.** Let $\alpha : [a, b] \to \mathbb{R}^2$ be a twice differentiable parametrization of a curve in the plane. Show that

$$\int_a^b \alpha''(t) \cdot \alpha'(t)\, dt = \frac{1}{2}\alpha'(b) \cdot \alpha'(b) - \frac{1}{2}\alpha'(a) \cdot \alpha'(a).$$

14. **Electrostatic Equilibrium.** If a solid conductor in electrostatic equilibrium carries a net charge, then the charge resides on the surface of the conductor and the electrostatic field is zero inside the conductor. On the surface of the conductor, the field is perpendicular to the surface of the conductor.

    (a) Show that every point on the surface of the conductor has the same potential. (*Hint:* If the potential energy is not constant on the surface, what can you say about the direction of the electric field?)
    (b) Show that the potential energy is constant inside the conductor. (*Hint:* If not, show the field is nonzero inside the conductor.)

15. **Central Force Fields.** A ***central force field*** is a field whose magnitude depends only on the distance between two particles and that acts in a direction along the line joining

the particles. Thus if $\mathbf{F}$ is a central force field and one particle is located at the origin and the other at $\mathbf{x}$, then the force on the particle at $\mathbf{x}$ is of the form

$$\mathbf{F}(\mathbf{x}) = \frac{g(\|\mathbf{x}\|)}{\|\mathbf{x}\|}\mathbf{x}$$

for some $g : \mathbb{R} \to \mathbb{R}$, where we assume that g is a continuous function. Let α be a parametrization of a curve from a point P to a point Q. Show that $\mathbf{F}$ is conservative by showing that it is a gradient vector field. (*Hint:* See Exercise 10 of Section 3.6.)

16. Diffusion in Equilibrium. In this exercise, we explore the diffusive flux of a diffusion process in equilibrium that follows Fick's law. Suppose that a solute is being introduced into a solvent at a constant rate at the origin in the plane so that on a small circle of radius r_1 centered at the origin a concentration Q_1 is maintained and at a distance r_2 from the origin a concentration of Q_2 is maintained. Let $\mathcal{R}$ denote the region that lies between the circle $\mathcal{C}_1$ of radius r_1 and the circle $\mathcal{C}_2$ of radius r_2 that are centered at the origin.

(a) The function $c = c(x, y)$ given by

$$c(x, y) = Q_1 + \frac{Q_2 - Q_1}{\ln(r_2/r_1)}\ln(\|\mathbf{x}\|/r_1)$$

models this equilibrium diffusion process. Show that $c(x, y) = Q_1$ if $\|\mathbf{x}\| = r_1$ and $c(x, y) = Q_2$ if $\|\mathbf{x}\| = r_2$.

(b) Compute the flux vector $\mathbf{J} = -\delta\nabla c$ of the diffusion process.

(c) Compute the total flux of $\mathbf{J}$ out of $\mathcal{R}$ by computing either the appropriate line integral or the appropriate double integral.

(d) What does your answer to (c) tell you about the flux *into* $\mathcal{R}$ across $\mathcal{C}_1$ and *out* of $\mathcal{R}$ across $\mathcal{C}_2$? Does this make sense for an equilibrium diffusion process? Explain.

17. Diffusion and Growth. A population that grows exponentially and spreads in the plane according to Fick's Law can be modeled by population density function

$$p(x, y, t) = \frac{M}{4\pi\delta t}e^{\alpha t - (x^2+y^2)/(4\delta t)},$$

where M is the initial population of the species, which is concentrated at the origin, δ is the diffusivity of the population, and α is the growth rate of the population.

(a) What is the relationship between the function p and the function c of Exercise 13 of Section 6.3?

(b) Compute the flux vector $\mathbf{J}(x, y, t) = -\delta\nabla p(x, y, t)$ of the diffusion process. (*Hint:* Why is part (a) relevant?)

(c) Compute the outward flux of **J** through a circle of radius 1 centered at the origin by computing either the appropriate line integral or double integral. (*Hint:* Think about part (a).)
(d) Setting $M = 1$, $\delta = 1$, and $\alpha = 1$, describe the behavior of the total flux as a function of time t. What does this tell us about the diffusion process as a function of time? Explain.
(e) How does your answer to (d) compare to your answer to part (c) of Exercise 13 of Section 6.3? In particular, explain the effect on the total flux of including the exponential growth of the population.