Chapter

2

Parametric Curves and Vector Fields

Having set the stage in the first chapter, we are now ready to begin the study of calculus. In Section 2.1, we introduce functions, called parametrizations, that are defined on subsets of $\mathbb{R}$ and take values in $\mathbb{R}^2$ or $\mathbb{R}^3$. The images of these functions are curves in the plane or in space, and their coordinates are themselves real-valued functions. If the coordinate functions are quantities associated with physical or biological systems, the function gives rise to a mathematical model of the system. In Section 2.2, we use the intuitive idea of velocity to motivate the definition of the derivative of a parametrization. We also show how we can use techniques from one-variable calculus to calculate derivatives. In Section 2.4, we use an intuitive understanding of the motion of a fluid flow to introduce vector fields. Vector fields are functions whose domain is a subset of the plane or space and whose image can be thought of as a set of vectors in the plane or vectors in space. We also introduce the concepts of a flow line of a vector field and of a critical point of a vector field, which connect the study of vector fields to parametrizations and their derivatives.

In Sections 2.3 and 2.5, we apply the concept of a parametrization to explore models from epidemiology, physics, physiology, and ecology. These are distinguished by the fact that they model phenomena that can be described by two or three quantities that vary in time. Our goal here is to use mathematics to understand the qualitative and quantitative behavior of these systems as they change in time. In each case, we provide sufficient background information to construct the model from the physical characteristics of the system. As is often the case when developing mathematical models, the descriptions of the systems lead to conditions on the rates of change of the functions, that is, on their derivatives. We will rely on a computer algebra system to generate symbolic, numerical, or graphical solutions to the equations we construct.

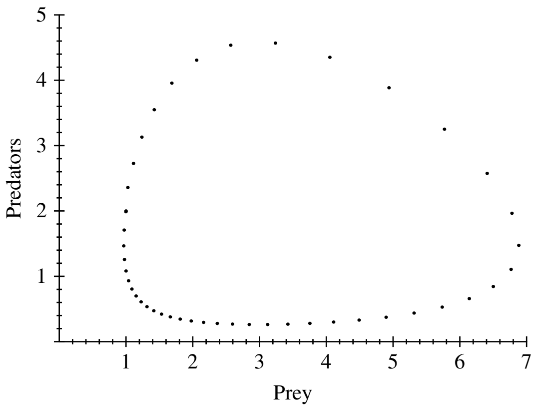

Figure 2.1 Changing populations of a predator–prey system. Consecutive points are separated by the same fixed time interval. (See Question 1.)

A Collaborative Exercise—Time-Dependent Biological Systems

Time-dependent plots of the parameters of interest are fundamental tools in the mathematical study of biological systems. Unfortunately, the data required to produce such plots are hard to come by. For example, in population biology, there is no way to know the size of two interacting wild populations at every instant in time. In human biology, specifically immunology, cell and pathogen counts in blood or tissue can only be obtained by invasive procedures, dramatically restricting one's ability to obtain data. Consequently, abstract or idealized models of biological systems can be important surrogates for working with real data. The following examples illustrate this.

A Predator–Prey System

The predator–prey model of Lotka and Volterra was one of the first theoretical models in population biology.[1] Here we focus on the output of their model. We suppose we have two interacting species that interact as predator and prey. Although much simplified—for example, the predator population only preys on this prey species, the prey is only preyed upon by this predator species, and the prey population's food supply is not taken into account—the model does shed some light on the population dynamics.

Figure 2.1 contains a simulation of a predator–prey system, where each dot represents the populations at an instant in time. The first coordinate represents the number of prey (in thousands), and the second represents the number of predators in (in tens). The dots should be read counterclockwise, and the time interval between consecutive dots

[1] Alfred J. Lotka, an American mathematician and physicist, and Vito Volterra, an Italian mathematician and physicist, developed this model independently in the 1920s.

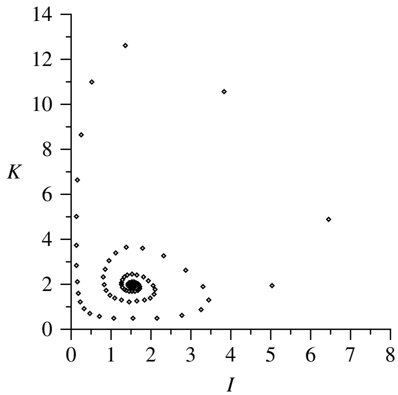

Figure 2.2 Changing cell counts for killer T-cell-infected cell system in units of 100 cells per ml. Consecutive points are separated by the same fixed time interval. The initial value is $(5, 2)$. (See Question 2.)

is assumed to be the same, but will not be specified. Here are several questions about this plot.

1. a. Where in the plot do the prey and predator populations reach their maxima and minima?
   b. Where in the plot are the prey and predator populations changing most rapidly? Least rapidly? Explain your answer.
   c. Describe the long-term simultaneous behavior of the two populations. (*Hint:* Use the language of increasing, decreasing, maximum, and minimum.)

A Killer T-Cell-Virus System

The human immune system is enormously complicated, but it is possible to construct reasonable models of particular features. An important example is that of the adaptive immune response of cytotoxic or killer T-cells to an antigen, for example, a virus. Briefly, killer T-cells are formed in bone marrow and migrate to the thymus (the T is for thymus), where they differentiate in order to react to a specific antigen. When a person is infected by a virus, through an intricate chain of cellular events, antigen-specific killer T-cells are signaled to replicate and to recognize and kill infected cells. If the immune response is successful, the virus will be eliminated. If the immune response is unsuccessful, the infected cell population will grow. A third possibility is that of chronic infection, where the virus is not eliminated, but the growth of the infected cell population is limited by the immune response.

Figure 2.2 contains a simulation of a killer T-cell-infected cell system.[2] The first coordinate of a point represents the number of infected cells per milliliter in hundreds, and

[2]See Section 2.2 of Dominik Wodarz, *Killer Cell Dynamics*, Springer, 2007.

the second represents the number of killer T-cells per milliliter in hundreds. As above, the dots should be read counterclockwise, and the time interval between consecutive dots is assumed to be the same and is not specified. The initial dot in the sequence is located at $(5, 2)$.

Here are several questions about this plot.

2. a. Where in the plot do the infected cell and killer T-cell counts reach their maxima and minima?
   b. Where in the plot are the infected cell and killer T-cell counts changing most rapidly? Least rapidly? Explain your answer.
   c. Describe the long-term simultaneous behavior of the two cell counts. (*Hint:* Use the language of increasing, decreasing, maximum, and minimum.)
   d. Would you characterize the immune response as successful, unsuccessful, or resulting in chronic infection? Explain your answer.

■ 2.1 Parametric Representations of Curves

In this section, we will introduce a way to use the language of functions to represent curves in the plane or in space. At first, we will think of curves that are traced by an object in motion. In this example, we will consider the motion of a satellite across the surface of the earth.

Example 2.1

Satellite Tracks. Figure 2.3 shows the path of the Hubble Space Telescope and MIR Space Station on a flat map of the earth. As time elapses, we see, for example, that the space telescope traces a curve that oscillates roughly between the latitude of the southern edge of Brazil and the latitude of the northern coast of Africa. At each time, we can specify the position of the space telescope by giving its coordinates in latitude and longitude. As time changes, the coordinates change, so that we can think of each coordinate as being a function of time. We will represent the position of the space telescope by an ordered pair of functions of time: the first, or x-coordinate, giving the longitude and the second, or y-coordinate, giving the latitude.

Now let us express the ideas of the example in symbolic terms. If an object moves through the plane with xy-coordinates over an interval of time $a \leq t \leq b$, we will describe the position of the object at time t by an ***ordered pair*** $(x(t), y(t))$, whose coordinates are functions of time. This defines a function α whose domain is the interval $[a, b]$ and which takes values in $\mathbb{R}^2$. We write $\alpha : [a, b] \to \mathbb{R}^2$, where $\alpha(t) = (x(t), y(t))$.

If an object moves through space with xyz-coordinates over a time interval $[a, b]$, we will describe the position of the object at time t by an ***ordered triple***, $(x(t), y(t), z(t))$, whose coordinates are functions of time. We will then write $\alpha : [a, b] \to \mathbb{R}^3$, where $\alpha(t) = (x(t), y(t), z(t))$.

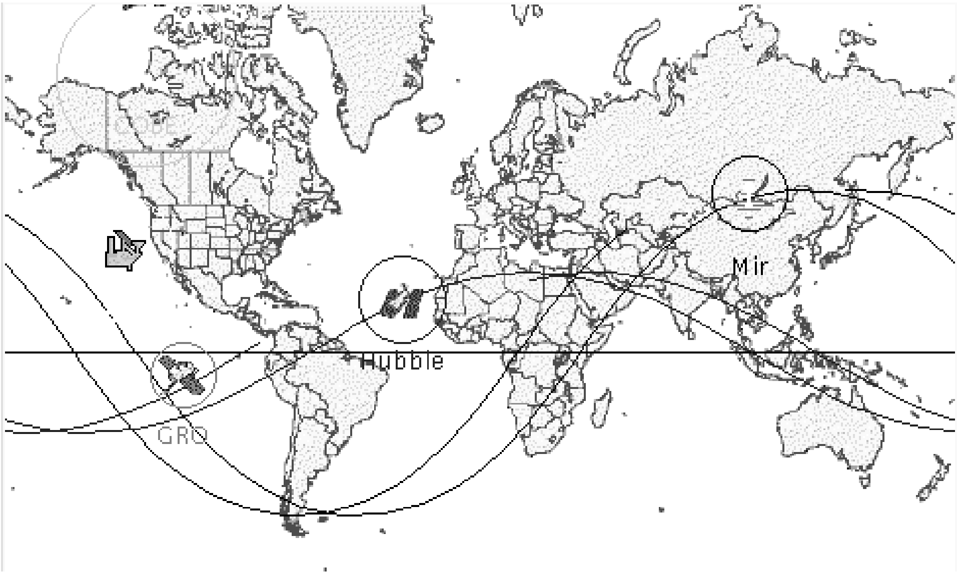

Figure 2.3 The track of the orbits of the Hubble Space Telescope and the MIR Space Station on a map of the earth. Image courtesy of National Aeronautics and Space Administration. (See Example 2.1.)

Example 2.2

Examples from Biology. As we saw in the collaborative exercise at the beginning of this chapter, it is also possible use time-dependent functions to model physical systems that are not related to motion. In population biology or ecology, the number of organisms or animals in a population is considered a function of time. Identifying a population with the coordinate of time-dependent function, we are able to study the interactions of two or more populations. Over time, the sizes of the population vary, tracing a curve in Euclidean space. Since populations are nonnegative, the curve will lie in the first quadrant (or octant). Similarly, in cell biology and immunology, cell or antigen concentrations per unit volume can be considered functions of time and can be identified with the coordinates of a time-dependent function that takes values in the first quadrant (or octant).

The following definition introduces the terminology we will use when referring to functions of time with values in $\mathbb{R}^2$ or $\mathbb{R}^3$.

Definition 2.1 A time-dependent function $\alpha : [a, b] \to \mathbb{R}^2$ given by $\alpha(t) = (x(t), y(t))$, or $\alpha : [a, b] \to \mathbb{R}^3$ given by $\alpha(t) = (x(t), y(t), z(t))$, that traces a portion of a curve in the plane or in space is called a ***parametrization*** or ***parametric representation*** of the curve.

The independent variable t is called the ***parameter*** of α. The ***image*** of α, which we denote by $\mathrm{Im}(\alpha)$, is the portion of the curve that is traced by α. In set notation,

$$\mathrm{Im}(\alpha) = \{(x, y) : (x, y) = (x(t), y(t)),\ t \in [a, b]\,\}$$

for curves in the plane. For curves in space,

$$\mathrm{Im}(\alpha) = \{(x, y, z) : (x, y, z) = (x(t), y(t), z(t)),\ t \in [a, b]\,\}.$$

The point $\alpha(a)$ is called the ***initial point*** of α and the point $\alpha(b)$ is called the ***final point*** or ***endpoint*** of α. ♦

We will assume throughout the text that our parametrizations are continuous in the sense that their coordinate functions are continuous. We will represent parametrizations of curves by the lowercase Greek letters alpha, beta, and gamma, which are written α, β, and γ, respectively.

It will be useful for us to have available a list of parametrizations of familiar and frequently used curves in the plane and in space. In the following examples, we will consider parametrizations of lines and line segments, circles and ellipses, and graphs of functions. Our first example, parametrizing lines and line segments, makes use of the language of vectors. Here we will work in the plane, so that a parametrization α has two coordinates. However, the construction also can be used in space, where the parametrization will have three coordinates.

Example 2.3 Linear Parametrizations

A. Lines. The parametric description of a line introduced in Section 1.4 with parameter t is a parametrization in the sense of Definition 2.1. Let $P = (1, -1)$ and $Q = (2, 3)$. Following Section 1.4, the line L through P and Q is the set of points corresponding to vectors of the form $\mathbf{p} + t\mathbf{v}$, where $\mathbf{p}$ is the vector corresponding to P, $\mathbf{q}$ is the vector corresponding to Q, $\mathbf{v} = \mathbf{q} - \mathbf{p}$, and $t \in \mathbb{R}$. Expanding $\mathbf{p} + t\mathbf{v}$ in coordinates, we obtain a parametrization for L,

$$\alpha(t) = \mathbf{p} + t\mathbf{v} = (1, -1) + t\,((2, 3) - (1, -1)) = (1 + t, -1 + 4t).$$

More generally, if $P = (x_0, y_0)$ and $Q = (x_1, y_1)$, the line through P and Q can be parametrized by

$$\alpha(t) = \mathbf{p} + t\mathbf{v} = (x_0, y_0) + t((x_1, y_1) - (x_0, y_0)) = (x_0 + t(x_1 - x_0),\ y_0 + t(y_1 - y_0)).$$

B. Line Segments. If we restrict the domain of the parametrization α of part A to the time interval $[0, 1]$, we obtain a parametrization of the line segment extending from P to Q. That is, $\mathrm{Im}(\alpha)$ is the line segment from P to Q. The initial point is $\alpha(0) = P$ and the endpoint is $\alpha(1) = Q$.

More generally, restricting α to the interval $[a, b]$ parametrizes the line segment in L extending from $\mathbf{p} + a\mathbf{v}$ to $\mathbf{p} + b\mathbf{v}$.

In our next example, we will make use of the trigonometric identity, $\cos^2 t + \sin^2 t = 1$ to parametrize circles and ellipses.

Example 2.4 **Trigonometric Parametrizations**

A. Circles. The circle of radius r_0 centered at the origin satisfies the polynomial equation $x^2 + y^2 = r_0^2$. We can parametrize this circle by

$$\alpha(t) = (r_0 \cos t, r_0 \sin t).$$

Notice that if we substitute the coordinates of α into the equation for the circle, the equation is satisfied,

$$(r_0 \cos t)^2 + (r_0 \sin t)^2 = r_0^2(\cos^2 t + \sin^2 t) = r_0^2.$$

The parametrization α traces the circle in a counterclockwise manner. If we restrict the domain to the interval $[0, 2\pi]$, then α traces the circle once and the initial point and final point are both equal to the point $(r_0, 0)$. (See Figure 2.4(a).) By choosing different intervals for the domain of α, we can parametrize different portions of the circle or we can trace it more than once.

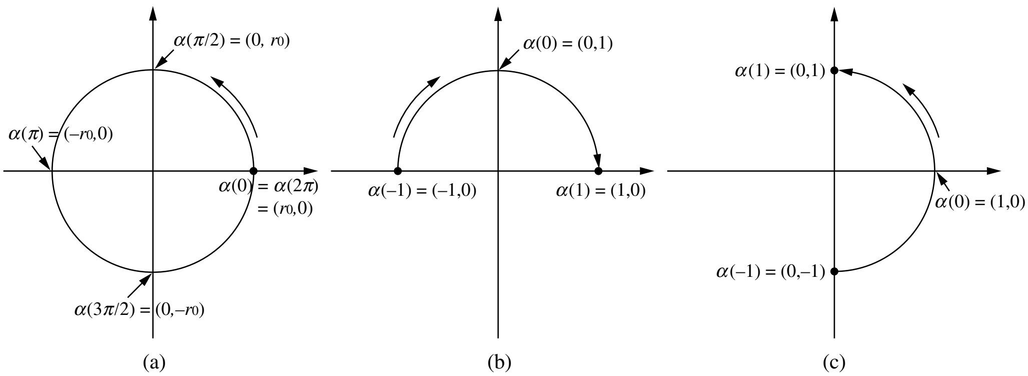

Figure 2.4 (a) The image of the function of Example 2.4A. Notice that the initial point and endpoint of α are $(r_0, 0)$ and the circle is traced counterclockwise in 2π units of time. (b) The image of the function of Example 2.5A. Notice that the initial point α is $(-1, 0)$ and the endpoint is $(1, 0)$. The semicircle is traced clockwise in 2 units of time. (c) The image of the function of Example 2.5B. Notice that the initial point α is $(0, -1)$ and the endpoint is $(0, 1)$. The semicircle is traced counterclockwise in 2 units of time.

B. Ellipses. The polynomial equation for an ellipse centered at the origin with axes of length $2a$ lying along the x-axis and length $2b$ lying along the y-axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

By modifying the parametrization α of the circle of part A, we can produce a parametrization of this ellipse. Define α by

$$\alpha(t) = (a\cos t, b\sin t).$$

A calculation, which we leave to the reader, shows that the coordinates of α satisfy the equation of the ellipse. (See Exercise 4.) As above, if we restrict the domain to the interval $[0, 2\pi]$, the ellipse is traced once in a counterclockwise manner with initial point and final point equal to $(a, 0)$.

Unlike the previous examples, which used particular functions for the coordinates of the parametrization, the following construction applies to the graph of any function.

Example 2.5 Graphs of Functions of One Variable

A. Graph of a Function $y = f(x)$. The graph of a function $y = f(x)$ of one variable is a curve in the plane. It can be parametrized by

$$\alpha(t) = (t, f(t)),$$

where t is restricted to the domain of f. Notice that this uses the x-coordinate as the parameter for α. For example, the top half of the unit circle centered at the origin is the graph of the function $f(x) = \sqrt{1 - x^2}$, which is defined on the interval $[-1, 1]$. Thus we can parametrize the top half of the unit circle by

$$\alpha(t) = (t, \sqrt{1 - t^2}) \text{ for } t \in [-1, 1].$$

This parametrizes the semicircle in a clockwise direction with initial point $(-1, 0)$ and final point $(1, 0)$. (See Figure 2.4(b).)

B. Graph of a Function $x = g(y)$. In a similar manner, we can parametrize the graph of a function $x = g(y)$ by

$$\alpha(t) = (g(t), t),$$

where t is restricted to the domain of g. This uses the y-coordinate as the parameter for α. For example, the portion of the unit circle centered at the origin lying in the first and fourth quadrants is the graph of $x = g(y) = \sqrt{1-y^2}$. It can be parametrized by

$$\alpha(t) = (\sqrt{1-t^2}, t) \text{ for } t \in [-1, 1].$$

This parametrizes this semicircle in a counterclockwise direction with initial point $(0, -1)$ and final point $(0, 1)$. (See Figure 2.4(c).)

As the last two examples show, there are different ways to parametrize the same curve. In fact, if we restrict the parametrization of Example 2.4A (with $r_0 = 1$) to the interval $[0, \pi/2]$, the parametrization of Example 2.5A to the interval $[0, 1]$, and the parametrization of Example 2.5B also to the interval $[0, 1]$, we have three different parametrizations of the quarter circle centered at the origin in the first quadrant.

In the following example, we will employ a parametrization of an ellipse to model the motion of a simple pendulum. In this case, the coordinates in the plane will represent the position and velocity of the pendulum. Keep in mind that the coordinate plane we use for the model is not the physical plane in which the pendulum is swinging.

Example 2.6

The Simple Pendulum. A simple pendulum is an idealized version of a real pendulum and is one of the first mechanical systems studied in physics. It consists of a mass suspended from a rod that is free to swing in a vertical plane under the influence of gravity. Physically, it is assumed that the mass is concentrated at a point, that the rod has no mass, and that the rod is free to swing without friction. This is why we say it is an "idealized" version of a pendulum rather than a real pendulum. Swinging back and forth, the pendulum traces an arc of a circle, and its position can be specified by giving the displacement of the pendulum mass from the vertical as a distance x along its arc. See Figure 2.5(a). The distance x is related to the angle θ by $x = l\theta$, where θ is measured in radians and l is the length of the pendulum rod. If the pendulum is displaced a distance x_0 from the vertical along its arc and released with no initial velocity, its position is given by $x(t) = x_0 \sin(\sqrt{\frac{9.8}{l}}t + \pi/2)$. (The coefficient of t ensures that Newton's second law is satisfied.) The velocity v is the derivative of x, so we have $v(t) = x_0\sqrt{\frac{9.8}{l}} \cos(\sqrt{\frac{9.8}{l}}t + \pi/2)$. A calculation shows that

$$\frac{x(t)^2}{x_0^2} + \frac{v(t)^2}{x_0^2(9.8/l)} = 1,$$

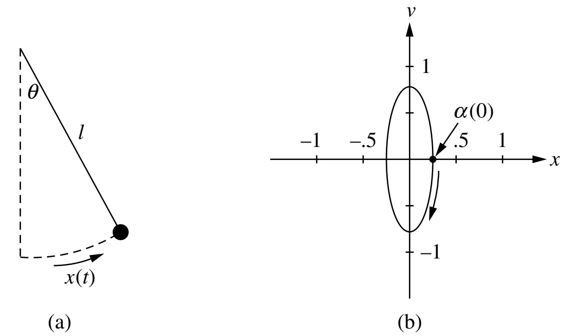

Figure 2.5 (a) A simple pendulum. The displacement of the pendulum is measured along the arc of its motion from the rest (vertical) position. Positive displacement is measured counterclockwise from the vertical, and negative displacement is measured clockwise from the vertical. (b) The ellipse in the xv-plane for a pendulum with length $l = 1$ and initial displacement $x_0 = 0.25$. It is the image of the parametric representation α for the motion of the pendulum in Example 2.29.

for all values of t. This is the equation of an ellipse in the xv-plane with axes of length $2x_0$ and $2x_0\sqrt{9.8/l}$. Figure 2.5(b) is a plot of the ellipse for $l = 1$ and $x_0 = 0.25$. The parametrization α of this ellipse is

$$\alpha(t) = (0.25\sin(\sqrt{9.8}\,t + \pi/2),\ 0.25\sqrt{9.8}\cos(\sqrt{9.8}\,t + \pi/2)).$$

The first coordinate of α gives the displacement of the pendulum from the vertical, and the second coordinate is the velocity of the pendulum mass along its arc. As t increases from 0, we see that $\sin(\sqrt{9.8}\,t + \pi/2)$ oscillates from 1 to -1: thus the position of the pendulum oscillates from 0.25 to -0.25. At the same time, the velocity of the pendulum changes from 0 to $-0.25\sqrt{9.8}$ to 0 to $0.25\sqrt{9.8}$ to 0, reflecting the motion of the pendulum to the left and then the right with changing speeds. (This example is explored further in the exercises.)

The formula that we used in Example 2.6 for the displacement of the pendulum is a mathematical model for the motion of a pendulum. This model is more accurate for small oscillations of the pendulum, and it becomes less accurate as the size of the oscillations increase. Nevertheless, it provides a good qualitative tool for understanding the motion of a pendulum.

The Algebra of Parametrizations

We will devote the remainder of this section to the algebra of parametrizations. For the most part, we will be concerned with manipulating a single parametrization, either by changing the parameter or by moving the image of the parametrization. First, we will look at two ways to alter the parameter.

Example 2.7 Transformations of the Parameter

A. Shifting. Consider the parametrization $\alpha(t) = (\cos(t), \sin(t))$, $0 \le t \le \pi$, of the top half of the unit circle centered at the origin. The initial point is $(1, 0)$ and the final point is $(-1, 0)$. If we define a new parametrization by

$$\beta(t) = \alpha(t + \pi/2) = (\cos(t + \pi/2), \sin(t + \pi/2)) \text{ for } 0 \le t \le \pi,$$

the initial point of β is $(0, 1)$ and the final point of β is $(0, -1)$. The image of β is the portion of the unit circle lying in the second and third quadrants. That is, adding $\pi/2$ to the parameter of α has the effect of shifting the image of α by a distance that is traversed in $\pi/2$ units of time.

More generally, if α is a parametrization of a curve, ***shifting*** the parameter of a parametrization by c to produce a parametrization $\beta(t) = \alpha(t + c)$ has the effect of shifting the image of α by a distance that is traversed in c units of time. If $c > 0$, this shift is forward along the curve: if $c < 0$, this shift is backward along the curve. Alternatively, we can think of β as a parametrization of the same curve over a different time interval. That is, if α traces a portion of a curve over the time interval $[a, b]$, then β traces the same portion over the time interval $[a - c, b - c]$.

B. Scaling. Starting with the same parametrization α, consider a parametrization β defined by

$$\beta(t) = \alpha(t/2) = (\cos(t/2), \sin(t/2)) \text{ for } 0 \le t \le \pi.$$

The initial point of β is $(1, 0)$ and the endpoint of β is $(0, 1)$. The image of β is the quarter-circle lying in the first quadrant. Multiplying the parameter by $1/2$ has the effect of slowing down the parametrization by a factor of 2 so that over the same time period β traces out half the image of α. Similarly, if we had multiplied the parameter by 2, the parametrization would speed up by a factor of 2. Over the same time interval, we would trace the entire circle.

More generally, if α is a parametrization of a curve defined on an interval $[0, b]$, ***scaling*** the parameter by a factor of c to produce a parametrization $\beta(t) = \alpha(ct)$ has the effect of stretching the image of the parametrization along the curve by a time factor of c.

It is worth noting that the parametrization of the ellipse that arose in the model for a simple pendulum in Example 2.6 is obtained from a parametrization of the form of Example 2.4B by shifting and scaling the parameter. (See Exercise 13.)

Both of the transformations we introduced in Example 2.7 are examples of the composition of functions. In each case, we composed a parametrization α with a function

$g = g(t)$ of one variable to produce a function $\beta(t) = \alpha(g(t))$, which is the ***composition*** of α with g. In coordinates, if $\alpha(t) = (x(t), y(t))$, then in coordinates, $\beta(t) = (x(g(t)), y(g(t)))$. We will denote the composition of α with g by $\alpha \circ g$. In Example 2.7A, $g(t) = t + c$, and in Example 2.7B, $g(t) = ct$.

It is also possible to move or translate the image of a parametrization by adding a constant vector. Here we will work with a parametrization of the unit circle centered at the origin.

Example 2.8

Transforming the Image: Translation. Let $\alpha(t) = (\cos t, \sin t)$, $0 \leq t \leq 2\pi$. Let $\mathbf{p} = (2, 1)$ and define a new parametrization β by

$$\beta(t) = \alpha(t) + \mathbf{p} = (\cos t, \sin t) + (2, 1) = (\cos t + 2, \sin t + 1)$$

for $0 \leq t \leq 2\pi$. A calculation, which we leave to the reader, shows that the coordinates of β satisfy the equation

$$(x - 2)^2 + (y - 1)^2 = 1,$$

which is the equation for a circle of radius 1 centered at the point $(2, 1)$. That is, adding $(2, 1)$ to α has the effect of moving or translating the image of α by $(2, 1)$.

In general, if β is related to α by $\beta(t) = \alpha(t) + \mathbf{p}$ for a vector $\mathbf{p}$, then the image of β is the ***translation*** of the image α by the vector $\mathbf{p}$.

Let us put the ideas of shifting, scaling, and translation together to construct a parametrization of a person walking around a running track at a constant speed. While the verbal description is simple, translating it into mathematics takes some effort.

Example 2.9

A Compound Motion. We would like to parametrize the motion of a person who walks with constant speed counterclockwise around a 400 m track in four minutes. The straightaways of the track are 100 m in length, and the ends of the track are semicircles 100 m in length. Thus the ends are semicircles of radius $100/\pi$ m. Choose the origin to be in the middle of the track and orient the track so that the sides of the track are parallel to the x-axis as in Figure 2.6. The track consists of four curves of equal length: the segments $\mathcal{C}_1$ and $\mathcal{C}_3$ and the semicircles $\mathcal{C}_2$ and $\mathcal{C}_4$. If the path is traveled at constant speed, it takes one minute to travel each portion of the track. Suppose that the motion begins at the right-hand endpoint of $\mathcal{C}_1$, $(50, 100/\pi)$ in our chosen coordinate system. We will give the details of the construction of the parametrization of $\mathcal{C}_1$ and $\mathcal{C}_2$ and leave the constructions for $\mathcal{C}_3$ and $\mathcal{C}_4$ for the exercises. (See Exercise 17.)

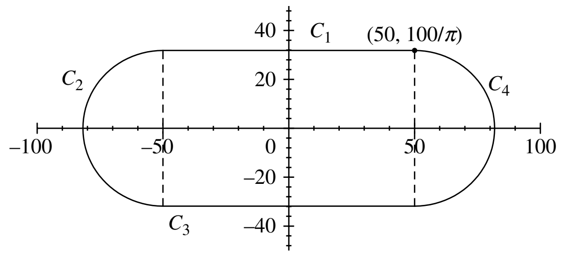

Figure 2.6 The image of the parametrization of Example 2.9.

$\mathcal{C}_1$: Since $\mathcal{C}_1$ is the line segment with endpoints $(50, 100/\pi)$ and $(-50, 100/\pi)$, we can apply the formula of Example 2.3B to produce a parametrization. Denoting this parametrization by α_1, we have that

$$\alpha_1(t) = (50, 100/\pi) + t(-100, 0) = (50 - 100t, 100/\pi)$$

for $t \in [0, 1]$. Notice that $\alpha_1(0) = (50, 100/\pi)$ and $\alpha_1(1) = (-50, 100/\pi)$; thus the first segment is traveled in the first minute.

$\mathcal{C}_2$: The curve $\mathcal{C}_2$ is the left-hand semicircle of a circle of radius $100/\pi$ centered at $(-50, 0)$. We will build the parametrization of $\mathcal{C}_2$ in several steps. Eventually, the domain of the parametrization must be $[1, 2]$ to reflect that this part of the track is walked during the second minute. To parametrize $\mathcal{C}_2$, we will first parametrize a semicircle centered at the origin, scale and shift the parameter to get the correct time interval, and then translate the parametrization to the left end of the track.

Using the formula from Example 2.4A, we start with the parametrization of the semicircle of radius $100/\pi$ in the third and fourth quadrants centered at the origin

$$((100/\pi)\cos t, (100/\pi)\sin t) \text{ for } \pi/2 \leq t \leq 3\pi/2.$$

Next, scale the parameter by π so that the time interval has length 1 rather than π,

$$((100/\pi)\cos(\pi t), (100/\pi)\sin(\pi t)) \text{ for } 1/2 \leq t \leq 3/2.$$

Then shift the time parameter by $-1/2$ so that the interval is $[1, 2]$,

$$((100/\pi)\cos(\pi(t - 1/2)), (100/\pi)\sin(\pi(t - 1/2))) \text{ for } 1 \leq t \leq 2.$$

Finally, to obtain the expression for α_2, translate the image so that it parametrizes the left half of a circle centered at $(-50, 0)$,

$$\alpha_2(t) = ((100/\pi)\cos(\pi(t - 1/2)) - 50, (100/\pi)\sin(\pi(t - 1/2))) \text{ for } 1 \leq t \leq 2.$$

This is the desired parametrization of $\mathcal{C}_2$.

$\mathcal{C}_3$: The line segment $\mathcal{C}_3$ can be parametrized the way we parametrized $\mathcal{C}_1$. However, we will have to shift the parameter so that the time interval is $[2, 3]$. Carrying out these calculations, we get

$$\alpha_3(t) = (-50, -100/\pi) + (t-2)(100, 0) = (-50 + (t-2)100, -100/\pi) \text{ for } 2 \le t \le 3.$$

$\mathcal{C}_4$: As we did for $\mathcal{C}_2$, we can parametrize a circle of radius $100/\pi$ centered at the origin, scale the parameter, shift the parameter, and translate the image. These steps yield the parametrization

$$\alpha_4(t) = ((100/\pi)\cos(\pi(t - 7/2)) + 50, (100/\pi)\sin(\pi(t - 7/2))), \ t \in [3, 4].$$

Putting these four pieces together, we can parametrize the walk around the track by $\alpha : [0, 4] \to \mathbb{R}^2$, where

$$\alpha(t) = \begin{cases} (50 - 100t, 100/\pi) & t \in [0, 1], \\ ((100/\pi)\cos(\pi(t - 1/2)) - 50, (100/\pi)\sin(\pi(t - 1/2))) & t \in [1, 2], \\ (-50 + (t-2)100, -100/\pi) & t \in [2, 3], \\ ((100/\pi)\cos(\pi(t - 7/2)) + 50, (100/\pi)\sin(\pi(t - 7/2))) & t \in [3, 4]. \end{cases}$$

Summary In this section, we introduced the language of ***parametrizations*** to represent the motion of an object along a curve in the plane or in space and to model the evolution of a physical system over time. In examples, we developed explicit parametrizations of ***lines***, ***line segments***, ***circles***, and ***ellipses***. We studied the effects of ***shifting*** and ***scaling*** the ***parameter*** of the parametrization and of ***translating*** the ***image*** of the parametrization. We also introduced a model for the motion of a ***simple pendulum***.

Section 2.1 Exercises

1. **Line Segments.** For each of the following pairs of points and intervals, give a parametrization of the line segment between P and Q whose domain is the given interval $[a, b]$. (*Hint:* Begin with the standard parametrization of a line segment given in Example 2.3, and then shift or scale the parameter as necessary.)

   (a) $P = (1, 0)$, $Q = (0, 1)$, and $[a, b] = [0, 1]$.
   (b) $P = (1, 0, 1)$, $Q = (0, 1, -1)$, and $[a, b] = [0, 2]$.
   (c) $P = (-2, -2)$, $Q = (1, 6)$, and $[a, b] = [3, 4]$.
   (d) $P = (-2, 2, 1)$, $Q = (3, -1, 2)$, and $[a, b] = [-1, 2]$.

2. **Lines.** Suppose that L is a line in the xy-plane that passes through the point (x_0, y_0) and has slope m.

   (a) Find a direction vector for the L.

(b) Find a parametrization α of L that satisfies $\alpha(0) = (x_0, y_0)$.
(c) Find a parametrization of the line with slope $m = 5/2$ that passes through $(x_0, y_0) = (-1, 0.5)$ at time $t = 0$.

3. Arcs of Circles. Sketch the images of the following parametrizations α. Label the endpoints of α and indicate the direction of α on the sketch.

(a) $\alpha(t) = (\cos t, \sin t)$ for $\pi/4 \le t \le 5\pi/4$.
(b) $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ for $0 \le t \le 1/2$.
(c) $\alpha(t) = (\cos 3t, \sin 3t)$ for $-\pi/6 \le t \le \pi/6$.
(d) $\alpha(t) = (\cos \pi t, \sin \pi t)$ for $1/2 \le t \le 1$.

4. Ellipses. Consider the parametrization $\alpha(t) = (a \cos t, b \sin t)$, $0 \le t \le 2\pi$, where $a > 0$ and $b > 0$. (See Example 2.4B.)

(a) Show that the coordinates of α satisfy the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(b) Use α to construct a parametrization β of the same ellipse with the property that $\beta(0) = (0, b)$.
(c) Use α to construct a parametrization β of the same ellipse so that the ellipse is traced once in 2 units of time.

5. Arcs of Ellipses. Sketch the images of the following parametrizations α. Label the endpoints of α and indicate the direction of the α on the sketch.

(a) $\alpha(t) = (2\cos t, \sin t)$ for $0 \le t \le 3\pi/4$.
(b) $\alpha(t) = (\cos \pi t, \frac{1}{2} \sin \pi t)$ for $1 \le t \le 2$.
(c) $\alpha(t) = (\frac{1}{3} \cos(2\pi t), \sin(2\pi t))$ for $1/4 \le t \le 3/4$.
(d) $\alpha(t) = (\frac{1}{2} \cos(3t), \frac{2}{3} \sin(3t))$ for $-\pi/2 \le t \le 0$.

6. More Ellipses. In this exercise, we want to introduce some terms concerning ellipses and consider another useful representation of an ellipse. Let us begin with an ellipse defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where a and b are positive constants. If $a > b$, we call the interval from $-a$ to a on the x-axis the ***major axis*** and the interval from $-b$ to b on the y-axis the ***minor axis*** of the ellipse. The ***semimajor axis*** is the interval from the origin to $(a, 0)$ and the ***semiminor axis*** is the interval from the origin to $(0, b)$. The ***eccentricity*** e is defined to be $e = \sqrt{1 - (b^2/a^2)}$. The ***foci*** of the ellipse are located a distance $\sqrt{a^2 - b^2}$ from its center on the major axis. (See Figure 2.7.)

(a) Suppose that the major axis of an ellipse has length 6 and lies along the x-axis and the minor axis has length 4 and lies along the y-axis. Using the above form for the ellipse, locate the foci.

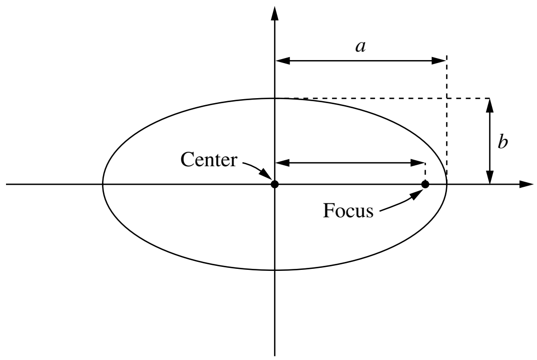

Figure 2.7 An ellipse with semimajor axis of length a and semiminor axis of length b. The center of the ellipse is at the origin, and the foci of the ellipse are located at a distance $\sqrt{a^2 - b^2}$ from the focus along the major axis. The ellipse satisfies the equation $x^2/a^2 + y^2/b^2 = 1$. (See Exercise 6.)

(b) By shifting the x-coordinate, rewrite the above equation to find the equation of an ellipse with a focus located at the origin. (This will make the minor axis parallel to the y-axis instead of on the y-axis.)
(c) Construct a parametrization of the shifted ellipse from part (b).

7. Circles and Ellipses. It is possible to parametrize circles and ellipses using a sine function for the first coordinate and a cosine function for the second coordinate. Sketch the image of each of the following parametrizations α. Label the endpoints of α and indicate the direction of the α on the sketch.

(a) $\alpha(t) = (\sin t, \cos t)$ for $0 \le t \le \pi/2$.
(b) $\alpha(t) = (\sin 2t, \cos 2t)$ for $-\pi/4 \le t \le \pi/2$.
(c) $\alpha(t) = (\sin t, 2\cos t)$ for $0 \le t \le \pi/2$.
(d) $\alpha(t) = (2\sin(t/2), 3\cos(t/2))$ for $-\pi \le t \le \pi$.

8. Parametrizing Arcs of Circles. Find a parametrization α for each of the following arcs of a circle that satisfies the given conditions.

(a) α parametrizes the unit circle centered at the origin counterclockwise with $\alpha(0) = (0, 1)$ and $\alpha(\pi) = (0, -1)$.
(b) α parametrizes the unit circle centered at the origin clockwise with $\alpha(0) = (-1, 0)$ and $\alpha(1) = (1, 0)$.
(c) α parametrizes the circle of radius 2 centered at the origin counterclockwise with $\alpha(0) = (-2, 0)$ and $\alpha(1) = (-2, 0)$.
(d) α parametrizes the circle of radius 3 centered at $(2, 0)$ counterclockwise with $\alpha(0) = (2, 3)$ and $\alpha(2) = (2, -3)$.

9. **Helices.** The curve in space parametrized by $\alpha(t) = (\cos(t), \sin(t), t)$ is called a ***helix***. Notice that the first two coordinates of α are the coordinates of a parametrization of a circle of radius 1 centered at the origin.

(a) Describe in words the image of α for the time interval $[0, 2\pi]$.
(b) For each of the following parametrizations β, describe the image of β.

(i) $\beta(t) = (2\cos t, 2\sin t, t)$ for $0 \le t \le 2\pi$.
(ii) $\beta(t) = \alpha(2t)$ for $0 \le t \le 2\pi$.
(iii) $\beta(t) = (\cos t, t, \sin t)$ for $0 \le t \le 2\pi$.

10. **Reversing the Direction of a Parametrization**

(a) The parametrization $\alpha(t) = (t, t^3)$, $1 \le t \le 3$, parametrizes the portion of the graph of $y = x^3$ from $(1, 1)$ to $(3, 27)$. By using shifting and/or scaling of the parameter of α, produce a parametrization β of the same curve with the same domain, but with $\beta(1) = (3, 27)$ and $\beta(3) = (1, 1)$.
(b) Suppose that $\alpha(t)$, $a \le t \le b$, parametrizes motion along a curve from $P = \alpha(a)$ to $Q = \alpha(b)$. Use α to construct a parametrization β of motion along the same curve in the opposite direction, that is, with $\beta(a) = Q$ and $\beta(b) = P$. Explain your answer.

11. **Graphs of Functions.** For each of the following parametrizations of curves in the plane, find an expression for the curve as a portion of the graph of a function $y = f(x)$ or a portion of the graph of a function $x = g(y)$. Be sure to say which form you found.

(a) $\alpha(t) = (e^t, e^t)$, $0 \le t < \infty$.
(b) $\alpha(t) = (e^t, e^{2t})$, $-\infty < t \le 0$.
(c) $\alpha(t) = (e^{-t}, e^t)$, $-\infty < t < \infty$.
(d) $\alpha(t) = (\sin t, \sin^3 t)$, $0 \le t \le \pi/2$.

12. **Interacting Populations.** In the collaborative exercise at the beginning of the chapter, we introduced a plot modeling the populations of two species that interact as predator and prey. Figure 2.8 shows a similar curve. Assume that it takes 3 years for the populations to trace this curve once and let $\alpha(t) = (x(t), y(t))$ denote a parametrization of this curve with initial point $\alpha(0) = (540, 15)$.

(a) Locate the points where the predator population reaches its maximum and minimum values and the points where the prey population reaches its maximum and minimum values. Approximately, what are the coordinates of these points?
(b) Describe what happens to the individual populations $x = x(t)$ and $y = y(t)$ as α traces the curve in Figure 2.8.
(c) Use your descriptions from part (a) to sketch the graphs of x and y as functions of t on the same set of coordinate axes. (Plot t on the horizontal axis and x and y on the vertical axis.)
(d) Why do you think the maximum values for x and y occur at different times?

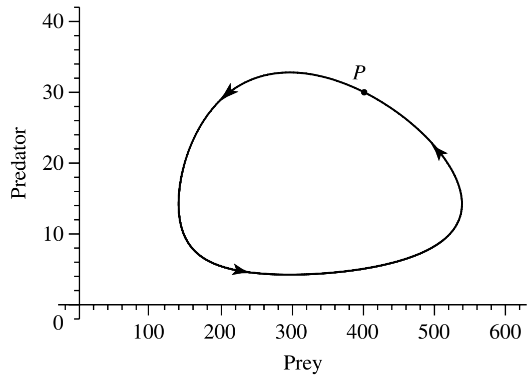

Figure 2.8 A curve whose coordinates are the number of prey and the number of predators in two interacting populations. For example, the point P represents the time when there are 400 prey and 30 predators. The arrows indicate the direction in which the curve is traced over time. (See Exercise 12.)

13. The Simple Pendulum. In Example 2.6, we parametrized the motion of a simple pendulum of length l with initial position and velocity $(x_0, 0)$ by

$$\alpha(t) = (x_0 \sin(\sqrt{\frac{9.8}{l}}t + \pi/2),\ x_0\sqrt{\frac{9.8}{l}}\cos(\sqrt{\frac{9.8}{l}}t + \pi/2)).$$

Use shifting and scaling to obtain α from the parametrization

$$\beta(t) = (x_0 \sin(t),\ x_0\sqrt{\frac{9.8}{l}}\cos(t)).$$

Explain.

14. The Period of a Simple Pendulum. When set in motion, a simple pendulum oscillates back and forth. The time it takes to complete one full oscillation and return to initial position is called the ***period*** of its motion.

(a) What is the period of a pendulum of length l? (Use the formula of Exercise 13.)
(b) What is the effect of changing the length of the pendulum on the motion of the pendulum? Explain.
(c) How does the image of α depend on l?

15. The Velocity of a Simple Pendulum. In Example 2.6, we parametrized the motion of a simple pendulum of length l with initial position and velocity $(x_0, 0)$.

(a) Describe in words the velocity of the pendulum through one period of its motion. In particular, where does it reach its minimum and maximum speeds?

(b) How does the motion of the pendulum depend on x_0? How is this reflected in the parametrization α? In particular, how does the image of α depend on x_0?

16. Motion under the Influence of Gravity. A simplified model of the motion of an object near the surface of the earth takes into account only the force of gravity and neglects other forces, for example, drag due to air resistance. Gravity acts in a downward direction with magnitude $g = 9.8$ m/s^2. Thus the vertical component of the force is $-g = -9.8$ m/s^2. Based on Newton's second law, the formula for the vertical component of the motion of an object is given by

$$y(t) = -\frac{g}{2}t^2 + v_0 t + y_0,$$

where v_0 is the initial vertical velocity of the object and y_0 is the initial vertical position of the object. If an object is launched so that its motion also has a horizontal component, $x(t)$, the motion can be modeled by the function $\alpha(t) = (x(t), y(t))$.

(a) Suppose an object is launched from the ground so that its horizontal velocity is 10 m/s, so that $x(t) = 10t$. How far will the object travel over the ground if its initial vertical velocity is also 10 m/s?

(b) Suppose an object is launched from the ground so that its horizontal velocity is 10 m/s and it travels a total of 100 m before hitting ground. What was its initial vertical velocity?

17. Compound Motion. Use shifting, scaling, and translation to construct the parametrizations of $\mathcal{C}_3$ and $\mathcal{C}_4$ of Example 2.9. Explain your construction.

18. Compound Motion—A Slide. A slide is in the shape of a helix of radius 5 ft and height 50 ft that makes five complete turns as it descends to the ground. To get to the top of the slide, you must climb a ladder that goes from the ground up through the center of the slide to a platform at the top. Suppose that it takes you 2 minutes to climb to the platform at the top of the slide, 15 sec to walk to the edge of the platform, and 30 sec to slide down the slide. Assume that each portion of the trip is made at a (different) constant rate. Construct a parametrization of this motion. (See Exercise 9 for a parametrization of a helix.)

■ 2.2 The Derivative of a Parametrization

If an object is in motion, then at each instant in time, the object will have a speed and a direction of motion. For example, we might imagine a car being driven along a winding mountain road. At each time, we can read the speed of the car off the speedometer and tell the direction of motion by looking straight ahead through the windshield. We would, of course, read the speed as a nonnegative number. On the other hand, it would be convenient to represent the direction by a vector having three coordinates, with the x-coordinate representing the east-west component of the direction, the y-coordinate representing the north-south component of the direction, and the z-coordinate representing the vertical component of the direction, that is, how much the car is ascending or descending. If we scale this vector to have magnitude equal to the speed of the car, then we will have constructed the velocity vector of the car at the instant in time. This leads us to make the following informal definition of velocity.

Definition 2.2 The ***velocity*** of the motion of an object at a particular time is the vector whose direction is the direction of motion of the object at that time and whose magnitude is the speed of the object at that time. ♦

If an object in motion traces a curve in the plane or in space, we might imagine sketching vectors that represent its velocity at points on the curve. At a particular point on the curve, the velocity can be represented by a vector that begins at the point, points tangent to the curve in the direction of motion, and has magnitude equal to the speed of the object. For example, Figure 2.9(a) shows velocity vectors for the motion of an object moving in a counterclockwise direction with speed 1 around a circle of radius 1 centered at the origin.

Our immediate goal is to develop a rigorous definition of velocity, which, among other features, will allow us to compute the velocity vector of an object from a parametrization of its motion. We begin with the notion of the average velocity of an object.

Suppose that the motion of an object is parametrized by α. In the plane, $\alpha(t) = (x(t), y(t))$, and in space, $\alpha(t) = (x(t), y(t), z(t))$. Let t_0 be the time at which we want

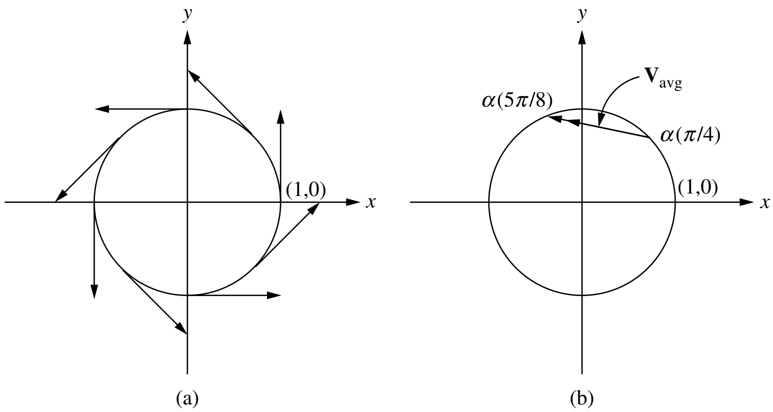

Figure 2.9 (a) Velocity vectors that represent motion around the unit circle in a counterclockwise direction at speed equal to 1 at all times. (b) The average velocity of $\alpha(t) = (\cos t, \sin t)$ at $t = \pi/4$.

to compute the velocity. The average velocity of the object over a time interval $[t_0, t]$ is the displacement of the object multiplied by $1/\Delta t$. Let us state this more formally.

Definition 2.3 If the motion of an object is parametrized by the function α, then ***average velocity***, $\mathbf{v}_{\text{avg}}$, of the object over the time interval $[t_0, t]$ of length $\Delta t = t - t_0$ is the vector

$$\mathbf{v}_{\text{avg}} = \frac{1}{\Delta t}(\alpha(t) - \alpha(t_0)) = \frac{\alpha(t) - \alpha(t_0)}{\Delta t}. \quad \blacklozenge$$

Note that $(\alpha(t) - \alpha(t_0))/\Delta t$ is called a ***difference quotient***. Let us consider an example of this calculation.

Example 2.10 **Average Velocity.** Suppose that the motion of an object is parametrized by $\alpha(t) = (\cos t, \sin t)$, so that the object traces the unit circle centered at the origin. Let us compute the average velocity of the object over the time interval $[\pi/4, 5\pi/8]$. This is

$$\begin{aligned} \mathbf{v}_{\text{avg}} &= \frac{\alpha(5\pi/8) - \alpha(\pi/4)}{5\pi/8 - \pi/4} \\ &= \frac{(\cos(5\pi/8), \sin(5\pi/8)) - (\cos(\pi/4), \sin(\pi/4))}{3\pi/8} \\ &\approx (-0.925, 0.184) \end{aligned}$$

The displacement vector $\alpha(5\pi/8) - \alpha(\pi/4)$ and the average velocity vector are shown on a plot of the image of α in Figure 2.9(b). Notice that since the average velocity is a positive multiple of the displacement vector, it points in the same direction as the displacement vector.

The average velocity at a point can be thought of as an ***approximation*** to the velocity at the point. By keeping t_0 fixed and choosing smaller time intervals, we will obtain better approximations to the velocity. For example, based upon Figure 2.9(b), we would expect that by keeping $\pi/4$ fixed and choosing times closer to $\pi/4$, we would obtain a vector that more accurately reflects the true velocity vector at $t = \pi/4$. To obtain an exact value for the velocity vector, we will take the limit of the average velocity as the length of the time interval $[t_0, t]$ approaches 0, that is, as Δt approaches 0. The length of this vector will be the speed of the object at time t_0. In order to express this limit in terms of t_0 and Δt alone, we will use the fact that $t = t_0 + \Delta t$. Writing this out carefully, we have the following definition.

Definition 2.4 If the motion of an object is parametrized by the function α, then the ***velocity*** of the object at time t_0 is the vector $\mathbf{v}(t_0)$ defined by

$$\mathbf{v}(t_0) = \lim_{\Delta t \to 0} \frac{\alpha(t_0 + \Delta t) - \alpha(t_0)}{\Delta t}.$$

The ***speed*** $s(t_0)$ of the object at time t_0 is $s(t_0) = ||\mathbf{v}(t_0)||$. ♦

Notice that since speed is the length of a vector, it will always be a nonnegative real number. Before we can carry out an explicit calculation of the velocity of an object from a parametrization of its motion, we will need to make the connection between velocity and differentiation. This is the subject of the next subsection.

Differentiable Functions and the Derivative

While our intuition tells us that at each instant in time a moving object must have a velocity, this does not follow automatically from the definition. The difficulty is that before the fact, we do not know if the limit of the average velocities will exist. For example, because the denominator of the expression for the average velocity is approaching zero, the entire expression might grow in magnitude without bound. That is, the limit might not converge to a vector with finite entries. Fortunately, there is a large collection of functions for which this limit exists. We say that these functions are differentiable. Let us put this in the form of a definition.

Definition 2.5 Let $\alpha : [a, b] \to \mathbb{R}^2$ (or $\mathbb{R}^3$) and let $t_0 \in (a, b)$. The function α is said to be ***differentiable*** at t_0 if the following limit exists:

$$\lim_{\Delta t \to 0} \frac{\alpha(t_0 + \Delta t) - \alpha(t_0)}{\Delta t}.$$

If this limit exists, we call it the ***derivative*** of α at t_0, or the ***tangent vector*** at t_0, and we denote it by $\alpha'(t_0)$.

If this limit exists for every t, $a < t < b$, we say that α is ***differentiable*** on (a, b), and we define the ***derivative*** of α on (a, b) to be the function whose value at each point $t \in (a, b)$ is this limit. ♦

If α parametrizes the motion of an object, the ***velocity vector*** of the motion at t_0 is the tangent vector at t_0.

It turns out that the derivative of α can be expressed in terms of the derivative of the coordinate functions of α. This reduces the problem of computing derivatives of parametrizations to a problem in one-variable calculus. In order to see why this is possible, let us apply the limit definition of the derivative to the coordinate form of α. Here we will work in $\mathbb{R}^2$, but the argument is also valid in $\mathbb{R}^3$.

Let us begin with a parametrization α that is differentiable at t_0, so that

$$\alpha'(t_0) = \lim_{\Delta t \to 0} \frac{\alpha(t_0 + \Delta t) - \alpha(t_0)}{\Delta t}$$

exists. This limit should be interpreted as the vector whose entries are the limits of the difference quotients of the coordinates:

$$\left(\lim_{\Delta t \to 0} \frac{x(t_0 + \Delta t) - x(t_0)}{\Delta t}, \lim_{\Delta t \to 0} \frac{y(t_0 + \Delta t) - y(t_0)}{\Delta t} \right).$$

The first entry of this vector is the derivative of the coordinate function x with respect to t at t_0, $x'(t_0)$, and the second coordinate is the derivative of the coordinate function y at t_0, $y'(t_0)$. So, we have shown that $\alpha'(t_0) = (x'(t_0), y'(t_0))$.

It is possible to reverse this argument. If $x = x(t)$ and $y = y(t)$ are differentiable functions, then $\alpha(t) = (x(t), y(t))$ is differentiable. These calculations are summarized in the following proposition.

Proposition 2.1 Let $\alpha : [a, b] \to \mathbb{R}^2$ be a function with coordinates $\alpha(t) = (x(t), y(t))$. Then

1. α is a differentiable function if and only if the coordinate functions of α, $x = x(t)$ and $y = y(t)$, are differentiable functions $[a, b] \to \mathbb{R}$.
2. If α is differentiable, then the derivative of α is given by

$$\alpha'(t) = (x'(t), y'(t)). \quad \blacklozenge$$

Returning to the discussion of derivatives, the proposition also holds for functions $\alpha : [a, b] \to \mathbb{R}^3$. If $\alpha(t) = (x(t), y(t), z(t))$, then

$$\alpha'(t) = (x'(t), y'(t), z'(t)).$$

If α parametrizes the motion of an object and is a differentiable function, we can now compute the velocity and the speed of the object in terms of the coordinates of α.

Definition 2.6 If the motion of an object is parametrized by α, the ***velocity*** $\mathbf{v}$ of the object is α', $\mathbf{v}(t) = \alpha'(t)$, and the ***speed*** s of the object is the magnitude of the velocity, $s(t) = \|\mathbf{v}(t)\|$. In coordinates in the plane,

$$\mathbf{v}(t) = (x'(t), y'(t)) \text{ and } s(t) = \sqrt{x'(t)^2 + y'(t)^2}.$$

In space,

$$\mathbf{v}(t) = (x'(t), y'(t), z'(t)) \text{ and } s(t) = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}. \quad \blacklozenge$$

Now let us carry out several explicit calculations of velocity and speed for motions that we have considered previously. In each case, the coordinates of the parametrization will be differentiable, so that the parametrization will also be differentiable. We can then calculate the derivative of the parametrization α by differentiating its coordinates.

Example 2.11 Velocity and Speed

A. Linear Motion. Suppose that an object moves along the line from $P = (x_0, y_0)$ to $Q = (x_1, y_1)$ according to the parametrization $\alpha(t) = \mathbf{p} + t\mathbf{v}_0$, where the vector $\mathbf{p}$ corresponds to the point P and $\mathbf{v}_0 = \overrightarrow{PQ}$ is the displacement vector from P to Q. (See Example 2.3.) In coordinates, $\alpha(t) = (x_0 + t(x_1 - x_0),\ y_0 + t(y_1 - y_0))$. Then

$$\begin{aligned}\alpha'(t) &= ((x_0 + t(x_1 - x_0))',\ (y_0 + t(y_1 - y_0))') \\ &= (x_1 - x_0,\ y_1 - y_0) \\ &= \mathbf{v}_0.\end{aligned}$$

This calculation shows that derivative of α is the direction vector $\mathbf{v}_0$ of the line. So, we also have that the velocity $\mathbf{v}(t) = \mathbf{v}_0$. This says that the velocity vector of this motion is constant. It follows that the speed is also constant, $s(t) = \|\mathbf{v}_0\|$. We will say that motion along a straight line that has ***constant speed*** is ***linear motion***.

B. Uniform Circular Motion. Suppose that an object moves around a circle of radius 1 centered at the origin and the motion is given by

$$\alpha(t) = (\cos(s_0 t), \sin(s_0 t)),$$

where s_0 is a positive constant. Then

$$\begin{aligned}\alpha'(t) &= (\cos(s_0 t)', \sin(s_0 t)') \\ &= (-s_0 \sin(s_0 t), s_0 \cos(s_0 t)).\end{aligned}$$

The velocity of the object is

$$\mathbf{v}(t) = (-s_0 \sin(s_0 t), s_0 \cos(s_0 t)),$$

and the speed of the object is

$$s(t) = \sqrt{(-s_0 \sin(s_0 t))^2 + (s_0 \cos(s_0 t))^2} = s_0.$$

This shows that the speed of the object is constant. We will say that motion around a circle that has ***constant speed*** is ***uniform circular*** motion. Velocity vectors for uniform circular motion around a circle of radius 1 with speed 1 are shown in Figure 2.9(a).

Now let's consider an example where the speed is not constant.

Example 2.12 **Motion Around an Ellipse.** Suppose that an object in motion in the plane moves according to the $\alpha(t) = (3\cos t, \sin t)$, so that it traces an ellipse with major axis of length 6 lying along the x-axis and minor axis of length 2 lying along the y-axis. Then $\mathbf{v}(t) = \alpha'(t) = (-3\sin t, \cos t)$ and $s(t) = \sqrt{9(\sin t)^2 + (\cos t)^2}$. (See Figure 2.10.) Since the speed is not constant, we would like to find out where the object is moving fastest and where it is moving slowest. That is, we want to find the maximum and minimum values of speed. This is a problem from one-variable calculus. First, we find the critical points of s by computing $s'(t)$ and finding solutions to $s'(t) = 0$. Differentiating s, we have that

$$s'(t) = \frac{8\sin(t)\cos(t)}{\sqrt{9(\sin t)^2 + (\cos t)^2}},$$

which is 0 if the numerator is zero. Thus $s'(t) = 0$ if $\sin(t)\cos(t) = 0$, which is the case for $t = 0, \pi/2, \pi, 3\pi/2, \ldots$. Computing the corresponding speeds, we have

$$s(0) = 1,\ s(\pi/2) = 3,\ s(\pi) = 1,\ s(3\pi/2) = 3, \ldots.$$

The object is moving fastest when $t = \pi/2$ and $t = 3\pi/2$, which occurs at the points $(0, 1)$ and $(0, -1)$, when it is closest to the origin. It is moving slowest when $t = 0$ and $t = \pi$, which occurs at the points $(3, 0)$ and $(-3, 0)$, when it is furthest from the origin.

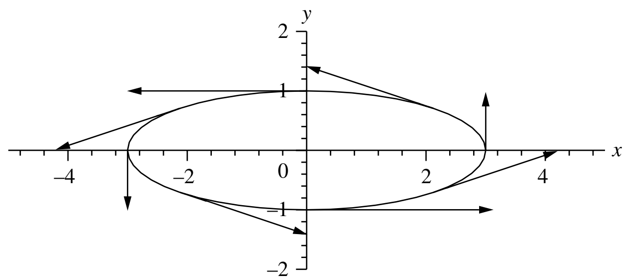

Figure 2.10 Velocity vectors for the parametrization $\alpha(t) = (3\cos t, \sin t)$ of the ellipse $x^2/9 + y^2 = 1$. (See Example 2.12.)

Before we present the next example, we must make the following definition.

Definition 2.7 If the motion of an object is parametrized by α, the ***acceleration*** $\mathbf{a}$ of the object is the derivative of the velocity of the object. That is, the acceleration of an object is the second derivative of the parametrization of its motion,

$$\mathbf{a}(t) = \mathbf{v}'(t) = \alpha''(t). \quad \blacklozenge$$

Combined with Proposition 2.1, this definition tells us that the coordinates of the acceleration of an object are obtained by twice differentiating the coordinates of the parametrization of the motion. If we know the acceleration of an object, we can integrate the coordinates of the acceleration to obtain the coordinates of the velocity. We can recover the coordinates of the parametrization by integrating the coordinates of the velocity. The following example illustrates an important application of this construction.

Example 2.13 **Motion under the Influence of Gravity.** Consider the motion of an object of unit mass in a vertical plane near the surface of the earth, so that there is one horizontal direction and one vertical direction. The force of gravity is given by the vector $\mathbf{a} = (0, -9.8)$ in units of meters per second squared. If we neglect other forces on the object in motion, like air resistance, then this is also the acceleration of the object. Since the velocity of the object and the acceleration are related by $\mathbf{v}'(t) = \mathbf{a}(t)$, we know that $\mathbf{v}'(t) = (0, -9.8)$. Integrating each coordinate, we have $\mathbf{v}(t) = (v_1, -9.8t + v_2)$, where v_1 and v_2 are constants of integration. Since $\mathbf{v}(0) = (v_1, v_2)$, these constants are the coordinates of the velocity of the object at time 0. We call $\mathbf{v}(0)$ the ***initial velocity*** of the object.

Since the parametrization α of the motion and the velocity $\mathbf{v}$ are related by $\alpha'(t) = \mathbf{v}(t)$, we know that $\alpha'(t) = (v_1, -9.8t + v_2)$. Integrating each coordinate, we can find α:

$$\alpha(t) = (v_1 t + x_1, -9.8t^2/2 + v_2 t + x_2),$$

where x_1 and x_2 are constants of integration. Since $\alpha(0) = (x_1, x_2)$, the constants are the position of the object at time 0. We call $\alpha(0)$ the ***initial position*** of the object.

We will explore this example further in Exercise 8.

Tangent Lines

The tangent vector of a parametrization at a point can be used to construct a line tangent to the curve at the point. In one-variable calculus, the tangent line to the graph of a function at a point is the line passing through the point with slope equal to the slope of the graph, or derivative of the function at the point. The analogous statement is true for parametrizations: The tangent line to the image of α at $\alpha(t_0)$ is the line passing through $\alpha(t_0)$ whose direction vector is equal to the tangent vector, or derivative of the parametrization at the point, $\alpha'(t_0)$. It is convenient to represent the line as the image of a linear parametrization β. We do this in the following definition.

Definition 2.8 Suppose that α is a differentiable parametrization, t_0 is in the domain of α, and $\alpha'(t_0) \neq \mathbf{0}$. We define the ***tangent line*** to the image of α at $\alpha(t_0)$ to be the line parametrized by

$$\beta(t) = \alpha(t_0) + t\,\alpha'(t_0), \quad t \in \mathbb{R}. \ \blacklozenge$$

Example 2.14 **The Tangent Line to a Helix.** The image of the parametrization $\alpha(t) = (\cos t, \sin t, t)$ is a helix that winds counterclockwise about the z-axis. (See Exercise 9 of Section 2.1.) The tangent vector is $\alpha'(t) = (-\sin t, \cos t, 1)$. It follows that the tangent line to the helix at the point $\alpha(t_0)$ is given by

$$\beta(t) = (\cos t_0, \sin t_0, t_0) + t(-\sin t_0, \cos t_0, 1).$$

For example, if we select the time $t_0 = 3\pi/2$, we have

$$\begin{aligned}\beta(t) &= (\cos(3\pi/2), \sin(3\pi/2), 3\pi/2) + t(-\sin(3\pi/2), \cos(3\pi/2), 1)\\ &= (0, -1, 3\pi/2) + t(-1, 0, 1).\end{aligned}$$

Figure 2.11(a) contains a plot of the image of α and this tangent line. Notice that the tangent line and the image of α come together at $\alpha(3\pi/2)$ and that they have the same direction there.

In order to apply the tangent line formula, we required the derivative to be nonzero. If $\alpha'(t_0) = \mathbf{0}$, the formula $\beta(t) = \alpha(t_0) + t\alpha'(t_0)$ yields $\beta(t) = \alpha(t_0)$. The image of β is

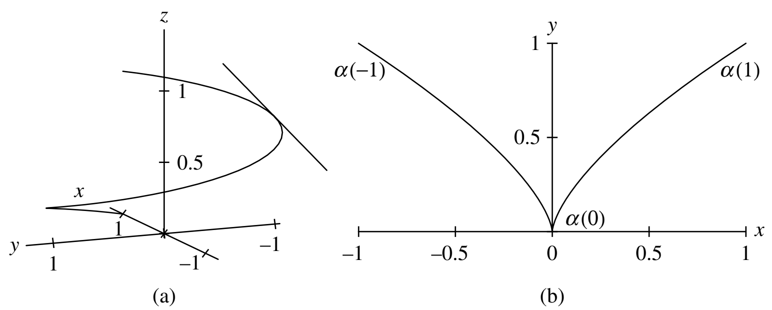

Figure 2.11 (a) The tangent line to the helix $\alpha(t) = (\cos t, \sin t, t)$ at $t = 3\pi/2$. (See Example 2.14.) (b) The image of $\alpha(t) = (t^3, t^2)$. (See Example 2.15.)

the point $\alpha(t_0)$, not a line. Even though α is differentiable at t_0, it does not guarantee that there is a tangent line. This should be contrasted with the one-variable case. Since the derivative of a function $y = f(x)$ at a point x_0 is the slope of the tangent line, being differentiable is equivalent to the graph of f having a tangent line. The following example shows what can "go wrong" when $\alpha'(t_0) = \mathbf{0}$.

Example 2.15 **The Tangent Behavior at a Cusp.** The parametrization $\alpha(t) = (t^3, t^2)$ is differentiable for all t. Computing the derivative, $\alpha'(t) = (3t^2, 2t)$, we see that $\alpha'(0) = (0, 0)$. The coordinates of α satisfy the equation $y = x^{2/3}$. Since $x = t^3$ takes all possible values, the image of α is the graph of the function $y = f(x) = x^{2/3}$, which, as a function of one variable, fails to be differentiable at $x = 0$. The graph of f, hence the image of α, comes to a sharp point, or ***cusp***, at the origin. (See Figure 2.11(b).) Since f is not differentiable at the origin, its graph does not have a tangent line at the origin. Consequently, the image of α does not have a tangent line at the origin, even though α is differentiable.

Chain Rule for Parametrizations

If a parametrization β is the composition of a function g and a parametrization α, $\beta = \alpha \circ g$, so that $\beta(t) = \alpha(g(t))$, the derivative of β can be expressed in terms of the derivative of α and the derivative of g. This result is known as the ***chain rule***, and its form is analogous to the chain rule for functions of one variable. In one variable, if $h(x) = f(g(x))$, then $h'(x) = f'(g(x))g'(x)$. We will state the chain rule as a proposition.

Proposition 2.2 **The Chain Rule.** Let $\alpha : R \to \mathbb{R}^2$ or $\alpha : R \to \mathbb{R}^3$ be a differentiable parametrization, and let $g : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then the composition $\alpha \circ g$ is a differentiable parametrization and

$$(\alpha \circ g)'(t) = \alpha'(g(t))g'(t). \quad \blacklozenge$$

The proof of the chain rule has a form common to many of the equalities involving vectors. Working from the side of the equality that we want to expand or simplify, $(\alpha \circ g)'$, we express it in coordinates, apply facts about functions of one variable in each of the coordinates, and then reorganize the coordinate form to obtain the vector form we want.

Proof: We will write out the steps for a parametrization of a curve in the plane. The steps for a parametrization of a curve in space are analogous. Let us begin with a differentiable parametrization $\alpha = \alpha(s)$, where $\alpha(s) = (x(s), y(s))$, and a differentiable function $g = g(t)$. The composition $\alpha \circ g$ is given by

$$(\alpha \circ g)(t) = (x(g(t)), y(g(t))).$$

Since α is differentiable, the coordinate functions x and y are differentiable. Then, since g is differentiable, the compositions $x(g(t))$ and $y(g(t))$ are differentiable. This means that when we compute the derivative of the composition, we may apply the one-variable chain rule to its entries.

$$\begin{aligned}(\alpha \circ g)'(t) &= (x(g(t))', y(g(t))') \\ &= (x'(g(t))g'(t),\ y'(g(t))g'(t)).\end{aligned}$$

Factoring out the term $g'(t)$, we have

$$(x'(g(t)),\ y'(g(t)))\, g'(t) = \alpha'(g(t))g'(t).$$

This is the desired result. ∎

Two instances of the chain rule are of particular importance for us: scaling and shifting the independent variable. (See Example 2.7.)

Example 2.16 **Transforming the Parameter**

A. Shifting Revisited. Suppose $\alpha : \mathbb{R} \to \mathbb{R}^3$ is differentiable and $g(t) = t + k$, where k is a constant. If $\beta = \alpha \circ g$, then

$$\beta'(t) = (\alpha \circ g)'(t) = \alpha'(g(t))g'(t) = \alpha'(t + k)1 = \alpha'(t + k).$$

Shifting the independent variable of a parametrization by k has the effect of shifting the tangent vectors so the tangent vector to β at t_0 corresponds to the tangent vector to α at $t_0 + k$. If the parameter represents time, this corresponds to our sense that shifting the parameter corresponds to shifting the behavior of the system k units forward or backward in time.

B. Scaling Revisited. Now let $g(t) = kt$, where k is a constant. If $\beta = \alpha \circ g$, then

$$\beta'(t) = (\alpha \circ g)'(t) = \alpha'(g(t))g'(t) = \alpha'(kt)k = k\alpha'(kt).$$

Scaling the independent variable of the parametrization by k also scales the derivative by a factor of k.

In terms of the motion, if α is a parametrization of the motion of an object, we know that $\alpha(kt)$ represents motion along the same curve that in 1 unit of time covers as much of the curve as the original motion does in k units of time. The chain rule says that the velocity of the new motion is $k\mathbf{v}(kt)$, so that the velocity is also k times as large as it is for the original motion.

Summary We defined the ***velocity*** of an object to be the ***limit*** of its ***average velocity***. The ***speed*** of an object is the length of its velocity vector. The discussion of velocity motivated the definition of the ***derivative of a parametrization***. At any point the derivative of a parametrization is a vector that is tangent to the image of the parametrization. It is called the ***tangent vector*** of the parametrization. If the parametrization represents motion, the derivative is the velocity of the motion and its length is the speed of the motion. We showed how to compute the derivative of a parametrization in terms of the derivatives of its coordinate functions. If $\alpha(t) = (x(t), y(t))$, then $\alpha'(t) = (x'(t), y'(t))$. We defined the ***acceleration*** of an object in motion to be the derivative of its velocity, and we defined the ***tangent line*** to the image of α at t_0 to be the image of $\beta(t) = \alpha(t_0) + t\alpha'(t_0)$. Finally, we stated and proved the ***chain rule*** for parametrizations, $(\alpha \circ g)'(t) = \alpha'(g(t))g'(t)$.

Section 2.2 Exercises

1. Velocity and Speed. Each of the following functions α is a parametrization for the motion of an object in the plane or in space. Find the velocity vector and the speed of the motion at the given time t_0.

(a) $\alpha(t) = (e^t, e^{3t})$ at $t_0 = 0$.
(b) $\alpha(t) = (2\cos t, 3\sin t)$ at $t_0 = \pi/2$.
(c) $\alpha(t) = (\cos 2t, t, \sin 2t)$ at $t_0 = \pi/4$.
(d) $\alpha(t) = (1, t, \sqrt{1+t^2})$ at $t_0 = 0$.

2. Tangent Lines. For each of the following parametrizations α, find the tangent vector and the tangent line to the curve parametrized by α at the point t_0.

(a) $\alpha(t) = (1 + \cos 2t, 2 + \sin 2t)$ at $t_0 = \pi/4$.
(b) $\alpha(t) = (te^{2t}, e^{t^2})$ at $t_0 = 1$.
(c) $\alpha(t) = (1 + 2t, 1 - t, 2 + t)$ at $t_0 = 1$.
(d) $\alpha(t) = ((2 + \sin 2t)\cos t, (2 + \sin 2t)\sin t, \cos 3t)$ at $t_0 = 0$.

3. The Algebra of Differentiation. Each of the following algebraic facts can be verified by the type of coordinate calculation that we used to prove the chain rule. Carry out the calculation for each of the following identities involving parametrizations. In each case, say which property of derivatives of functions of one variable you use to verify the identify. Assume that all the parametrizations are differentiable and take values in $\mathbb{R}^3$:

(a) $(c\alpha(t))' = c\alpha'(t)$.
(b) $(\alpha(t) + \beta(t))' = \alpha'(t) + \beta'(t)$.
(c) $(\alpha(t) \cdot \beta(t))' = \alpha'(t) \cdot \beta(t) + \alpha(t) \cdot \beta'(t)$.
(d) $(\alpha(t) \times \beta(t))' = \alpha'(t) \times \beta(t) + \alpha(t) \times \beta'(t)$.

4. The Tangent Line to a Graph. Suppose $\mathcal{C}$ is a curve in $\mathbb{R}^2$ that is the graph of a differentiable function $f : R \to \mathbb{R}$.

(a) What is the point-slope equation for the tangent line to this curve at the point $(x_0, f(x_0))$?
(b) Parametrize this curve and then use the techniques developed in this section to find a parametrization of the tangent line to this curve at the point $(x_0, f(x_0))$.
(c) Show that the parametrization of part (b) is a parametrization of the line from part (a).

5. The Cycloid. Consider a wheel of radius 1 that rests on the x-axis and rolls without slipping in the positive direction along the x-axis with unit speed. If we mark a point on the wheel, it will trace a curve in the plane as the wheel rolls. This curve is called a ***cycloid***. If we assume that the motion starts with the wheel resting on the origin and that the marked point is at the top of the wheel, one unit from the center at the beginning of the motion, the motion of the point is parametrized by

$$\alpha(t) = (t + \sin t, 1 + \cos t).$$

Figure 2.12 is a plot of the image of α for $0 \leq t \leq 4\pi$.

(a) Compute the velocity vector $\mathbf{v}(t)$ and the speed $s(t)$ for the motion of the point.
(b) At which time t is the velocity vector horizontal? At which time t is the velocity vector vertical?
(c) Where is the point moving fastest? Where is it moving slowest?
(d) Describe what is happening at the points $t = \pi$ and $t = 3\pi$.

6. Acceleration. Suppose $\alpha(t)$ is a differentiable function that parametrizes the motion of an object and that $\mathbf{v}(t)$ and $\mathbf{a}(t)$ are the corresponding velocity and acceleration vectors.

(a) Suppose $\alpha(t)$ parametrizes linear motion along the line segment from $P = (x_0, y_0, z_0)$ to $Q = (x_1, y_1, z_1)$, with

$$\alpha(t) = (x_0, y_0, z_0) + ct(x_1 - x_0, y_1 - y_0, z_1 - z_0),$$

where c is a positive constant. What is the acceleration vector $\mathbf{a}(t)$ for this motion?

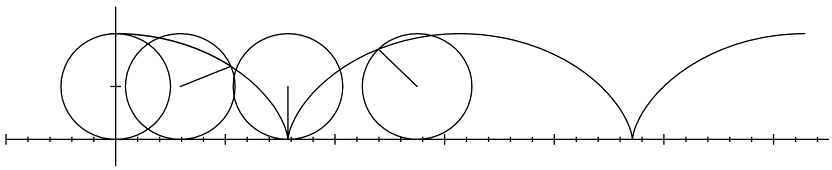

Figure 2.12 The cycloid of Exercise 5.

(b) Suppose $\alpha(t) = (R\cos(ct), R\sin(ct))$, so that α parametrizes uniform circular motion around a circle of radius R centered at the origin. What is the acceleration vector $\mathbf{a}(t)$ for this motion?

(c) Suppose $\alpha(t) = (R\cos t, R\sin t, ct)$ parametrizes motion along a helix. What is the acceleration vector $\mathbf{a}(t)$ for this motion?

7. **Constant Speed Motion.** Suppose α is a differentiable function that parametrizes the motion of an object that is traveling at a constant speed. That is, $s(t) = k$. What is the angle between the velocity vector $\mathbf{v}(t)$ and the acceleration vector $\mathbf{a}(t)$ for this motion? (*Hint:* Use the facts that (i) $s(t) = \sqrt{\mathbf{v}(t) \cdot \mathbf{v}(t)}$ and (ii) $s'(t) = 0$.)

8. **Projectile Motion.** Let us consider the motion of an object near the surface of the earth that is subject only to the force of gravity. As we saw in Example 2.13, the motion of the object can be parametrized by

$$\alpha(t) = (v_1 t + x_1, -9.8t^2/2 + v_2 t + x_2),$$

where $\alpha(0) = (x_1, x_2)$ is the initial position of the object and (v_1, v_2) is its initial velocity.

(a) Suppose the initial position of the object is $\alpha(0) = (0, 0)$ and the initial velocity of the object is $\mathbf{v}(0) = (5, 10)$. How long will the object remain in motion, how high will it travel, and how far will it travel?

(b) Suppose an object is launched with initial speed of 50 m/s and it reached a maximum height of 20 m. How far did it travel?

9. **More Projectile Motion.** Suppose the motion of an object is parametrized by the function α from Exercise 8.

(a) If the initial position of the object is $(0, 0)$ and the initial velocity is (v_1, v_2), when will the object hit the ground? What horizontal distance does it travel before hitting the ground?

(b) If the initial position of the object is $(0, 0)$, when will it reach its maximum height? What is the maximum height?

(c) At which angle θ should the object be launched in order to maximize the horizontal distance traveled? (*Hint:* Express $\mathbf{v}(0)$ as $v_0(\cos\theta, \sin\theta)$, where v_0 is a constant and $(\cos\theta, \sin\theta)$ is a unit vector.)

10. **Halley's Comet.** In this exercise, we will investigate the velocity of Halley's comet. Although its orbit is an ellipse, it does not have a simple parametrization. Thus we will work directly with position data. We will use astronomical units (AU) in our calculations. An astronomical unit is approximately 1.496×10^8 km, the average distance from the earth to the sun.

The orbit of the comet is an ellipse with eccentricity 0.9674 with the sun located at one of its foci. (See Exercise 6 in Section 2.1.) The distance at perihelion, the closest point to the sun, is approximately 0.59 astronomical units. The comet takes 76 years to orbit the sun. The table in Figure 2.13 shows the location of Halley's comet at four-year

t	x	y	t	x	y	t	x	y	t	x	y
0	−0.59	0	20	30.16	3.28	40	35.54	−0.38	60	27.24	−3.86
4	11.77	4.37	24	32.38	2.61	44	35.03	−1.15	64	23.46	−4.33
8	18.53	4.58	28	33.98	1.90	48	33.98	−1.90	68	18.53	−4.58
12	23.46	4.33	32	35.03	1.15	52	32.38	−2.61	72	11.77	−4.35
16	27.24	3.86	36	35.54	.38	56	30.16	−3.28	76	−0.59	0

Figure 2.13 The location of Halley's comet at four-year intervals in xy-coordinates. The sun is located at the origin of the coordinate system, and the major axis coincides with the x-axis. Distances are in astronomical units. (See Exercise 10.)

intervals using a Cartesian coordinate system with the sun at the origin and the x-axis aligned with the major axis of the orbit. Figure 2.14 contains a plot of the data points.

(a) How would you use the data to approximate the velocity of Halley's comet?
(b) Carry out your approximation of the velocity at four-year intervals.
(c) Where is Halley's comet moving fastest? Where is it moving slowest?
(d) Compare the motion of Halley's comet to the motion corresponding to a parametrization of the type given in Example 2.4B. Explain how they differ.

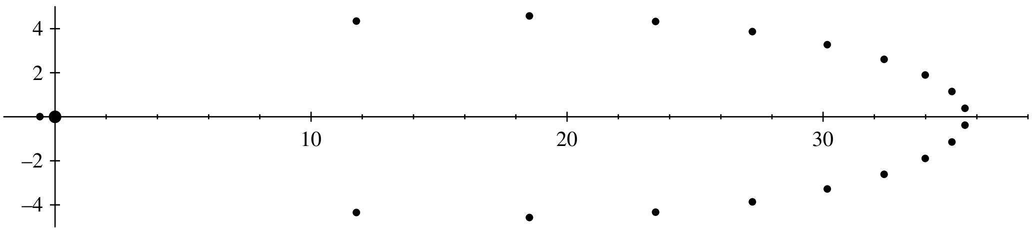

Figure 2.14 The plot of the data points for the orbit of Halley's comet given in Figure 2.13. The sun is located at the origin. (See Exercise 10.)

■ 2.3 Modeling with Parametric Curves

In this section, we would like to introduce two different uses of parametric curves. The first concerns the velocity of a robotic arm. It builds on the collaborative exercise at the beginning of Chapter 1. The other is a model for the spread of influenza. This develops ideas introduced in the collaborative exercise at the beginning of Chapter 2.

Velocity and Inverse-Velocity Kinematics of a Robotic Arm

In the collaborative exercise at the beginning of Chapter 1, we considered the forward- and inverse-position kinematic problems for a two-link (two-segment) robotic arm. As in Chapter 1, the robot will consist of links, one of length 1 extending from a point O on the x-axis to a point P, and the second, also of length 1, extending from P to Q. (See Figure 2.15.) The *effector* of the robot is at the point Q. Knowing the lengths of the links and the properties of the joints, it was possible to determine the working envelope of the robot. That is, it was possible to determine all possible positions of the effector of the robot. This is the forward-position kinematics problem. We were also able to provide a qualitative answer to the question of how many positions of the links give the same position of the effector. This is the inverse-position kinematics problem.

Here we want to use parametric curves to answer similar questions for the velocity and speed of this robot. In particular, the very practical problem we will solve is how to make the effector move at constant speed. Regardless of the configuration of the joints, we must first construct a parametrization for the motion of the effector. We can then determine the velocity of the motion of the effector by computing the derivative of this parametrization. Of course, the speed is the length of the velocity. The parametrization of the motion of the effector will be a sum of a parametrization of the motion of P and a parametrization of the motion of Q relative to P. We can further modify either of these summands by composition with a function of t. This will allow us to adjust the speed of P or Q and thus to determine how to make Q (the effector) move at a constant speed.

In each case, it will be a sum of a parametrization of the motion of P and a parametrization of the motion of Q relative to P. We can further modify either of these summands by composing it with a function of t. This will allow us to adjust the speed of P or Q. To find the speed of the effector, we must first find its velocity. This can be accomplished by combining the rule for the derivative of the sum and the chain rule. We begin with a single hinge joint.

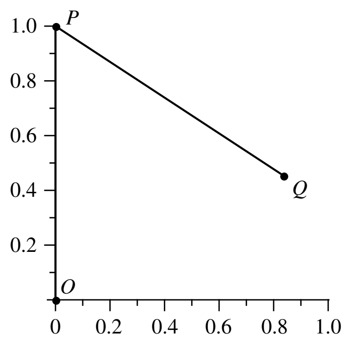

Figure 2.15 A robot with two arms of length 1. (See Example 2.17.)

Example 2.17

A Single Hinge Joint. If $\overline{OP}$ is fixed as shown in Figure 2.15 and the joint at P is a hinge joint free to move through an angle of 2π, then the working envelope of the robot is the circle of radius 1 centered at $(0, 1)$. Let $g = g(t)$ be the angle made by the segment $\overline{PQ}$ with the segment $\overline{OP}$ measured counterclockwise from the position where Q coincides with O. Then the position of the effector is given by

$$\alpha(t) = (0, 1) + (\sin(g(t)), -\cos(g(t))) = (\sin(g(t)), 1 - \cos(g(t))).$$

Using the chain rule, the speed of the effector is

$$s(t) = \|\alpha'(t)\| = \|(\cos(g(t)), \sin(g(t)))g'(t)\| = 1 \cdot |g'(t)| = |g'(t)|.$$

If g is an increasing function of t, that is, $g'(t) \geq 0$, then the speed of the effector is simply $g'(t)$. Consequently, for the effector to move at constant speed, the rate of change of the angle must be constant, $g'(t) = k$, so that g must be a linear function of t, $g(t) = kt + \theta_0$, where θ_0 is the initial angle at the hinge joint.

Now let us consider a more complicated example. Suppose there is a slider joint located at O allowing O to slide along the x-axis while $\overline{OP}$ remains perpendicular to the axis, in addition to a hinge joint located at P that is free to rotate through an angle of 2π. At time t, the hinge joint at P will be located at $(f(t), 1)$, where we assume that f is a differentiable function of t. If $g = g(t)$ again represents the angle made by $\overline{PQ}$ with $\overline{OP}$, then position of the effector is given by

$$\alpha(t) = (f(t), 1) + (\sin(g(t)), -\cos(g(t))) = (f(t) + \sin(g(t)), 1 - \cos(g(t))).$$

In Example 2.17, f was held constant. In the general case, we can calculate $\alpha'(t)$ and $s(t)$. First,

$$\alpha'(t) = (f'(t) + \cos(g(t))g'(t), \sin(g(t))g'(t)).$$

Then, the speed of the effector is

$$\begin{aligned} s(t) &= \|\alpha'(t)\| = \|(f'(t) + \cos(g(t))g'(t), \sin(g(t))g'(t))\| \\ &= \sqrt{f'(t)^2 + 2f'(t)\cos(g(t))g'(t) + \cos^2(g(t))g'(t)^2 + \sin^2(g(t))g'(t)^2} \\ &= \sqrt{f'(t)^2 + 2f'(t)\cos(g(t))g'(t) + g'(t)^2}. \end{aligned}$$

Let us consider the special case when f and g are linear functions of t.

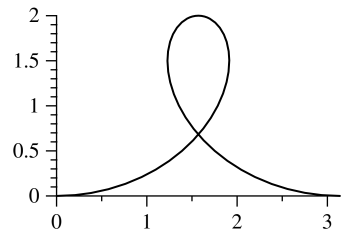

Figure 2.16 The path of the effector for a slide joint moving linearly and a hinge joint rotating with constant angular speed. (See Example 2.18.)

Example 2.18

A Hinge Plus Slider Forward-Velocity Kinematics Problem. Assume that $f(t) = \frac{1}{2}t$ and $g(t) = t$. It follows from our calculation that the speed of the effector is $s(t) = \sqrt{\frac{5}{4} + \cos(t)}$. Figure 2.16 shows the path traced by the effector.

Notice the speed is a maximum when $t = 0$ and a minimum when $t = \pi$. This makes intuitive sense because when $t = 0$, the velocity of the base and effector are in the same direction, and the magnitude of the sum is the sum of the magnitudes. On the other hand, when $t = \pi$, the base is moving to the right while the effector has reached its highest position and is moving to the left, and the magnitude of the sum is the difference of magnitudes.

An interesting and natural question is how to move the effector at constant speed. Or, we might say, what conditions does the requirement that $s(t) = s_0$ place on f and g? This problem does not have a unique answer. However, if f is required to be linear, it is possible to derive an ordinary differential equation for g, which can be solved using numerical techniques from calculus. The following example demonstrates this.

Example 2.19

A Hinge Plus Slider Inverse-Velocity Kinematics Problem. To begin, assume that f is linear, $f(t) = v_0 t$, so that the slider joint moves with constant velocity $v_0 > 0$. Then $s(t) = \sqrt{v_0^2 + 2v_0 \cos(g(t))g'(t) + g'(t)^2}$. Squaring both sides, it suffices to work with the equation:

$$s_0^2 = v_0^2 + 2v_0 \cos(g(t))g'(t) + g'(t)^2.$$

To simplify this further, we write s_0 as a product, $s_0 = s_1 v_0$. Using this substitution, moving the $s_0^2 = (s_1 v_0)^2$ term to the right and rearranging, we obtain the following quadratic equation in $g'(t)$:

$$0 = g'(t)^2 + 2v_0 \cos(g(t))g'(t) + v_0^2(1 - s_1^2).$$

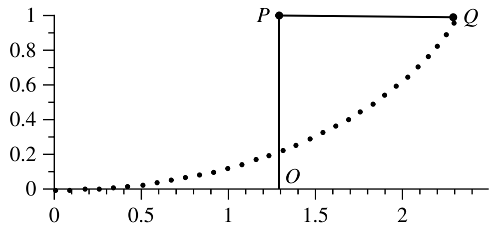

Figure 2.17 A constant speed solution to moving the effector from the origin to height 1 as the slider moves at constant speed 1 to the right. The robot is shown in its final position. (See Example 2.19.)

Applying the quadratic formula gives

$$\begin{aligned} g'(t) &= \frac{1}{2}\left(-2v_0\cos(g(t)) \pm \sqrt{4v_0^2\cos^2(g(t)) + 4v_0^2(s_1^2 - 1)}\right) \\ &= -v_0\cos(g(t)) \pm v_0\sqrt{\cos^2(g(t)) + s_1^2 - 1}. \end{aligned}$$

Choosing the positive square root yields a formula for $g'(t)$ that is always positive. This is a first order differential equation for g:

$$\frac{dg}{dt} = -v_0\cos(g(t)) + v_0\sqrt{\cos^2(g(t)) + s_1^2 - 1}.$$

This equation cannot be solved explicitly, but if we choose values for v_0 and s_1, we can use Euler's method or another numerical method from one-variable calculus to solve for g. We can then substitute the numerically computed values for $g(t)$ into the formula for α to plot the position of the effector. Figure 2.17 shows a point plot of the position of the effector for $v_0 = 1$, $s_0 = 2$, and $g(0) = 0$ (the initial position of the effector at the origin). The displayed points are 0.042 units of time apart, and the final time is approximately 1.29 units of time. This represents a constant speed solution to the problem of lifting the effector from the origin to height 1 as the slider joint moves to the right at constant speed. The final location of O is approximately 1.3 and of Q is approximately $(2.3, 1)$.

Alternately, it is also possible to require that g be linear and solve for f. This is explored in Exercise 5.

SIR Epidemic Model

An ***SIR*** model for the spread of an infectious disease can be used to model diseases, like measles or a particular strain of influenza, for which the memory feature of the immune system prevents reinfection. Thus the population can be divided into three categories:

the ***susceptible*** people, the ***infected*** people, and the ***recovered*** people.[3] If a susceptible person contracts the disease, he or she will move to the infected group, and then, after a period of time, to the recovered group. Since recovery from the disease confers immunity, the recovered population is distinct from the susceptible population. For diseases that do not confer immunity, an SI model with only susceptible and infected populations would be appropriate. Here we will focus on a disease that confers immunity.

We will think of the sizes of each of the three populations as being functions of time alone. Let $S = S(t)$ be the number of susceptible individuals at time t, $I = I(t)$ be the number of infected individuals at time t, and $R = R(t)$ be the number of recovered individuals at time t. If the time period for the model is relatively short, which it is for the outbreak we will model, we can assume that the overall population is constant. Thus for any t, the three populations satisfy the equation

$$S(t) + I(t) + R(t) = N,$$

where the total number of people N in the population is constant. If we know S, I, and N, we can recover R from the fact that $R(t) = N - S(t) - I(t)$. This allows us to concentrate on the relationship between S and I and compute R if we need to.

Since susceptible individuals contract the disease from infected individuals and then move out of the susceptible category into the infected category, there is a relationship between these two categories. It makes sense, then, to plot the size of the susceptible population, $S(t)$, on one coordinate axis in the plane, and the size of the infected population, $I(t)$, on the other. For a particular outbreak of a disease, as time evolves, the values of S and I will trace a curve in the SI-plane. Denote this curve by α, where $\alpha : [0, b] \to \mathbb{R}^2$ and $\alpha(t) = (S(t), I(t))$.

If we knew the values of S and I for each t, we could immediately plot the image of α. Of course, to do so, we would have to wait until the end of an epidemic to produce a mathematical model for it! Instead we are going to assume that ***the spread of the disease behaves according to a simple set of rules***. Our job will be to come up with mathematical versions of these rules. Since the rules are for the spread of the disease, these will involve S' and I', the rates of change of S and I.

We will concentrate on formulating rules for influenza. Influenza is transmitted through contact between a susceptible person and an infected person. The change in the susceptible population is due entirely to individuals becoming infected, that is, to individuals moving from the susceptible population to the infected population. It follows that S', the rate of change of S, is always negative. We will assume that this rate is jointly proportional to the number of susceptible individuals and the number of infected individuals.

[3]For a general reference on SIR models, see Chapter 19 of J.D. Murray, *Mathematical Biology*, Springer-Verlag, 1993.

Symbolically,

$$S'(t) = -rS(t)I(t),$$

where $r > 0$ is a constant, which we call the ***rate of transmission***. This equation makes sense because if there are more infected people, we expect that an individual susceptible person should come into contact with more infected people, and so should have a greater chance of becoming infected. On the other hand, if there are more susceptible people, there are more people who could contract the disease from contact. In both cases, the rate should increase.

Now consider I', the rate of change of the infected population. Since every decrease in the susceptible population results in an equal increase in the infected population, I' must contain the term $rI(t)S(t)$. As people recover, they move from the infected population to the recovered population. We will assume that this rate is proportional to the size of the infected population. This will be reflected by a term of the form $-aI(t)$, where $a > 0$ is a constant, which we call the ***rate of recovery***. (See Exercise 11.) Assuming there are no other factors to consider, I' is given by

$$I'(t) = rS(t)I(t) - aI(t).$$

Below is a summary of this discussion.

SIR Epidemic Model

Denote the susceptible population by $S = S(t)$, the infected population by $I = I(t)$, and the recovered population by $R = R(t)$, and assume the total population is constant, then

$$S(t) + I(t) + R(t) = N.$$

The ***SIR***, or ***susceptible-infected-recovered***, model of an epidemic is given by the pair of differential equations for the rate of change of S and the rate of change of I,

$$\begin{aligned} S'(t) &= -rS(t)I(t) \\ I'(t) &= rS(t)I(t) - aI(t), \end{aligned}$$

where the constant $r > 0$ is the ***rate of transmission*** and the constant $a > 0$ is the ***rate of recovery***.

We would like to solve this pair of equations for I and for S. Here is one way to do this. First, we can eliminate t from these equation by applying the chain rule from one-variable calculus. It says that

$$\frac{dI}{dS} = \frac{dI/dt}{dS/dt}.$$

Substituting the equations of the model into this quotient, we have

$$\frac{dI}{dS} = \frac{rSI - aI}{-rSI} = -\frac{rS - a}{rS} = -1 + \frac{a}{rS}.$$

Separating variables, we obtain

$$dI = (-1 + \frac{a}{rS})dS.$$

Integrating both sides of this equation, we obtain an equation for I in terms of S,

$$I = -S + \frac{a}{r}\ln S + k,$$

where k is a constant of integration. If we start the model at time $t = 0$, we can express k in terms of the initial sizes of the susceptible and infected populations.

$$k = I(0) + S(0) - \frac{a}{r}\ln S(0).$$

Let us apply this model for I to the data from a real epidemic.

Example 2.20 Influenza at a British Boarding School

A. The Parametric Curve. In 1978, the *British Medical Journal*[4] reported on an outbreak of influenza at a British boys' boarding school. There were 763 students at the school, and the outbreak began with one infected student, so that $S(0) = 762$ and $I(0) = 1$. In the report it was determined that $r \approx 2.18 \times 10^{-3}$/day and $a \approx 4.40 \times 10^{-1}$/day, so that $a/r \approx 202$ and $k \approx 763 - 202\ln(762) \approx -577$. The equation for I becomes

$$I = -S + 202\ln S - 577.$$

[4]News and Notes, *British Medical Journal*, 1:6112.586, March 4, 1978.

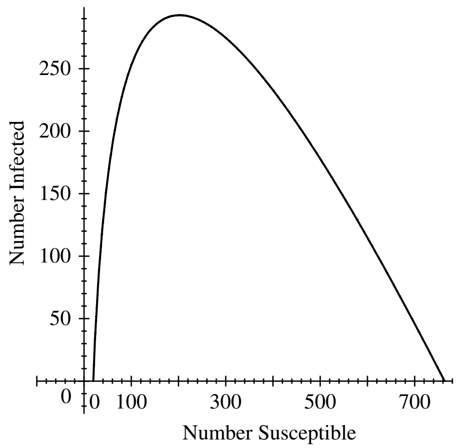

Figure 2.18 The plot of α , $\alpha(t) = (S(t), I(t))$, for the SIR model of Example 2.20.

This defines a curve in the first quadrant of the SI-plane that contains the image of α. (See Figure 2.18.)

B. An Analysis of the Model. The initial point of α is $\alpha(0) = (762, 1)$, the right endpoint of the curve. From the model, $S' < 0$ as long as there are susceptible *and* infected people remaining. Thus S must be decreasing as long as there are susceptible and infected people remaining, so that α traces the curve in Figure 2.18 from right to left.

From the model, $I'(t) = I(t)\,(rS(t) - a)$. If $I(t) > 0$, then $I'(t) > 0$ when $rS(t) - a > 0$, or when $S(t) > a/r \approx 202$. So as the susceptible population decreases from 762 to 202, the infected population increases. When the susceptible population falls below 202, $I'(t) < 0$ and the infected population decreases. That is, the epidemic begins to subside. This is because the number of susceptible people has decreased to the point that the rate at which the susceptible population becomes infected, $rS(t)I(t)$, is exceeded by the rate at which the infected population recovers, $aI(t)$. This epidemic ends when $I = 0$. This occurs with 19 people still uninfected.

It is important to note that we learned a good deal about the spread of the epidemic in Example 2.20 without having an explicit formula for I or S in terms of t. The equations of the model by themselves were sufficient for us to determine when the populations were increasing or decreasing and when they had a maximum or a minimum.

Summary

Our primary goals in developing these models is to illustrate different ways in which parametric curves may be used to analyze the behavior of a real-world system.

In modeling the motion of a simple robot, we used ***sums*** and ***compositions*** to construct a formula for the position of the effector at time t. We computed the velocity and speed of the effector. We also demonstrated how to find a constant speed motion of the effector by reducing the problem to solving an ordinary differential equation. Since the equation could not be solved symbolically, it was necessary to use Euler's method or a similar method to produce a numerical solution. It was then possible to plot the position of the effector at the numerically computed times.

The second application concerned an ***SIR***, or ***susceptible-infected-recovered***, model for the spread of an influenza-like epidemic. We developed a model for the ***relationship between the susceptible and infected populations***. By making assumptions about ***the spread of the disease***, we constructed a pair of differential equations for the ***rates of change of these populations***. Using techniques from one-variable calculus, we were able to solve these equations to find the image of the parametrization and were able to analyze the data from a particular outbreak of influenza.

Section 2.3 Exercises

1. **Forward-Velocity Kinematics.** In Example 2.18, we considered the motion of the effector of a simple robot when the slider joint moved with constant positive velocity and the hinge joint rotated with constant positive angular velocity. Using the notation of the example, the effector moves according to

$$\alpha(t) = (f(t) + \sin(g(t)), 1 - \cos(g(t))),$$

where $f(t) = k_1 t$ and $g(t) = k_2 t$ for constants k_1 and k_2. In Example 2.18, $k_1 = \frac{1}{2}$ and $k_2 = 1$. Here we consider other possibilities for these coefficients. In each case, describe the motion of the effector on the interval $0 \le t \le 2\pi$. (*Hint:* Examine the signs $x'(t)$ and $y'(t)$ on $[0, 2\pi]$.)

(a) Suppose $k_1 > k_2$, for example, $k_1 = 2$ and $k_2 = 1$.
(b) Suppose $k_1 = k_2$, for example, $k_1 = k_2 = 1$.
(c) Suppose $k_1 > |k_2|$, for example, $k_1 = 2$ and $k_2 = -1$.
(d) Suppose $k_1 = -k_2$, for example, $k_1 = 1$ and $k_2 = -1$.
(e) Suppose $k_1 < |k_2|$, for example, $k_1 = 1$ and $k_2 = -2$.

2. **Inverse-Velocity Kinematics I.** In Example 2.19, we considered a solution to the problem of moving the effector of a simple robot at constant speed $s_0 = 2$ to height 1 for horizontal velocity $v_0 = 1$.

(a) Suppose that $v_0 = 1.5$ while s_0 remains equal to 2. Without doing any calculations, how would this change the solution? In particular:

(i) Would the final position of O be to the right or left of 1.3 (the final position in Example 2.19)? Explain.
(ii) Would the values of g be greater or less than in Example 2.19? Explain.

(b) Suppose that $v_0 = 0.5$ while s_0 remains equal to 2. Without doing any calculations, how would this change the solution? In particular:

(i) Would the final position of O be to the right or left of 1.3 (the final position in Example 2.19)? Explain.
(ii) Would the values of g be greater or less than in Example 2.19? Explain.

3. Inverse-Velocity Kinematics II. Consider the robot of Example 2.19. The example exhibited a constant speed motion of the effector in which the final position of the effector was approximately $(2.3, 1)$. How would the robot have to move from the initial position of both O and Q at the origin (so the angle is 0) in order to lift the effector to the following points? That is, should f' and g' be positive or negative? Explain your answer.

(a) $(\frac{1}{2}, \frac{1}{2})$. (b) $(0, \frac{3}{2})$. (c) $(-\frac{1}{2}, 1)$.

4. A Numerical Solution I. Consider the robot of Example 2.19. Here we consider what happens if the other root is chosen.

(a) Use a numerical differential equation solver to solve the differential equation using the negative square root:

$$\frac{dg}{dt} = v_0 \cos(g(t)) - v_0\sqrt{\cos^2(g(t)) + s_1^2 - 1}.$$

Use the same speed for $\overline{OP}$, $v_0 = 1$, the same speed for the effector, $s_0 = 2$, and the same initial condition, $g(0) = 0$. The result should be a list of times and values for g at those times.

(b) The motion of the effector can be obtained from the numerical solution in (a) by the following procedure. If the times are $(t_0, t_1, \ldots, t_n)$ and the corresponding g values are $(g_0, g_1, \ldots, g_n)$, then plot the list of points

$$(v_0 t_i + \sin(g_i), 1 - \cos(g_i)), \quad i = 1, \ldots, n.$$

Describe the resulting plot and motion.

5. A Numerical Solution II. In Example 2.19, the horizontal velocity of the slider was held constant. This allowed us to find a differential equation for g', which we solved numerically. Suppose instead that g' is held constant. That is, set $g(t) = v_0 t$.

(a) Following the steps in Example 2.19, derive a differential equation for $f'(t)$ whose solution will solve the constant speed problem $s(t) = s_0$.
(b) How does this equation differ from the one for $g'(t)$ in Example 2.19?
(c) Choosing the positive square root [your solution to (a) should involve a square root], integrate the differential equation for $f(t)$ using a computer algebra system to find $f(t)$ for the values $s_0 = 2$ and $v_0 = 1$.
(d) How long does it take the effector to reach height 1?

6. **A Cartesian Robot.** A Cartesian or gantry robot consists of an effector suspended from a beam. (See Figure 2.19.) The effector is able to move vertically, and its distance from the beam is denoted by z with $z = 0$ corresponding to the effector in the raised position and $z = h_0$ corresponding to the effector on the floor (or ground). The point of suspension of the effector is free to slide horizontally along the beam. This position is denoted by x with $0 \leq x \leq w_0$. The beam is in turn free to slide along two rails. The position of the beam along the rails is denoted by y with $0 \leq y \leq l_0$. We will say the robot has height h_0, width w_0, and length l_0.

(a) The possible positions of the effector lie in a box in the first octant (all coordinates are nonnegative) of a Euclidean space with coordinates $\{(z, x, y) : 0 \leq x \leq w_0, 0 \leq y \leq l_0, \text{ and } 0 \leq z \leq h_0\}$. Sketch and label this box and the coordinate axes so that positions with the effector retracted ($z = 0$) are at the top of the box as shown in Figure 2.19.
(b) Are the coordinates in part (a) right-handed or left-handed? Explain.
(c) One method of moving the effector at constant speed between two locations at the same height, say from (z_1, x_1, y_1) to (z_1, x_2, y_2), is to use a linear parametrization of the motion. Write a formula for such a parametrization so that it has constant speed v_0 ft/s.
(d) Suppose the dimensions of a Cartesian robot in feet are $h_0 = 20$, $w_0 = 50$, and $l_0 = 100$. (Cartesian robots used in manufacturing can be quite large.) Construct a parametrization in three parts to lift an object on the floor at location $(20, 20, 0)$ to a height of 10 ft with unit speed, move it horizontally and linearly to $(10, 30, 100)$

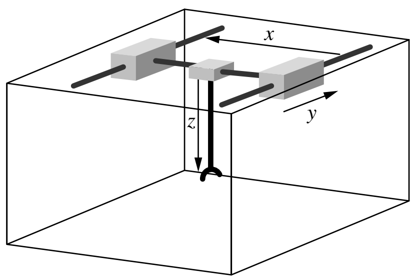

Figure 2.19 A schematic diagram of a Cartesian or gantry robot. (See Exercise 6.)

with constant speed 2 ft/s, and then lower the object to height of 5 ft above the floor, again with unit speed.

(e) Assuming no time is spent accelerating to constant speed or changing velocity at the "corners" of the motion, how long will the entire motion last?

7. A SCARA Robot I. Figure 2.20(a) shows a simplified rendering of a Selective Compliant Articulated Robot Arm, or SCARA robot. The joints at P and Q are hinge joints (ideally) free to move through 2π, and the effector at R moves vertically on a slider joint. SCARA robots are capable of the same tasks as a Cartesian robot, but have a smaller footprint. However, to move the effector of a SCARA robot in a straight line at a fixed height, it is necessary to solve the inverse kinematics problem.

In this problem, we will assume the horizontal links have length 1 and will work in a horizontal plane containing P and Q with coordinates as shown in Figure 2.20(b). The angle between the link $\overline{PQ}$ and the positive x-axis is θ measured counterclockwise, and the angle between the link $\overline{PQ}$ and the link $\overline{QR}$ is ϕ measured clockwise. Thus, for example, when $\theta = 0$ and $\phi = 0$, the projection of R to this plane is located at the origin. With this notation, the projection of R to this plane has coordinates

$$R(\theta, \phi) = (\cos\theta - \cos(\theta - \phi), \sin\theta - \sin(\theta - \phi)).$$

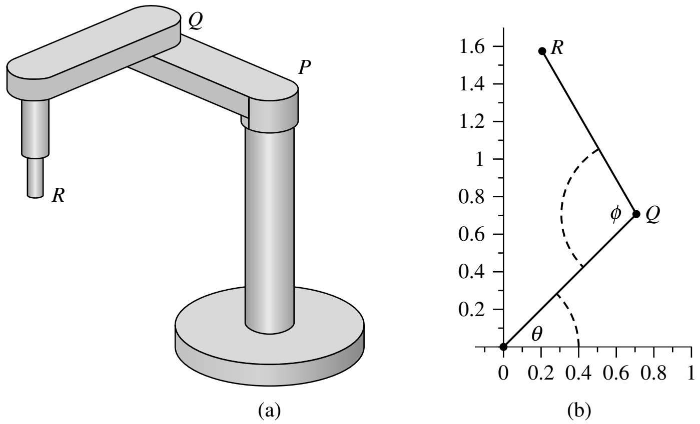

Figure 2.20 (a) A SCARA robot with hinge joint at P and Q and a slider joint moving the effector R. (b) xy-coordinates for the effector of a SCARA robot projected to the horizontal plane containing P and Q. (See Exercise 7.)

(a) Using either Figure 2.20(b) or the above formula, what are the xy-coordinates for

(i) $R(0, \pi/2)$. (ii) $R(\pi/4, \pi/2)$. (iii) $R(\pi/2, \pi/2)$.

(b) For a given angle $\theta < \pi/2$, use the formula to show that $\phi = 2\theta$ locates the projection of the effector R on the y-axis.
(c) If the effector is to be positioned on a line that makes an angle τ with the positive x-axis, what is the relationship between θ and ϕ?

8. A SCARA Robot II. Here we solve the inverse kinematics problem for the SCARA robot of Exercise 7 when the projection of the effector moves along the y-axis.

(a) Suppose that the angle θ is a function of time, $\theta = \theta(t)$. Using Exercise 7(b) and the formula given in Exercise 7, write a parametric formula $r(t)$ for the projection of the projector in terms of $\theta(t)$ when the projection of the effector is located on the y-axis.
(b) Find the velocity and speed of $r(t)$ in terms of θ.
(c) Suppose that the speed of the projection of the effector along the y-axis is a constant s_0. Use your formula for speed in part (b) to derive a differential equation for $\theta(t)$.
(d) Solve your differential equation in part (c) for θ. Note: Since the solution may be rotated, this solves the inverse-velocity kinematics problem for a SCARA robot moving in a radial direction at constant speed.

9. When Does an Epidemic End? In Example 2.20, we noted that when the epidemic ended there were 19 individuals remaining who had not contracted influenza.

(a) If there were still susceptible individuals, why did the epidemic end? Refer to the SIR model in your explanation.
(b) This behavior is a common behavior of epidemics. In general, why is it that not all susceptible individuals become infected in an epidemic?
(c) What does this imply about childhood diseases like the measles and chicken pox? Why is this a health concern?

10. Initial Conditions. The SIR model in Example 2.20 described a flu epidemic at a boarding school of 763 students that began with initial populations of 762 susceptible students and 1 infected student. We analyzed the progress of the epidemic by considering a parametric curve whose coordinates gave the susceptible and infected populations at any time t. Figure 2.21(a) is a plot of the parametric curves for this epidemic that can be derived from the SIR model if we assume different initial populations of susceptibles and infected and assume that the total population is 763. The initial populations used are $(762, 1)$ (this gives the curve that was analyzed in Example 2.20), $(700, 63)$, $(600, 163)$, $(500, 263)$, $(400, 363)$, $(300, 463)$, and $(200, 563)$. For each curve, the infection rate is $r = 2.18 \times 10^{-3}$ and the recovery rate is $a = 4.40 \times 10^{-1}$ as in Example 2.20.

(a) Generally describe the spread of the disease as modeled by these curves.
(b) In general, what is the effect of changing the number of students initially infected by the flu?

(c) For each of these curves, at what point does the epidemic begin to subside? How does that point depend on the initial point? Use the equation for $I'(t)$ and the implicit equation for the curve to explain your answer.

11. Rate of Recovery. The rate of recovery, a, in the SIR model is the coefficient of $I(t)$ in the differential equation for $I'(t)$. The quantity $1/a$ can be interpreted as the average length of time it takes to recover from the disease. Does this make sense for the particular model of Example 2.20? Explain why this makes sense generally.

12. Rate of Transmission. In the construction of the SIR model, we assumed that the rate of change of the susceptible population depended on the size of the susceptible population and the size of the infected population, $S'(t) = -rS(t)I(t)$, where the constant r is the rate of transmission. Another way to read this equation is that S' depends on nothing else about the populations other than their size. Is this a realistic assumption for all outbreaks of a disease? What other factors might affect the rate of change of the susceptible population? Explain.

13. SIR with Immigration. The SIR model in Example 2.20 assumed that the total population of students was constant. We would like to change the basic model to include the regular infusion of new members to the susceptible population. For example, if the disease were a long-lasting one, the model should reflect new people moving into the community.

(a) The plot in Figure 2.21(b) is the parametric curve that represents the susceptible and infected populations for this modified scenario. Describe the spread of the disease as modeled by this curve.

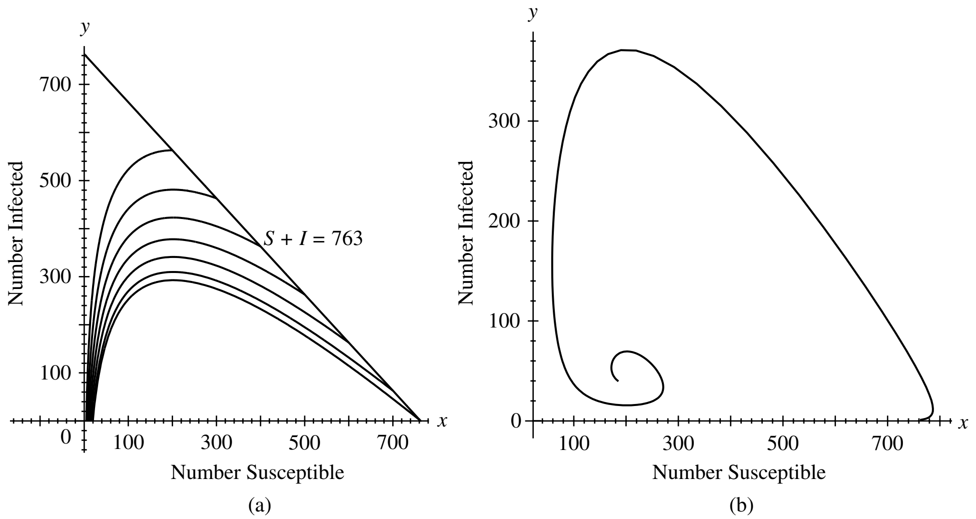

Figure 2.21 (a) The SIR model for different initial conditions. (See Exercise 10.) (b) An SIR model with immigration. (See Exercise 13.)

(b) How might the equations of the SIR model be modified to take into account immigration? Explain your answer.

14. Vaccination. Suppose that a vaccination for the disease is available, which is the case for some strains of influenza. Further, suppose that a community is able to vaccinate a fixed number of people each day. How would you take this into account in your SIR model? Explain your answer.

15. Transmission Without Symptoms. Suppose that individuals who are infected with a disease are contagious before they show any symptoms, but once they display symptoms they are quarantined for two weeks. How might you incorporate this into the basic SIR model? Explain your answer.

■ 2.4 Vector Fields

In this section, we develop the concept of a ***vector field***. Our motivation is to model ***the motion of a moving fluid or gas***, which we call a ***flow***. In the next section, we use vector fields to model other physical phenomena.

We will assume that fluid or gas is moving through a region of the plane or space so that at each location the velocity of the flow does not vary in time. For example, we might think of the steady flow of water in a stream, the prevailing winds in a geographic region, or the regular flow of air across the surface of an airplane wing. In each of these examples, we can imagine focusing on one location and observing that the speed and direction of the water or air do not vary in time. In contrast, there are flows where the direction and speed of the water at a location will vary in time and, in some cases, change unpredictably from one instant to another. These more complicated flows are beyond the scope of our discussion.

For the moment, consider the movement of water on the surface of a stream. First, choose a coordinate system on the surface of the water so that a point on the surface is represented by an ordered pair (x, y). The velocity of the stream at a point (x, y) can be represented by a vector (u, v). Since the coordinates of (u, v) depend on the location and *not* the time, we will think of u and v as being functions of x and y, $u = u(x, y)$, and $v = v(x, y)$. Further, we will think of the collection of velocity vectors as the values of a function $\mathbf{F}$, $\mathbf{F}(x, y) = (u(x, y), v(x, y))$, so that $\mathbf{F}(x, y)$ is the velocity vector of the flow at (x, y). If the fluid is moving through space, the velocity vector is in $\mathbb{R}^3$, and we will write $\mathbf{F}(x, y, z) = (u(x, y, z), v(x, y, z), w(x, y, z))$. However, our primary interest in this chapter is in flows in the plane. Based on this discussion, we have the following definition.

Definition 2.9 A ***vector field*** on a subset $\mathcal{D}$ of the plane, $\mathcal{D} \subset \mathbb{R}^2$, is a function $\mathbf{F} : \mathcal{D} \to \mathbb{R}^2$. In coordinates, $\mathbf{F}(x, y) = (u(x, y), v(x, y))$, where u and v are real-valued functions, $u : \mathcal{D} \to \mathbb{R}$ and $v : \mathcal{D} \to \mathbb{R}$.

If the vector field consists of velocity vectors of a flow, we call it a ***velocity field***. ♦

To "plot" a vector field, we will plot sufficiently many of the vectors $\mathbf{F}(x, y)$ to indicate the qualitative behavior of the corresponding flow. Naturally, when we plot a particular vector $\mathbf{F}(x, y)$, we will place the arrow that represents the vector so that its initial point is located at (x, y). The plots of vector fields that you see in the figures are computer generated. In order to keep the plots from becoming hopelessly cluttered, the plotting software plots a regular pattern or grid of vectors and scales the lengths of the vectors so that the vectors do not overlap. The following example illustrates how we might interpret the plot of a vector field.

Example 2.21

The Jet Stream. The jet stream is a fast-moving current of air high in the atmosphere that determines large-scale weather patterns. Figure 2.22 contains a plot of the velocity of

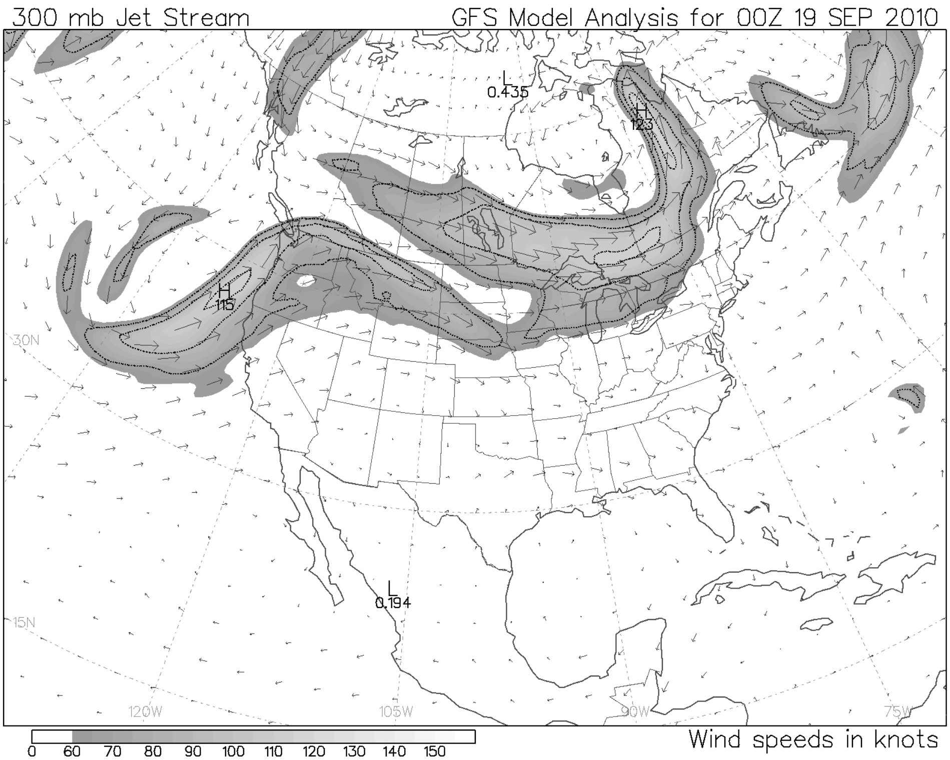

Figure 2.22 A plot of the velocity vector field of the jet stream over North America. Courtesy of the Meteorology program at San Francisco State University. (See Example 2.21.)

the flow of air in the upper atmosphere over North America. The jet stream corresponds to the swatch of longer vectors that meanders from west to east across Canada and the northeastern United States. From the plot, we can see that it moves north over the Pacific Ocean, dips south over British Columbia, moves north to the Arctic, south over Hudson Bay into the United States, and then sweeps north along the Eastern seaboard. The maximum speed of the jet stream, which is represented by the longest vectors, is 145 knots per hour.

We will also find it useful to work with a symbolic expression for the coordinates of a vector field. For example, the following gives an idealized model for the flow of fluid in a pipe.

Example 2.22 **Flow Through a Pipe**. When fluid moves through a pipe at a constant rate, that is, at a constant volume per unit time, the velocity of the fluid is not constant. Because fluid adheres to the walls of the pipe, fluid near the center of the pipe moves more rapidly than fluid near the walls of the pipe. A flow of this type is called a ***Poiseuille flow***. If we choose a coordinate system with the z-axis aligned with the center of the pipe, there is a simple symbolic expression for the velocity vector field of the flow. If the pipe has radius r_0, the speed of the fluid at the center of the pipe is v_0, and the fluid is moving in the direction of positive z direction, the velocity field of the flow is

$$\mathbf{F}(x,y,z) = \left(0, 0, v_0(1 - \frac{x^2+y^2}{r_0^2})\right)$$

for $x^2 + y^2 \le r_0^2$. The speed of the fluid at a point (x, y, z) is $\|\mathbf{F}(x, y, z)\|$. In this case, $\|\mathbf{F}(x, y, z)\| = v_0(1 - (x^2 + y^2)/r_0^2)$.

Flow Lines of Vector Fields

A useful way to understand the motion of a fluid is to think about the motion of an individual particle in the flow. This time, imagine placing a cork in a moving stream at point $P = (x_0, y_0)$ at time $t = 0$. The cork will move downstream with the flow, and the velocity of the cork at a point will be the velocity of the fluid at the point. If the motion of the cork is parametrized by a function α, its velocity at time t is $\alpha'(t)$. Since the cork is located at $\alpha(t) = (x(t), y(t))$, the vector $\alpha'(t)$ must equal $\mathbf{F}(x(t), y(t))$. Focusing on the mathematical content of this discussion, we are led to the following definition.

Definition 2.10 A ***flow line*** of a vector field $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ with ***initial point*** (x_0, y_0) is a parametrization $\alpha : [0, b] \to \mathbb{R}^2$ such that $\alpha(0) = (x_0, y_0)$ and

$$\alpha'(t) = \mathbf{F}(\alpha(t)), \quad t \in [0, b].$$

In coordinates, if $\alpha(t) = (x(t), y(t))$, then $\alpha(0) = (x(0), y(0)) = (x_0, y_0)$ and

$$(x'(t), y'(t)) = (u(x(t), y(t)),\ v(x(t), y(t))). \blacklozenge$$

Intuitively, if α is a flow line of $\mathbf{F}$, the tangent vector α' must be a vector of the vector field. In order to verify that a parametrization α is a flow line of $\mathbf{F}$, it is necessary to show that $\alpha'(t) = \mathbf{F}(\alpha(t))$ for all $t \in [0, b]$. That is, we must show that the coordinates of α satisfy the equations

$$x'(t) = u(x(t), y(t)) \quad \text{and} \quad y'(t) = v(x(t), y(t)).$$

Let us apply these ideas to a flow in the plane.

Example 2.23 **A Circular Flow.** Here we investigate the flow lines of the vector field $\mathbf{F}(x, y) = (-y, x)$. A plot of $\mathbf{F}$ near the origin is contained in Figure 2.23(a).

A. An Intuitive Argument. Imagine that $\mathbf{F}$ is the velocity field of a fluid flow. Based on the plot, we might conjecture that the fluid is circulating around the origin in a counterclockwise manner. This is supported by a simple calculation. The dot product of $\mathbf{F}(x, y) = (-y, x)$ and the direction vector (x, y) from the origin is always zero, $(-y, x) \cdot (x, y) = 0$. This says that $\mathbf{F}$ is always perpendicular to the vector (x, y), so that at each location in the plane the fluid is moving in a direction perpendicular to the direction to the origin, which is the case when a particle moves in a circle about the origin. (See Figure 2.23(b).)

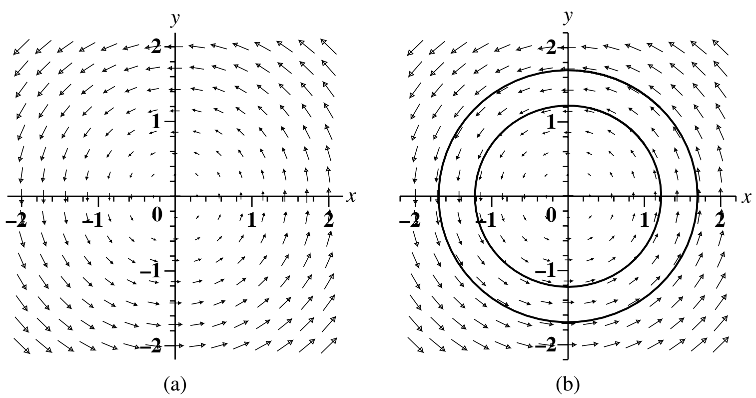

Figure 2.23 (a) The vector field $\mathbf{F}(x, y) = (-y, x)$. (b) Flow lines of the vector field $\mathbf{F}(x, y) = (-y, x)$. (See Example 2.23.)

B. A Calculation. Our conjecture is, in fact, correct and can be verified symbolically. Define, for example, $\alpha(t) = (R\cos t, R\sin t)$, for $t \in \mathbb{R}$, where $R > 0$. We know from our work with curves that α is a constant speed parametrization of a circle of radius R centered at the origin. It remains to show that α satisfies the equations of a flow line of $\mathbf{F}$. First, $\alpha'(t) = (-R\sin t, R\cos t)$. Substituting $\alpha(t)$ into the formula for $\mathbf{F}$, we have

$$\begin{aligned}\mathbf{F}(\alpha(t)) &= (-y(t), x(t)) \\ &= (-R\sin t, R\cos t) \\ &= \alpha'(t).\end{aligned}$$

Further, $\alpha(0) = (R, 0)$. These calculations show that α is a flow line of $\mathbf{F}$ with initial point $\alpha(0) = (R, 0)$.

As the example demonstrates, given formulas for α and $\mathbf{F}$, it is possible to check if α is a flow line of $\mathbf{F}$. We need only substitute the formulas for $x = x(t)$ and $y = y(t)$ into the formulas for u and v and check that the resulting expressions are equal to x' and y'. Unfortunately, it is more often the case that we need to analyze flow lines of $\mathbf{F}$ knowing only the formulas for u and v. In order to find x and y, we have to solve the differential equations

$$\begin{aligned}x'(t) &= u(x(t), y(t)) \\ y'(t) &= v(x(t), y(t))\end{aligned}$$

for x and for y. We will call these equations the ***flow line equations*** of the vector field $\mathbf{F}$. If the expressions for u and v are sufficiently simple, it is possible to solve these equations explicitly using ideas from calculus. The next example demonstrates this for the vector field $\mathbf{F}(x, y) = (2x, y)$.

Example 2.24 **A Symbolic Solution to the Flow Line Equations.** Let $\mathbf{F}(x, y) = (2x, y)$. A plot of $\mathbf{F}$ is given in Figure 2.24. Let us find the flow line α of $\mathbf{F}$ whose initial point is $(\frac{1}{4}, \frac{1}{2})$. The coordinate functions $x = x(t)$ and $y = y(t)$ of α must satisfy $\alpha(0) = (x(0), y(0)) = (\frac{1}{4}, \frac{1}{2})$ and the flow line equations for $\mathbf{F}$:

$$\begin{aligned}x'(t) &= 2x(t) \\ y'(t) &= y(t).\end{aligned}$$

Each one is a differential equation for exponential growth. To solve these equations, we use the fact from one variable calculus that if a function f satisfies the differential equation $f'(t) = kf(t)$ for $k \neq 0$, then $f(t) = f(0)e^{kt}$. The solution to the first equation

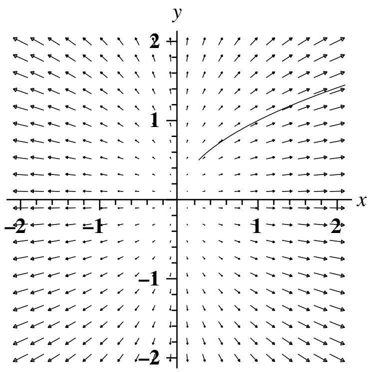

Figure 2.24 The vector field $\mathbf{F}(x, y) = (2x, y)$ and the flow line with initial point $(\frac{1}{4}, \frac{1}{2})$. (See Example 2.24.)

is $x(t) = x(0)e^{2t}$, so that $x(t) = \frac{1}{4}e^{2t}$, and the solution to the second equation is $y(t) = y(0)e^t$ or $y(t) = \frac{1}{2}e^t$. The flow line is given by

$$\alpha(t) = (\tfrac{1}{4}e^{2t}, \tfrac{1}{2}e^t).$$

Notice that as t approaches ∞, each of the coordinates of α approaches ∞. In this case, the x-coordinate of α is the square of the y-coordinate, $x(t) = y(t)^2$. This means that the image of the flow line α traces a portion of the graph of the parabola $x = y^2$. The image of α is shown in Figure 2.24.

For most vector fields, it is not possible to find symbolic solutions to the flow line equations. Nevertheless, we can use intuitive graphical techniques to sketch the curves traced by flow lines. The idea is to sketch a curve starting at the initial point whose tangent vectors are vectors of the vector field. Since vector field plots contain a relatively small number of vectors, this is a rough process when done by hand. It will not give us quantitative information, that is, it will not tell us the location of $\alpha(t)$ for particular values of t, but it is potentially useful when trying to sort out the qualitative behavior of a flow. We can also use computer software packages to generate pictures of flow lines that give qualitative information about the flow.

Example 2.25

Computer-Generated Flow Lines. Consider the vector field $\mathbf{F}(x, y) = (-y+0.1(x^2-1), x^2 - 1 + 0.1y)$, which is plotted in Figure 2.25(a). The vectors of the field near $(1, 0)$ indicate a counterclockwise flow around this point. Figure 2.25(b) is a plot of this vector field along with the computer-generated flow lines for the initial points $(1.5, 0)$, $(0, 1.2)$, $(0, 1.044)$, and $(-1.75, -1.25)$. Initially, the flow line beginning at $(1.5, 0)$ circles the point $(1, 0)$ in a counterclockwise manner. However, as t increases, the flow line moves into the

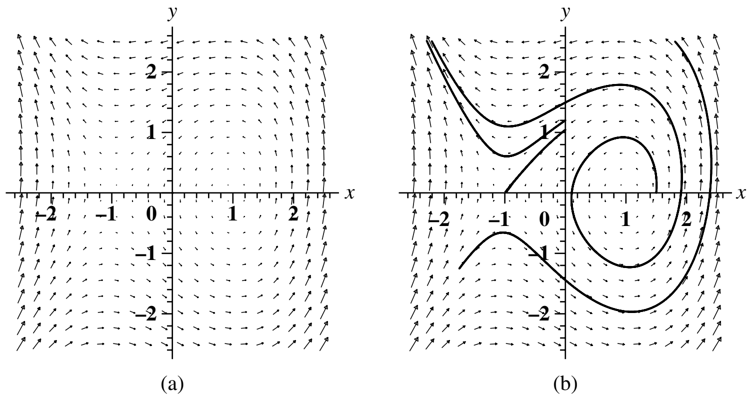

Figure 2.25 (a) The vector field $\mathbf{F}(x, y) = (-y + 0.1(x^2 - 1), x^2 - 1 + 0.1y)$. (b) Flow lines of $\mathbf{F}$ with initial points $(1.5, 0)$, $(0, 1.2)$, $(0, 1.044)$, and $(-1.75, -1.25)$. (See Examples 2.25 and 2.26.)

second quadrant and eventually moves toward ∞. The flow line beginning at $(0, 1.2)$ begins by moving toward the point $(-1, 0)$ but then moves away from this point and off toward ∞ in the second quadrant. The flow line beginning at $(0, 1.044)$ appears to approach the point $(-1, 0)$. The flow line beginning at $(-1.75, -1.25)$ moves toward the point $(-1, 0)$ and then moves off into the fourth quadrant, following the first flow line. Since the length of $\mathbf{F}$ increases further from the origin, the flow lines are traversed more rapidly further from the origin. Notice that each of the flow lines is tangent to very few of the plotted vectors $\mathbf{F}(x, y)$. Nevertheless, the flow lines appear to closely follow the directions of the plotted vectors.

As useful as a computer plot of flow line can be, the plot itself does not give the quantitative information of the location of $\alpha(t)$ for particular t values. Depending on the software, however, it is possible to generate values for the coordinates of $\alpha(t)$ for particular t.

Critical Points of Vector Fields

In trying to understand the behavior of a flow, points where the velocity of the flow are zero play a special role. First, let us make a definition.

Definition 2.11 A ***critical point*** or ***equilibrium point*** of a vector field $\mathbf{F}(x, y)$ is a point (x_0, y_0) such that $\mathbf{F}(x_0, y_0) = (0, 0)$. ♦

Intuitively, a critical point of fluid flow corresponds to a point where the fluid is not moving. If we were to place a cork in a fluid flow at a critical point, it would remain at that point. For this reason, critical points are also called equilibrium points. More formally, if (x_0, y_0) is a critical point of $\mathbf{F}$, we can define a parametrization α by $\alpha(t) = (x_0, y_0)$ for all t. Then $\alpha'(t) = (0, 0)$ for all t. Substituting into the flow line equations, we have $\mathbf{F}(\alpha(t)) = \mathbf{F}(x_0, y_0) = (0, 0) = \alpha'(t)$. This says that α is a flow line of $\mathbf{F}$ for the initial point (x_0, y_0). Of course, this α describes the motion of an object at rest at the point (x_0, y_0). In contrast, if we drop a cork *near* a critical point, its subsequent behavior can be quite interesting. Let's return to Example 2.25 for a moment.

Example 2.26

The Flow near a Critical Point. Consider the flow that gives rise to the vector field $\mathbf{F}(x, y) = (-y + 0.1(x^2 - 1), x^2 - 1 + 0.1y)$ of Example 2.25. To find where $\mathbf{F}$ is $(0, 0)$, we must solve the equations

$$\begin{aligned} -y + 0.1(x^2 - 1) &= 0 \\ x^2 - 1 + 0.1y &= 0. \end{aligned}$$

Solving the first equation for y, we get $y = 0.1(x^2 - 1)$. Substituting this into the second equation, we get $x^2 - 1 + 0.1(0.1(x^2 - 1)) = 0$, which simplifies to $1.01(x^2 - 1) = 0$. The solutions to this equation are $x = 1$ and $x = -1$. Substituting these into the first equation and solving for y, we find that this vector field is zero at $(1, 0)$ and $(-1, 0)$. If we were to place a cork in the flow at either of these points, it would remain there forever. From Example 2.25, if we were to place a cork in the flow at $(0, 1.044)$ it would approach the critical point at $(-1, 0)$. If we were to place the cork at nearby points, for example, $(0, 1.2)$ the cork would approach $(-1, 0)$ and then veer off. In contrast, if we put a cork into the flow at any point near $(1, 0)$, it would trace a spiral-like curve around $(1, 0)$ and then move off to infinity. (See Figure 2.25(b).)

Intuitively, if we choose an initial point near a critical point, the long-term behavior of the flow line falls into two categories. It might be the case that the flow line remains close to the critical point for all time, or it might be the case that the flow line eventually moves away from the critical point. These two types of behavior can also be used to assign critical points to two categories, that is, to ***classify*** critical points. We will say that a critical point is ***stable*** if every flow line that starts near the critical point remains near the critical point for all time. If this is not the case, we say that a critical point is ***unstable***. In other words, a critical point is unstable if for at least some initial points near the critical point, the flow lines eventually move away from the critical point.

Returning to our cork–fluid analogy, if a cork is placed in a fluid flow near a stable critical point, it will remain near the critical point for all time. On the other hand, it can be placed near an unstable critical point so that it will eventually move away from the critical point.

In order to determine if a critical point is stable or unstable, we will plot flow lines for initial points near the critical point. In most of the examples we will consider, it will be sufficient to examine only a few flow lines in order to make a determination about the critical point.

Example 2.27 Stable and Unstable Critical Points

A. The vector field $\mathbf{F}(x, y) = (-y, x)$ of Example 2.23 has only one critical point, which is located at the origin, $(0, 0)$. The flow lines of $\mathbf{F}$ are circles centered at $(0, 0)$ that are traversed counterclockwise. Since flow lines that start near $(0, 0)$ remain near $(0, 0)$ for all time, it is a stable critical point.

B. The vector field $\mathbf{F}(x, y) = (2x, y)$ of Example 2.24 also has a critical point at the origin. In this case, since the flow lines of $\mathbf{F}$ that begin near the origin flow away from the origin, it is an unstable critical point. (See Exercise 3.)

C. In Examples 2.25 and 2.26, we considered the vector field $\mathbf{F}(x, y) = (-y + 0.1(x^2 - 1), x^2 - 1 + 0.1y)$. It has two critical points, $(-1, 0)$ and $(1, 0)$. We saw that some flow lines that start near $(-1, 0)$ flow away from the point, while one flow line appears to flow toward it. On the other hand, all flow lines that start near $(1, 0)$ flow away from the point. (See Figure 2.25(b).) It follows that each critical point is unstable.

D. Consider the vector field, $\mathbf{F}(x, y) = (-2x - y, 4x - 7y)$, which we have not considered before. If we solve the system of equations

$$-2x - y = 0$$
$$4x - 7y = 0,$$

we see that there is a single critical point at $(0, 0)$. Figure 2.26 shows the flow lines of $\mathbf{F}$ for the initial points $(0.5, 2)$, $(0.5, -2)$, $(-0.5, -2)$, $(2, 0)$, $(-2, 0)$, and $(-1, 2)$. Each flow line moves toward the critical point at the origin. Since all the vectors near the origin point toward the origin, it is reasonable to conclude that every flow line of $\mathbf{F}$ that starts near the origin will move toward the origin. Thus we conclude that $(0, 0)$ is a stable critical point.

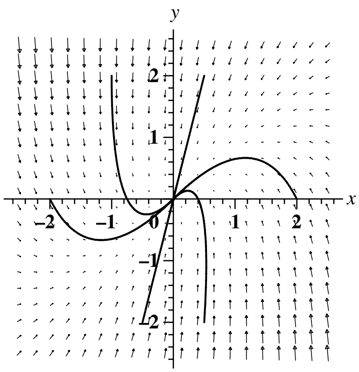

Figure 2.26 Flow lines of the vector field $\mathbf{F}(x, y) = (-2x - y, 4x - 7y)$. (See Example 2.27D.)

Based on Example 2.27, we can refine our classification of critical points by dividing each category, stable and unstable, into two subcategories. In Example 2.27A, the flow lines of $\mathbf{F}(x, y) = (-y, x)$ near the stable critical point at the origin circle the origin, but do not flow in toward the origin. In Example 2.27D, the flow lines near the stable critical point at the origin all flow toward the origin. For unstable critical points, we also see two options. In Example 2.27C, all the flow lines that start near $(1, 0)$ flow away from $(1, 0)$, while only some of the flow lines that start near $(-1, 0)$ flow away from $(-1, 0)$. These distinctions are summarized in the following definition.

Definition 2.12 A stable critical point is called a ***sink*** if all flow lines that begin near the critical point flow toward the critical point. A stable critical point is called a ***center*** if all flow lines that begin near the critical point trace closed curves around the critical point.

An unstable critical point is called a ***source*** if all flow lines that begin near the critical point flow away from the critical point. An unstable critical point is called a ***saddle*** if some flow lines that begin near the critical point flow away from the critical point and others flow toward the critical point. ♦

Using this language, we see that the stable critical point of $\mathbf{F}(x, y) = (-y, x)$ at $(0, 0)$ is a center and the stable critical point of $\mathbf{F}(x, y) = (-2x - y, 4x - 7y)$ at $(0, 0)$ is a sink. The unstable critical point of $\mathbf{F}(x, y) = (-y + 0.1(x^2 - 1), x^2 - 1 + 0.1y)$ at $(1, 0)$ is a source, and the unstable critical point at $(-1, 0)$ is a saddle.

Summary

The discussion in this section was motivated by an intuitive understanding of the behavior of a moving fluid, that is, of a ***fluid flow***. Focusing on flows in the plane, we defined a ***vector field*** to be a function $\mathbf{F}$ that assigns a vector, $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ to points (x, y) in the plane. A ***flow line*** α of a vector field $\mathbf{F}$ is a parametrization whose tangent vector α' is a vector of the field $\mathbf{F}$ at all times. If the vector field is the velocity field of a fluid flow, a flow line describes the motion of a particle placed in the flow at the initial point of the flow line.

To understand the behavior of a vector field on an intuitive or geometric level, we studied the behavior of flow lines that start near ***critical*** or ***equilibrium points*** of the vector field. These are points (x, y) where the field is zero, that is, where $\mathbf{F}(x, y) = (0, 0)$. Based on the behavior of nearby flow lines, we ***classified*** critical points into two categories, ***stable*** and ***unstable***. Then we further classified stable critical points into ***sinks*** and ***centers*** and unstable critical points into ***sources*** and ***saddles***.

Section 2.4 Exercises

1. **Flow Lines of Vector Fields.** For each of the following vector fields, proceed as follows. (i) Sketch the flow lines starting at the initial points $(\pm 1, 0)$, $(0, \pm 1)$, $(1, \pm 1)$, and $(-1, \pm 1)$. (These may also be plotted using a computer algebra system, if one is available

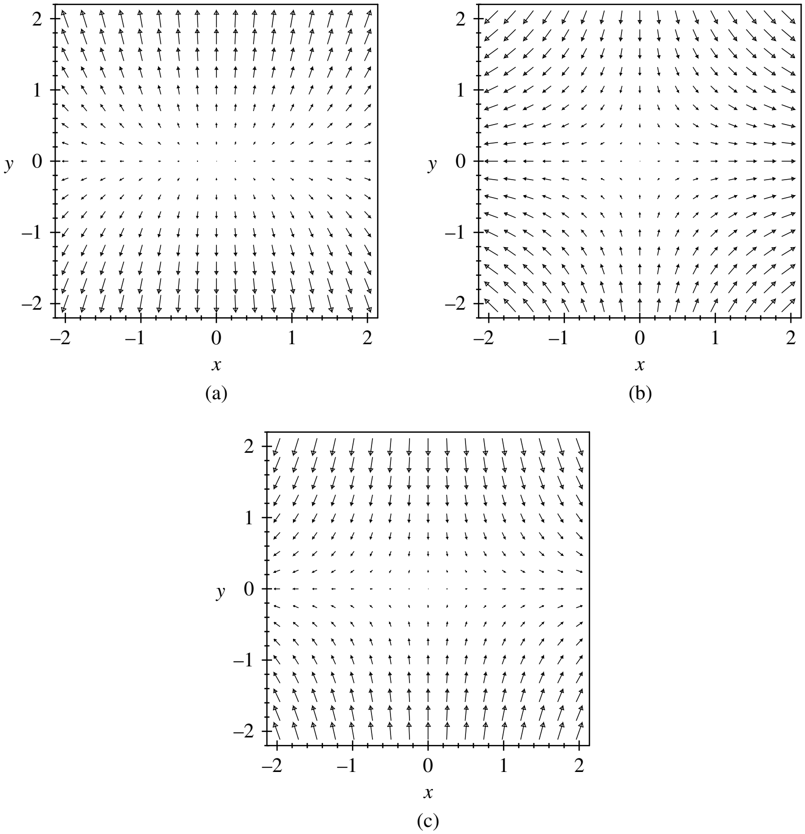

Figure 2.27 Vector fields for Exercise 1 (a)–(c).

to you.) (ii) Use your sketch to classify the critical point of $\mathbf{F}$ at the origin as stable or unstable and as a center, sink, source, or saddle.

(a) $\mathbf{F}(x, y) = (x, 3y)$. (See Figure 2.27(a).)
(b) $\mathbf{F}(x, y) = (x, -y)$. (See Figure 2.27(b).)
(c) $\mathbf{F}(x, y) = (x, -3y)$. (See Figure 2.27(c).)

2. **Verifying Parametrizations of Flow Lines.** For each of the following vector fields and parametrizations, verify by a direct calculation that the function α parametrizes a flow line of the vector field and find the initial point of the flow line.

(a) $\mathbf{F}(x, y) = (3x, 2y)$.

(i) $\alpha(t) = (e^{3t}, e^{2t})$.
(ii) $\alpha(t) = (-2e^{3t}, 4e^{2t})$.

(b) $\mathbf{F}(x, y) = (x - 1, y)$.

(i) $\alpha(t) = (2e^t + 1, 3e^t)$.
(ii) $\alpha(t) = (1 - e^t, -e^t)$.

3. Symbolic Solutions to the Flow Line Equations. Let $\mathbf{F}(x, y) = (2x, y)$ be the vector field of Example 2.24.

(a) Following the method of Example 2.24, use techniques from one-variable calculus to construct parametrizations of the flow lines of $\mathbf{F}$ with initial points at $(-2, 1)$, $(-1, 3)$, and $(0, 1)$.
(b) Each of the flow lines from part (a) traces a portion of the graph of a function of y. For each flow line in (a), write out the expression for this function in the form $x = g(y)$.
(c) Does every flow line of $\mathbf{F}$ trace part of the graph of a function $x = g(y)$? Explain your answer.

4. Flow Lines and Initial Points. In each of the following parts, the general form for a flow line α of the vector field $\mathbf{F}$ is given in terms of constants A and B. Verify by a direct calculation that the function α parametrizes a flow line of the vector field. For each of the given initial points (x_0, y_0), find the values of A and B that give the flow line with (x_0, y_0) as the initial point. (*Hint:* Set $\alpha(0) = (x_0, y_0)$ and solve for A and B.)

(a) $\mathbf{F}(x, y) = (-x, 2y)$. $\alpha(t) = (Ae^{-t}, Be^{2t})$. Initial points: (i) $(x_0, y_0) = (2, 1)$. (ii) $(x_0, y_0) = (-3, 3)$.
(b) $\mathbf{F}(x, y) = (x + 2, y + 1)$. $\alpha(t) = (Ae^t - 2, Be^t - 1)$. Initial points: (i) $(x_0, y_0) = (1, 1)$. (ii) $(x_0, y_0) = (0, 1)$.
(c) $\mathbf{F}(x, y) = (-2x + y, -x)$. $\alpha(t) = (Ae^{-t} + Bte^{-t}, (A + B)e^{-t} + Bte^{-t})$. Initial points: (i) $(x_0, y_0) = (1, 1)$. (ii) $(x_0, y_0) = (1, -1)$.
(d) $\mathbf{F}(x, y) = (2x + 2y, -x)$. $\alpha(t) = (Ae^t(\cos t + \sin t) + Be^t(\sin t - \cos t), -Ae^t \sin t + Be^t \cos t)$. Initial points: (i) $(x_0, y_0) = (1, 0)$. (ii) $(x_0, y_0) = (-1, 1)$.

5. Critical Points of Vector Fields. Determine the critical points of each of the following vector fields on the indicated domain, and based on the accompanying plot, determine the type of the critical points. (If a computer algebra system is available to you, you may find it helpful to plot several flow lines of the given vector fields.)

(a) $\mathbf{F}(x, y) = (2x + y - 1, -y + x)$, $[-1.5, 1.5] \times [-1.5, 1.5]$. (See Figure 2.28(a).)
(b) $\mathbf{F}(x, y) = (xy - x, xy - y)$, $[-1.5, 1.5] \times [-1.5, 1.5]$. (See Figure 2.28(b).)
(c) $\mathbf{F}(x, y) = (\cos(x - y), xy)$, $[-2, 2] \times [-2, 2]$. (See Figure 2.28(c).)

6. Exponential Functions and Flow Lines. From one-variable calculus, we know that the function $f(t) = Ae^{kt}$ is the unique solution to the differential equation $f'(t) = kf(t)$ subject to the initial condition $f(0) = A$.

(a) Use this fact to construct the flow line of the vector field $\mathbf{F}(x, y) = (3x, 2y)$ with initial point $(x_0, y_0) = (1, -2)$.

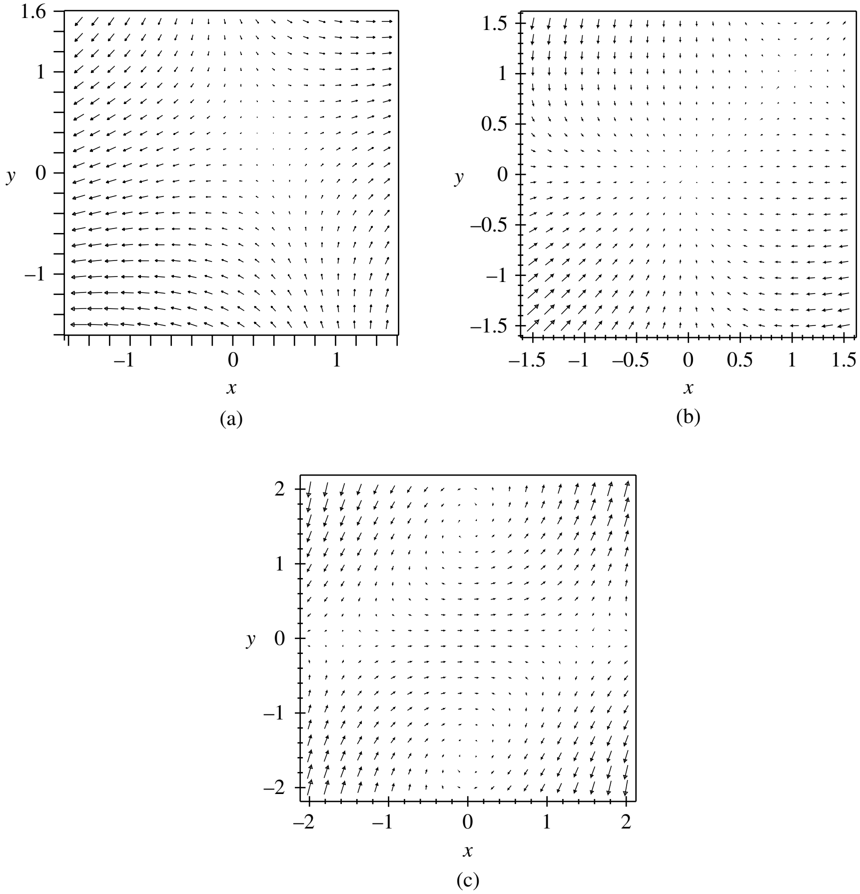

Figure 2.28 Vector fields for Exercise 5.

(b) Use this fact to construct the general form for a flow line of the vector field $\mathbf{F}(x, y) = (kx, ly)$, where k and l are constants, with initial point (x_0, y_0).
(c) Give a qualitative description of the behavior of the flow lines of $\mathbf{F}(x, y) = (kx, ly)$ as $t \to \infty$, if (i) $k > 0,\ l > 0$, (ii) $k > 0,\ l < 0$, (iii) $k < 0,\ l > 0$, and (iv) $k < 0,\ l < 0$.

7. Shifting and Scaling. In Example 2.23, we saw that $\alpha(t) = (R\cos t, R\sin t)$ is a flow line of the vector field $\mathbf{F}(x, y) = (-y, x)$.

(a) Verify that $\beta(t) = \alpha(t + k)$, where k is a constant is also a flow line of $\mathbf{F}$.
(b) Suppose that α is replaced by $\beta(t) = \alpha(ct)$, where c is a constant, $c \neq 0,\ 1$. Is β a flow line of $\mathbf{F}$? Why or why not?

8. Shifting and Scaling: The General Case. Suppose that $\alpha(t) = (x(t), y(t))$ is a flow line of the vector field $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ with initial point (x_0, y_0).

(a) Suppose that α is replaced by $\beta(t) = \alpha(t + k)$, where k is a constant. Is β a flow line of $\mathbf{F}$? Why or why not? If so, what is the initial point of β? If not, is β the flow line of a vector field related to $\mathbf{F}$?

(b) Suppose that α is replaced by $\beta(t) = \alpha(ct)$, where c is a constant, $c \neq 0,\ 1$. Is β a flow line of $\mathbf{F}$? Why or why not? If so, what is the initial point of β? If not, is β the flow line of a vector field related to $\mathbf{F}$?

9. The Vector Field $-\mathbf{F}$. Consider the vector field $\mathbf{F}(x, y) = (2x, y)$ of Example 2.24 and the vector field $\mathbf{G}(x, y) = -\mathbf{F}(x, y) = (-2x, -y)$.

(a) Sketch the vector field $\mathbf{G}$ and several of its flow lines. How do the vector fields $\mathbf{F}$ and $\mathbf{G}$ differ?

(b) What, if any, is the relationship between the flow lines of $\mathbf{F}$ and $\mathbf{G}$? Explain. (These may also be plotted using a computer algebra system, if one is available to you.)

(c) What, if any, is the relationship between the critical points of $\mathbf{F}$ and the critical points of $\mathbf{G}$?

10. $-\mathbf{F}$, the General Case. Consider an arbitrary vector field $\mathbf{F}(x, y) = (u(x, y), v(x, y))$ and the vector field $\mathbf{G}(x, y) = (-u(x, y), -v(x, y))$.

(a) How do the vector fields $\mathbf{F}$ and $\mathbf{G}$ differ?

(b) What, if any, is the relationship between the flow lines of $\mathbf{F}$ and $\mathbf{G}$? Explain.

11. Critical Points of $-\mathbf{F}$. Suppose that $\mathbf{F}(x, y)$ has a critical point at (x_0, y_0).

(a) Show that $\mathbf{G}(x, y) = -\mathbf{F}(x, y)$ also has a critical point at (x_0, y_0).

(b) For each of the types for the critical point of $\mathbf{F}$ at (x_0, y_0) (sink, center, source, saddle), what is the type of the critical point of $\mathbf{G}$ at (x_0, y_0)? Explain.

12. Intersecting Flow Lines? Let α and β be parametrizations of two different flow lines of a vector field $\mathbf{F}$. Is it possible for the image of α and the image of β to intersect? Explain why or why not.

13. Limit Cycles. Define a vector field $\mathbf{F}$ by

$$\begin{aligned}\mathbf{F}(x, y) = (&x \sin(\pi\sqrt{x^2 + y^2}) + y \cos(\pi\sqrt{x^2 + y^2}),\\ &y \sin(\pi\sqrt{x^2 + y^2}) - x \cos(\pi\sqrt{x^2 + y^2})).\end{aligned}$$

(a) Show that unit circle parametrized $\alpha(t) = \cos(\pi t), \sin(\pi t))$ is a flow line of $\mathbf{F}$.

(b) $\mathbf{F}$ has a critical point at the origin. Use a computer algebra system to plot $\mathbf{F}$. What is the type of critical point at the origin?

(c) Use a computer algebra system to plot flow lines of $\mathbf{F}$ with initial point near the origin. The unit circle is called a *limit cycle* for this vector field. How does your polot justify this terminology? (*Hint:* Be sure to use a sufficiently large time interval.)

2.5 Modeling with Vector Fields

In Section 2.4, we used fluid flows to motivate our discussion of vector fields, flow lines, and critical points. Here we would like to turn things around and use the mathematical objects to help us to think about real-world phenomena. For example, we already know that the velocity field of a fluid flow can be used to understand the motion of the fluid. In this section, we will use vector fields to construct mathematical models of two "real-world" systems that evolve in time: a predator–prey model and a pendulum model.

In each case, we will use two quantities to describe the behavior of the system. We will think of these quantities as coordinates of a coordinate system in the plane. For now, call them x and y. Since x and y depend on time, we will write them as functions of time, $x = x(t)$ and $y = y(t)$. If we know values of x and y at time t, we will say that we know the ***state of the system at time*** t. In each case, we will start with enough information to write out formulas for x' and y', the rates of change of x and y, but not with enough information to write out formulas for x and y.

In order to proceed, we will have to make an important assumption about x' and y'. We will assume that at time t the rates of change x' and y' can be expressed in terms of the values of x and y at time t. We express this by writing x' and y' as functions of x and y, which in turn depend on t:

$$\begin{aligned} x'(t) &= u(x(t), y(t)) \\ y'(t) &= v(x(t), y(t)). \end{aligned}$$

These are the flow line equations for the vector field $\mathbf{F}$ whose coordinate functions are u and v, $\mathbf{F}(x, y) = (u(x, y), v(x, y))$. We will call the differential equations for x' and y' a ***mathematical model*** for the physical system described by x and y.

As in Section 2.4, a flow line α for the initial point (x_0, y_0) will tell us what happens to the quantities x and y if they start at the values $x = x_0$ and $y = y_0$ and time is allowed to evolve. Since x and y describe the original physical system, knowledge of the flow line starting at (x_0, y_0) will tell us what happens to the system when it starts in the state (x_0, y_0). For example, if (x_0, y_0) is a critical point of $\mathbf{F}$, we know that a flow line that starts at (x_0, y_0) remains there for all time since its velocity is always zero. In physical terms, (x_0, y_0) describes an ***equilibrium state of the system***. If the system starts in an equilibrium state, it remains there for all time. As we explore the models in this section, we will see that the type of a critical point has important implications for the behavior of the physical system.

As we did in Section 2.3, we will describe the physical system in some detail before we construct our model. This is necessary in order for us to be able to draw meaningful conclusions about the physical system.

Predator–Prey Model

Here we want to construct a mathematical model for the population sizes of two species that interact as predators and their prey. (See the collaborative exercise at the beginning of Chapter 2.) For example, we might think of sharks and fish or coyote and prairie dogs. Let $x = x(t)$ represent the number of prey and $y = y(t)$ represent the number of predators. We will begin by making several assumptions about these populations and their interactions. For each rate of change, we will end up with a sum of positive and negative terms. The positive terms represent factors that cause the population to increase, and the negative terms represent factors that cause the population to decrease.

Our primary assumption is that the only factors affecting the rates of growth of predator and prey can be expressed in terms of the number of predators and the number of prey. This means that we will neglect interactions with other species or with the surrounding environment. Now let us focus on x', the rate of growth of the prey population.

We will assume that in the absence of predators, the number of prey grows at a rate proportional to the number of prey,

$$x'(t) = ax(t).$$

This is the differential equation that describes exponential growth. If $x(0) = x_0$, the solution to this equation is given by $x(t) = x_0e^{at}$. Thus in the absence of predators, we are assuming that the prey population will grow exponentially. In the presence of predators, the rate of change of the prey must reflect the number of incidents of predation in a unit of time. Intuitively, we would expect that there will be more contacts between predator and prey, hence more predation, if either the number of prey increases or the number of predators increases. The simplest way to express this is to say that the number of contacts is proportional to the product of the number of the predators and the number of prey. Since this should cause a decrease in the rate of growth of the prey, we conclude that the presence of predators contributes a term of the form $-bxy$ to the total rate of change of the prey. Assuming that there are no other factors affecting the rate of change of the prey population, we have

$$x'(t) = ax(t) - bx(t)y(t),$$

where a and b are positive constants.

The situation for predators is somewhat different. The predator population should increase as the number of contacts between predator and prey increases. Thus the expression for y' should include a term of the form dxy, where d is a positive constant. On the other hand, since predators compete for the same food source—the prey—the more predators there are, the less food that is available for each predator, thus driving down the population of predators. Thus the rate of change of predators should also contain a negative term that increases as the predator population increases. We will assume that this term is proportional to the predator population and is of the form $-cy$, where c is a positive constant. Assuming there are no other factors affecting the rate of change of the predator population, we have

$$y'(t) = -cy(t) + dx(t)y(t),$$

where c and d are positive constants.

Below is a summary of this discussion.

The Predator–Prey Model

Denote the prey population by $x = x(t)$ and the predator population by $y = y(t)$. The ***predator–prey*** model for the interaction of the two species is given by the pair of differential equations for the rate of change of x and the rate of change of y,

$$\begin{aligned} x'(t) &= \quad ax(t) - bx(t)y(t) \\ y'(t) &= -cy(t) + dx(t)y(t), \end{aligned}$$

where a, b, c, and d are positive constants.

It is important to note that these equations are ***autonomous***. That is, the time variable does not appear other than as the independent variable for x and y. For example, in one variable, the differential equation $x'(t) = (t+1)x(t)$, is ***nonautonomous***. Autonomous systems of equations are represented by vector fields that do not vary in time.

The vector field $\mathbf{F}$ corresponding to the predator–prey equations is

$$\mathbf{F}(x, y) = (ax - bxy, -cy + dxy).$$

Since x and y must be nonnegative, we will focus on the first quadrant. A flow line α of $\mathbf{F}$ with initial point $\alpha(0) = (x_0, y_0)$ will represent the behavior of the two populations for all time if the initial value of the prey population is x_0 and the initial value of the predator population is y_0.

In the next example, we assign numerical values to the coefficients so that we can analyze the plot and flow lines of $\mathbf{F}$.

Example 2.28

A Predator-Prey Vector Field. Figure 2.29 is a plot of the predator–prey vector field $\mathbf{F}(x, y) = (1.5x - xy, -3y + xy)$, where the prey or x-axis is marked in units of 1000 and the predator or y-axis is marked in units of 100. The initial point of the flow line is $(1, 2)$, which corresponds to 1000 prey and 200 predators.

Beginning with populations of 1000 prey and 200 predators, we see that the predator population decreases initially while the prey population remains fairly stable. This indicates that there were insufficient prey to support a population of 200 predators. The prey population begins to increase in response to the decrease in the predator population, when the predator population dips below 100. The prey population continues to increase until approximately 6800. During this period, the predator population decreases until the

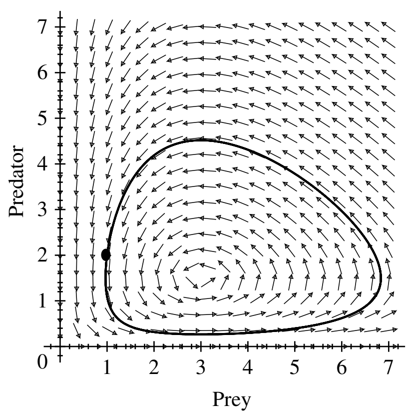

Figure 2.29 The predator–prey model of Example 2.28. The horizontal axis is marked in units of 1000 and the vertical axis is marked in units of 100. The flow line shown is for the initial conditions $(x_0, y_0) = (1, 2)$ or 1000 prey and 200 predators. (Note the vectors are not to scale.)

prey population reaches approximately 3000, when it begins to respond to the increase in prey. The increase in predators is at first gradual, but then becomes more rapid. At the point of maximum prey population, the predator population is approximately 150. After this point, the predator population continues to increase, which causes the prey population to decrease. As the prey population decreases, the predator population increases more slowly and reaches a maximum of approximately 440, when it begins to decrease. At this time the prey population is down to 3000 and continues to decrease even as the number of predators decrease. Both populations continue to decrease and return to their initial values and the cycle begins again.

Notice that the predator and prey populations change in response to each other but that there is a time delay in the response of one population to changes in the other.

If we examine the plot in Figure 2.29, we can see that the vector field **F** appears to have a critical point when there are approximately 3000 prey and 150 predators. Let us return to the general equations to analyze the critical points of **F**. The critical points are solutions to the pair of equations given by $\mathbf{F}(x, y) = (0, 0)$:

$$\begin{aligned} ax - bxy &= 0 \\ -cy + dxy &= 0, \end{aligned}$$

where a, b, c, and d are positive constants. Since $ax - bxy = x(a - by)$, either $x = 0$ or $y = a/b$. If $x = 0$, then from the second equation we must also have $y = 0$. If $y = a/b$, then from the second equation, $x = c/d$. We have found two critical points, $(0, 0)$ and $(c/d, a/b)$. In the above example, we speculated on the presence of the second critical point, $(3/1, 1.5/1) = (3, 1.5)$, which corresponds to 3000 prey and 150 predators.

Let us analyze the critical point at the origin. If there are no predators initially, we would expect the prey population to increase exponentially. In fact, if $y_0 = 0$, then the flow line α of $\mathbf{F}$ with initial point $(x_0, 0)$ is given by $\alpha(t) = (x_0 e^{at}, 0)$, which parametrizes the x-axis in the direction of increasing x. No matter how small the initial value of x_0, the flow line will eventually move away from the origin. This means that the origin is an unstable critical point of $\mathbf{F}$. On the other hand, if there are no prey initially, it can be shown that the number of predators will decrease to zero. (See Exercise 3.) This would model a scenario when the entire prey population is eradicated, say by disease.

A symbolic analysis of the type of the second critical point is more involved, so we will instead rely on a graphical analysis. It appears in Figure 2.29 and is indeed the case that every flow line that has positive initial conditions, $x_0 > 0$ and $y_0 > 0$, is a closed curve that circles the critical point $(c/d, a/b)$. Further, if a flow line starts close to the critical point, it will remain close to the critical point for all t. We conclude that $(c/d, a/b)$ is a stable critical point and is a center. (See Exercise 1 for an analysis of these closed curves.)

In the exercises, you will have the opportunity to explore this and other models of interacting populations. Vito Volterra used the predator–prey model to explain the populations of selachians (sharks) and fish in the Mediterranean Sea during World War I.[5] It has been used to model a variety of predator–prey systems since that time. It is worth noting that the cycles that are seen in the predator–prey model occur in epidemiology where microbes are the predators and humans are the prey.[6]

The Simple Pendulum

Here we want to use the language of vector fields to reconsider the simple pendulum, which we introduced in Example 2.6 of Section 2.1. The pendulum is an example of a mechanical system, that is, it consists of a finite number of objects or parts that are subject to forces that behave according to Newton's second law. The second law says that the vector sum of all the forces acting on an object is equal to the scalar product of the mass of the object and its acceleration vector. In symbols,

$$\mathbf{F} = m\mathbf{a}.$$

(Note that we will use $\mathbf{F}$ to represent the force vector and $\mathbf{G}$ to represent the vector field of the model.) A consequence of Newton's second law is that if we are given the initial position and velocity of the objects in a mechanical system, we can, in principle, determine the behavior of the system from Newton's equation. To construct the vector field that models the system, we must first write out Newton's equation and then turn

[5] Vito Volterra, "Fluctuations in the Abundance of a Species Considered Mathematically," *Nature*, Vol. 118, pp. 558-560.

[6] In their comprehensive text *Infectious Diseases of Humans*, Oxford University Press, 1992, Roy M. Anderson and Robert M. May comment (p. 128) that "sustained host-microparasite cycles ... are the clearest examples of predator-prey cycles that ecologists are likely to find."

Newton's equation into a pair of equations for position and velocity. As above, we will have to make simplifying assumptions about the nature of the physical system, in this case the action of the forces, in order to write out the equations. We will illustrate this and the construction of the vector field with the simple pendulum.

We will assume that the pendulum rod has no mass, so that it can be neglected when analyzing the forces acting on the pendulum. We will also assume that the pendulum swings without friction or air resistance, so that the only force acting on the pendulum is gravity. Gravity acts vertically in a downward direction with magnitude equal to mg, where g is the scalar acceleration due to gravity. If we place the pendulum in a coordinate plane with the first coordinate representing the horizontal direction and the second coordinate representing the vertical direction, the force acting on the pendulum is given by $\mathbf{F} = (0, -mg)$. We will express the position of the pendulum in terms of its displacement x along its arc. If the pendulum has length l, $x = l\theta$, where θ is the angle the pendulum makes with the vertical. (See Figure 2.30(a).)

We are interested in the component of $\mathbf{F}$ in the direction of motion of the pendulum, which is equal to $\mathbf{F} \cdot \mathbf{t}$, where $\mathbf{t}$ is the unit tangent vector to the motion. Since $\mathbf{F} = m\mathbf{a}$, it follows that $\mathbf{F} \cdot \mathbf{t} = m\mathbf{a} \cdot \mathbf{t}$, where $\mathbf{a}$ is the acceleration of the pendulum. A calculation shows that this equality implies that

$$-g\sin(x(t)/l) = x''(t).$$

(See Exercise 5.) Since $v(t) = x'(t)$, we can rewrite this equation as a pair of first order equations in position and velocity:

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= -g\sin(x(t)/l). \end{aligned}$$

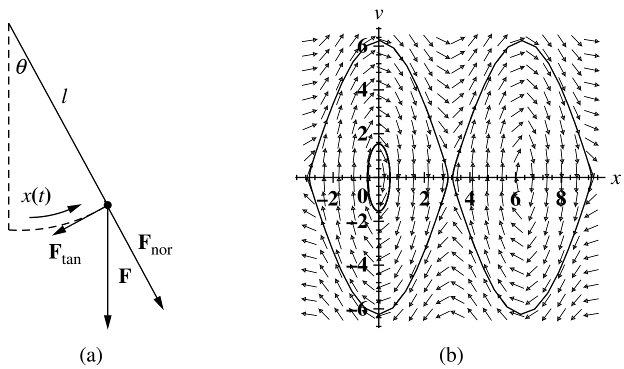

Figure 2.30 (a) The forces acting on a pendulum. (b) The vector field $\mathbf{G}$ for Example 2.29 showing flow lines with initial points $(0.5, 0)$, $(\pi - 0.1, 0)$, and $(\pi + 0.1, 0)$.

These are the equations that we have been seeking. Notice that, like the system of equations for the predator–prey system, this system is autonomous.

Let us summarize what we have done so far.

The Simple Pendulum Model

The ***simple pendulum model*** for a pendulum of length l and mass m moving under the influence of gravity and no other forces is

$$\begin{aligned} x'(t) &= v(t) \\ v'(t) &= -g\sin(x(t)/l), \end{aligned}$$

where $x = x(t)$ is the displacement of the pendulum bob along its arc and $v = v(t)$ is the velocity of the pendulum bob. The constant g is the acceleration due to gravity.

The simple pendulum equations give rise to a vector field $\mathbf{G}$ in the plane with position-velocity coordinates. It is given by

$$\mathbf{G}(x, v) = (v, -g\sin(x/l)).$$

A flow line α of $\mathbf{G}$ with initial point (x_0, v_0) models the motion of the pendulum when it is set in motion with initial displacement x_0 and initial velocity v_0. For a mechanical system, the position-velocity plane is called the ***phase plane*** of the mechanical system and a flow line is called a ***phase plane trajectory*** of the system. In the following example, we will analyze a particular $\mathbf{G}$.

Example 2.29

A Simple Pendulum. Figure 2.30(b) contains a plot of $\mathbf{G}$ for a pendulum of length $l = 1$ m. (Note $g = 9.8\text{m/s}^2$.) Since $l = 1$, $x = \theta$.

The critical points of $\mathbf{G}$ satisfy the equations

$$\begin{aligned} v &= 0 \\ -g\sin(x) &= 0. \end{aligned}$$

The second equation is satisfied when $x = k\pi$, where k is an integer. It follows that the critical points of $\mathbf{G}$ are of the form $(k\pi, 0)$ for k, an integer. These correspond to two distinct states of the pendulum: when the pendulum is positioned vertically downward and at rest and when the pendulum is positioned vertically upward and at rest.

We will use the plot in Figure 2.30(b) to determine the type of each critical point. Let us start at the origin. Initial points near the origin correspond to giving the pendulum a small initial displacement and a small initial velocity, which will cause the pendulum to undergo small periodic oscillations close to the vertical downward position. The corresponding flow line α is a closed curve that encircles the origin. (See Figure 2.30(b).) Since this is the case for all initial points near the origin, the origin is a stable critical point and is a center. (See Exercise 9.) Now let us analyze the critical point at $(\pi, 0)$. An initial point $(x_0, 0)$ with x_0 close to π corresponds to releasing the pendulum from near the upward vertical position with zero initial velocity. The corresponding flow line α is a closed curve around the origin, or $(2\pi, 0)$. No matter how close the initial point is to the critical point, the flow line will leave a small neighborhood of $(\pi, 0)$. Thus $(\pi, 0)$ is an unstable critical point of $\mathbf{G}$. In Exercise 9, you will be asked determine whether this is a source or saddle.

It is worth noting that we were able to model the simple pendulum by a vector field in the plane because the pendulum had "one degree of freedom" in which to move. Thus the position and velocity could each be described by a single real number, and the position and velocity together could be described by a pair of real numbers. This, of course, is not always the case. For example, a Foucault pendulum, which can be used to demonstrate the rotation of the earth, is free to swing in any direction. So its motion has two degrees of freedom, and its position is described by giving two angles rather than one. Similarly, the velocity is described by two real numbers, the velocity in each of the angular directions. Together, then, it takes four real numbers to describe the position and velocity of a Foucault pendulum.

Summary

In this section, we developed the idea of a ***mathematical model*** for a time-dependent physical system that can be described by two time-dependent quantities. The values of the two quantities at a particular time give the ***state of the system*** at that time. A model consists of a ***pair of first order ordinary differential equations*** for the two quantities. The pair of equations can be ***represented*** by a ***vector field*** in the plane. If the system of equations is ***autonomous***, the vector field will not depend on time. A ***flow line*** of the vector field with initial point (x_0, y_0) describes the behavior of the system when it is placed in the state (x_0, y_0) at time $t = 0$.

We introduced mathematical models for a ***predator–prey*** system of interacting species and for a ***simple pendulum.*** We then used the ideas that we developed in Section 2.4 to give a qualitative analysis of the behavior of each system. In particular, we used ***graphical methods to analyze flow lines*** and ***classify the critical points*** of the corresponding vector field.

Section 2.5 Exercises

1. **Extreme Values of the Populations.** The nonconstant flow lines in the first quadrant of the predator–prey model are closed curves that enclose the critical point. On each flow line, there are points where the populations reach maximum and minimum values.

   (a) How would you characterize the points where prey population is at a maximum or a minimum? Explain your answer. (*Hint:* Think about the flow line equations.)

(b) How would you characterize the points where the predator population is at a maximum or a minimum? Explain your answer. (*Hint:* Think about the flow line equations.)
(c) Is it possible to find the coordinates of these points exactly? Explain your answer.
(d) What, if any, is the relationship between these points and the equilibrium point? Explain your answer.

2. Average Population Values. Figure 2.29 contains the plot of the flow line for the particular predator–prey model of Example 2.28 with initial populations of 1000 prey and 200 predators.

(a) Sketch the flow lines for initial populations of 500 prey and 200 predators and of 2000 prey and 200 predators. (These may also be plotted using a computer algebra system, if one is available to you.) In each case, describe how the evolution of the predator system differs from the evolution of the system with the original initial conditions.
(b) Let us assume that one cycle of the predator–prey model takes T units of time to complete. The *average population of prey* over the cycle is $x_{\text{avg}} = \frac{1}{T}\int_0^T x(t)dt$. Solving the predator equation for $x(t)$, we see that

$$x(t) = \frac{1}{d}\left(\frac{y'(t)}{y(t)} + c\right).$$

Substitute this expression for $x(t)$ into the integral for x_{avg} and evaluate the integral.
(c) Similarly, the *average population of predators* is given by the integral $y_{\text{avg}} = \frac{1}{T}\int_0^T y(t)dt$. Solve for $y(t)$ in the prey equation, and use this expression to compute y_{avg}.
(d) Do these results make sense in light of flow lines you sketched? Explain why or why not.

3. Predators Without Prey. In the general predator–prey model, suppose that the initial number of prey is zero and the initial number of predators is y_0.

(a) Find a symbolic form for the flow line of the predator–prey vector field $\mathbf{F}$ with initial conditions $\alpha(0) = (0, y_0)$.
(b) We have already concluded that the critical point at the origin is unstable. What is the type of the unstable critical point at the origin?

4. Competition for Resources. A more realistic model of interacting populations of predator and prey should include competition within a species for resources, which would depend on the number of interactions between members of the same species. We can incorporate this into our basic model by including a degree 2 term, x^2, with a negative

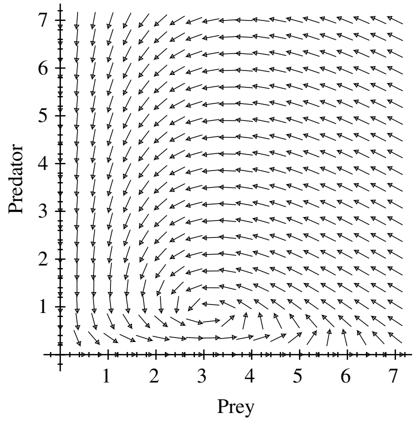

Figure 2.31 The predator–prey model with competition within species. The axes are labeled in units of 1000 for prey and units of 100 for predators. (See Exercise 4.)

coefficient in the prey equation, and a degree 2 term, y^2, with a negative coefficient in the predator equation. This gives us

$$x' = ax - bxy - ex^2,$$
$$y' = -cy + dxy - fy^2,$$

where $e,\ f > 0$. The vector field for this model is plotted in Figure 2.31, with $a = 1.5$, $b = d = 1, c = 3$, and $e = f = 0.2$.

(a) Sketch the flow lines of this vector field for initial populations of 6000 prey and 0, 200, 400, and 600 predators. (These may also be plotted using a computer algebra system, if one is available to you.)
(b) Describe what happens to the predator–prey system for each of the initial populations of part (a).

5. **The Tangential Component of Force.** In this exercise, we will verify that the equation

$$-g\sin(x(t)/l) = x''(t)$$

for the motion of a simple pendulum is a consequence of Newton's law. (See Figure 2.30(a).) Place the pendulum of length l in the xy-plane so that its pivot is at the origin and the downward vertical direction is aligned with the negative y-axis. Since $\theta = x/l$, the position of the pendulum bob displaced through an angle θ at time t is given by

$$\beta(t) = (l\cos((x(t)/l) - \pi/2), l\sin((x(t)/l) - \pi/2)).$$

Notice that when $x(t) = 0$, $\beta(t) = (0, -l)$ and that $\beta(t) \cdot \beta(t) = l^2$.

(a) Differentiate the equation $\beta(t) \cdot \beta(t) = l^2$ and use the result to show that $\beta'(t) \cdot \beta(t) = 0$.

(b) Show that the acceleration $\mathbf{a} = \beta''$ of the pendulum can be written

$$\mathbf{a}(t) = -\left(\frac{x'(t)}{l}\right)^2 \beta(t) + \frac{x''(t)}{x'(t)}\beta'(t)$$

if $x'(t) \neq 0$.

(c) Let

$$\mathbf{t}(t) = \frac{1}{\|\beta'(t)\|}\beta'(t)$$

be the unit tangent vector to the motion. Show that

(i) $m\mathbf{a}(t) \cdot \mathbf{t}(t) = mx'(t)x''(t)/\|\beta'(t)\|$.

(ii) $\mathbf{F} \cdot \mathbf{t}(t) = -mg\cos((x(t)/l) - \pi/2)x'(t)/\|\beta'(t)\|$.

(d) Use Newton's law, $\mathbf{F} = m\mathbf{a}$, and the two parts of (c) to show that $-g\cos((x(t)/l) - \pi/2) = x''(t)$.

(e) Finally, using part (d) and the fact that $\cos(\theta - \pi/2) = \sin\theta$, show that we arrive at the equation $-g\sin(x(t)/l) = x''(t)$ used in the text.

6. The Small Displacement Approximation I. In Example 2.6 of Section 2.1, we modeled the motion of a simple pendulum by the function

$$\alpha(t) = (x_0 \sin(\sqrt{\frac{9.8}{l}}t + \pi/2), x_0\sqrt{\frac{9.8}{l}}\cos(\sqrt{\frac{9.8}{l}}t + \pi/2)),$$

and we commented that this function is *an approximation to the motion of the pendulum that is better when the displacement is small.* The model we have developed in this section is not an approximation, but you should keep in mind that a simple pendulum is not a real pendulum. Based on the discussion of the simple pendulum in this section, is the comment about Example 2.6 of Section 2.1 justified? Explain your answer.

7. The Small Displacement Approximation II. Using material from one-variable calculus, we can derive the approximation α in Example 2.6 of Section 2.1 from the equation $x''(t) = -g\sin(x(t)/l)$. We will start with the facts that the Taylor series for the sine function is given by

$$\sin(u) = \sum_{j=0}^{\infty} \frac{(-1)^j u^{2j+1}}{(2j+1)!}$$

and that the first Taylor polynomial of sine is just the first term of this series, $P_1(u) = u$. If we replace $\sin(u)$ by u in the differential equation for x, we obtain the simpler equation

$$x''(t) = -(g/l)x(t).$$

(a) Show that the first coordinate of α given in Exercise 6 satisfies this equation.
(b) Use this construction of the first coordinate of α to explain why α is a good approximation when x is small. (*Hint:* Think about the relationship between sine and its Taylor series for small values of u.)

8. **Nonclosed Flow Lines.** Are all the flow lines of the vector field **G** of Example 2.29 closed curves? Explain your answer and describe the behavior of the pendulum that corresponds to any flow lines that are not closed curves.

9. **Stable and Unstable Critical Points**

(a) In Example 2.29, we claimed that the origin is a stable critical point of the vector field **G**. Using your physical understanding about the motion of the pendulum, explain why this satisfies the definition of a stable critical point.
(b) In Example 2.29, we argued that the critical point of **G** at $(\pi, 0)$ is unstable. Is this critical point a saddle or a source? Explain your answer by referring to Figure 2.30(b).

10. **Critical Points of Mechanical Systems.** In the predator–prey model, the vector field **F** that we constructed had a critical point in the interior of the first quadrant, in particular, not on the horizontal axis. In physical terms, what would it mean for a vector field that models a mechanical system to have a critical point that is not on the x-axis? Explain your answer.

11. **Nonautonomous Systems.** All the vector fields that we have studied so far have represented *autonomous* systems. This means that the coordinate functions of the vector field do not depend explicitly on time. For example, we would say that $\sin(x(t))$ does not depend explicitly on time, but that t^2 or $\sin(t)$ does. Suppose that a vector field **F** is of the form

$$\mathbf{F}(x, y) = (u(x, y, t), v(x, y, t)),$$

meaning that the vector field depends explicitly on time. Vector fields of this form are called *nonautonomous*.

(a) Describe a physical system that can only be modeled by a nonautonomous vector field. What necessitates that your example be nonautonomous as opposed to autonomous?
(b) How would you define flow line of a nonautonomous vector field?
(c) Intuitively, what are the differences between flow lines of nonautonomous and autonomous vector fields? Explain your answer.

2.6 End of Chapter Exercises

1. Arcs of Circles and Ellipses. Sketch the images of the following parametrizations α and indicate the direction of α on the sketch.

(a) $\alpha(t) = (\cos 2\pi t, \sin 2\pi t)$ for $1 \le t \le 3/2$.
(b) $\alpha(t) = (\sin \pi t, 4\cos \pi t)$ for $-1/4 \le t \le 1/4$.
(c) $\alpha(t) = (\cos(2t - \pi), \sin(2t - \pi))$ for $0 \le t \le \pi/2$.
(d) $\alpha(t) = (2\sin(2\pi(t-1)), \cos(2\pi(t-1)))$ for $0 \le t \le 2$.

2. Parametrizing Arcs of Circles. Find a parametrization α of each of the following arcs of a circle that satisfies the given conditions.

(a) α parametrizes the unit circle centered at the origin clockwise with $\alpha(0) = (0,1)$ and $\alpha(\pi) = (0,-1)$.
(b) α parametrizes the unit circle centered at the origin counterclockwise with $\alpha(0) = (\sqrt{2}/2, \sqrt{2}/2)$ and $\alpha(1) = (-\sqrt{2}/2, \sqrt{2}/2)$.
(c) α parametrizes the circle of radius 1 centered at $(0,1)$ counterclockwise with $\alpha(0) = (0,0)$ and $\alpha(1) = (2,0)$.
(d) α parametrizes the circle of radius 2 centered at $(-1,-1)$ clockwise with $\alpha(0) = (-3,-1)$ and $\alpha(1) = (1,-1)$.

3. A Simple Pendulum. Consider a simple pendulum of length 1 (as in Example 2.6 of Section 2.1) whose motion is parametrized by

$$\alpha(t) = (A\sin(\sqrt{9.8}t + 0.246),\ A\sqrt{9.8}\cos(\sqrt{9.8}t + 0.246)),$$

where $A = (\pi/4)(1/\sin(0.246))$. The image of the parametrization is shown in Figure 2.32.

(a) What are the initial position and the initial velocity of the pendulum?
(b) What is the period of the pendulum?
(c) Describe the motion of the pendulum from time $t = 0$ until the pendulum completes one period of its motion.

4. Vector Sum of Parametrizations. Certain motions of objects can be written as the vector sum of two different motions. For example, if α parametrizes the motion of a person walking in a straight line and β parametrizes the motion of a yo-yo being spun in a vertical plane at the end of its string, $\gamma = \alpha + \beta$ will represent the motion of the yo-yo being spun in a vertical plane as the person spinning the yo-yo walks in a straight line. In coordinates in the plane, $\alpha(t) = (x_1(t), y_1(t))$ and $\beta(t) = (x_2(t), y_2(t))$,

$$\gamma(t) = \alpha(t) + \beta(t) = (x_1(t), y_1(t)) + (x_2(t), y_2(t)) = (x_1(t) + x_2(t), y_1(t) + y_2(t)).$$

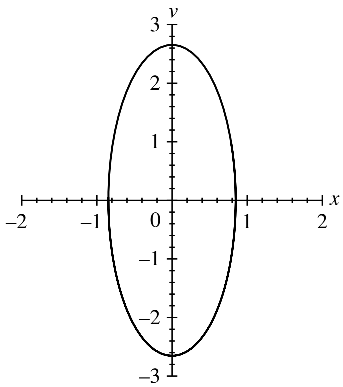

Figure 2.32 The image of the parametrization α of Exercise 3.

(a) Suppose that a yo-yo is being spun in a circle on the end of its string with constant speed so that it traces a circle in a vertical plane (the xy-plane). If one rotation takes 1 s, the string has length 0.75 m, and the string is being held at a point 1 m off the ground, how would you parametrize the motion?
(b) Suppose a person walks at a speed of 1 m/s along the x-axis in the xy-plane. How would you parametrize the motion?
(c) How would you parametrize the motion of a yo-yo being spun as in part (a) by a person walking at a speed of 1 m/s?

5. **Vector Sum of Parametrizations.** A carousel is 40 ft in diameter and makes one revolution every 30 s. The carousel horses move up and down a distance of 1 ft twice during each revolution of the carousel. Parametrize the motion of a person riding the carousel on a horse that is 2 ft from the edge of the carousel and whose seat is 4 ft off the ground at its lowest.

6. **The Orbit of the Moon about the Sun.** This exercise investigates the motion of the moon about the sun. For the purposes of this exercise, we will assume that the orbit of the earth about the sun is a circle and the orbit of the moon around the earth is a circle.

(a) Construct a parametrization of the orbit of the earth about the sun that places the sun at the origin. Assume that the orbit is a circle of radius 149.5×10^8 km. It takes the earth approximately 365.26 days to orbit the sun.
(b) Construct a parametrization of the orbit of the moon about the earth that places the earth at the origin. Assume that the orbit is circle of radius $384,405$ km (the mean distance from the earth to the moon). It takes the moon approximately 27.32 days to orbit the earth.
(c) Construct a parametrization of the motion of the moon about the sun that places the sun at the origin. (Neglect the 5.1 deg tilt in the plane of the orbit of the moon relative to the plane of the orbit of the earth about the sun.) Describe this motion.

7. **Retrograde Motion of Mars.** In this exercise, we will analyze the direction from the earth to Mars as a function of time. If α parametrizes the orbit of the earth and β parametrizes the orbit of Mars, the direction vector at time t is equal to $(\beta - \alpha)(t) = \beta(t) - \alpha(t)$. The motion represented by this direction is the motion of Mars relative to the backdrop of the stars in the sky. For this problem, let us assume that the orbits of the earth and Mars are both circles centered at the origin with the sun located at the origin and that these circles lie in the xy-plane. The radius of the orbit of the earth is approximately 92.9 million miles and that of Mars is approximately 141.5 million miles. Mars takes 686.8 earth days to orbit the sun.

   (a) Construct parametrizations α of the orbit of the earth and β of the orbit of Mars.
   (b) Use a computer to plot the parametrization $\beta - \alpha$ of the direction vector from the earth to Mars for a period of at least 10 years.
   (c) Use the plot in (b) to analyze the direction of Mars from the earth. In particular, describe how Mars would move as it crossed the sky over this time period, were we to look at Mars from earth. Explain your answer.

8. **Velocity and Speed.** Each of the following functions α is a parametrization for the motion of an object in the plane or in space. Find the velocity vector and the speed of the motion at the given time t_0.

   (a) $\alpha(t) = (t^2 - 1, 4t^3)$ at $t_0 = 0$.
   (b) $\alpha(t) = (\cos 2t, \sin t, t - \pi)$ at $t_0 = \pi/2$.
   (c) $\alpha(t) = (e^{2t}, e^{-2t}, 2t)$ at $t_0 = 1$.
   (d) $\alpha(t) = (\frac{1}{1+t^2}, t)$ at $t_0 = 0$.

9. **Tangent Lines.** For each of the following parametrizations α, find the tangent vector and the tangent line to the curve parametrized by α at the point t_0.

   (a) $\alpha(t) = (\cos(2\pi t), 3\sin(2\pi t))$ at $t_0 = 1$.
   (b) $\alpha(t) = (e^{t/2}, e^{t^2/4})$ at $t_0 = 0$.
   (c) $\alpha(t) = (t - 1, 3t + 1, -4t)$ at $t_0 = 2$.
   (d) $\alpha(t) = (2 - \sin(\pi t), 2 + \sin(\pi t), t)$ at $t_0 = -1$.

10. **Cycloids.** As in Exercise 5 of Section 2.2, assume that a wheel of radius 1 rests on the x-axis and rolls without slipping in the positive direction along the x-axis with unit speed. Assume that the marked point lies a distance b units from the center of the wheel along a radial segment (or spoke). If the motion starts with the wheel resting on the origin and the radial segment lies along the y-axis at the beginning of the motion, the curve traced by the point is parametrized by $\alpha(t) = (t + b\sin t, 1 + b\cos t)$. If $b = 1$, the curve is the cycloid of Exercise 5 of Section 2.2.

    (a) Compute the velocity vector $\mathbf{v}(t)$ for the motion of the point.
    (b) Suppose $b < 1$. At which time(s) t is the velocity vector horizontal? At which time(s) t is the velocity vector vertical?

(c) Suppose $b > 1$. At which time(s) t is the velocity vector horizontal? At which time(s) t is the velocity vector vertical?
(d) Sketch the image of $\alpha(t)$ for $b = 2$ and $b = 1/2$.

11. Verifying Parametrizations of Flow Lines. For each of the following vector fields and parametrizations, verify by a direct calculation that the function α parametrizes a flow line of the vector field and find the initial point of the flow line.

(a) $\mathbf{F}(x, y) = (y, x)$.

(i) $\alpha = ((e^t + e^{-t})/2, (e^t - e^{-t})/2)$.
(ii) $\alpha = ((3e^t + e^{-t})/2, (3e^t - e^{-t})/2)$.

(b) $\mathbf{F}(x, y) = (y, -4x)$.

(i) $\alpha(t) = (\cos(2t)/2, -\sin(2t))$.
(ii) $\alpha(t) = (\cos(2t) + \sin(2t)/2, -2\sin(2t) + \cos(2t))$.

12. Flow Lines of Vector Fields. For each of the following vector fields, proceed as follows. (i) Sketch the flow lines starting at the initial points $(\pm 1, 0)$, $(0, \pm 1)$, $(1, \pm 1)$, and $(-1, \pm 1)$. (These may also be plotted using a computer algebra system, if one is available to you.) (ii) Use your sketch to classify the critical point of $\mathbf{F}$ at the origin as stable or unstable and as a center, sink, source, or saddle.

(a) $\mathbf{F}(x, y) = (y, 2x)$. (See Figure 2.33(a).)
(b) $\mathbf{F}(x, y) = (3y, x)$. (See Figure 2.33(b).)
(c) $\mathbf{F}(x, y) = (x + y, y)$. (See Figure 2.33(c).)

13. Critical Points of Vector Fields. Determine the critical points of each of the following vector fields on the indicated domain, and based on the accompanying plot, determine the type of the critical points. (If a computer algebra system is available to you, you may find it helpful to plot several flow lines of the given vector fields.)

(a) $\mathbf{F}(x, y) = (x + y - 1, y - x - 1)$, $[-1.5, 1.5] \times [-1.5, 1.5]$. (See Figure 2.34(a).)
(b) $\mathbf{F}(x, y) = (y - x + x^2 - 1, y - x)$, $[-1.5, 1.5] \times [-1.5, 1.5]$. (See Figure 2.34(b).)
(c) $\mathbf{F}(x, y) = (e^{xy}((y - x)^2 + x^2 - 0.25), e^{xy}(y - x)^2)$, $[-1, 1] \times [-1, 1]$. (See Figure 2.34(c).)

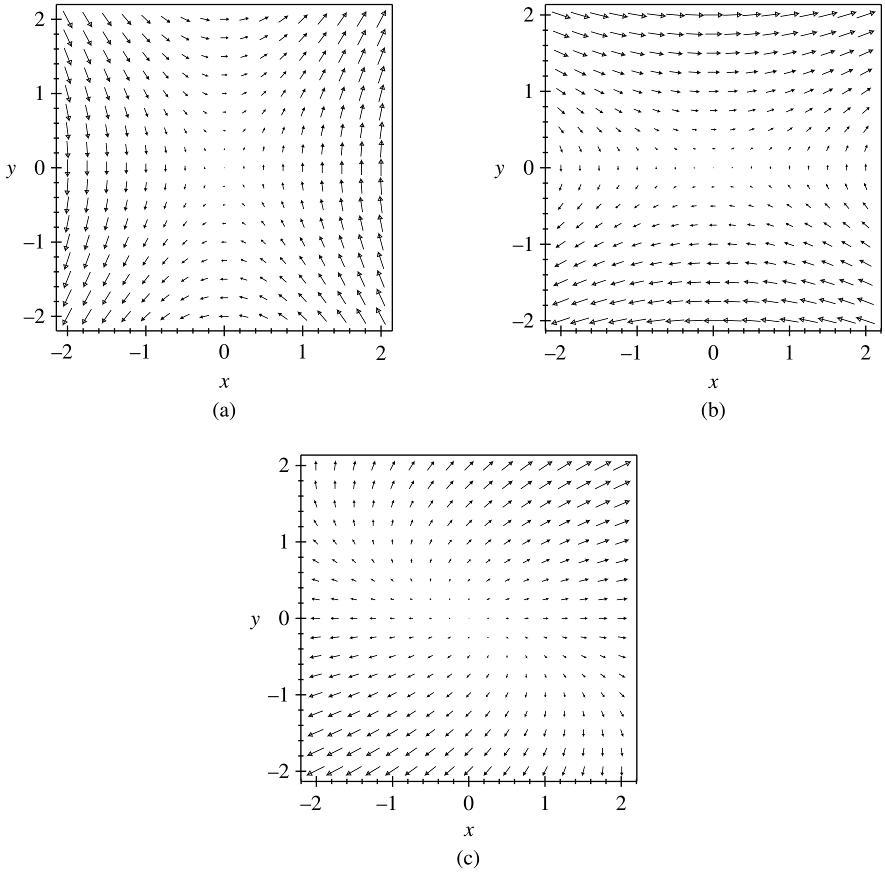

Figure 2.33 Vector fields for Exercise 12.

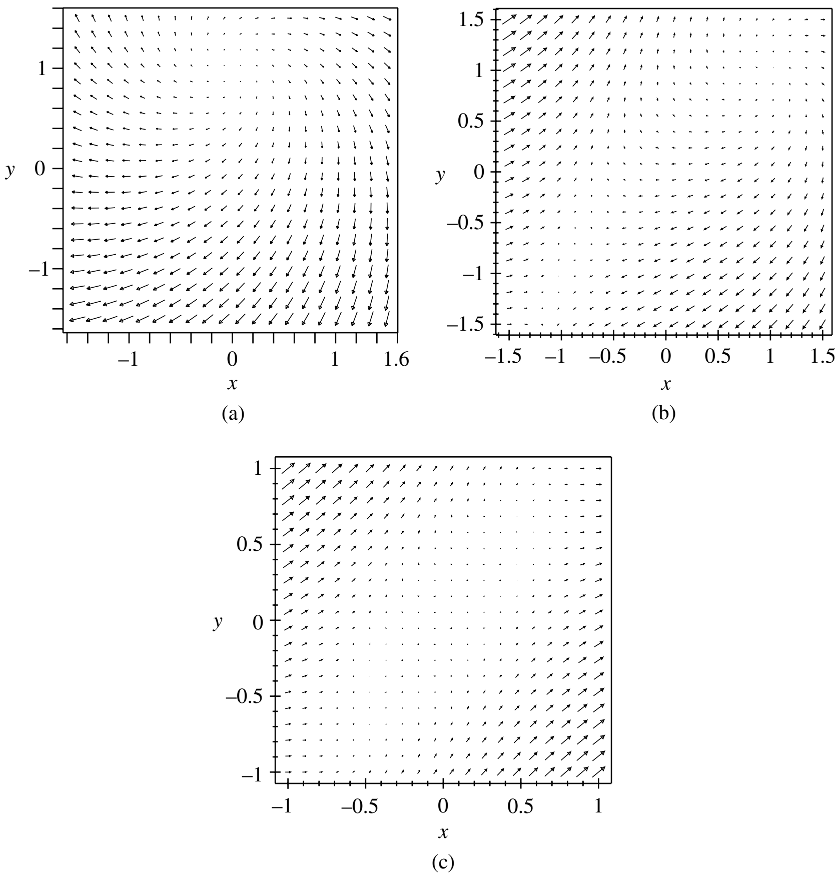

Figure 2.34 Vector fields for Exercise 13.