HIGHER-ORDER DIFFERENTIAL EQUATIONS

3

Chapter Contents

We turn now to DEs of order two and higher. In the first six sections of this chapter we examine some of the underlying theory of **linear DEs** and methods for solving certain kinds of linear equations. The difficulties that surround higher-order **nonlinear DEs** and the few methods that yield analytic solutions of such equations are examined next (Section 3.7). The chapter concludes with higher-order linear and nonlinear mathematical models (Sections 3.8, 3.9, and 3.11) and the first of several methods to be considered on solving systems of linear DEs (Section 3.12).

3.1 Theory of Linear Equations

Introduction We turn now to differential equations of order two or higher. In this section we will examine some of the underlying theory of linear DEs. Then in the five sections that follow we learn how to solve linear higher-order differential equations.

3.1.1 Initial-Value and Boundary-Value Problems

Initial-Value Problem In Section 1.2 we defined an initial-value problem for a general nth-order differential equation. For a linear differential equation, an **nth-order initial-value problem** is

$$\begin{aligned} &\textit{Solve:} && a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \\ &\textit{Subject to:} && y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}. \end{aligned} \tag{1}$$

Recall that for a problem such as this, we seek a function defined on some interval I containing x_0 that satisfies the differential equation and the n initial conditions specified at x_0: $y(x_0) = y_0$, $y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1}$. We have already seen that in the case of a second-order initial-value problem, a solution curve must pass through the point (x_0, y_0) and have slope y_1 at this point.

Existence and Uniqueness In Section 1.2 we stated a theorem that gave conditions under which the existence and uniqueness of a solution of a first-order initial-value problem were guaranteed. The theorem that follows gives sufficient conditions for the existence of a unique solution of the problem in (1).

Theorem 3.1.1 Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x), a_0(x)$, and $g(x)$ be continuous on an interval I, and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

EXAMPLE 1 Unique Solution of an IVP

The initial-value problem

$$3y''' + 5y'' - y' + 7y = 0, \quad y(1) = 0, \quad y'(1) = 0, \quad y''(1) = 0$$

possesses the trivial solution $y = 0$. Since the third-order equation is linear with constant coefficients, it follows that all the conditions of Theorem 3.1.1 are fulfilled. Hence $y = 0$ is the *only* solution on any interval containing $x = 1$. ≡

EXAMPLE 2 Unique Solution of an IVP

You should verify that the function $y = 3e^{2x} + e^{-2x} - 3x$ is a solution of the initial-value problem $y'' - 4y = 12x$, $y(0) = 4$, $y'(0) = 1$. Now the differential equation is linear, the coefficients as well as $g(x) = 12x$ are continuous, and $a_2(x) = 1 \neq 0$ on any interval I containing $x = 0$. We conclude from Theorem 3.1.1 that the given function is the unique solution on I. ≡

The requirements in Theorem 3.1.1 that $a_i(x)$, $i = 0, 1, 2, \ldots, n$ be continuous and $a_n(x) \neq 0$ for every x in I are both important. Specifically, if $a_n(x) = 0$ for some x in the interval, then the solution of a linear initial-value problem may not be unique or even exist. For example, you should verify that the function $y = cx^2 + x + 3$ is a solution of the initial-value problem

$$x^2y'' - 2xy' + 2y = 6, \quad y(0) = 3, \quad y'(0) = 1$$

on the interval $(-\infty, \infty)$ for any choice of the parameter c. In other words, there is no unique solution of the problem. Although most of the conditions of Theorem 3.1.1 are satisfied, the obvious difficulties are that $a_2(x) = x^2$ is zero at $x = 0$ and that the initial conditions are also imposed at $x = 0$.

Boundary-Value Problem Another type of problem consists of solving a linear differential equation of order two or greater in which the dependent variable y or its derivatives are specified at *different points.* A problem such as

$$\textit{Solve:} \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\textit{Subject to:} \quad y(a) = y_0, \quad y(b) = y_1$$

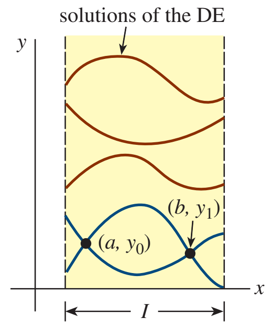

FIGURE 3.1.1 Colored curves are solutions of a BVP

is called a **two-point boundary-value problem**, or simply a **boundary-value problem (BVP)**. The prescribed values $y(a) = y_0$ and $y(b) = y_1$ are called **boundary conditions (BC)**. A solution of the foregoing problem is a function satisfying the differential equation on some interval I, containing a and b, whose graph passes through the two points (a, y_0) and (b, y_1). See **FIGURE 3.1.1**.

For a second-order differential equation, other pairs of boundary conditions could be

$$\begin{aligned} y'(a) &= y_0, & y(b) &= y_1 \\ y(a) &= y_0, & y'(b) &= y_1 \\ y'(a) &= y_0, & y'(b) &= y_1, \end{aligned}$$

where y_0 and y_1 denote arbitrary constants. These three pairs of conditions are just special cases of the general boundary conditions

$$\begin{aligned} A_1y(a) + B_1y'(a) &= C_1 \\ A_2y(b) + B_2y'(b) &= C_2. \end{aligned}$$

The next example shows that even when the conditions of Theorem 3.1.1 are fulfilled, a boundary-value problem may have several solutions (as suggested in Figure 3.1.1), a unique solution, or no solution at all.

EXAMPLE 3 A BVP Can Have Many, One, or No Solutions

In Example 4 of Section 1.1 we saw that the two-parameter family of solutions of the differential equation $x'' + 16x = 0$ is

$$x = c_1 \cos 4t + c_2 \sin 4t. \tag{2}$$

(a) Suppose we now wish to determine that solution of the equation that further satisfies the boundary conditions $x(0) = 0$, $x(\pi/2) = 0$. Observe that the first condition $0 = c_1 \cos 0 + c_2 \sin 0$ implies $c_1 = 0$, so that $x = c_2 \sin 4t$. But when $t = \pi/2$, $0 = c_2 \sin 2\pi$ is satisfied for any choice of c_2 since $\sin 2\pi = 0$. Hence the boundary-value problem

$$x'' + 16x = 0, \quad x(0) = 0, \quad x(\pi/2) = 0 \tag{3}$$

has infinitely many solutions. **FIGURE 3.1.2** shows the graphs of some of the members of the one-parameter family $x = c_2 \sin 4t$ that pass through the two points (0, 0) and $(\pi/2, 0)$.

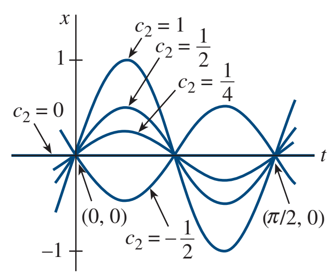

FIGURE 3.1.2 The BVP in (3) of Example 3 has many solutions

(b) If the boundary-value problem in (3) is changed to

$$x'' + 16x = 0, \quad x(0) = 0, \quad x(\pi/8) = 0, \tag{4}$$

then $x(0) = 0$ still requires $c_1 = 0$ in the solution (2). But applying $x(\pi/8) = 0$ to $x = c_2 \sin 4t$ demands that $0 = c_2 \sin(\pi/2) = c_2 \cdot 1$. Hence $x = 0$ is a solution of this new boundary-value problem. Indeed, it can be proved that $x = 0$ is the *only* solution of (4).

(c) Finally, if we change the problem to

$$x'' + 16x = 0, \quad x(0) = 0, \quad x(\pi/2) = 1, \tag{5}$$

we find again that $c_1 = 0$ from $x(0) = 0$, but that applying $x(\pi/2) = 1$ to $x = c_2 \sin 4t$ leads to the contradiction $1 = c_2 \sin 2\pi = c_2 \cdot 0 = 0$. Hence the boundary-value problem (5) has no solution. ≡

3.1.2 Homogeneous Equations

A linear nth-order differential equation of the form

Note $y = 0$ is always a solution of a homogeneous linear equation.

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \tag{6}$$

is said to be **homogeneous**, whereas an equation

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{7}$$

with $g(x)$ not identically zero, is said to be **nonhomogeneous**. For example, $2y'' + 3y' - 5y = 0$ is a homogeneous linear second-order differential equation, whereas $x^2y''' + 6y' + 10y = e^x$ is a nonhomogeneous linear third-order differential equation. The word *homogeneous* in this context does not refer to coefficients that are homogeneous functions as in Section 2.5; rather, the word has exactly the same meaning as in Section 2.3.

We shall see that in order to solve a nonhomogeneous linear equation (7), we must first be able to solve the **associated homogeneous equation** (6).

To avoid needless repetition throughout the remainder of this section, we shall, as a matter of course, make the following important assumptions when stating definitions and theorems about the linear equations (6) and (7). On some common interval I,

Remember these assumptions in the definitions and theorems of this chapter.

- the coefficients $a_i(x)$, $i = 0, 1, 2, \ldots, n$, are continuous;
- the right-hand member $g(x)$ is continuous; and
- $a_n(x) \neq 0$ for every x in the interval.

■ Differential Operators In calculus, differentiation is often denoted by the capital letter D; that is, $dy/dx = Dy$. The symbol D is called a **differential operator** because it transforms a differentiable function into another function. For example, $D(\cos 4x) = -4 \sin 4x$, and $D(5x^3 - 6x^2) = 15x^2 - 12x$. Higher-order derivatives can be expressed in terms of D in a natural manner:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2} = D(Dy) = D^2y \quad \text{and in general} \quad \frac{d^ny}{dx^n} = D^ny,$$

where y represents a sufficiently differentiable function. Polynomial expressions involving D, such as $D + 3$, $D^2 + 3D - 4$, and $5x^3D^3 - 6x^2D^2 + 4xD + 9$, are also differential operators. In general, we define an ***n*th-order differential operator** to be

$$L = a_n(x)D^n + a_{n-1}(x)D^{n-1} + \cdots + a_1(x)D + a_0(x). \tag{8}$$

As a consequence of two basic properties of differentiation, $D(cf(x)) = c\,Df(x)$, c a constant, and $D\{f(x) + g(x)\} = Df(x) + Dg(x)$, the differential operator L possesses a linearity property; that is, L operating on a linear combination of two differentiable functions is the same as the linear combination of L operating on the individual functions. In symbols, this means

$$L\{\alpha f(x) + \beta g(x)\} = \alpha L(f(x)) + \beta L(g(x)), \tag{9}$$

where α and β are constants. Because of (9) we say that the nth-order differential operator L is a **linear operator**.

■ Differential Equations Any linear differential equation can be expressed in terms of the D notation. For example, the differential equation $y'' + 5y' + 6y = 5x - 3$ can be written as

$D^2y + 5Dy + 6y = 5x - 3$ or $(D^2 + 5D + 6)y = 5x - 3$. Using (8), the nth-order linear differential equations (6) and (7) can be written compactly as

$$L(y) = 0 \quad \text{and} \quad L(y) = g(x),$$

respectively.

Superposition Principle In the next theorem we see that the sum, or **superposition**, of two or more solutions of a homogeneous linear differential equation is also a solution.

Theorem 3.1.2 Superposition Principle—Homogeneous Equations

Let $y_1, y_2, \ldots, y_k$ be solutions of the homogeneous nth-order differential equation (6) on an interval I. Then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x),$$

where the c_i, $i = 1, 2, \ldots, k$ are arbitrary constants, is also a solution on the interval.

PROOF

We prove the case $k = 2$. Let L be the differential operator defined in (8), and let $y_1(x)$ and $y_2(x)$ be solutions of the homogeneous equation $L(y) = 0$. If we define $y = c_1y_1(x) + c_2y_2(x)$, then by linearity of L we have

$$L(y) = L\{c_1y_1(x) + c_2y_2(x)\} = c_1L(y_1) + c_2L(y_2) = c_1 \cdot 0 + c_2 \cdot 0 = 0. \equiv$$

Corollaries to Theorem 3.1.2

(a) A constant multiple $y = c_1y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear differential equation is also a solution.

(b) A homogeneous linear differential equation always possesses the trivial solution $y = 0$.

EXAMPLE 4 Superposition—Homogeneous DE

The functions $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of the homogeneous linear equation $x^3y''' - 2xy' + 4y = 0$ on the interval $(0, \infty)$. By the superposition principle, the linear combination

$$y = c_1x^2 + c_2x^2 \ln x$$

is also a solution of the equation on the interval. $\equiv$

The function $y = e^{7x}$ is a solution of $y'' - 9y' + 14y = 0$. Since the differential equation is linear and homogeneous, the constant multiple $y = ce^{7x}$ is also a solution. For various values of c we see that $y = 9e^{7x}$, $y = 0$, $y = -\sqrt{5}e^{7x}, \ldots$, are all solutions of the equation.

Linear Dependence and Linear Independence The next two concepts are basic to the study of linear differential equations.

Definition 3.1.1 Linear Dependence/Independence

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that

$$c_1f_1(x) + c_2f_2(x) + \cdots + c_nf_n(x) = 0$$

for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**.

In other words, a set of functions is linearly independent on an interval if the only constants for which

$$c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$$

for every x in the interval are $c_1 = c_2 = \cdots = c_n = 0$.

It is easy to understand these definitions in the case of two functions $f_1(x)$ and $f_2(x)$. If the functions are linearly dependent on an interval, then there exist constants c_1 and c_2 that are not both zero such that for every x in the interval $c_1 f_1(x) + c_2 f_2(x) = 0$. Therefore, if we assume that $c_1 \neq 0$, it follows that $f_1(x) = (-c_2/c_1) f_2(x)$; that is

If two functions are linearly dependent, then one is simply a constant multiple of the other.

Conversely, if $f_1(x) = c_2 f_2(x)$ for some constant c_2, then $(-1) \cdot f_1(x) + c_2 f_2(x) = 0$ for every x on some interval. Hence the functions are linearly dependent, since at least one of the constants (namely, $c_1 = -1$) is not zero. We conclude that

Two functions are linearly independent when neither is a constant multiple of the other

on an interval. For example, the functions $f_1(x) = \sin 2x$ and $f_2(x) = \sin x \cos x$ are linearly dependent on $(-\infty, \infty)$ because $f_1(x)$ is a constant multiple of $f_2(x)$. Recall from the double angle formula for the sine that $\sin 2x = 2 \sin x \cos x$. On the other hand, the functions $f_1(x) = x$ and $f_2(x) = |x|$ are linearly independent on $(-\infty, \infty)$. Inspection of **FIGURE 3.1.3** should convince you that neither function is a constant multiple of the other on the interval.

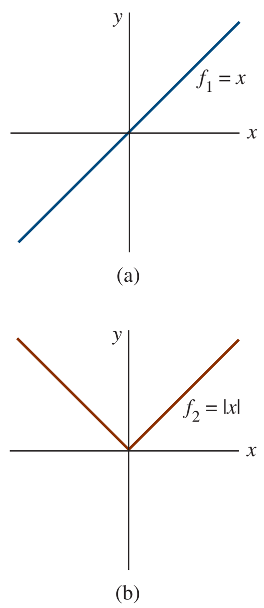

FIGURE 3.1.3 The set consisting of f_1 and f_2 is linearly independent on $(-\infty, \infty)$

It follows from the preceding discussion that the ratio $f_2(x)/f_1(x)$ is not a constant on an interval on which $f_1(x)$ and $f_2(x)$ are linearly independent. This little fact will be used in the next section.

EXAMPLE 5 Linearly Dependent Functions

The functions $f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, $f_3(x) = \sec^2 x$, $f_4(x) = \tan^2 x$ are linearly dependent on the interval $(-\pi/2, \pi/2)$ since

$$c_1 \cos^2 x + c_2 \sin^2 x + c_3 \sec^2 x + c_4 \tan^2 x = 0,$$

when $c_1 = c_2 = 1$, $c_3 = -1$, $c_4 = 1$. We used here $\cos^2 x + \sin^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$. ≡

A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is linearly dependent on an interval if at least one function can be expressed as a linear combination of the remaining functions.

EXAMPLE 6 Linearly Dependent Functions

The functions $f_1(x) = \sqrt{x} + 5$, $f_2(x) = \sqrt{x} + 5x$, $f_3(x) = x - 1$, $f_4(x) = x^2$ are linearly dependent on the interval $(0, \infty)$ since f_2 can be written as a linear combination of f_1, f_3, and f_4. Observe that

$$f_2(x) = 1 \cdot f_1(x) + 5 \cdot f_3(x) + 0 \cdot f_4(x)$$

for every x in the interval $(0, \infty)$. ≡

■ Solutions of Differential Equations We are primarily interested in linearly independent functions or, more to the point, linearly independent solutions of a linear differential equation. Although we could always appeal directly to Definition 3.1.1, it turns out that the question of whether n solutions $y_1, y_2, \ldots, y_n$ of a homogeneous linear nth-order differential equation (6) are linearly independent can be settled somewhat mechanically using a determinant.

Definition 3.1.2 Wronskian

Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n - 1$ derivatives. The determinant

$$W(f_1, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions.

Theorem 3.1.3 Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \ldots, y_n) \neq 0$ for every x in the interval.

It follows from Theorem 3.1.3 that when $y_1, y_2, \ldots, y_n$ are n solutions of (6) on an interval I, the Wronskian $W(y_1, y_2, \ldots, y_n)$ is either identically zero or never zero on the interval.

A set of n linearly independent solutions of a homogeneous linear nth-order differential equation is given a special name.

Definition 3.1.3 Fundamental Set of Solutions

Any set $y_1, y_2, \ldots, y_n$ of n linearly independent solutions of the homogeneous linear nth-order differential equation (6) on an interval I is said to be a **fundamental set of solutions** on the interval.

The basic question of whether a fundamental set of solutions exists for a linear equation is answered in the next theorem.

Theorem 3.1.4 Existence of a Fundamental Set

There exists a fundamental set of solutions for the homogeneous linear nth-order differential equation (6) on an interval I.

Analogous to the fact that any vector in three dimensions can be expressed uniquely as a linear combination of the *linearly independent* vectors **i**, **j**, **k**, any solution of an nth-order homogeneous linear differential equation on an interval I can be expressed uniquely as a linear combination of n linearly independent solutions on I. In other words, n linearly independent solutions $y_1, y_2, \ldots, y_n$ are the basic building blocks for the general solution of the equation.

Theorem 3.1.5 General Solution—Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (6) on an interval I. Then the **general solution** of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x),$$

where c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Theorem 3.1.5 states that if $Y(x)$ is any solution of (6) on the interval, then constants $C_1, C_2, \ldots, C_n$ can always be found so that

$$Y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x).$$

We will prove the case when $n = 2$.

PROOF

Let Y be a solution and y_1 and y_2 be linearly independent solutions of $a_2 y'' + a_1 y' + a_0 y = 0$ on an interval I. Suppose $x = t$ is a point in I for which $W(y_1(t), y_2(t)) \neq 0$. Suppose also that $Y(t) = k_1$ and $Y'(t) = k_2$. If we now examine the equations

$$C_1 y_1(t) + C_2 y_2(t) = k_1$$
$$C_1 y_1'(t) + C_2 y_2'(t) = k_2,$$

it follows that we can determine C_1 and C_2 uniquely, provided that the determinant of the coefficients satisfies

$$\begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix} \neq 0.$$

But this determinant is simply the Wronskian evaluated at $x = t$, and, by assumption, $W \neq 0$. If we define $G(x) = C_1 y_1(x) + C_2 y_2(x)$, we observe that $G(x)$ satisfies the differential equation, since it is a superposition of two known solutions; $G(x)$ satisfies the initial conditions

$$G(t) = C_1 y_1(t) + C_2 y_2(t) = k_1 \quad \text{and} \quad G'(t) = C_1 y_1'(t) + C_2 y_2'(t) = k_2;$$

$Y(x)$ satisfies the *same* linear equation and the *same* initial conditions. Since the solution of this linear initial-value problem is unique (Theorem 3.1.1), we have $Y(x) = G(x)$ or $Y(x) = C_1 y_1(x) + C_2 y_2(x)$. ≡

EXAMPLE 7 General Solution of a Homogeneous DE

The functions $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval $(-\infty, \infty)$. By inspection, the solutions are linearly independent on the x-axis. This fact can be corroborated by observing that the Wronskian

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every x. We conclude that y_1 and y_2 form a fundamental set of solutions, and consequently $y = c_1 e^{3x} + c_2 e^{-3x}$ is the general solution of the equation on the interval. ≡

EXAMPLE 8 A Solution Obtained from a General Solution

The function $y = 4 \sinh 3x - 5e^{3x}$ is a solution of the differential equation in Example 7. (Verify this.) In view of Theorem 3.1.5, we must be able to obtain this solution from the general solution $y = c_1 e^{3x} + c_2 e^{-3x}$. Observe that if we choose $c_1 = 2$ and $c_2 = -7$, then $y = 2e^{3x} - 7e^{-3x}$ can be rewritten as

$$y = 2e^{3x} - 2e^{-3x} - 5e^{-3x} = 4\left(\frac{e^{3x} - e^{-3x}}{2}\right) - 5e^{-3x}.$$

The last expression is recognized as $y = 4 \sinh 3x - 5e^{-3x}$. ≡

EXAMPLE 9 General Solution of a Homogeneous DE

The functions $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ satisfy the third-order equation $y''' - 6y'' + 11y' - 6y = 0$. Since

$$W(e^x, e^{2x}, e^{3x}) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$$

for every real value of x, the functions y_1, y_2, and y_3 form a fundamental set of solutions on $(-\infty, \infty)$. We conclude that $y = c_1e^x + c_2e^{2x} + c_3e^{3x}$ is the general solution of the differential equation on the interval. ≡

3.1.3 Nonhomogeneous Equations

Any function y_p free of arbitrary parameters that satisfies (7) is said to be a **particular solution** of the equation. For example, it is a straightforward task to show that the constant function $y_p = 3$ is a particular solution of the nonhomogeneous equation $y'' + 9y = 27$.

Now if $y_1, y_2, \ldots, y_k$ are solutions of (6) on an interval I and y_p is any particular solution of (7) on I, then the linear combination

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x) + y_p \tag{10}$$

is also a solution of the nonhomogeneous equation (7). If you think about it, this makes sense, because the linear combination $c_1y_1(x) + c_2y_2(x) + \cdots + c_ky_k(x)$ is mapped into 0 by the operator $L = a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$, whereas y_p is mapped into $g(x)$. If we use $k = n$ linearly independent solutions of the nth-order equation (6), then the expression in (10) becomes the general solution of (7).

Theorem 3.1.6 General Solution—Nonhomogeneous Equations

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (7) on an interval I, and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (6) on I. Then the **general solution** of the equation on the interval is

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p,$$

where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

PROOF

Let L be the differential operator defined in (8), and let $Y(x)$ and $y_p(x)$ be particular solutions of the nonhomogeneous equation $L(y) = g(x)$. If we define $u(x) = Y(x) - y_p(x)$, then by linearity of L we have

$$L(u) = L\{Y(x) - y_p(x)\} = L(Y(x)) - L(y_p(x)) = g(x) - g(x) = 0.$$

This shows that $u(x)$ is a solution of the homogeneous equation $L(y) = 0$. Hence, by Theorem 3.1.5, $u(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$, and so

$$Y(x) - y_p(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$$

or

$$Y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x).$$ ≡

Complementary Function We see in Theorem 3.1.6 that the general solution of a nonhomogeneous linear equation consists of the sum of two functions:

$$y = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) + y_p(x) = y_c(x) + y_p(x).$$

The linear combination $y_c(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$, which is the general solution of (6), is called the **complementary function** for equation (7). In other words, to solve a nonhomogeneous linear differential equation we first solve the associated homogeneous equation and then find any particular solution of the nonhomogeneous equation. The general solution of the nonhomogeneous equation is then

$$y = \textit{complementary function} + \textit{any particular solution}.$$

EXAMPLE 10 General Solution of a Nonhomogeneous DE

By substitution, the function $y_p = -\frac{11}{12} - \frac{1}{2}x$ is readily shown to be a particular solution of the nonhomogeneous equation

$$y''' - 6y'' + 11y' - 6y = 3x. \tag{11}$$

In order to write the general solution of (11), we must also be able to solve the associated homogeneous equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

But in Example 9 we saw that the general solution of this latter equation on the interval $(-\infty, \infty)$ was $y_c = c_1e^x + c_2e^{2x} + c_3e^{3x}$. Hence the general solution of (11) on the interval is

$$y = y_c + y_p = c_1e^x + c_2e^{2x} + c_3e^{3x} - \frac{11}{12} - \frac{1}{2}x.$$ ≡

Another Superposition Principle The last theorem of this discussion will be useful in Section 3.4, when we consider a method for finding particular solutions of nonhomogeneous equations.

Theorem 3.1.7 Superposition Principle—Nonhomogeneous Equations

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order differential equation (7) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \ldots, g_k$. That is, suppose y_{p_i} denotes a particular solution of the corresponding differential equation

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_i(x), \tag{12}$$

where $i = 1, 2, \ldots, k$. Then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \cdots + y_{p_k}(x) \tag{13}$$

is a particular solution of

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \cdots + g_k(x). \tag{14}$$

PROOF

We prove the case $k = 2$. Let L be the differential operator defined in (8), and let $y_{p_1}(x)$ and $y_{p_2}(x)$ be particular solutions of the nonhomogeneous equations $L(y) = g_1(x)$ and $L(y) = g_2(x)$, respectively. If we define $y_p = y_{p_1}(x) + y_{p_2}(x)$, we want to show that y_p is a particular solution of $L(y) = g_1(x) + g_2(x)$. The result follows again by the linearity of the operator L:

$$L(y_p) = L\{y_{p_1}(x) + y_{p_2}(x)\} = L(y_{p_1}(x)) + L(y_{p_2}(x)) = g_1(x) + g_2(x). \qquad \equiv$$

EXAMPLE 11 Superposition—Nonhomogeneous DE

You should verify that

$$\begin{aligned} y_{p_1} &= -4x^2 && \text{is a particular solution of} && y'' - 3y' + 4y = -16x^2 + 24x - 8, \\ y_{p_2} &= e^{2x} && \text{is a particular solution of} && y'' - 3y' + 4y = 2e^{2x}, \\ y_{p_3} &= xe^{x} && \text{is a particular solution of} && y'' - 3y' + 4y = 2xe^{x} - e^{x}. \end{aligned}$$

It follows from Theorem 3.1.7 that the superposition of y_{p_1}, y_{p_2}, and y_{p_3},

$$y = y_{p_1} + y_{p_2} + y_{p_3} = -4x^2 + e^{2x} + xe^{x},$$

is a solution of

$$y'' - 3y' + 4y = \underbrace{-16x^2 + 24x - 8}_{g_1(x)} + \underbrace{2e^{2x}}_{g_2(x)} + \underbrace{2xe^{x} - e^{x}}_{g_3(x)}. \qquad \equiv$$

If the y_{p_i} are particular solutions of (12) for $i = 1, 2, \ldots, k$, then the linear combination

This sentence is a generalization of Theorem 3.1.7.

$$y_p = c_1 y_{p_1} + c_2 y_{p_2} + \cdots + c_k y_{p_k},$$

where the c_i are constants, is also a particular solution of (14) when the right-hand member of the equation is the linear combination

$$c_1 g_1(x) + c_2 g_2(x) + \cdots + c_k g_k(x).$$

Before we actually start solving homogeneous and nonhomogeneous linear differential equations, we need one additional bit of theory presented in the next section.

Remarks

This remark is a continuation of the brief discussion of dynamical systems given at the end of Section 1.3.

A dynamical system whose rule or mathematical model is a linear nth-order differential equation

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t)$$

is said to be a **linear system**. The set of n time-dependent functions $y(t), y'(t), \ldots, y^{(n-1)}(t)$ are the **state variables** of the system. Recall, their values at some time t give the **state of the system**. The function g is variously called the **input function**, **forcing function**, or **excitation function**. A solution $y(t)$ of the differential equation is said to be the **output** or **response of the system**. Under the conditions stated in Theorem 3.1.1, the output or response $y(t)$ is uniquely determined by the input and the state of the system prescribed at a time t_0; that is, by the initial conditions $y(t_0), y'(t_0), \ldots, y^{(n-1)}(t_0)$.

In order that a dynamical system be a linear system, it is necessary that the superposition principle (Theorem 3.1.7) hold in the system; that is, the response of the system to a superposition of inputs is a superposition of outputs. We have already examined some simple linear systems in Section 2.7 (linear first-order equations); in Section 3.8 we examine linear systems in which the mathematical models are second-order differential equations.

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3.1 Exercises Answers to selected odd-numbered problems begin on page ANS-000.

3.1.1 Initial-Value and Boundary-Value Problems

In Problems 1–4, the given family of functions is the general solution of the differential equation on the indicated interval. Find a member of the family that is a solution of the initial-value problem.

1. $y = c_1e^x + c_2e^{-x}$, $(-\infty, \infty)$; $y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$

2. $y = c_1e^{4x} + c_2e^{-x}$, $(-\infty, \infty)$; $y'' - 3y' - 4y = 0$, $y(0) = 1$, $y'(0) = 2$

3. $y = c_1x + c_2x \ln x$, $(0, \infty)$; $x^2y'' - xy' + y = 0$, $y(1) = 3$, $y'(1) = -1$

4. $y = c_1 + c_2 \cos x + c_3 \sin x$, $(-\infty, \infty)$; $y''' + y' = 0$, $y(\pi) = 0$, $y'(\pi) = 2$, $y''(\pi) = -1$

5. Given that $y = c_1 + c_2x^2$ is a two-parameter family of solutions of $xy'' - y' = 0$ on the interval $(-\infty, \infty)$, show that constants c_1 and c_2 cannot be found so that a member of the family satisfies the initial conditions $y(0) = 0$, $y'(0) = 1$. Explain why this does not violate Theorem 3.1.1.

6. Find two members of the family of solutions in Problem 5 that satisfy the initial conditions $y(0) = 0$, $y'(0) = 0$.

7. Given that $x(t) = c_1 \cos \omega t + c_2 \sin \omega t$ is the general solution of $x'' + \omega^2x = 0$ on the interval $(-\infty, \infty)$, show that a solution satisfying the initial conditions $x(0) = x_0$, $x'(0) = x_1$, is given by

$$x(t) = x_0 \cos \omega t + \frac{x_1}{\omega} \sin \omega t.$$

8. Use the general solution of $x'' + \omega^2x = 0$ given in Problem 7 to show that a solution satisfying the initial conditions $x(t_0) = x_0$, $x'(t_0) = x_1$, is the solution given in Problem 7 shifted by an amount t_0:

$$x(t) = x_0 \cos \omega (t - t_0) + \frac{x_1}{\omega} \sin \omega (t - t_0).$$

In Problems 9 and 10, find an interval centered about $x = 0$ for which the given initial-value problem has a unique solution.

9. $(x - 2)y'' + 3y = x$, $y(0) = 0$, $y'(0) = 1$

10. $y'' + (\tan x)y = e^x$, $y(0) = 1$, $y'(0) = 0$

11. **(a)** Use the family in Problem 1 to find a solution of $y'' - y = 0$ that satisfies the boundary conditions $y(0) = 0$, $y(1) = 1$.
(b) The DE in part (a) has the alternative general solution $y = c_3 \cosh x + c_4 \sinh x$ on $(-\infty, \infty)$. Use this family to find a solution that satisfies the boundary conditions in part (a).
(c) Show that the solutions in parts (a) and (b) are equivalent.

12. Use the family in Problem 5 to find a solution of $xy'' - y' = 0$ that satisfies the boundary conditions $y(0) = 1$, $y'(1) = 6$.

In Problems 13 and 14, the given two-parameter family is a solution of the indicated differential equation on the interval $(-\infty, \infty)$. Determine whether a member of the family can be found that satisfies the boundary conditions.

13. $y = c_1e^x \cos x + c_2e^x \sin x$; $y'' - 2y' + 2y = 0$
(a) $y(0) = 1$, $y'(\pi) = 0$ **(b)** $y(0) = 1$, $y(\pi) = -1$
(c) $y(0) = 1$, $y(\pi/2) = 1$ **(d)** $y(0) = 0$, $y(\pi) = 0$

14. $y = c_1x^2 + c_2x^4 + 3$; $x^2y'' - 5xy' + 8y = 24$
(a) $y(-1) = 0$, $y(1) = 4$ **(b)** $y(0) = 1$, $y(1) = 2$
(c) $y(0) = 3$, $y(1) = 0$ **(d)** $y(1) = 3$, $y(2) = 15$

3.1.2 Homogeneous Equations

In Problems 15–22, determine whether the given set of functions is linearly dependent or linearly independent on the interval $(-\infty, \infty)$.

15. $f_1(x) = x$, $f_2(x) = x^2$, $f_3(x) = 4x - 3x^2$

16. $f_1(x) = 0$, $f_2(x) = x$, $f_3(x) = e^x$

17. $f_1(x) = 5$, $f_2(x) = \cos^2 x$, $f_3(x) = \sin^2 x$

18. $f_1(x) = \cos 2x$, $f_2(x) = 1$, $f_3(x) = \cos^2 x$

19. $f_1(x) = x$, $f_2(x) = x - 1$, $f_3(x) = x + 3$

20. $f_1(x) = 2 + x$, $f_2(x) = 2 + |x|$

21. $f_1(x) = 1 + x$, $f_2(x) = x$, $f_3(x) = x^2$

22. $f_1(x) = e^x$, $f_2(x) = e^{-x}$, $f_3(x) = \sinh x$

In Problems 23−30, verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

23. $y'' - y' - 12y = 0$; e^{-3x}, e^{4x}, $(-\infty, \infty)$

24. $y'' - 4y = 0$; $\cosh 2x$, $\sinh 2x$, $(-\infty, \infty)$

25. $y'' - 2y' + 5y = 0$; $e^x \cos 2x$, $e^x \sin 2x$, $(-\infty, \infty)$

26. $4y'' - 4y' + y = 0$; $e^{x/2}$, $xe^{x/2}$, $(-\infty, \infty)$

27. $x^2y'' - 6xy' + 12y = 0$; x^3, x^4, $(0, \infty)$

28. $x^2y'' + xy' + y = 0$; $\cos(\ln x)$, $\sin(\ln x)$, $(0, \infty)$

29. $x^3y''' + 6x^2y'' + 4xy' - 4y = 0$; x, x^{-2}, $x^{-2} \ln x$, $(0, \infty)$

30. $y^{(4)} + y'' = 0$; 1, x, $\cos x$, $\sin x$, $(-\infty, \infty)$

3.1.3 Nonhomogeneous Equations

In Problems 31–34, verify that the given two-parameter family of functions is the general solution of the nonhomogeneous differential equation on the indicated interval.

31. $y'' - 7y' + 10y = 24e^x$;
$y = c_1e^{2x} + c_2e^{5x} + 6e^x$, $(-\infty, \infty)$

32. $y'' + y = \sec x$;
$y = c_1 \cos x + c_2 \sin x + x \sin x + (\cos x) \ln(\cos x)$, $(-\pi/2, \pi/2)$

33. $y'' - 4y' + 4y = 2e^{2x} + 4x - 12$;
$y = c_1e^{2x} + c_2xe^{2x} + x^2e^{2x} + x - 2$, $(-\infty, \infty)$

34. $2x^2y'' + 5xy' + y = x^2 - x$;
$y = c_1x^{-1/2} + c_2x^{-1} + \frac{1}{15}x^2 - \frac{1}{6}x$, $(0, \infty)$

35. **(a)** Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$ are, respectively, particular solutions of

$$y'' - 6y' + 5y = -9e^{2x}$$

and $$y'' - 6y' + 5y = 5x^2 + 3x - 16.$$

(b) Use part (a) to find particular solutions of

$$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}$$

and $$y'' - 6y' + 5y = -10x^2 - 6x + 32 + e^{2x}.$$

36. **(a)** By inspection, find a particular solution of

$$y'' + 2y = 10.$$

(b) By inspection, find a particular solution of

$$y'' + 2y = -4x.$$

(c) Find a particular solution of $y'' + 2y = -4x + 10$.

(d) Find a particular solution of $y'' + 2y = 8x + 5$.

Discussion Problems

37. Let $n = 1, 2, 3, \ldots$. Discuss how the observations $D^nx^{n-1} = 0$ and $D^nx^n = n!$ can be used to find the general solutions of the given differential equations.

(a) $y'' = 0$ **(b)** $y''' = 0$
(c) $y^{(4)} = 0$ **(d)** $y'' = 2$
(e) $y''' = 6$ **(f)** $y^{(4)} = 24$

38. Suppose that $y_1 = e^x$ and $y_2 = e^{-x}$ are two solutions of a homogeneous linear differential equation. Explain why $y_3 = \cosh x$ and $y_4 = \sinh x$ are also solutions of the equation.

39. **(a)** Verify that $y_1 = x^3$ and $y_2 = |x|^3$ are linearly independent solutions of the differential equation $x^2y'' - 4xy' + 6y = 0$ on the interval $(-\infty, \infty)$.

(b) Show that $W(y_1, y_2) = 0$ for every real number x. Does this result violate Theorem 3.1.3? Explain.

(c) Verify that $Y_1 = x^3$ and $Y_2 = x^2$ are also linearly independent solutions of the differential equation in part (a) on the interval $(-\infty, \infty)$.

(d) Find a solution of the differential equation satisfying $y(0) = 0$, $y'(0) = 0$.

(e) By the superposition principle, Theorem 3.1.2, both linear combinations $y = c_1y_1 + c_2y_2$ and $Y = c_1Y_1 + c_2Y_2$ are solutions of the differential equation. Discuss whether one, both, or neither of the linear combinations is a general solution of the differential equation on the interval $(-\infty, \infty)$.

40. Is the set of functions $f_1(x) = e^{x+2}$, $f_2(x) = e^{x-3}$ linearly dependent or linearly independent on the interval $(-\infty, \infty)$? Discuss.

41. Suppose $y_1, y_2, \ldots, y_k$ are k linearly independent solutions on $(-\infty, \infty)$ of a homogeneous linear nth-order differential equation with constant coefficients. By Theorem 3.1.2 it follows that $y_{k+1} = 0$ is also a solution of the differential equation. Is the set of solutions $y_1, y_2, \ldots, y_k, y_{k+1}$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

42. Suppose that $y_1, y_2, \ldots, y_k$ are k nontrivial solutions of a homogeneous linear nth-order differential equation with constant coefficients and that $k = n + 1$. Is the set of solutions $y_1, y_2, \ldots, y_k$ linearly dependent or linearly independent on $(-\infty, \infty)$? Discuss.

3.2 Reduction of Order

Introduction In Section 3.1 we saw that the general solution of a homogeneous linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{1}$$

was a linear combination $y = c_1y_1 + c_2y_2$, where y_1 and y_2 are solutions that constitute a linearly independent set on some interval I. Beginning in the next section we examine a method for determining these solutions when the coefficients of the DE in (1) are constants. This method, which is a straightforward exercise in algebra, breaks down in a few cases and yields only a single solution y_1 of the DE. It turns out that we can construct a second solution y_2 of a homogeneous equation (1) (even when the coefficients in (1) are variable) provided that we know one nontrivial solution y_1 of the DE. The basic idea described in this section is that the linear *second-order* equation (1) can be reduced to a linear *first-order* DE by means of a substitution involving the known solution y_1. A second solution, y_2 of (1), is apparent after this first-order DE is solved.

Reduction of Order Suppose $y(x)$ denotes a known solution of equation (1). We seek a second solution $y_2(x)$ of (1) so that y_1 and y_2 are linearly independent on some interval I. Recall that if

y_1 and y_2 are linearly independent, then their ratio y_2/y_1 is nonconstant on I; that is, $y_2/y_1 = u(x)$ or $y_2(x) = u(x)y_1(x)$. The idea is to find $u(x)$ by substituting $y_2(x) = u(x)y_1(x)$ into the given differential equation. This method is called **reduction of order** since we must solve a first-order equation to find u.

The first example illustrates the basic technique.

EXAMPLE 1 Finding a Second Solution

Given that $y_1 = e^x$ is a solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$, use reduction of order to find a second solution y_2.

Solution If $y = u(x)y_1(x) = u(x)e^x$, then the first two derivatives of y are obtained from the product rule:

$$y' = ue^x + e^x u', \qquad y'' = ue^x + 2e^x u' + e^x u''.$$

By substituting y and y'' into the original DE, it simplifies to

$$y'' - y = e^x(u'' + 2u') = 0.$$

Since $e^x \neq 0$, the last equation requires $u'' + 2u' = 0$. If we make the substitution $w = u'$, this linear second-order equation in u becomes $w' + 2w = 0$, which is a linear first-order equation in w. Using the integrating factor e^{2x}, we can write $d/dx\,[e^{2x}w] = 0$. After integrating we get $w = c_1e^{-2x}$ or $u' = c_1e^{-2x}$. Integrating again then yields $u = -\frac{1}{2}c_1e^{-2x} + c_2$. Thus

$$y = u(x)e^x = -\frac{c_1}{2}e^{-x} + c_2e^x. \tag{2}$$

By picking $c_2 = 0$ and $c_1 = -2$ we obtain the desired second solution, $y_2 = e^{-x}$. Because $W(e^x, e^{-x}) \neq 0$ for every x, the solutions are linearly independent on $(-\infty, \infty)$. ≡

Since we have shown that $y_1 = e^x$ and $y_2 = e^{-x}$ are linearly independent solutions of a linear second-order equation, the expression in (2) is actually the general solution of $y'' - y = 0$ on the interval $(-\infty, \infty)$.

General Case Suppose we divide by $a_2(x)$ in order to put equation (1) in the **standard form**

$$y'' + P(x)y' + Q(x)y = 0, \tag{3}$$

where $P(x)$ and $Q(x)$ are continuous on some interval I. Let us suppose further that $y_1(x)$ is a known solution of (3) on I and that $y_1(x) \neq 0$ for every x in the interval. If we define $y = u(x)y_1(x)$, it follows that

$$y' = uy_1' + y_1u', \qquad y'' = uy_1'' + 2y_1'u' + y_1u''$$

$$y'' + Py' + Qy = u[\underbrace{y''_1 + Py'_1 + Qy_1}_{\text{zero}}] + y_1u'' + (2y_1' + Py_1)u' = 0.$$

This implies that we must have

$$y_1u'' + (2y_1' + Py_1)u' = 0 \qquad \text{or} \qquad y_1w' + (2y_1' + Py_1)w = 0, \tag{4}$$

where we have let $w = u'$. Observe that the last equation in (4) is both linear and separable. Separating variables and integrating, we obtain

$$\frac{dw}{w} + 2\frac{y_1'}{y_1}dx + P\,dx = 0$$

$$\ln|wy_1^2| = -\int P\,dx + c \quad \text{or} \quad wy_1^2 = c_1e^{-\int P dx}.$$

We solve the last equation for w, use $w = u'$, and integrate again:

$$u = c_1 \int \frac{e^{-\int P dx}}{y_1^2} dx + c_2.$$

By choosing $c_1 = 1$ and $c_2 = 0$, we find from $y = u(x)y_1(x)$ that a second solution of equation (3) is

$$y_2 = y_1(x) \int \frac{e^{-\int P(x) dx}}{y_1^2(x)} dx. \tag{5}$$

It makes a good review of differentiation to verify that the function $y_2(x)$ defined in (5) satisfies equation (3) and that y_1 and y_2 are linearly independent on any interval on which $y_1(x)$ is not zero.

EXAMPLE 2 A Second Solution by Formula (5)

The function $y_1 = x^2$ is a solution of $x^2y'' - 3xy' + 4y = 0$. Find the general solution on the interval $(0, \infty)$.

Solution From the standard form of the equation

$$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = 0,$$

we find from (5)

$$y_2 = x^2 \int \frac{e^{3\int dx/x}}{x^4} dx \quad \leftarrow e^{3\int dx/x} = e^{\ln x^3} = x^3$$

$$= x^2 \int \frac{dx}{x} = x^2 \ln x.$$

The general solution on the interval $(0, \infty)$ is given by $y = c_1y_1 + c_2y_2$; that is, $y = c_1x^2 + c_2x^2 \ln x$. ≡

Remarks

We have derived and illustrated how to use (5) because this formula appears again in the next section and in Section 5.2. We use (5) simply to save time in obtaining a desired result. Your instructor will tell you whether you should memorize (5) or whether you should know the first principles of reduction of order.

3.2 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

In Problems 1–16, the indicated function $y_1(x)$ is a solution of the given equation. Use reduction of order or formula (5), as instructed, to find a second solution $y_2(x)$.

1. $y'' - 4y' + 4y = 0$; $y_1 = e^{2x}$
2. $y'' + 2y' + y = 0$; $y_1 = xe^{-x}$
3. $y'' + 16y = 0$; $y_1 = \cos 4x$
4. $y'' + 9y = 0$; $y_1 = \sin 3x$
5. $y'' - y = 0$; $y_1 = \cosh x$
6. $y'' - 25y = 0$; $y_1 = e^{5x}$
7. $9y'' - 12y' + 4y = 0$; $y_1 = e^{2x/3}$
8. $6y'' + y' - y = 0$; $y_1 = e^{x/3}$
9. $x^2y'' - 7xy' + 16y = 0$; $y_1 = x^4$
10. $x^2y'' + 2xy' - 6y = 0$; $y_1 = x^2$
11. $xy'' + y' = 0$; $y_1 = \ln x$
12. $4x^2y'' + y = 0$; $y_1 = x^{1/2} \ln x$
13. $x^2y'' - xy' + 2y = 0$; $y_1 = x \sin(\ln x)$
14. $x^2y'' - 3xy' + 5y = 0$; $y_1 = x^2 \cos(\ln x)$
15. $(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0$; $y_1 = x + 1$
16. $(1 - x^2)y'' + 2xy' = 0$; $y_1 = 1$

In Problems 17–20, the indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction of order to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution of the given nonhomogeneous equation.

17. $y'' - 4y = 2; \quad y_1 = e^{-2x}$
18. $y'' + y' = 1; \quad y_1 = 1$
19. $y'' - 3y' + 2y = 5e^{3x}; \quad y_1 = e^x$
20. $y'' - 4y' + 3y = x; \quad y_1 = e^x$

22. Verify that $y_1(x) = x$ is a solution of $xy'' - xy' + y = 0$. Use reduction of order to find a second solution $y_2(x)$ in the form of an infinite series. Conjecture an interval of definition for $y_2(x)$.

≡ Discussion Problems

21. **(a)** Give a convincing demonstration that the second-order equation $ay'' + by' + cy = 0$, a, b, and c constants, always possesses at least one solution of the form $y_1 = e^{m_1 x}$, m_1 a constant.

(b) Explain why the differential equation in part (a) must then have a second solution, either of the form $y_2 = e^{m_2 x}$, or of the form $y_2 = xe^{m_1 x}$, m_1 and m_2 constants.

(c) Reexamine Problems 1–8. Can you explain why the statements in parts (a) and (b) above are not contradicted by the answers to Problems 3–5?

≡ Computer Lab Assignments

23. **(a)** Verify that $y_1(x) = e^x$ is a solution of

$$xy'' - (x + 10)y' + 10y = 0.$$

(b) Use (5) to find a second solution $y_2(x)$. Use a CAS to carry out the required integration.

(c) Explain, using Corollary (a) of Theorem 3.1.2, why the second solution can be written compactly as

$$y_2(x) = \sum_{n=0}^{10} \frac{1}{n!} x^n.$$

3.3 Homogeneous Linear Equations with Constant Coefficients

Introduction We have seen that the linear first-order DE $y' + ay = 0$, where a is a constant, possesses the exponential solution $y = c_1 e^{-ax}$ on the interval $(-\infty, \infty)$. Therefore, it is natural to ask whether exponential solutions exist for homogeneous linear higher-order DEs

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0, \tag{1}$$

where the coefficients a_i, $i = 0, 1, \ldots, n$ are real constants and $a_n \neq 0$. The surprising fact is that *all* solutions of these higher-order equations are either exponential functions or are constructed out of exponential functions.

Auxiliary Equation We begin by considering the special case of a second-order equation

$$ay'' + by' + cy = 0. \tag{2}$$

If we try a solution of the form $y = e^{mx}$, then after substituting $y' = me^{mx}$ and $y'' = m^2 e^{mx}$ equation (2) becomes

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0 \quad \text{or} \quad e^{mx}(am^2 + bm + c) = 0.$$

Since e^{mx} is never zero for real values of x, it is apparent that the only way that this exponential function can satisfy the differential equation (2) is to choose m as a root of the quadratic equation

$$am^2 + bm + c = 0. \tag{3}$$

This last equation is called the **auxiliary equation** of the differential equation (2). Since the two roots of (3) are $m_1 = (-b + \sqrt{b^2 - 4ac})/2a$ and $m_2 = (-b - \sqrt{b^2 - 4ac})/2a$, there will be three forms of the general solution of (1) corresponding to the three cases:

- m_1 and m_2 are real and distinct $(b^2 - 4ac > 0)$,
- m_1 and m_2 are real and equal $(b^2 - 4ac = 0)$, and
- m_1 and m_2 are conjugate complex numbers $(b^2 - 4ac < 0)$.

We discuss each of these cases in turn.

Case I*:** ***Distinct Real Roots Under the assumption that the auxiliary equation (3) has two unequal real roots m_1 and m_2, we find two solutions, $y_1 = e^{m_1 x}$ and $y_2 = e^{m_2 x}$, respectively. We see that these functions are linearly independent on $(-\infty, \infty)$ and hence form a fundamental set. It follows that the general solution of (2) on this interval is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}. \tag{4}$$

Case II*:** ***Repeated Real Roots When $m_1 = m_2$ we necessarily obtain only one exponential solution, $y_1 = e^{m_1 x}$. From the quadratic formula we find that $m_1 = -b/2a$ since the only way to have $m_1 = m_2$ is to have $b^2 - 4ac = 0$. It follows from the discussion in Section 3.2 that a second solution of the equation is

$$y_2 = e^{m_1 x}\int \frac{e^{2m_1 x}}{e^{2m_1 x}}dx = e^{m_1 x}\int dx = xe^{m_1 x}. \tag{5}$$

In (5) we have used the fact that $-b/a = 2m_1$. The general solution is then

$$y = c_1 e^{m_1 x} + c_2 x e^{m_1 x}. \tag{6}$$

Case III*:** ***Conjugate Complex Roots If m_1 and m_2 are complex, then we can write $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real and $i^2 = -1$. Formally, there is no difference between this case and Case I, hence

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}.$$

However, in practice we prefer to work with real functions instead of complex exponentials. To this end we use Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

where θ is any real number.* It follows from this formula that

$$e^{i\beta x} = \cos\beta x + i\sin\beta x \quad \text{and} \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x, \tag{7}$$

where we have used $\cos(-\beta x) = \cos\beta x$ and $\sin(-\beta x) = -\sin\beta x$. Note that by first adding and then subtracting the two equations in (7), we obtain, respectively,

$$e^{i\beta x} + e^{-i\beta x} = 2\cos\beta x \quad \text{and} \quad e^{i\beta x} - e^{-i\beta x} = 2i\sin\beta x.$$

Since $y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$ is a solution of (2) for any choice of the constants C_1 and C_2, the choices $C_1 = C_2 = 1$ and $C_1 = 1$, $C_2 = -1$ give, in turn, two solutions:

$$y_1 = e^{(\alpha + i\beta)x} + e^{(\alpha - i\beta)x} \quad \text{and} \quad y_2 = e^{(\alpha + i\beta)x} - e^{(\alpha - i\beta)x}.$$

But $$y_1 = e^{\alpha x}(e^{i\beta x} + e^{-i\beta x}) = 2e^{\alpha x}\cos\beta x$$

and $$y_2 = e^{\alpha x}(e^{i\beta x} - e^{-i\beta x}) = 2ie^{\alpha x}\sin\beta x.$$

---

*A formal derivation of Euler's formula can be obtained from the Maclaurin series $e^x = \sum_{n=0}^{\infty} x^n/n!$ by substituting $x = i\theta$, using $i^2 = -1$, $i^3 = -i$, ..., and then separating the series into real and imaginary parts. The plausibility thus established, we can adopt $\cos\theta + i\sin\theta$ as the *definition* of $e^{i\theta}$.

Hence from Corollary (a) of Theorem 3.1.2 the last two results show that $e^{\alpha x} \cos \beta x$ and $e^{\alpha x} \sin \beta x$ are *real* solutions of (2). Moreover, these solutions form a fundamental set on $(-\infty, \infty)$. Consequently, the general solution is

$$y = c_1 e^{\alpha x} \cos \beta x + c_2 e^{\alpha x} \sin \beta x = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x). \qquad (8)$$

EXAMPLE 1 Second-Order DEs

Solve the following differential equations.

(a) $2y'' - 5y' - 3y = 0$ (b) $y'' - 10y' + 25y = 0$ (c) $y'' + 4y' + 7y = 0$

Solution We give the auxiliary equations, the roots, and the corresponding general solutions.

(a) $2m^2 - 5m - 3 = (2m + 1)(m - 3)$, $m_1 = -\frac{1}{2}$, $m_2 = 3$. From (4),

$$y = c_1 e^{-x/2} + c_2 e^{3x}.$$

(b) $m^2 - 10m + 25 = (m - 5)^2$, $m_1 = m_2 = 5$. From (6),

$$y = c_1 e^{5x} + c_2 x e^{5x}.$$

(c) $m^2 + 4m + 7 = 0$, $m_1 = -2 + \sqrt{3}\, i$, $m_2 = -2 - \sqrt{3}\, i$. From (8) with $\alpha = -2$, $\beta = \sqrt{3}$, we have

$$y = e^{-2x}(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x).$$

≡

EXAMPLE 2 An Initial-Value Problem

Solve the initial-value problem $4y'' + 4y' + 17y = 0$, $y(0) = -1$, $y'(0) = 2$.

Solution By the quadratic formula we find that the roots of the auxiliary equation $4m^2 + 4m + 17 = 0$ are $m_1 = -\frac{1}{2} + 2i$ and $m_2 = -\frac{1}{2} - 2i$. Thus from (8) we have $y = e^{-x/2}(c_1 \cos 2x + c_2 \sin 2x)$. Applying the condition $y(0) = -1$, we see from $e^0(c_1 \cos 0 + c_2 \sin 0) = -1$ that $c_1 = -1$. Differentiating $y = e^{-x/2}(-\cos 2x + c_2 \sin 2x)$ and then using $y'(0) = 2$ gives $2c_2 + \frac{1}{2} = 2$ or $c_2 = \frac{3}{4}$. Hence the solution of the IVP is $y = e^{-x/2}(-\cos 2x + \frac{3}{4} \sin 2x)$. In **FIGURE 3.3.1** we see that the solution is oscillatory but $y \to 0$ as $x \to \infty$. ≡

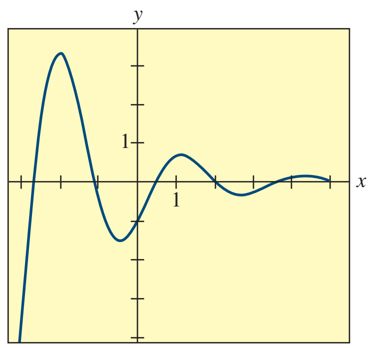

FIGURE 3.3.1 Graph of solution of IVP in Example 2

Two Equations Worth Knowing

The two differential equations

$$y'' + k^2 y = 0 \quad \text{and} \quad y'' - k^2 y = 0,$$

k real, are important in applied mathematics. For $y'' + k^2 y = 0$, the auxiliary equation $m^2 + k^2 = 0$ has imaginary roots $m_1 = ki$ and $m_2 = -ki$. With $\alpha = 0$ and $\beta = k$ in (8), the general solution of the DE is seen to be

$$y = c_1 \cos kx + c_2 \sin kx. \qquad (9)$$

On the other hand, the auxiliary equation $m^2 - k^2 = 0$ for $y'' - k^2 y = 0$ has distinct real roots $m_1 = k$ and $m_2 = -k$ and so by (4) the general solution of the DE is

$$y = c_1 e^{kx} + c_2 e^{-kx}. \qquad (10)$$

Notice that if we choose $c_1 = c_2 = \frac{1}{2}$ and $c_1 = \frac{1}{2}$, $c_2 = -\frac{1}{2}$ in (10), we get the particular solutions $y = \frac{1}{2}(e^{kx} + e^{-kx}) = \cosh kx$ and $y = \frac{1}{2}(e^{kx} - e^{-kx}) = \sinh kx$. Since $\cosh kx$ and $\sinh kx$ are linearly independent on any interval of the x-axis, an alternative form for the general solution of $y'' - k^2 y = 0$ is

$$y = c_1 \cosh kx + c_2 \sinh kx. \qquad (11)$$

See Problems 41, 42, and 53 in Exercises 3.3.

Higher-Order Equations In general, to solve an nth-order differential equation

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \tag{12}$$

where the a_i, $i = 0, 1, \ldots, n$ are real constants, we must solve an nth-degree polynomial equation

$$a_n m^n + a_{n-1}m^{n-1} + \cdots + a_2 m^2 + a_1 m + a_0 = 0. \tag{13}$$

If all the roots of (13) are real and distinct, then the general solution of (12) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}.$$

It is somewhat harder to summarize the analogues of Cases II and III because the roots of an auxiliary equation of degree greater than two can occur in many combinations. For example, a fifth-degree equation could have five distinct real roots, or three distinct real and two complex roots, or one real and four complex roots, or five real but equal roots, or five real roots but with two of them equal, and so on. When m_1 is a root of multiplicity k of an nth-degree auxiliary equation (that is, k roots are equal to m_1), it can be shown that the linearly independent solutions are

$$e^{m_1 x},\ xe^{m_1 x},\ x^2 e^{m_1 x}, \ldots,\ x^{k-1}e^{m_1 x}$$

and the general solution must contain the linear combination

$$c_1 e^{m_1 x} + c_2 xe^{m_1 x} + c_3 x^2 e^{m_1 x} + \cdots + c_k x^{k-1} e^{m_1 x}.$$

Lastly, it should be remembered that when the coefficients are real, complex roots of an auxiliary equation always appear in conjugate pairs. Thus, for example, a cubic polynomial equation can have at most two complex roots.

EXAMPLE 3 Third-Order DE

Solve $y''' + 3y'' - 4y = 0$.

Solution It should be apparent from inspection of $m^3 + 3m^2 - 4 = 0$ that one root is $m_1 = 1$ and so $m - 1$ is a factor of $m^3 + 3m^2 - 4$. By division we find

$$m^3 + 3m^2 - 4 = (m - 1)(m^2 + 4m + 4) = (m - 1)(m + 2)^2,$$

and so the other roots are $m_2 = m_3 = -2$. Thus the general solution is

$$y = c_1 e^{x} + c_2 e^{-2x} + c_3 xe^{-2x}.$$ ≡

EXAMPLE 4 Fourth-Order DE

Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = 0$.

Solution The auxiliary equation $m^4 + 2m^2 + 1 = (m^2 + 1)^2 = 0$ has roots $m_1 = m_3 = i$ and $m_2 = m_4 = -i$. Thus from Case II the solution is

$$y = C_1 e^{ix} + C_2 e^{-ix} + C_3 xe^{ix} + C_4 xe^{-ix}.$$

By Euler's formula the grouping $C_1 e^{ix} + C_2 e^{-ix}$ can be rewritten as $c_1 \cos x + c_2 \sin x$ after a relabeling of constants. Similarly, $x(C_3 e^{ix} + C_4 e^{-ix})$ can be expressed as $x(c_3 \cos x + c_4 \sin x)$. Hence the general solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 x \cos x + c_4 x \sin x.$$ ≡

Example 4 illustrates a special case when the auxiliary equation has repeated complex roots. In general, if $m_1 = \alpha + i\beta, \beta > 0$, is a complex root of multiplicity k of an auxiliary equation with real coefficients, then its conjugate $m_2 = \alpha - i\beta$ is also a root of multiplicity k. From the $2k$ complex-valued solutions

$$e^{(\alpha+i\beta)x}, \quad xe^{(\alpha+i\beta)x}, \quad x^2e^{(\alpha+i\beta)x}, \ldots, \quad x^{k-1}e^{(\alpha+i\beta)x}$$
$$e^{(\alpha-i\beta)x}, \quad xe^{(\alpha-i\beta)x}, \quad x^2e^{(\alpha-i\beta)x}, \ldots, \quad x^{k-1}e^{(\alpha-i\beta)x}$$

we conclude, with the aid of Euler's formula, that the general solution of the corresponding differential equation must then contain a linear combination of the $2k$ real linearly independent solutions

$$e^{\alpha x}\cos\beta x, \quad xe^{\alpha x}\cos\beta x, \quad x^2e^{\alpha x}\cos\beta x, \ldots, \quad x^{k-1}e^{\alpha x}\cos\beta x$$
$$e^{\alpha x}\sin\beta x, \quad xe^{\alpha x}\sin\beta x, \quad x^2e^{\alpha x}\sin\beta x, \ldots, \quad x^{k-1}e^{\alpha x}\sin\beta x.$$

In Example 4 we identify $k = 2$, $\alpha = 0$, and $\beta = 1$.

■ Rational Roots Of course the most difficult aspect of solving constant–coefficient differential equations is finding roots of auxiliary equations of degree greater than two. For example, to solve $3y''' + 5y'' + 10y' - 4y = 0$ we must solve $3m^3 + 5m^2 + 10m - 4 = 0$. Something we can try is to test the auxiliary equation for rational roots. Recall, if $m_1 = p/q$ is a rational root (expressed in lowest terms) of an auxiliary equation $a_nm^n + \cdots + a_1m + a_0 = 0$ with integer coefficients, then p is a factor of a_0 and q is a factor of a_n. For our specific cubic auxiliary equation, all the factors of $a_0 = -4$ and $a_n = 3$ are p: $\pm1, \pm2, \pm4$ and q: $\pm1, \pm3$, so the possible rational roots are p/q: $\pm1, \pm2, \pm4, \pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{4}{3}$. Each of these numbers can then be tested, say, by synthetic division. In this way we discover both the root $m_1 = \frac{1}{3}$ and the factorization

$$3m^3 + 5m^2 + 10m - 4 = (m - \tfrac{1}{3})(3m^2 + 6m + 12).$$

The quadratic formula then yields the remaining roots $m_2 = -1 + \sqrt{3}i$ and $m_3 = -1 - \sqrt{3}i$. Therefore the general solution of $3y''' + 5y'' + 10y' - 4y = 0$ is $y = c_1e^{x/3} + e^{-x}(c_2\cos\sqrt{3}x + c_3\sin\sqrt{3}x)$.

■ Use of Computers Finding roots or approximations of roots of polynomial equations is a routine problem with an appropriate calculator or computer software. The computer algebra systems *Mathematica* and *Maple* can solve polynomial equations (in one variable) of degree less than five in terms of algebraic formulas. For the auxiliary equation in the preceding paragraph, the commands

Solve[3 m∧3 + 5 m∧2 + 10 m – 4 = = 0, m] (in *Mathematica*)

solve(3*m∧3 + 5*m∧2 + 10*m – 4, m); (in *Maple*)

yield immediately their representations of the roots $\frac{1}{3}, -1 + \sqrt{3}i, -1 - \sqrt{3}i$. For auxiliary equations of higher degree it may be necessary to resort to numerical commands such as **NSolve** and **FindRoot** in *Mathematica*. Because of their capability of solving polynomial equations, it is not surprising that some computer algebra systems are also able to give explicit solutions of homogeneous linear constant-coefficient differential equations. For example, the inputs

DSolve [y″[x] + 2 y′[x] + 2 y[x] = = 0, y[x], x] (in *Mathematica*)

dsolve(diff(y(x),x$2) + 2*diff(y(x),x) + 2*y(x) = 0, y(x)); (in *Maple*)

give, respectively,

$$\mathbf{y[x] \;{-}{>}\; \frac{C[2]\,Cos\,[x] \;-\; C[1]\,Sin\,[x]}{E^x}} \tag{14}$$

and **y(x) = _C1 exp(–x) sin(x) – _C2 exp(–x) cos(x)**

Translated, this means $y = c_2e^{-x}\cos x + c_1e^{-x}\sin x$ is a solution of $y'' + 2y' + 2y = 0$.

In the classic text *Differential Equations* by Ralph Palmer Agnew* (used by the author as a student), the following statement is made:

> *It is not reasonable to expect students in this course to have computing skill and equipment necessary for efficient solving of equations such as*

$$4.317\frac{d^4y}{dx^4} + 2.179\frac{d^3y}{dx^3} + 1.416\frac{d^2y}{dx^2} + 1.295\frac{dy}{dx} + 3.169y = 0. \qquad (15)$$

Although it is debatable whether computing skills have improved in the intervening years, it is a certainty that technology has. If one has access to a computer algebra system, equation (15) could be considered reasonable. After simplification and some relabeling of the output, *Mathematica* yields the (approximate) general solution

$$\begin{aligned} y = c_1e^{-0.728852x}\cos(0.618605x) + c_2e^{-0.728852x}\sin(0.618605x) \\ + c_3e^{-0.476478x}\cos(0.759081x) + c_4e^{-0.476478x}\sin(0.759081x). \end{aligned}$$

We note in passing that the **DSolve** and **dsolve** commands in *Mathematica* and *Maple,* like most aspects of any CAS, have their limitations.

Finally, if we are faced with an initial-value problem consisting of, say, a fourth-order differential equation, then to fit the general solution of the DE to the four initial conditions we must solve a system of four linear equations in four unknowns (the c_1, c_2, c_3, c_4 in the general solution). Using a CAS to solve the system can save lots of time. See Problems 35, 36, 61, and 62 in Exercises 3.3.

*McGraw-Hill, New York, 1960.

Remarks

In case you are wondering, the method of this section also works for homogeneous linear *first-order* differential equations $ay' + by = 0$ with constant coefficients. For example, to solve, say, $2y' + 7y = 0$, we substitute $y = e^{mx}$ into the DE to obtain the auxiliary equation $2m + 7 = 0$. Using $m = -\frac{7}{2}$, the general solution of the DE is then $y = c_1e^{-7x/2}$.

3.3 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

In Problems 1–14, find the general solution of the given second-order differential equation.

1. $4y'' + y' = 0$
2. $y'' - 36y = 0$
3. $y'' - y' - 6y = 0$
4. $y'' - 3y' + 2y = 0$
5. $y'' + 8y' + 16y = 0$
6. $y'' - 10y' + 25y = 0$
7. $12y'' - 5y' - 2y = 0$
8. $y'' + 4y' - y = 0$
9. $y'' + 9y = 0$
10. $3y'' + y = 0$
11. $y'' - 4y' + 5y = 0$
12. $2y'' + 2y' + y = 0$
13. $3y'' + 2y' + y = 0$
14. $2y'' - 3y' + 4y = 0$

In Problems 15–28, find the general solution of the given higher-order differential equation.

15. $y''' - 4y'' - 5y' = 0$
16. $y''' - y = 0$
17. $y''' - 5y'' + 3y' + 9y = 0$
18. $y''' + 3y'' - 4y' - 12y = 0$
19. $\dfrac{d^3u}{dt^3} + \dfrac{d^2u}{dt^2} - 2u = 0$
20. $\dfrac{d^3x}{dt^3} - \dfrac{d^2x}{dt^2} - 4x = 0$
21. $y''' + 3y'' + 3y' + y = 0$
22. $y''' - 6y'' + 12y' - 8y = 0$
23. $y^{(4)} + y''' + y'' = 0$
24. $y^{(4)} - 2y'' + y = 0$

25. $16\dfrac{d^4y}{dx^4} + 24\dfrac{d^2y}{dx^2} + 9y = 0$

26. $\dfrac{d^4y}{dx^4} - 7\dfrac{d^2y}{dx^2} - 18y = 0$

27. $\dfrac{d^5u}{dr^5} + 5\dfrac{d^4u}{dr^4} - 2\dfrac{d^3u}{dr^3} - 10\dfrac{d^2u}{dr^2} + \dfrac{du}{dr} + 5u = 0$

28. $2\dfrac{d^5x}{ds^5} - 7\dfrac{d^4x}{ds^4} + 12\dfrac{d^3x}{ds^3} + 8\dfrac{d^2x}{ds^2} = 0$

In Problems 29–36, solve the given initial-value problem.

29. $y'' + 16y = 0,\ y(0) = 2, y'(0) = -2$

30. $\dfrac{d^2y}{d\theta^2} + y = 0,\ y(\pi/3) = 0, y'(\pi/3) = 2$

31. $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} - 5y = 0,\ y(1) = 0, y'(1) = 2$

32. $4y'' - 4y' - 3y = 0,\ y(0) = 1, y'(0) = 5$

33. $y'' + y' + 2y = 0,\ y(0) = y'(0) = 0$

34. $y'' - 2y' + y = 0,\ y(0) = 5, y'(0) = 10$

35. $y''' + 12y'' + 36y' = 0,\ y(0) = 0, y'(0) = 1, y''(0) = -7$

36. $y''' + 2y'' - 5y' - 6y = 0,\ y(0) = y'(0) = 0, y''(0) = 1$

In Problems 37−40, solve the given boundary-value problem.

37. $y'' - 10y' + 25y = 0,\ y(0) = 1, y(1) = 0$

38. $y'' + 4y = 0,\ y(0) = 0,\ y(\pi) = 0$

39. $y'' + y = 0,\ y'(0) = 0,\ y'(\pi/2) = 0$

40. $y'' - 2y' + 2y = 0,\ y(0) = 1, y(\pi) = 1$

In Problems 41 and 42, solve the given problem first using the form of the general solution given in (10). Solve again, this time using the form given in (11).

41. $y'' - 3y = 0,\ y(0) = 1, y'(0) = 5$

42. $y'' - y = 0,\ y(0) = 1, y'(1) = 0$

In Problems 43–48, each figure represents the graph of a particular solution of one of the following differential equations:

(a) $y'' - 3y' - 4y = 0$	**(b)** $y'' + 4y = 0$
(c) $y'' + 2y' + y = 0$	**(d)** $y'' + y = 0$
(e) $y'' + 2y' + 2y = 0$	**(f)** $y'' - 3y' + 2y = 0$

Match a solution curve with one of the differential equations. Explain your reasoning.

43.

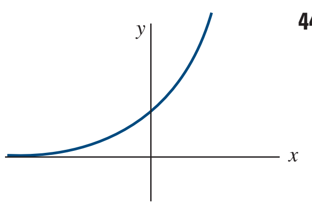

FIGURE 3.3.2 Graph for Problem 43

44.

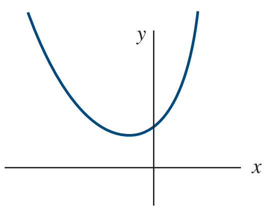

FIGURE 3.3.3 Graph for Problem 44

45.

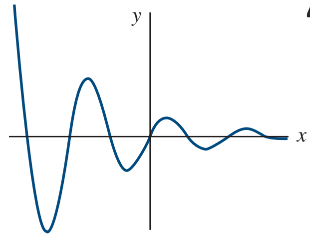

FIGURE 3.3.4 Graph for Problem 45

46.

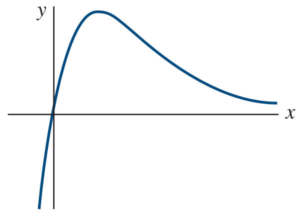

FIGURE 3.3.5 Graph for Problem 46

47.

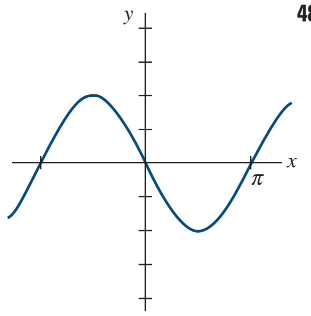

FIGURE 3.3.6 Graph for Problem 47

48.

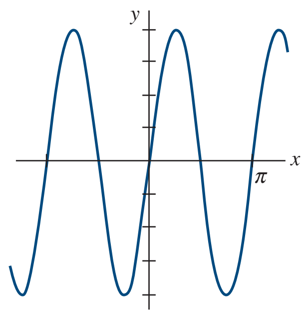

FIGURE 3.3.7 Graph for Problem 48

≡ Discussion Problems

49. The roots of a cubic auxiliary equation are $m_1 = 4$ and $m_2 = m_3 = -5$. What is the corresponding homogeneous linear differential equation? Discuss: Is your answer unique?

50. Two roots of a cubic auxiliary equation with real coefficients are $m_1 = -\frac{1}{2}$ and $m_2 = 3 + i$. What is the corresponding homogeneous linear differential equation?

51. Find the general solution of $y''' + 6y'' + y' - 34y = 0$ if it is known that $y_1 = e^{-4x}\cos x$ is one solution.

52. To solve $y^{(4)} + y = 0$ we must find the roots of $m^4 + 1 = 0$. This is a trivial problem using a CAS, but it can also be done by hand working with complex numbers. Observe that $m^4 + 1 = (m^2 + 1)^2 - 2m^2$. How does this help? Solve the differential equation.

53. Verify that $y = \sinh x - 2\cos(x + \pi/6)$ is a particular solution of $y^{(4)} - y = 0$. Reconcile this particular solution with the general solution of the DE.

54. Consider the boundary-value problem $y'' + \lambda y = 0, y(0) = 0, y(\pi/2) = 0$. Discuss: Is it possible to determine values of λ so that the problem possesses (a) trivial solutions? (b) nontrivial solutions?

55. In the study of techniques of integration in calculus, certain indefinite integrals of the form $\int e^{ax} f(x)\,dx$ could be evaluated by applying integration by parts twice, recovering the original integral on the right-hand side, solving for the original integral, and obtaining a constant multiple $k\int e^{ax} f(x)\,dx$ on the left-hand side. Then the value of the integral is found by dividing by k. Discuss: For what kinds of functions f does the described procedure work? Your solution should lead to a differential equation. Carefully analyze this equation and solve for f.

Mathematical Model

56. Slipping Chain Reread the discussion on the slipping chain in Section 1.3 and illustrated in Figure 1.3.6 on page 22.

(a) Use the form of the solution given in (11) of this section to find the general solution of equation (16) of Section 1.3:

$$\frac{d^2x}{dt^2} - \frac{64}{L}x = 0.$$

(b) Find a particular solution that satisfies the initial conditions stated in the discussion on pages 22–23.

(c) Suppose that the total length of the chain is $L = 20$ ft and that $x_0 = 1$. Find the velocity at which the slipping chain will leave the supporting peg.

Computer Lab Assignments

In Problems 57–60, use a computer either as an aid in solving the auxiliary equation or as a means of directly obtaining the general solution of the given differential equation. If you use a CAS to obtain the general solution, simplify the output and, if necessary, write the solution in terms of real functions.

57. $y''' - 6y'' + 2y' + y = 0$

58. $6.11y''' + 8.59y'' + 7.93y' + 0.778y = 0$

59. $3.15y^{(4)} - 5.34y'' + 6.33y' - 2.03y = 0$

60. $y^{(4)} + 2y'' - y' + 2y = 0$

In Problems 61 and 62, use a CAS as an aid in solving the auxiliary equation. Form the general solution of the differential equation. Then use a CAS as an aid in solving the system of equations for the coefficients c_i, $i = 1, 2, 3, 4$ that result when the initial conditions are applied to the general solution.

61. $2y^{(4)} + 3y''' - 16y'' + 15y' - 4y = 0,$
$y(0) = -2, y'(0) = 6, y''(0) = 3, y'''(0) = \frac{1}{2}$

62. $y^{(4)} - 3y''' + 3y'' - y' = 0,$
$y(0) = y'(0) = 0, y''(0) = y'''(0) = 1$

3.4 Undetermined Coefficients

Introduction To solve a nonhomogeneous linear differential equation

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x) \tag{1}$$

we must do two things: (*i*) find the complementary function y_c; and (*ii*) find *any* particular solution y_p of the nonhomogeneous equation. Then, as discussed in Section 3.1, the general solution of (1) on an interval I is $y = y_c + y_p$.

The complementary function y_c is the general solution of the associated homogeneous DE of (1); that is

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0.$$

In the last section we saw how to solve these kinds of equations when the coefficients were constants. Our goal then in the present section is to examine a method for obtaining particular solutions.

Method of Undetermined Coefficients The first of two ways we shall consider for obtaining a particular solution y_p is called the **method of undetermined coefficients**. The underlying idea in this method is a conjecture, an educated guess really, about the form of y_p motivated by the kinds of functions that make up the input function $g(x)$. The general method is limited to nonhomogeneous linear DEs such as (1) where

- the coefficients, a_i, $i = 0, 1, \ldots, n$ are constants, and
- where $g(x)$ is a constant, a polynomial function, exponential function $e^{\alpha x}$, sine or cosine functions $\sin \beta x$ or $\cos \beta x$, or finite sums and products of these functions.

Strictly speaking, $g(x) = k$ (a constant) is a polynomial function. Since a constant function is probably not the first thing that comes to mind when you think of polynomial functions, for emphasis we shall continue to use the redundancy "constant functions, polynomial functions, ..."

A constant k is a polynomial function.

The following functions are some examples of the types of inputs $g(x)$ that are appropriate for this discussion:

$$g(x) = 10, \quad g(x) = x^2 - 5x, \quad g(x) = 15x - 6 + 8e^{-x}$$
$$g(x) = \sin 3x - 5x \cos 2x, \quad g(x) = xe^x \sin x + (3x^2 - 1)e^{-4x}.$$

That is, $g(x)$ is a linear combination of functions of the type

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \quad P(x)e^{\alpha x}, \quad P(x)e^{\alpha x} \sin \beta x, \quad \text{and} \quad P(x)e^{\alpha x} \cos \beta x,$$

where n is a nonnegative integer and α and β are real numbers. The method of undetermined coefficients is not applicable to equations of form (1) when

$$g(x) = \ln x, \quad g(x) = \frac{1}{x}, \quad g(x) = \tan x, \quad g(x) = \sin^{-1} x,$$

and so on. Differential equations in which the input $g(x)$ is a function of this last kind will be considered in Section 3.5.

The set of functions that consists of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines has the remarkable property that derivatives of their sums and products are again sums and products of constants, polynomials, exponentials $e^{\alpha x}$, sines, and cosines. Since the linear combination of derivatives $a_n y_p^{(n)} + a_{n-1} y_p^{(n-1)} + \cdots + a_1 y_p' + a_0 y_p$ must be identical to $g(x)$, it seems reasonable to assume that y_p *has the same form as* $g(x)$.

The next two examples illustrate the basic method.

EXAMPLE 1 General Solution Using Undetermined Coefficients

Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6.$ (2)

Solution **Step 1** We first solve the associated homogeneous equation $y'' + 4y' - 2y = 0$. From the quadratic formula we find that the roots of the auxiliary equation $m^2 + 4m - 2 = 0$ are $m_1 = -2 - \sqrt{6}$ and $m_2 = -2 + \sqrt{6}$. Hence the complementary function is

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

Step 2 Now, since the function $g(x)$ is a quadratic polynomial, let us assume a particular solution that is also in the form of a quadratic polynomial:

$$y_p = Ax^2 + Bx + C.$$

We seek to determine *specific* coefficients A, B, and C for which y_p is a solution of (2). Substituting y_p and the derivatives $y_p' = 2Ax + B$ and $y_p'' = 2A$ into the given differential equation (2), we get

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C$$
$$= 2x^2 - 3x + 6.$$

Since the last equation is supposed to be an identity, the coefficients of like powers of x must be equal:

equal

$$\boxed{-2A}\,x^2 + \boxed{8A - 2B}\,x + \boxed{2A + 4B - 2C} = 2x^2 - 3x + 6.$$

That is,

$$-2A = 2, \quad 8A - 2B = -3, \quad 2A + 4B - 2C = 6.$$

Solving this system of equations leads to the values $A = -1$, $B = -\frac{5}{2}$, and $C = -9$. Thus a particular solution is

$$y_p = -x^2 - \frac{5}{2}x - 9.$$

Step 3 The general solution of the given equation is

$$y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \frac{5}{2}x - 9. \quad \equiv$$

EXAMPLE 2 Particular Solution Using Undetermined Coefficients

Find a particular solution of $y'' - y' + y = 2 \sin 3x$.

Solution A natural first guess for a particular solution would be $A \sin 3x$. But since successive differentiations of $\sin 3x$ produce $\sin 3x$ *and* $\cos 3x$, we are prompted instead to assume a particular solution that includes both of these terms:

$$y_p = A \cos 3x + B \sin 3x.$$

Differentiating y_p and substituting the results into the differential equation give, after regrouping,

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2 \sin 3x$$

or

equal

$$\boxed{-8A - 3B}\cos 3x + \boxed{3A - 8B}\sin 3x = 0 \cos 3x + 2 \sin 3x.$$

From the resulting system of equations,

$$-8A - 3B = 0, \qquad 3A - 8B = 2,$$

we get $A = \frac{6}{73}$ and $B = -\frac{16}{73}$. A particular solution of the equation is

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x. \quad \equiv$$

As we mentioned, the form that we assume for the particular solution y_p is an educated guess; it is not a blind guess. This educated guess must take into consideration not only the types of functions that make up $g(x)$ but also, as we shall see in Example 4, the functions that make up the complementary function y_c.

EXAMPLE 3 Forming y_p by Superposition

Solve $\quad y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$. (3)

Solution **Step 1** First, the solution of the associated homogeneous equation $y'' - 2y' - 3y = 0$ is found to be $y_c = c_1 e^{-x} + c_2 e^{3x}$.

Step 2 Next, the presence of $4x - 5$ in $g(x)$ suggests that the particular solution includes a linear polynomial. Furthermore, since the derivative of the product xe^{2x} produces $2xe^{2x}$ and e^{2x}, we also assume that the particular solution includes both xe^{2x} and e^{2x}. In other words, g is the sum of two basic kinds of functions:

$$g(x) = g_1(x) + g_2(x) = \textit{polynomial} + \textit{exponentials}.$$

How to use Theorem 3.1.7 in the solution of Example 3.

Correspondingly, the superposition principle for nonhomogeneous equations (Theorem 3.1.7) suggests that we seek a particular solution

$$y_p = y_{p_1} + y_{p_2},$$

where $y_{p_1} = Ax + B$ and $y_{p_2} = Cxe^{2x} + Ee^{2x}$. Substituting

$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

into the given equation (3) and grouping like terms gives

$$y_p'' - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}. \tag{4}$$

From this identity we obtain the four equations

$$-3A = 4, \qquad -2A - 3B = -5, \qquad -3C = 6, \qquad 2C - 3E = 0.$$

The last equation in this system results from the interpretation that the coefficient of e^{2x} in the right member of (4) is zero. Solving, we find $A = -\frac{4}{3}$, $B = \frac{23}{9}$, $C = -2$, and $E = -\frac{4}{3}$. Consequently,

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}.$$

Step 3 The general solution of the equation is

$$y = c_1e^{-x} + c_2e^{3x} - \frac{4}{3}x + \frac{23}{9} - \left(2x + \frac{4}{3}\right)e^{2x}.$$ ≡

In light of the superposition principle (Theorem 3.1.7), we can also approach Example 3 from the viewpoint of solving two simpler problems. You should verify that substituting

$$y_{p_1} = Ax + B \qquad \text{into} \qquad y'' - 2y' - 3y = 4x - 5$$

and

$$y_{p_2} = Cxe^{2x} + Ee^{2x} \qquad \text{into} \qquad y'' - 2y' - 3y = 6xe^{2x}$$

yield, in turn, $y_{p_1} = -\frac{4}{3}x + \frac{23}{9}$ and $y_{p_2} = -\left(2x + \frac{4}{3}\right)e^{2x}$. A particular solution of (3) is then $y_p = y_{p_1} + y_{p_2}$.

The next example illustrates that sometimes the "obvious" assumption for the form of y_p is not a correct assumption.

EXAMPLE 4 A Glitch in the Method

Find a particular solution of $\quad y'' - 5y' + 4y = 8e^x$.

Solution Differentiation of e^x produces no new functions. Thus, proceeding as we did in the earlier examples, we can reasonably assume a particular solution of the form $y_p = Ae^x$. But substitution of this expression into the differential equation yields the contradictory statement $0 = 8e^x$, and so we have clearly made the wrong guess for y_p.

The difficulty here is apparent upon examining the complementary function $y_c = c_1e^x + c_2e^{4x}$. Observe that our assumption Ae^x is already present in y_c. This means that e^x is a solution of the associated homogeneous differential equation, and a constant multiple Ae^x when substituted into the differential equation necessarily produces zero.

What then should be the form of y_p? Inspired by Case II of Section 3.3, let's see whether we can find a particular solution of the form

$$y_p = Axe^x.$$

Substituting $y_p' = Axe^x + Ae^x$ and $y_p'' = Axe^x + 2Ae^x$ into the differential equation and simplifying gives

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x.$$

From the last equality we see that the value of A is now determined as $A = -\frac{8}{3}$. Therefore a particular solution of the given equation is $y_p = -\frac{8}{3}xe^x$. ≡

The difference in the procedures used in Examples 1–3 and in Example 4 suggests that we consider two cases. The first case reflects the situation in Examples 1–3.

***Case I*:** No function in the assumed particular solution is a solution of the associated homogeneous differential equation.

In Table 3.4.1 we illustrate some specific examples of $g(x)$ in (1) along with the corresponding form of the particular solution. We are, of course, taking for granted that no function in the assumed particular solution y_p is duplicated by a function in the complementary function y_c.

TABLE 3.4.1 Trial Particular Solutions

$g(x)$	Form of y_p
1. 1 (any constant)	A
2. $5x + 7$	$Ax + B$
3. $3x^2 - 2$	$Ax^2 + Bx + C$
4. $x^3 - x + 1$	$Ax^3 + Bx^2 + Cx + E$
5. $\sin 4x$	$A \cos 4x + B \sin 4x$
6. $\cos 4x$	$A \cos 4x + B \sin 4x$
7. e^{5x}	Ae^{5x}
8. $(9x - 2)e^{5x}$	$(Ax + B)e^{5x}$
9. x^2e^{5x}	$(Ax^2 + Bx + C)e^{5x}$
10. $e^{3x} \sin 4x$	$Ae^{3x} \cos 4x + Be^{3x} \sin 4x$
11. $5x^2 \sin 4x$	$(Ax^2 + Bx + C) \cos 4x + (Ex^2 + Fx + G) \sin 4x$
12. $xe^{3x} \cos 4x$	$(Ax + B)e^{3x} \cos 4x + (Cx + E)e^{3x} \sin 4x$

EXAMPLE 5 Forms of Particular Solutions—Case I

Determine the form of a particular solution of

(a) $y'' - 8y' + 25y = 5x^3e^{-x} - 7e^{-x}$ (b) $y'' + 4y = x \cos x$.

Solution (a) We can write $g(x) = (5x^3 - 7)e^{-x}$. Using entry 9 in Table 3.4.1 as a model, we assume a particular solution of the form

$$y_p = (Ax^3 + Bx^2 + Cx + E)e^{-x}.$$

Note that there is no duplication between the terms in y_p and the terms in the complementary function $y_c = e^{4x}(c_1 \cos 3x + c_2 \sin 3x)$.

(b) The function $g(x) = x \cos x$ is similar to entry 11 in Table 3.4.1 except, of course, that we use a linear rather than a quadratic polynomial and $\cos x$ and $\sin x$ instead of $\cos 4x$ and $\sin 4x$ in the form of y_p:

$$y_p = (Ax + B) \cos x + (Cx + E) \sin x.$$

Again observe that there is no duplication of terms between y_p and $y_c = c_1 \cos 2x + c_2 \sin 2x$. ≡

If $g(x)$ consists of a sum of, say, m terms of the kind listed in the table, then (as in Example 3) the assumption for a particular solution y_p consists of the sum of the trial forms $y_{p_1}, y_{p_2}, \ldots, y_{p_m}$ corresponding to these terms:

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m}.$$

The foregoing sentence can be put another way.

Form Rule for Case I: *The form of y_p is a linear combination of all linearly independent functions that are generated by repeated differentiations of $g(x)$.*

EXAMPLE 6 Forming y_p by Superposition—Case I

Determine the form of a particular solution of

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}.$$

Solution

Corresponding to $3x^2$ we assume $\quad y_{p_1} = Ax^2 + Bx + C.$

Corresponding to $-5\sin 2x$ we assume $\quad y_{p_2} = E\cos 2x + F\sin 2x.$

Corresponding to $7xe^{6x}$ we assume $\quad y_{p_3} = (Gx + H)e^{6x}.$

The assumption for the particular solution is then

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}.$$

No term in this assumption duplicates a term in $y_c = c_1e^{2x} + c_2e^{7x}$. ≡

Case II: A function in the assumed particular solution is also a solution of the associated homogeneous differential equation.

The next example is similar to Example 4.

EXAMPLE 7 Particular Solution—Case II

Find a particular solution of $\quad y'' - 2y' + y = e^x.$

Solution The complementary function is $y_c = c_1e^x + c_2xe^x$. As in Example 4, the assumption $y_p = Ae^x$ will fail since it is apparent from y_c that e^x is a solution of the associated homogeneous equation $y'' - 2y' + y = 0$. Moreover, we will not be able to find a particular solution of the form $y_p = Axe^x$ since the term xe^x is also duplicated in y_c. We next try

$$y_p = Ax^2e^x.$$

Substituting into the given differential equation yields $2Ae^x = e^x$ and so $A = \frac{1}{2}$. Thus a particular solution is $y_p = \frac{1}{2}x^2e^x$. ≡

Suppose again that $g(x)$ consists of m terms of the kind given in Table 3.4.1, and suppose further that the usual assumption for a particular solution is

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m},$$

where the y_{p_i}, $i = 1, 2, \ldots, m$ are the trial particular solution forms corresponding to these terms. Under the circumstances described in Case II, we can make up the following general rule.

Multiplication Rule for Case II: *If any y_{pi} contains terms that duplicate terms in y_c, then that y_{pi} must be multiplied by x^n, where n is the smallest positive integer that eliminates that duplication.*

EXAMPLE 8 An Initial-Value Problem

Solve the initial-value problem $y'' + y = 4x + 10 \sin x$, $y(\pi) = 0$, $y'(\pi) = 2$.

Solution The solution of the associated homogeneous equation $y'' + y = 0$ is $y_c = c_1 \cos x + c_2 \sin x$. Since $g(x) = 4x + 10 \sin x$ is the sum of a linear polynomial and a sine function, our normal assumption for y_p, from entries 2 and 5 of Table 3.4.1, would be the sum of $y_{p_1} = Ax + B$ and $y_{p_2} = C \cos x + E \sin x$:

$$y_p = Ax + B + C \cos x + E \sin x. \tag{5}$$

But there is an obvious duplication of the terms cos x and sin x in this assumed form and two terms in the complementary function. This duplication can be eliminated by simply multiplying y_{p_2} by x. Instead of (5) we now use

$$y_p = Ax + B + Cx \cos x + Ex \sin x.$$

Differentiating this expression and substituting the results into the differential equation gives

$$y_p'' + y_p = Ax + B - 2C \sin x + 2E \cos x = 4x + 10 \sin x. \tag{6}$$

Differentiating this expression and substituting the results into the differential equation give

$$y_p'' + y_p = Ax + B - 2C \sin x + 2E \cos x = 4x + 10 \sin x,$$

and so $A = 4$, $B = 0$, $-2C = 10$, $2E = 0$. The solutions of the system are immediate: $A = 4$, $B = 0$, $C = -5$, and $E = 0$. Therefore from (6) we obtain $y_p = 4x - 5x \cos x$. The general solution of the given equation is

$$y = y_c + y_p = c_1 \cos x + c_2 \sin x + 4x - 5x \cos x.$$

We now apply the prescribed initial conditions to the general solution of the equation. First, $y(\pi) = c_1 \cos \pi + c_2 \sin \pi + 4\pi - 5\pi \cos \pi = 0$ yields $c_1 = 9\pi$ since $\cos \pi = -1$ and $\sin \pi = 0$. Next, from the derivative

$$y' = -9\pi \sin x + c_2 \cos x + 4 + 5x \sin x - 5 \cos x$$

and $$y'(\pi) = -9\pi \sin \pi + c_2 \cos \pi + 4 + 5\pi \sin \pi - 5 \cos \pi = 2$$

we find $c_2 = 7$. The solution of the initial value is then

$$y = 9\pi \cos x + 7 \sin x + 4x - 5x \cos x.$$ ≡

EXAMPLE 9 Using the Multiplication Rule

Solve $y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$.

Solution The complementary function is $y_c = c_1e^{3x} + c_2xe^{3x}$. And so, based on entries 3 and 7 of Table 3.4.1, the usual assumption for a particular solution would be

$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{Ee^{3x}}_{y_{p_2}}.$$

Inspection of these functions shows that the one term in y_{p_2} is duplicated in y_c. If we multiply y_{p_2} by x, we note that the term xe^{3x} is still part of y_c. But multiplying y_{p_2} by x^2 eliminates all duplications. Thus the operative form of a particular solution is

$$y_p = Ax^2 + Bx + C + Ex^2e^{3x}.$$

Differentiating this last form, substituting into the differential equation, and collecting like terms gives

$$y_p'' - 6y_p' + 9y_p = 9Ax^2 + (-12A + 9B)x + 2A - 6B + 9C + 2Ee^{3x} = 6x^2 + 2 - 12e^{3x}.$$

It follows from this identity that $A = \frac{2}{3}$, $B = \frac{8}{9}$, $C = \frac{2}{3}$, and $E = -6$. Hence the general solution $y = y_c + y_p$ is

$$y = c_1e^{3x} + c_2xe^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2e^{3x}. \equiv$$

EXAMPLE 10 Third-Order DE—Case I

Solve $\quad y''' + y'' = e^x \cos x$.

Solution From the characteristic equation $m^3 + m^2 = 0$ we find $m_1 = m_2 = 0$ and $m_3 = -1$. Hence the complementary function of the equation is $y_c = c_1 + c_2x + c_3e^{-x}$. With $g(x) = e^x\cos x$, we see from entry 10 of Table 3.4.1 that we should assume

$$y_p = Ae^x \cos x + Be^x \sin x.$$

Since there are no functions in y_p that duplicate functions in the complementary solution, we proceed in the usual manner. From

$$y_p''' + y_p'' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x = e^x \cos x$$

we get $-2A + 4B = 1$, $-4A - 2B = 0$. This system gives $A = -\frac{1}{10}$ and $B = \frac{1}{5}$, so that a particular solution is $y_p = -\frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x$. The general solution of the equation is

$$y = y_c + y_p = c_1 + c_2x + c_3e^{-x} - \frac{1}{10}e^x \cos x + \frac{1}{5}e^x \sin x. \equiv$$

EXAMPLE 11 Fourth-Order DE—Case II

Determine the form of a particular solution of $\quad y^{(4)} + y''' = 1 - x^2e^{-x}$.

Solution Comparing $y_c = c_1 + c_2x + c_3x^2 + c_4e^{-x}$ with our normal assumption for a particular solution

$$y_p = \underbrace{A}_{y_{p_1}} + \underbrace{Bx^2e^{-x} + Cxe^{-x} + Ee^{-x}}_{y_{p_2}},$$

we see that the duplications between y_c and y_p are eliminated when y_{p_1} is multiplied by x^3 and y_{p_2} is multiplied by x. Thus the correct assumption for a particular solution is

$$y_p = Ax^3 + Bx^3e^{-x} + Cx^2e^{-x} + Exe^{-x}. \equiv$$

Remarks

(*i*) In Problems 27–36 of Exercises 3.4, you are asked to solve initial-value problems, and in Problems 37–40 boundary-value problems. As illustrated in Example 8, be sure to apply the initial conditions or the boundary conditions to the general solution $y = y_c + y_p$. Students often make the mistake of applying these conditions only to the complementary function y_c since it is that part of the solution that contains the constants.

(*ii*) From the "Form Rule for Case I" on page 124 of this section you see why the method of undetermined coefficients is not well suited to nonhomogeneous linear DEs when the input function $g(x)$ is something other than the four basic types listed in blue on page 120. If $P(x)$ is a polynomial, continued differentiation of $P(x)e^{\alpha x}\sin\beta x$ will generate an independent set containing only a *finite* number of functions—all of the same type, namely, polynomials times $e^{\alpha x}\sin\beta x$ or $e^{\alpha x}\cos\beta x$. On the other hand, repeated differentiations of input functions such as $g(x) = \ln x$ or $g(x) = \tan^{-1}x$ generate an independent set containing an *infinite* number of functions:

$$\textit{derivatives of } \ln x: \quad \frac{1}{x}, \frac{-1}{x^2}, \frac{2}{x^3}, \ldots,$$

$$\textit{derivatives of } \tan^{-1}x: \quad \frac{1}{1+x^2}, \frac{-2x}{(1+x^2)^2}, \frac{-2+6x^2}{(1+x^2)^3}, \ldots.$$

3.4 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

In Problems 1–26, solve the given differential equation by undetermined coefficients.

1. $y'' + 3y' + 2y = 6$
2. $4y'' + 9y = 15$
3. $y'' - 10y' + 25y = 30x + 3$
4. $y'' + y' - 6y = 2x$
5. $\frac{1}{4}y'' + y' + y = x^2 - 2x$
6. $y'' - 8y' + 20y = 100x^2 - 26xe^x$
7. $y'' + 3y = -48x^2e^{3x}$
8. $4y'' - 4y' - 3y = \cos 2x$
9. $y'' - y' = -3$
10. $y'' + 2y' = 2x + 5 - e^{-2x}$
11. $y'' - y' + \frac{1}{4}y = 3 + e^{x/2}$
12. $y'' - 16y = 2e^{4x}$
13. $y'' + 4y = 3\sin 2x$
14. $y'' - 4y = (x^2 - 3)\sin 2x$
15. $y'' + y = 2x\sin x$
16. $y'' - 5y' = 2x^3 - 4x^2 - x + 6$
17. $y'' - 2y' + 5y = e^x\cos 2x$
18. $y'' - 2y' + 2y = e^{2x}(\cos x - 3\sin x)$
19. $y'' + 2y' + y = \sin x + 3\cos 2x$
20. $y'' + 2y' - 24y = 16 - (x + 2)e^{4x}$
21. $y''' - 6y'' = 3 - \cos x$
22. $y''' - 2y'' - 4y' + 8y = 6xe^{2x}$
23. $y''' - 3y'' + 3y' - y = x - 4e^x$
24. $y''' - y'' - 4y' + 4y = 5 - e^x + e^{2x}$
25. $y^{(4)} + 2y'' + y = (x - 1)^2$
26. $y^{(4)} - y'' = 4x + 2xe^{-x}$

In Problems 27–36, solve the given initial-value problem.

27. $y'' + 4y = -2, \ y(\pi/8) = \frac{1}{2}, y'(\pi/8) = 2$
28. $2y'' + 3y' - 2y = 14x^2 - 4x - 11,$ $y(0) = 0, y'(0) = 0$
29. $5y'' + y' = -6x, y(0) = 0, \ y'(0) = -10$
30. $y'' + 4y' + 4y = (3 + x)e^{-2x}, \ y(0) = 2, y'(0) = 5$
31. $y'' + 4y' + 5y = 35e^{-4x}, \ y(0) = -3, y'(0) = 1$
32. $y'' - y = \cosh x, \ \ y(0) = 2, y'(0) = 12$
33. $\dfrac{d^2x}{dt^2} + \omega^2x = F_0\sin\omega t, \ x(0) = 0, x'(0) = 0$
34. $\dfrac{d^2x}{dt^2} + \omega^2x = F_0\cos\gamma t, \ x(0) = 0, x'(0) = 0$
35. $y''' - 2y'' + y' = 2 - 24e^x + 40e^{5x}, \ y(0) = \frac{1}{2}, y'(0) = \frac{5}{2},$ $y''(0) = -\frac{9}{2}$
36. $y''' + 8y = 2x - 5 + 8e^{-2x}, \ y(0) = -5, y'(0) = 3, y''(0) = -4$

In Problems 37–40, solve the given boundary-value problem.

37. $y'' + y = x^2 + 1, \ y(0) = 5, y(1) = 0$
38. $y'' - 2y' + 2y = 2x - 2, \ y(0) = 0, y(\pi) = \pi$
39. $y'' + 3y = 6x, \ y(0) = 0, y(1) + y'(1) = 0$
40. $y'' + 3y = 6x, \ y(0) + y'(0) = 0, y(1) = 0$

In Problems 41 and 42, solve the given initial-value problem in which the input function $g(x)$ is discontinuous. [*Hint*: Solve each problem on two intervals, and then find a solution so that y and y' are continuous at $x = \pi/2$ (Problem 41) and at $x = \pi$ (Problem 42).]

41. $y'' + 4y = g(x), y(0) = 1, y'(0) = 2$, where

$$g(x) = \begin{cases} \sin x, & 0 \le x \le \pi/2 \\ 0, & x > \pi/2 \end{cases}$$

42. $y'' - 2y' + 10y = g(x)$, $y(0) = 0$, $y'(0) = 0$, where

$$g(x) = \begin{cases} 20, & 0 \le x \le \pi \\ 0, & x > \pi \end{cases}$$

≡ Discussion Problems

43. Consider the differential equation $ay'' + by' + cy = e^{kx}$, where a, b, c, and k are constants. The auxiliary equation of the associated homogeneous equation is

$$am^2 + bm + c = 0.$$

(a) If k is not a root of the auxiliary equation, show that we can find a particular solution of the form $y_p = Ae^{kx}$, where $A = 1/(ak^2 + bk + c)$.

(b) If k is a root of the auxiliary equation of multiplicity one, show that we can find a particular solution of the form $y_p = Axe^{kx}$, where $A = 1/(2ak + b)$. Explain how we know that $k \neq -b/(2a)$.

(c) If k is a root of the auxiliary equation of multiplicity two, show that we can find a particular solution of the form $y = Ax^2e^{kx}$, where $A = 1/(2a)$.

44. Discuss how the method of this section can be used to find a particular solution of $y'' + y = \sin x \cos 2x$. Carry out your idea.

45. Without solving, match a solution curve of $y'' + y = f(x)$ shown in the figure with one of the following functions:

(i) $f(x) = 1$, **(ii)** $f(x) = e^{-x}$,
(iii) $f(x) = e^x$, **(iv)** $f(x) = \sin 2x$,
(v) $f(x) = e^x \sin x$, **(vi)** $f(x) = \sin x$.

Briefly discuss your reasoning.

(a)

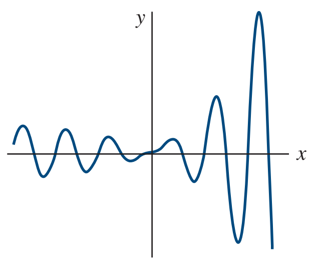

FIGURE 3.4.1 Solution curve

(b)

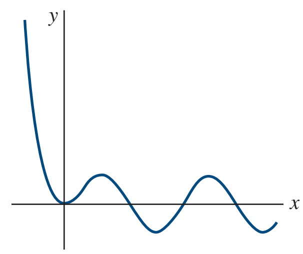

FIGURE 3.4.2 Solution curve

(c)

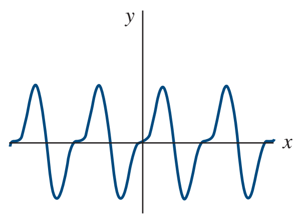

FIGURE 3.4.3 Solution curve

(d)

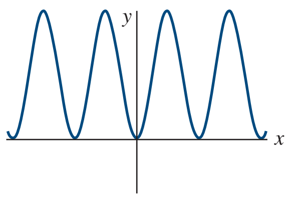

FIGURE 3.4.4 Solution curve

≡ Computer Lab Assignments

In Problems 46 and 47, find a particular solution of the given differential equation. Use a CAS as an aid in carrying out differentiations, simplifications, and algebra.

46. $y'' - 4y' + 8y = (2x^2 - 3x)e^{2x}\cos 2x + (10x^2 - x - 1)e^{2x}\sin 2x$

47. $y^{(4)} + 2y'' + y = 2\cos x - 3x\sin x$

3.5 Variation of Parameters

Introduction The **method of variation of parameters** used in Section 2.3 to find a particular solution of a linear first-order differential equation is applicable to linear higher-order equations as well. Variation of parameters has a distinct advantage over the method of the preceding section in that it *always* yields a particular solution y_p provided the associated homogeneous equation can be solved. In addition, the method presented in this section, unlike undetermined coefficients, is *not* limited to cases where the input function is a combination of the four types of functions listed on page 120, nor is it limited to differential equations with constant coefficients.

Some Assumptions To adapt the method of variation of parameters to a linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x), \qquad (1)$$

we begin as we did in Section 3.2—we put (1) in the standard form

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2}$$

by dividing through by the lead coefficient $a_2(x)$. Equation (2) is the second-order analogue of the linear first-order equation $dy/dx + P(x)y = f(x)$. In (2) we shall assume $P(x)$, $Q(x)$, and $f(x)$ are continuous on some common interval I. As we have already seen in Section 3.3, there is no difficulty in obtaining the complementary function y_c of (2) when the coefficients are constants.

Method of Variation of Parameters Corresponding to the substitution $y_p = u_1(x)y_1(x)$ that we used in Section 2.3 to find a particular solution y_p of $dy/dx + P(x)y = f(x)$, for the linear second-order DE (2) we seek a solution of the form

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x), \tag{3}$$

where y_1 and y_2 form a fundamental set of solutions on I of the associated homogeneous form of (1). Using the Product Rule to differentiate y_p twice, we get

$$y_p' = u_1y_1' + y_1u_1' + u_2y_2' + y_2u_2'$$

$$y_p'' = u_1y_1'' + y_1'u_1' + y_1u_1'' + u_1'y_1' + u_2y_2'' + y_2'u_2' + y_2u_2'' + u_2'y_2'.$$

Substituting (3) and the foregoing derivatives into (2) and grouping terms yields

$$\begin{aligned} y_p'' + P(x)y_p' + Q(x)y_p &= u_1\overbrace{[y_1'' + Py_1' + Qy_1]}^{\text{zero}} + u_2\overbrace{[y_2'' + Py_2' + Qy_2]}^{\text{zero}} \\ &\quad + y_1u_1'' + u_1'y_1' + y_2u_2'' + u_2'y_2' + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' \\ &= \frac{d}{dx}[y_1u_1'] + \frac{d}{dx}[y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' \\ &= \frac{d}{dx}[y_1u_1' + y_2u_2'] + P[y_1u_1' + y_2u_2'] + y_1'u_1' + y_2'u_2' = f(x). \end{aligned} \tag{4}$$

Because we seek to determine two unknown functions u_1 and u_2, reason dictates that we need two equations. We can obtain these equations by making the further assumption that the functions u_1 and u_2 satisfy $y_1u_1' + y_2u_2' = 0$. This assumption does not come out of the blue but is prompted by the first two terms in (4), since, if we demand that $y_1u_1' + y_2u_2' = 0$, then (4) reduces to $y_1'u_1' + y_2'u_2' = f(x)$. We now have our desired two equations, albeit two equations for determining the derivatives u_1' and u_2'. By Cramer's rule, the solution of the system

$$y_1u_1' + y_2u_2' = 0$$

$$y_1'u_1' + y_2'u_2' = f(x)$$

can be expressed in terms of determinants:

$$u_1' = \frac{W_1}{W} = -\frac{y_2f(x)}{W} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{y_1f(x)}{W} \tag{5}$$

If you are unfamiliar with Cramer's rule see Section 8.7.

where $$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}. \tag{6}$$

The functions u_1 and u_2 are found by integrating the results in (5). The determinant W is recognized as the Wronskian of y_1 and y_2. By linear independence of y_1 and y_2 on I, we know that $W(y_1(x), y_2(x)) \neq 0$ for every x in the interval.

Summary of the Method Usually it is not a good idea to memorize formulas in lieu of understanding a procedure. However, the foregoing procedure is too long and complicated to use each time we wish to solve a differential equation. In this case it is more efficient to simply use the formulas in (5). Thus to solve $a_2y'' + a_1y' + a_0y = g(x)$, first find the complementary function $y_c = c_1y_1 + c_2y_2$ and then compute the Wronskian $W(y_1(x), y_2(x))$. By dividing by a_2, we put the equation into the standard form $y'' + Py' + Qy = f(x)$ to determine $f(x)$. We find u_1 and u_2 by integrating $u_1' = W_1/W$ and $u_2' = W_2/W$, where W_1 and W_2 are defined as in (6). A particular solution is $y_p = u_1y_1 + u_2y_2$. The general solution of the equation is then $y = y_c + y_p$.

EXAMPLE 1 General Solution Using Variation of Parameters

Solve $\quad y'' - 4y' + 4y = (x + 1)e^{2x}$.

Solution From auxiliary equation $m^2 - 4m + 4 = (m - 2)^2 = 0$ we have $y_c = c_1e^{2x} + c_2xe^{2x}$. With the identifications $y_1 = e^{2x}$ and $y_2 = xe^{2x}$, we next compute the Wronskian:

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = e^{4x}.$$

Since the given differential equation is already in form (2) (that is, the coefficient of y'' is 1), we identify $f(x) = (x + 1)e^{2x}$. From (6) we obtain

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = -(x + 1)xe^{4x}, \quad W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix} = (x + 1)e^{4x},$$

and so from (5)

$$u_1' = -\frac{(x + 1)xe^{4x}}{e^{4x}} = -x^2 - x, \quad u_2' = \frac{(x + 1)e^{4x}}{e^{4x}} = x + 1.$$

It follows that $u_1 = -\frac{1}{3}x^3 - \frac{1}{2}x^2$ and $u_2 = \frac{1}{2}x^2 + x$. Hence

$$y_p = \left(-\frac{1}{3}x^3 - \frac{1}{2}x^2\right)e^{2x} + \left(\frac{1}{2}x^2 + x\right)xe^{2x} = \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}$$

and
$$y = y_c + y_p = c_1e^{2x} + c_2xe^{2x} + \frac{1}{6}x^3e^{2x} + \frac{1}{2}x^2e^{2x}.$$
≡

EXAMPLE 2 General Solution Using Variation of Parameters

Solve $\quad 4y'' + 36y = \csc 3x$.

Solution We first put the equation in the standard form (2) by dividing by 4:

$$y'' + 9y = \frac{1}{4}\csc 3x.$$

Since the roots of the auxiliary equation $m^2 + 9 = 0$ are $m_1 = 3i$ and $m_2 = -3i$, the complementary function is $y_c = c_1 \cos 3x + c_2 \sin 3x$. Using $y_1 = \cos 3x$, $y_2 = \sin 3x$, and $f(x) = \frac{1}{4}\csc 3x$, we obtain

$$W(\cos 3x, \sin 3x) = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix} = 3,$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\csc 3x & 3\cos 3x \end{vmatrix} = -\frac{1}{4}, \quad W_2 = \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\csc 3x \end{vmatrix} = \frac{1}{4}\frac{\cos 3x}{\sin 3x}.$$

Integrating

$$u_1' = \frac{W_1}{W} = -\frac{1}{12} \quad \text{and} \quad u_2' = \frac{W_2}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$$

gives $u_1 = -\frac{1}{12}x$ and $u_2 = \frac{1}{36}\ln|\sin 3x|$. Thus a particular solution is

$$y_p = -\frac{1}{12}x\cos 3x + \frac{1}{36}(\sin 3x)\ln|\sin 3x|.$$

The general solution of the equation is

$$y = y_c + y_p = c_1 \cos 3x + c_2 \sin 3x - \frac{1}{12} x \cos 3x + \frac{1}{36} (\sin 3x) \ln|\sin 3x|. \qquad (7)$$ ≡

Equation (7) represents the general solution of the differential equation on, say, the interval $(0, \pi/6)$.

Constants of Integration When computing the indefinite integrals of u_1' and u_2', we need not introduce any constants. This is because

$$\begin{aligned} y = y_c + y_p &= c_1y_1 + c_2y_2 + (u_1 + a_1)y_1 + (u_2 + b_1)y_2 \\ &= (c_1 + a_1)y_1 + (c_2 + b_1)y_2 + u_1y_1 + u_2y_2 \\ &= C_1y_1 + C_2y_2 + u_1y_1 + u_2y_2. \end{aligned}$$

EXAMPLE 3 General Solution Using Variation of Parameters

Solve $y'' - y = 1/x$.

Solution The auxiliary equation $m^2 - 1 = 0$ yields $m_1 = -1$ and $m_2 = 1$. Therefore $y_c = c_1e^x + c_2e^{-x}$. Now $W(e^x, e^{-x}) = -2$ and

$$\begin{aligned} u_1' &= -\frac{e^{-x}(1/x)}{-2}, & u_1 &= \frac{1}{2}\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt, \\ u_2' &= \frac{e^{x}(1/x)}{-2}, & u_2 &= -\frac{1}{2}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt. \end{aligned}$$

Since the foregoing integrals are nonelementary, we are forced to write

$$y_p = \frac{1}{2}e^x\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt,$$

and so

$$y = y_c + y_p = c_1e^x + c_2e^{-x} + \frac{1}{2}e^x\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt - \frac{1}{2}e^{-x}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt.$$ ≡

In Example 3 we can integrate on any interval $[x_0, x]$ not containing the origin. Also see Examples 2 and 3 in Section 3.10.

Higher-Order Equations The method we have just examined for nonhomogeneous second-order differential equations can be generalized to linear nth-order equations that have been put into the standard form

$$y^{(n)} + P_{n-1}(x)y^{(n-1)} + \cdots + P_1(x)y' + P_0(x)y = f(x). \qquad (8)$$

If $y_c = c_1y_1 + c_2y_2 + \cdots + c_ny_n$ is the complementary function for (8), then a particular solution is

$$y_p = u_1(x)y_1(x) + u_2(x)y_2(x) + \cdots + u_n(x)y_n(x),$$

where the u_k', $k = 1, 2, \ldots, n$, are determined by the n equations

$$\begin{aligned} y_1u_1' + \quad y_2u_2' + \cdots + \quad y_nu_n' &= 0 \\ y_1'u_1' + \quad y_2'u_2' + \cdots + \quad y_n'u_n' &= 0 \\ \vdots \qquad\qquad\qquad \vdots \quad & \\ y_1^{(n-1)}u_1' + y_2^{(n-1)}u_2' + \cdots + y_n^{(n-1)}u_n' &= f(x). \end{aligned} \qquad (9)$$

The first $n - 1$ equations in this system, like $y_1u_1' + y_2u_2' = 0$ in (4), are assumptions made to simplify the resulting equation after $y_p = u_1(x)y_1(x) + \cdots + u_n(x)y_n(x)$ is substituted in (8). In this case, Cramer's rule gives

$$u_k' = \frac{W_k}{W}, \qquad k = 1, 2, \ldots, n,$$

where W is the Wronskian of $y_1, y_2, \ldots, y_n$ and W_k is the determinant obtained by replacing the kth column of the Wronskian by the column consisting of the right-hand side of (9); that is, the column $(0, 0, \ldots, f(x))$. When $n = 2$ we get (5). When $n = 3$, the particular solution is $y_p = u_1y_1 + u_2y_2 + u_3y_3$, where y_1, y_2, and y_3 constitute a linearly independent set of solutions of the associated homogeneous DE, and u_1, u_2, u_3 are determined from

$$u_1' = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}, \quad u_3' = \frac{W_3}{W}, \tag{10}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{vmatrix}, \; W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{vmatrix}, \; W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{vmatrix}, \text{ and } W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}.$$

See Problems 25 and 26 in Exercises 3.5.

Remarks

In the problems that follow do not hesitate to simplify the form of y_p. Depending on how the antiderivatives of u_1' and u_2' are found, you may not obtain the same y_p as given in the answer section. For example, in Problem 3 in Exercises 3.5, both $y_p = \frac{1}{2}\sin x - \frac{1}{2}x\cos x$ and $y_p = \frac{1}{4}\sin x - \frac{1}{2}x\cos x$ are valid answers. In either case the general solution $y = y_c + y_p$ simplifies to $y = c_1\cos x + c_2\sin x - \frac{1}{2}x\cos x$. Why?

3.5 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

In Problems 1–18, solve each differential equation by variation of parameters.

1. $y'' + y = \sec x$
2. $y'' + y = \tan x$
3. $y'' + y = \sin x$
4. $y'' + y = \sec\theta\tan\theta$
5. $y'' + y = \cos^2 x$
6. $y'' + y = \sec^2 x$
7. $y'' - y = \cosh x$
8. $y'' - y = \sinh 2x$
9. $y'' - 4y = \dfrac{e^{2x}}{x}$
10. $y'' - 9y = \dfrac{9x}{e^{3x}}$
11. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}$
12. $y'' - 2y' + y = \dfrac{e^x}{1 + x^2}$
13. $y'' + 3y' + 2y = \sin e^x$
14. $y'' - 2y' + y = e^t \arctan t$
15. $y'' + 2y' + y = e^{-t}\ln t$
16. $2y'' + 2y' + y = 4\sqrt{x}$
17. $3y'' - 6y' + 6y = e^x \sec x$
18. $4y'' - 4y' + y = e^{x/2}\sqrt{1 - x^2}$

In Problems 19–22, solve each differential equation by variation of parameters subject to the initial conditions $y(0) = 1$, $y'(0) = 0$.

19. $4y'' - y = xe^{x/2}$
20. $2y'' + y' - y = x + 1$
21. $y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$
22. $y'' - 4y' + 4y = (12x^2 - 6x)e^{2x}$

In Problems 23 and 24, the indicated functions are known linearly independent solutions of the associated homogeneous differential equation on the interval $(0, \infty)$. Find the general solution of the given nonhomogeneous equation.

23. $x^2y'' + xy' + (x^2 - \frac{1}{4})y = x^{3/2}$; $\quad y_1 = x^{-1/2}\cos x$, $y_2 = x^{-1/2}\sin x$
24. $x^2y'' + xy' + y = \sec(\ln x)$; $\quad y_1 = \cos(\ln x)$, $y_2 = \sin(\ln x)$

In Problems 25 and 26, solve the given third-order differential equation by variation of parameters.

25. $y''' + y' = \tan x$

26. $y''' + 4y' = \sec 2x$

Discussion Problems

In Problems 27 and 28, discuss how the methods of undetermined coefficients and variation of parameters can be combined to solve the given differential equation. Carry out your ideas.

27. $3y'' - 6y' + 30y = 15 \sin x + e^x \tan 3x$

28. $y'' - 2y' + y = 4x^2 - 3 + x^{-1}e^x$

29. What are the intervals of definition of the general solutions in Problems 1, 7, 9, and 18? Discuss why the interval of definition of the general solution in Problem 24 is *not* $(0, \infty)$.

30. Find the general solution of $x^4y'' + x^3y' - 4x^2y = 1$ given that $y_1 = x^2$ is a solution of the associated homogeneous equation.

Computer Lab Assignments

In Problems 31 and 32, the indefinite integrals of the equations in (5) are nonelementary. Use a CAS to find the first four nonzero terms of a Maclaurin series of each integrand and then integrate the result. Find a particular solution of the given differential equation.

31. $y'' + y = \sqrt{1 + x^2}$

32. $4y'' - y = e^{x^2}$

3.6 Cauchy–Euler Equation

Introduction The relative ease with which we were able to find explicit solutions of linear higher-order differential equations with *constant coefficients* in the preceding sections does not, in general, carry over to linear equations with *variable coefficients*. We shall see in Chapter 5 that when a linear differential equation has variable coefficients, the best that we can usually expect is to find a solution in the form of an infinite series. However, the type of differential equation considered in this section is an exception to this rule; it is an equation with variable coefficients whose general solution can always be expressed in terms of powers of x, sines, cosines, logarithmic, and exponential functions. Moreover, its method of solution is quite similar to that for constant equations.

Cauchy–Euler Equation Any linear differential equation of the form

$$a_nx^n\frac{d^ny}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1x\frac{dy}{dx} + a_0y = g(x),$$

where the coefficients $a_n, a_{n-1}, \ldots, a_0$ are constants, is known diversely as a **Cauchy–Euler equation**, an **Euler–Cauchy equation**, an **Euler equation**, or an **equidimensional equation**. The observable characteristic of this type of equation is that the degree $k = n, n - 1, \ldots, 1, 0$ of the monomial coefficients x^k matches the order k of differentiation d^ky/dx^k:

$$\overset{\text{same}}{\downarrow\ \downarrow}\qquad\qquad\overset{\text{same}}{\downarrow\ \downarrow}$$
$$a_nx^n\frac{d^ny}{dx^n} + a_{n-1}x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \cdots.$$

As in Section 4.3, we start the discussion with a detailed examination of the forms of the general solutions of the homogeneous second-order equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = 0.$$

The solution of higher-order equations follows analogously. Also, we can solve the nonhomogeneous equation $ax^2y'' + bxy' + cy = g(x)$ by variation of parameters, once we have determined the complementary function $y_c(x)$.

The coefficient of d^2y/dx^2 is zero at $x = 0$. Hence, in order to guarantee that the fundamental results of Theorem 3.1.1 are applicable to the Cauchy–Euler equation, we confine our attention to finding the general solution on the interval $(0, \infty)$. Solutions on the interval $(-\infty, 0)$ can be obtained by substituting $t = -x$ into the differential equation. See Problems 37 and 38 in Exercises 3.6.

Lead coefficient being zero at $x = 0$ could cause a problem.

Method of Solution We try a solution of the form $y = x^m$, where m is to be determined. Analogous to what happened when we substituted e^{mx} into a linear equation with constant coefficients, after substituting x^m each term of a Cauchy–Euler equation becomes a polynomial in m times x^m since

$$a_k x^k \frac{d^k y}{dx^k} = a_k x^k m(m-1)(m-2)\cdots(m-k+1)x^{m-k}$$
$$= a_k m(m-1)(m-2)\cdots(m-k+1)x^m.$$

For example, by substituting $y = x^m$ the second-order equation becomes

$$ax^2 \frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = am(m-1)x^m + bmx^m + cx^m = (am(m-1) + bm + c)x^m.$$

Thus $y = x^m$ is a solution of the differential equation whenever m is a solution of the **auxiliary equation**

$$am(m-1) + bm + c = 0 \quad \text{or} \quad am^2 + (b-a)m + c = 0. \tag{1}$$

There are three different cases to be considered, depending on whether the roots of this quadratic equation are real and distinct, real and equal, or complex. In the last case the roots appear as a conjugate pair.

Case I*: *Distinct Real Roots Let m_1 and m_2 denote the real roots of (1) such that $m_1 \neq m_2$. Then $y_1 = x^{m_1}$ and $y_2 = x^{m_2}$ form a fundamental set of solutions. Hence the general solution is

$$y = c_1 x^{m_1} + c_2 x^{m_2}. \tag{2}$$

EXAMPLE 1 Distinct Roots

Solve $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = 0.$

Solution Rather than just memorizing equation (1), it is preferable to assume $y = x^m$ as the solution a few times in order to understand the origin and the difference between this new form of the auxiliary equation and that obtained in Section 3.3. Differentiate twice,

$$\frac{dy}{dx} = mx^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2},$$

and substitute back into the differential equation:

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^2 \cdot m(m-1)x^{m-2} - 2x \cdot mx^{m-1} - 4x^m$$
$$= x^m(m(m-1) - 2m - 4) = x^m(m^2 - 3m - 4) = 0$$

if $m^2 - 3m - 4 = 0$. Now $(m+1)(m-4) = 0$ implies $m_1 = -1$, $m_2 = 4$ and so (2) yields the general solution $y = c_1x^{-1} + c_2x^4$. ≡

Case II*: *Repeated Real Roots If the roots of (1) are repeated (that is, $m_1 = m_2$), then we obtain only one solution; namely; $y = x^{m_1}$. When the roots of the quadratic equation $am^2 + (b-a)m + c = 0$ are equal, the discriminant of the coefficients is necessarily zero. It follows from the quadratic formula that the root must be $m_1 = -(b-a)/2a$.

Now we can construct a second solution y_2, using (5) of Section 3.2. We first write the Cauchy–Euler equation in the standard form

$$\frac{d^2y}{dx^2} + \frac{b}{ax}\frac{dy}{dx} + \frac{c}{ax^2}y = 0$$

and make the identifications $P(x) = b/ax$ and $\int(b/ax)\,dx = (b/a)\ln x$. Thus

$$\begin{aligned} y_2 &= x^{m_1}\int \frac{e^{-(b/a)\ln x}}{x^{2m_1}}\,dx \\ &= x^{m_1}\int x^{-b/a}\cdot x^{-2m_1}\,dx \qquad \leftarrow e^{-(b/a)\ln x} = e^{\ln x^{-(b/a)}} = x^{-b/a} \\ &= x^{m_1}\int x^{-b/a}\cdot x^{(b-a)/a}\,dx \qquad \leftarrow -2m_1 = (b-a)/a \\ &= x^{m_1}\int \frac{dx}{x} = x^{m_1}\ln x. \end{aligned}$$

The general solution is then

$$y = c_1x^{m_1} + c_2x^{m_1}\ln x. \tag{3}$$

EXAMPLE 2 Repeated Roots

Solve $4x^2\dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} + y = 0.$

Solution The substitution $y = x^m$ yields

$$4x^2\frac{d^2y}{dx^2} + 8x\frac{dy}{dx} + y = x^m(4m(m-1) + 8m + 1) = x^m(4m^2 + 4m + 1) = 0$$

when $4m^2 + 4m + 1 = 0$ or $(2m+1)^2 = 0$. Since $m_1 = -\frac{1}{2}$ is a repeated root, (3) gives the general solution $y = c_1x^{-1/2} + c_2x^{-1/2}\ln x$. ≡

For higher-order equations, if m_1 is a root of multiplicity k, then it can be shown that

$$x^{m_1},\quad x^{m_1}\ln x,\quad x^{m_1}(\ln x)^2,\quad \ldots,\quad x^{m_1}(\ln x)^{k-1}$$

are k linearly independent solutions. Correspondingly, the general solution of the differential equation must then contain a linear combination of these k solutions.

Case III*: *Conjugate Complex Roots If the roots of (1) are the conjugate pair $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, where α and $\beta > 0$ are real, then a solution is $y = C_1x^{\alpha+i\beta} + C_2x^{\alpha-i\beta}$. But when the roots of the auxiliary equation are complex, as in the case of equations with constant coefficients, we wish to write the solution in terms of real functions only. We note the identity

$$x^{i\beta} = (e^{\ln x})^{i\beta} = e^{i\beta\ln x},$$

which, by Euler's formula, is the same as

$$x^{i\beta} = \cos(\beta\ln x) + i\sin(\beta\ln x).$$

Similarly, $$x^{-i\beta} = \cos(\beta\ln x) - i\sin(\beta\ln x).$$

Adding and subtracting the last two results yields

$$x^{i\beta} + x^{-i\beta} = 2\cos(\beta\ln x) \quad\text{and}\quad x^{i\beta} - x^{-i\beta} = 2i\sin(\beta\ln x),$$

respectively. From the fact that $y = C_1x^{\alpha+i\beta} + C_2x^{\alpha-i\beta}$ is a solution for any values of the constants, we see, in turn, for $C_1 = C_2 = 1$ and $C_1 = 1$, $C_2 = -1$ that

$$y_1 = x^{\alpha}(x^{i\beta} + x^{-i\beta}) \quad\text{and}\quad y_2 = x^{\alpha}(x^{i\beta} - x^{-i\beta})$$

or $\qquad y_1 = 2x^{\alpha}\cos(\beta \ln x) \quad$ and $\quad y_2 = 2ix^{\alpha}\sin(\beta \ln x)$

are also solutions. Since $W(x^{\alpha}\cos(\beta \ln x), x^{\alpha}\sin(\beta \ln x)) = \beta x^{2\alpha-1} \neq 0$, $\beta > 0$, on the interval $(0, \infty)$, we conclude that

$$y_1 = x^{\alpha}\cos(\beta \ln x) \quad \text{and} \quad y_2 = x^{\alpha}\sin(\beta \ln x)$$

constitute a fundamental set of real solutions of the differential equation. Hence the general solution is

$$y = x^{\alpha}[c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x)]. \tag{4}$$

EXAMPLE 3 An Initial-Value Problem

Solve the initial-value problem $\quad 4x^2y'' + 17y = 0,\ y(1) = -1, y'(1) = -\frac{1}{2}$.

Solution The y' term is missing in the given Cauchy–Euler equation; nevertheless, the substitution $y = x^m$ yields

$$4x^2y'' + 17y = x^m(4m(m-1) + 17) = x^m(4m^2 - 4m + 17) = 0$$

when $4m^2 - 4m + 17 = 0$. From the quadratic formula we find that the roots are $m_1 = \frac{1}{2} + 2i$ and $m_2 = \frac{1}{2} - 2i$. With the identifications $\alpha = \frac{1}{2}$ and $\beta = 2$, we see from (4) that the general solution of the differential equation is

$$y = x^{1/2}\,[c_1 \cos(2 \ln x) + c_2 \sin(2 \ln x)].$$

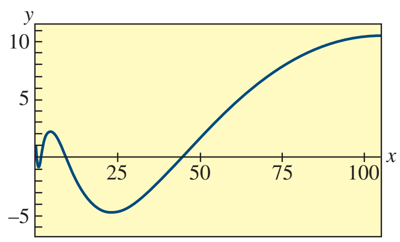

FIGURE 3.6.1 Graph of solution of IVP in Example 3

By applying the initial conditions $y(1) = -1$, $y'(1) = 0$ to the foregoing solution and using $\ln 1 = 0$ we then find, in turn, that $c_1 = -1$ and $c_2 = 0$. Hence the solution of the initial-value problem is $y = -x^{1/2}\cos(2 \ln x)$. The graph of this function, obtained with the aid of computer software, is given in **FIGURE 3.6.1** The particular solution is seen to be oscillatory and unbounded as $x \to \infty$. ≡

The next example illustrates the solution of a third-order Cauchy–Euler equation.

EXAMPLE 4 Third-Order Equation

Solve $\quad x^3\dfrac{d^3y}{dx^3} + 5x^2\dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 8y = 0.$

Solution The first three derivatives of $y = x^m$ are

$$\frac{dy}{dx} = mx^{m-1}, \quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}, \quad \frac{d^3y}{dx^3} = m(m-1)(m-2)x^{m-3}$$

so that the given differential equation becomes

$$\begin{aligned} x^3\frac{d^3y}{dx^3} + 5x^2\frac{d^2y}{dx^2} + 7x\frac{dy}{dx} + 8y &= x^3m(m-1)(m-2)x^{m-3} + 5x^2m(m-1)x^{m-2} + 7xmx^{m-1} + 8x^m \\ &= x^m\big(m(m-1)(m-2) + 5m(m-1) + 7m + 8\big) \\ &= x^m(m^3 + 2m^2 + 4m + 8) = x^m(m+2)(m^2+4) = 0. \end{aligned}$$

In this case we see that $y = x^m$ will be a solution of the differential equation for $m_1 = -2$, $m_2 = 2i$, and $m_3 = -2i$. Hence the general solution is

$$y = c_1x^{-2} + c_2 \cos(2 \ln x) + c_3 \sin(2 \ln x).$$ ≡

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The method of undetermined coefficients as described in Section 3.4 does not carry over, *in general,* to linear differential equations with variable coefficients. Consequently, in the following example the method of variation of parameters is employed.

EXAMPLE 5 Variation of Parameters

Solve $x^2y'' - 3xy' + 3y = 2x^4e^x$.

Solution Since the equation is nonhomogeneous, we first solve the associated homogeneous equation. From the auxiliary equation $(m-1)(m-3) = 0$ we find $y_c = c_1x + c_2x^3$. Now before using variation of parameters to find a particular solution $y_p = u_1y_1 + u_2y_2$, recall that the formulas $u_1' = W_1/W$ and $u_2' = W_2/W$, where W_1, W_2, and W are the determinants defined on page 129, and were derived under the assumption that the differential equation has been put into the standard form $y'' + P(x)y' + Q(x)y = f(x)$. Therefore we divide the given equation by x^2, and from

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 2x^2e^x$$

we make the identification $f(x) = 2x^2e^x$. Now with $y_1 = x$, $y_2 = x^3$ and

$$W = \begin{vmatrix} x & x^3 \\ 1 & 3x^2 \end{vmatrix} = 2x^3, \quad W_1 = \begin{vmatrix} 0 & x^3 \\ 2x^2e^x & 3x^2 \end{vmatrix} = -2x^5e^x, \quad W_2 = \begin{vmatrix} x & 0 \\ 1 & 2x^2e^x \end{vmatrix} = 2x^3e^x$$

we find $\quad u_1' = -\dfrac{2x^5e^x}{2x^3} = -x^2e^x \quad$ and $\quad u_2' = \dfrac{2x^3e^x}{2x^3} = e^x.$

The integral of the latter function is immediate, but in the case of u_1' we integrate by parts twice. The results are $u_1 = -x^2e^x + 2xe^x - 2e^x$ and $u_2 = e^x$. Hence

$$y_p = u_1y_1 + u_2y_2 = (-x^2e^x + 2xe^x - 2e^x)x + e^xx^3 = 2x^2e^x - 2xe^x.$$

Finally we have $y = y_c + y_p = c_1x + c_2x^3 + 2x^2e^x - 2xe^x$. ≡

Remarks

The similarity between the forms of solutions of Cauchy–Euler equations and solutions of linear equations with constant coefficients is not just a coincidence. For example, when the roots of the auxiliary equations for $ay'' + by' + cy = 0$ and $ax^2y'' + bxy' + cy = 0$ are distinct and real, the respective general solutions are

$$y = c_1e^{m_1x} + c_2e^{m_2x} \quad \text{and} \quad y = c_1x^{m_1} + c_2x^{m_2}, \quad x > 0. \tag{5}$$

In view of the identity $e^{\ln x} = x$, $x > 0$, the second solution given in (5) can be expressed in the same form as the first solution:

$$y = c_1e^{m_1 \ln x} + c_2e^{m_2 \ln x} = c_1e^{m_1t} + c_2e^{m_2t},$$

where $t = \ln x$. This last result illustrates another fact of mathematical life: Any Cauchy–Euler equation can *always* be rewritten as a linear differential equation with constant coefficients by means of the substitution $x = e^t$. The idea is to solve the new differential equation in terms of the variable t, using the methods of the previous sections, and once the general solution is obtained, resubstitute $t = \ln x$. Since this procedure provides a good review of the Chain Rule of differentiation, you are urged to work Problems 31–36 in Exercises 3.6.

3.6 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

In Problems 1–18, solve the given differential equation.

1. $x^2y'' - 2y = 0$
2. $4x^2y'' + y = 0$
3. $xy'' + y' = 0$
4. $xy'' - 3y' = 0$
5. $x^2y'' + xy' + 4y = 0$
6. $x^2y'' + 5xy' + 3y = 0$
7. $x^2y'' - 3xy' - 2y = 0$
8. $x^2y'' + 3xy' - 4y = 0$
9. $25x^2y'' + 25xy' + y = 0$
10. $4x^2y'' + 4xy' - y = 0$
11. $x^2y'' + 5xy' + 4y = 0$
12. $x^2y'' + 8xy' + 6y = 0$
13. $3x^2y'' + 6xy' + y = 0$
14. $x^2y'' - 7xy' + 41y = 0$
15. $x^3y''' - 6y = 0$
16. $x^3y''' + xy' - y = 0$
17. $xy^{(4)} + 6y''' = 0$
18. $x^4y^{(4)} + 6x^3y''' + 9x^2y'' + 3xy' + y = 0$

In Problems 19–24, solve the given differential equation by variation of parameters.

19. $xy'' - 4y' = x^4$
20. $2x^2y'' + 5xy' + y = x^2 - x$
21. $x^2y'' - xy' + y = 2x$
22. $x^2y'' - 2xy' + 2y = x^4e^x$
23. $x^2y'' + xy' - y = \ln x$
24. $x^2y'' + xy' - y = \dfrac{1}{x + 1}$

In Problems 25–30, solve the given initial-value problem. Use a graphing utility to graph the solution curve.

25. $x^2y'' + 3xy' = 0,\ \ y(1) = 0,\ y'(1) = 4$
26. $x^2y'' - 5xy' + 8y = 0,\ \ y(2) = 32,\ y'(2) = 0$
27. $x^2y'' + xy' + y = 0,\ \ y(1) = 1,\ y'(1) = 2$
28. $x^2y'' - 3xy' + 4y = 0,\ \ y(1) = 5,\ y'(1) = 3$
29. $xy'' + y' = x,\ \ y(1) = 1,\ y'(1) = -\frac{1}{2}$
30. $x^2y'' - 5xy' + 8y = 8x^6,\ \ y(\frac{1}{2}) = 0,\ y'(\frac{1}{2}) = 0$

In Problems 31–36, use the substitution $x = e^t$ to transform the given Cauchy–Euler equation to a differential equation with constant coefficients. Solve the original equation by solving the new equation using the procedure in Sections 3.3–3.5.

31. $x^2y'' + 9xy' - 20y = 0$
32. $x^2y'' - 9xy' + 25y = 0$
33. $x^2y'' + 10xy' + 8y = x^2$
34. $x^2y'' - 4xy' + 6y = \ln x^2$
35. $x^2y'' - 3xy' + 13y = 4 + 3x$
36. $x^3y''' - 3x^2y'' + 6xy' - 6y = 3 + \ln x^3$

In Problems 37 and 38, solve the given initial-value problem on the interval $(-\infty, 0)$.

37. $4x^2y'' + y = 0,\ \ y(-1) = 2,\ y'(-1) = 4$
38. $x^2y'' - 4xy' + 6y = 0,\ \ y(-2) = 8,\ y'(-2) = 0$

Discussion Problems

39. How would you use the method of this section to solve

$$(x + 2)^2y'' + (x + 2)y' + y = 0?$$

Carry out your ideas. State an interval over which the solution is defined.

40. Can a Cauchy–Euler differential equation of lowest order with real coefficients be found if it is known that 2 and $1 - i$ are two roots of its auxiliary equation? Carry out your ideas.
41. The initial conditions $y(0) = y_0$, $y'(0) = y_1$, apply to each of the following differential equations:

$$x^2y'' = 0,$$
$$x^2y'' - 2xy' + 2y = 0,$$
$$x^2y'' - 4xy' + 6y = 0.$$

For what values of y_0 and y_1 does each initial-value problem have a solution?

42. What are the x-intercepts of the solution curve shown in Figure 3.6.1? How many x-intercepts are there in the interval defined by $0 < x < \frac{1}{2}$?

Computer Lab Assignments

In Problems 43–46, solve the given differential equation by using a CAS to find the (approximate) roots of the auxiliary equation.

43. $2x^3y''' - 10.98x^2y'' + 8.5xy' + 1.3y = 0$
44. $x^3y''' + 4x^2y'' + 5xy' - 9y = 0$
45. $x^4y^{(4)} + 6x^3y''' + 3x^2y'' - 3xy' + 4y = 0$
46. $x^4y^{(4)} - 6x^3y''' + 33x^2y'' - 105xy' + 169y = 0$
47. Solve $x^3y''' - x^2y'' - 2xy' + 6y = x^2$ by variation of parameters. Use a CAS as an aid in computing roots of the auxiliary equation and the determinants given in (10) of Section 3.5.

3.7 Nonlinear Equations

Introduction The difficulties that surround higher-order *nonlinear* DEs and the few methods that yield analytic solutions are examined next.

Some Differences There are several significant differences between linear and nonlinear differential equations. We saw in Section 3.1 that homogeneous linear equations of order two or higher have the property that a linear combination of solutions is also a solution (Theorem 3.1.2). Nonlinear equations do not possess this property of superposability. For example, on the interval $(-\infty, \infty)$, $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = \cos x$, and $y_4 = \sin x$ are four linearly independent solutions of the nonlinear second-order differential equation $(y'')^2 - y^2 = 0$. But linear combinations such as $y = c_1e^x + c_3 \cos x$, $y = c_2e^{-x} + c_4 \sin x$, $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$ are not solutions of the equation for arbitrary nonzero constants c_i. See Problem 1 in Exercises 3.7.

In Chapter 2 we saw that we could solve a few nonlinear first-order differential equations by recognizing them as separable, exact, homogeneous, or perhaps Bernoulli equations. Even though the solutions of these equations were in the form of a one-parameter family, this family did not, as a rule, represent the general solution of the differential equation. On the other hand, by paying attention to certain continuity conditions, we obtained general solutions of linear first-order equations. Stated another way, nonlinear first-order differential equations can possess singular solutions whereas linear equations cannot. But the major difference between linear and nonlinear equations of order two or higher lies in the realm of solvability. Given a linear equation there is a chance that we can find some form of a solution that we can look at, an explicit solution or perhaps a solution in the form of an infinite series. On the other hand, nonlinear higher-order differential equations virtually defy solution. This does not mean that a nonlinear higher-order differential equation has no solution but rather that there are no analytical methods whereby either an explicit or implicit solution can be found.

Although this sounds disheartening, there are still things that can be done; we can always analyze a nonlinear DE qualitatively and numerically.

Let us make it clear at the outset that nonlinear higher-order differential equations are important—dare we say even more important than linear equations?—because as we fine-tune the mathematical model of, say, a physical system, we also increase the likelihood that this higher-resolution model will be nonlinear.

We begin by illustrating an analytical method that occasionally enables us to find explicit/implicit solutions of special kinds of nonlinear second-order differential equations.

Reduction of Order Nonlinear second-order differential equations $F(x, y', y'') = 0$, where the dependent variable y is missing, and $F(y, y', y'') = 0$, where the independent variable x is missing, can sometimes be solved using first-order methods. Each equation can be reduced to a first-order equation by means of the substitution $u = y'$.

The next example illustrates the substitution technique for an equation of the form $F(x, y', y'') = 0$. If $u = y'$, then the differential equation becomes $F(x, u, u') = 0$. If we can solve this last equation for u, we can find y by integration. Note that since we are solving a second-order equation, its solution will contain two arbitrary constants.

EXAMPLE 1 Dependent Variable y Is Missing

Solve $\quad y'' = 2x(y')^2$.

Solution If we let $u = y'$, then $du/dx = y''$. After substituting, the second-order equation reduces to a first-order equation with separable variables; the independent variable is x and the dependent variable is u:

$$\frac{du}{dx} = 2xu^2 \quad \text{or} \quad \frac{du}{u^2} = 2x\,dx$$

$$\int u^{-2}du = \int 2x\,dx$$

$$-u^{-1} = x^2 + c_1^2.$$

The constant of integration is written as c_1^2 for convenience. The reason should be obvious in the next few steps. Since $u^{-1} = 1/y'$, it follows that

$$\frac{dy}{dx} = -\frac{1}{x^2 + c_1^2}$$

and so $$y = -\int \frac{dx}{x^2 + c_1^2} \quad \text{or} \quad y = -\frac{1}{c_1}\tan^{-1}\frac{x}{c_1} + c_2.$$ ≡

Next we show how to solve an equation that has the form $F(y, y', y'') = 0$. Once more we let $u = y'$, but since the independent variable x is missing, we use this substitution to transform the differential equation into one in which the independent variable is y and the dependent variable is u. To this end we use the Chain Rule to compute the second derivative of y:

$$y'' = \frac{du}{dx} = \frac{du}{dy}\frac{dy}{dx} = u\frac{du}{dy}.$$

In this case the first-order equation that we must now solve is $F(y, u, u\, du/dy) = 0$.

EXAMPLE 2 Independent Variable x Is Missing

Solve $yy'' = (y')^2$.

Solution With the aid of $u = y'$, the Chain Rule shown above, and separation of variables, the given differential equation becomes

$$y\left(u\frac{du}{dy}\right) = u^2 \quad \text{or} \quad \frac{du}{u} = \frac{dy}{y}.$$

Integrating the last equation then yields $\ln|u| = \ln|y| + c_1$, which, in turn, gives $u = c_2 y$, where the constant $\pm e^{c_1}$ has been relabeled as c_2. We now resubstitute $u = dy/dx$, separate variables once again, integrate, and relabel constants a second time:

$$\int\frac{dy}{y} = c_2\int dx \quad \text{or} \quad \ln|y| = c_2 x + c_3 \quad \text{or} \quad y = c_4 e^{c_2 x}.$$ ≡

Use of Taylor Series In some instances a solution of a nonlinear initial-value problem, in which the initial conditions are specified at x_0, can be approximated by a Taylor series centered at x_0.

EXAMPLE 3 Taylor Series Solution of an IVP

Let us assume that a solution of the initial-value problem

$$y'' = x + y - y^2, \quad y(0) = -1, \quad y'(0) = 1 \tag{1}$$

exists. If we further assume that the solution $y(x)$ of the problem is analytic at 0, then $y(x)$ possesses a Taylor series expansion centered at 0:

$$y(x) = y(0) + \frac{y'(0)}{1!}x + \frac{y''(0)}{2!}x^2 + \frac{y'''(0)}{3!}x^3 + \frac{y^{(4)}(0)}{4!}x^4 + \frac{y^{(5)}(0)}{5!}x^5 + \cdots. \tag{2}$$

Note that the value of the first and second terms in the series (2) are known since those values are the specified initial conditions $y(0) = -1$, $y'(0) = 1$. Moreover, the differential equation itself defines the value of the second derivative at 0: $y''(0) = 0 + y(0) - y(0)^2 = 0 + (-1) - (-1)^2 = -2$. We can then find expressions for the higher derivatives y''', $y^{(4)}$, …, by calculating the successive derivatives of the differential equation:

$$y'''(x) = \frac{d}{dx}(x + y - y^2) = 1 + y' - 2yy' \tag{3}$$

$$y^{(4)}(x) = \frac{d}{dx}(1 + y' - 2yy') = y'' - 2yy'' - 2(y')^2 \tag{4}$$

$$y^{(5)}(x) = \frac{d}{dx}(y'' - 2yy'' - 2(y')^2) = y''' - 2yy''' - 6y'y'' \tag{5}$$

and so on. Now using $y(0) = -1$ and $y'(0) = 1$ we find from (3) that $y'''(0) = 4$. From the values $y(0) = -1$, $y'(0) = 1$, and $y''(0) = -2$, we find $y^{(4)}(0) = -8$ from (4). With the additional information that $y'''(0) = 4$, we then see from (5) that $y^{(5)}(0) = 24$. Hence from (2), the first six terms of a series solution of the initial-value problem (1) are

$$y(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5 + \cdots.$$ ≡

■ **Use of a Numerical Solver** Numerical methods, such as Euler's method or a Runge–Kutta method, are developed solely for first-order differential equations and then are extended to systems of first-order equations. In order to analyze an nth-order initial-value problem numerically, we express the nth-order ODE as a system of n first-order equations. In brief, here is how it is done for a second-order initial-value problem: First, solve for y''; that is, put the DE into normal form $y'' = f(x, y, y')$, and then let $y' = u$. For example, if we substitute $y' = u$ in

$$\frac{d^2y}{dx^2} = f(x, y, y'), \qquad y(x_0) = y_0, \qquad y'(x_0) = u_0, \tag{6}$$

then $y'' = u'$ and $y'(x_0) = u(x_0)$ so that the initial-value problem (6) becomes

$$\textit{Solve}: \quad \begin{cases} y' = u \\ u' = f(x, y, u) \end{cases}$$

$$\textit{Subject to}: \quad y(x_0) = y_0, \quad u(x_0) = u_0.$$

However, it should be noted that a commercial numerical solver *may not* require* that you supply the system.

EXAMPLE 4 Graphical Analysis of Example 3

Following the foregoing procedure, the second-order initial-value problem in Example 3 is equivalent to

$$\frac{dy}{dx} = u$$

$$\frac{du}{dx} = x + y - y^2$$

with initial conditions $y(0) = -1$, $u(0) = 1$. With the aid of a numerical solver we get the solution curve shown in blue in **FIGURE 3.7.1**. For comparison, the curve shown in red is the graph of the fifth-degree Taylor polynomial $T_5(x) = -1 + x - x^2 + \frac{2}{3}x^3 - \frac{1}{3}x^4 + \frac{1}{5}x^5$. Although we do not know the interval of convergence of the Taylor series obtained in Example 3, the closeness of the two curves in a neighborhood of the origin suggests that the power series may converge on the interval $(-1, 1)$. ≡

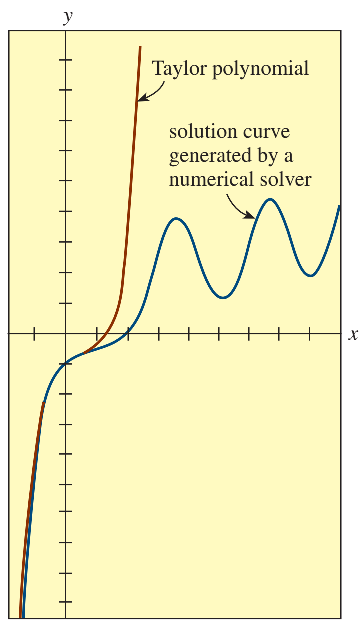


FIGURE 3.7.1 Comparison of two approximate solutions in Example 4

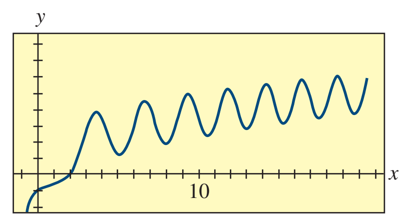


FIGURE 3.7.2 Numerical solution curve of IVP in (1) of Example 3

■ **Qualitative Questions** The colored graph in Figure 3.7.1 raises some questions of a qualitative nature: Is the solution of the original initial-value problem oscillatory as $x \to \infty$? The graph generated by a numerical solver on the larger interval shown in **FIGURE 3.7.2** would seem to *suggest* that the answer is yes. But this single example, or even an assortment of examples, does not answer the basic question of whether *all* solutions of the differential equation $y'' = x + y - y^2$ are oscillatory in nature. Also, what is happening to the solution curves in Figure 3.7.2 when

*Some numerical solvers require only that a second-order differential equation be expressed in normal form $y'' = f(x, y, y')$. The translation of the single equation into a system of two equations is then built into the computer program, since the first equation of the system is always $y' = u$ and the second equation is $u' = f(x, y, u)$.

x is near -1? What is the behavior of solutions of the differential equation as $x \to -\infty$? Are solutions bounded as $x \to \infty$? Questions such as these are not easily answered, in general, for nonlinear second-order differential equations. But certain kinds of second-order equations lend themselves to a systematic qualitative analysis, and these, like their first-order relatives encountered in Section 2.1, are the kind that have no explicit dependence on the independent variable. Second-order ODEs of the form

$$F(y, y', y'') = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} = f(y, y');$$

that is, equations free of the independent variable x, are called **autonomous**. The differential equation in Example 2 is autonomous, and because of the presence of the x term on its right side, the equation in Example 3 is nonautonomous. For an in-depth treatment of the topic of stability of autonomous second-order differential equations and autonomous systems of differential equations, the reader is referred to Chapter 11.

3.7 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

In Problems 1 and 2, verify that y_1 and y_2 are solutions of the given differential equation but that $y = c_1y_1 + c_2y_2$ is, in general, not a solution.

1. $(y'')^2 = y^2$; $y_1 = e^x$, $y_2 = \cos x$

2. $yy'' = \frac{1}{2}(y')^2$; $y_1 = 1$, $y_2 = x^2$

In Problems 3–8, solve the given differential equation by using the substitution $u = y'$.

3. $y'' + (y')^2 + 1 = 0$

4. $y'' = 1 + (y')^2$

5. $x^2y'' + (y')^2 = 0$

6. $(y + 1)y'' = (y')^2$

7. $y'' + 2y(y')^3 = 0$

8. $y^2y'' = y'$

9. Consider the initial-value problem

$$y'' + yy' = 0, \quad y(0) = 1, \quad y'(0) = -1.$$

(a) Use the DE and a numerical solver to graph the solution curve.

(b) Find an explicit solution of the IVP. Use a graphing utility to graph this solution.

(c) Find an interval of definition for the solution in part (b).

10. Find two solutions of the initial-value problem

$$(y'')^2 + (y')^2 = 1, \quad y(\pi/2) = \tfrac{1}{2}, \quad y'(\pi/2) = \sqrt{3}/2.$$

Use a numerical solver to graph the solution curves.

In Problems 11 and 12, show that the substitution $u = y'$ leads to a Bernoulli equation. Solve this equation (see Section 2.5).

11. $xy'' = y' + (y')^3$

12. $xy'' = y' + x(y')^2$

In Problems 13–16, proceed as in Example 3 and obtain the first six nonzero terms of a Taylor series solution, centered at 0, of the given initial-value problem. Use a numerical solver and a graphing utility to compare the solution curve with the graph of the Taylor polynomial.

13. $y'' = x + y^2$, $y(0) = 1$, $y'(0) = 1$

14. $y'' + y^2 = 1$, $y(0) = 2$, $y'(0) = 3$

15. $y'' = x^2 + y^2 - 2y'$, $y(0) = 1$, $y'(0) = 1$

16. $y'' = e^y$, $y(0) = 0$, $y'(0) = -1$

17. In calculus, the curvature of a curve that is defined by a function $y = f(x)$ is defined as

$$\kappa = \frac{y''}{[1 + (y')^2]^{3/2}}.$$

Find $y = f(x)$ for which $\kappa = 1$. [*Hint*: For simplicity, ignore constants of integration.]

Discussion Problems

18. In Problem 1 we saw that $\cos x$ and e^x were solutions of the nonlinear equation $(y'')^2 - y^2 = 0$. Verify that $\sin x$ and e^{-x} are also solutions. Without attempting to solve the differential equation, discuss how these explicit solutions can be found by using knowledge about linear equations. Without attempting to verify, discuss why the linear combinations $y = c_1e^x + c_2e^{-x} + c_3 \cos x + c_4 \sin x$ and $y = c_2e^{-x} + c_4 \sin x$ are not, in general, solutions, but the two special linear combinations $y = c_1e^x + c_2e^{-x}$ and $y = c_3 \cos x + c_4 \sin x$ *must* satisfy the differential equation.

19. Discuss how the method of reduction of order considered in this section can be applied to the third-order differential equation $y''' = \sqrt{1 + (y'')^2}$. Carry out your ideas and solve the equation.

20. Discuss how to find an alternative two-parameter family of solutions for the nonlinear differential equation $y'' = 2x(y')^2$ in Example 1. [*Hint*: Suppose that $-c_1^2$ is used as the constant of integration instead of $+c_1^2$.]

Mathematical Models

21. **Motion in a Force Field** A mathematical model for the position $x(t)$ of a body moving rectilinearly on the x-axis in an inverse-square force field is given by

$$\frac{d^2x}{dt^2} = -\frac{k^2}{x^2}.$$

Suppose that at $t = 0$ the body starts from rest from the position $x = x_0$, $x_0 > 0$. Show that the velocity of the body at time t is given by $v^2 = 2k^2(1/x - 1/x_0)$. Use the last expression and a CAS to carry out the integration to express time t in terms of x.

22. A mathematical model for the position $x(t)$ of a moving object is

$$\frac{d^2x}{dt^2} + \sin x = 0.$$

Use a numerical solver to graphically investigate the solutions of the equation subject to $x(0) = 0, x'(0) = x_1, x_1 \geq 0$. Discuss the motion of the object for $t \geq 0$ and for various choices of x_1. Investigate the equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + \sin x = 0$$

in the same manner. Give a possible physical interpretation of the dx/dt term.

3.8 Linear Models: Initial-Value Problems

Introduction In this section we are going to consider several linear dynamical systems in which each mathematical model is a linear second-order differential equation with constant coefficients along with initial conditions specified at time t_0:

$$a_2\frac{d^2y}{dt^2} + a_1\frac{dy}{dt} + a_0y = g(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1.$$

Recall, the function g is the **input**, **driving**, or **forcing function** of the system. The **output** or **response** of the system is a function $y(t)$ defined on an I interval containing t_0 that satisfies both the differential equation and the initial conditions on the interval I.

3.8.1 Spring/Mass Systems: Free Undamped Motion

Hooke's Law Suppose a flexible spring is suspended vertically from a rigid support and then a mass m is attached to its free end. The amount of stretch, or elongation, of the spring will, of course, depend on the mass; masses with different weights stretch the spring by differing amounts. By Hooke's law, the spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount of elongation s. Simply stated, $F = ks$, where k is a constant of proportionality called the **spring constant**. The spring is essentially characterized by the number k. For example, if a mass weighing 10 lb stretches a spring $\frac{1}{2}$ ft, then $10 = k(\frac{1}{2})$ implies $k = 20$ lb/ft. Necessarily then, a mass weighing, say, 8 lb stretches the same spring only $\frac{1}{2}$ ft.

Newton's Second Law After a mass m is attached to a spring, it stretches the spring by an amount s and attains a position of equilibrium at which its weight W is balanced by the restoring force ks. Recall that weight is defined by $W = mg$, where mass is measured in slugs, kilograms, or grams and $g = 32$ ft/s^2, 9.8 m/s^2, or 980 cm/s^2, respectively. As indicated in FIGURE 3.8.1(b), the condition of equilibrium is $mg = ks$ or $mg - ks = 0$. If the mass is displaced by an amount x from its equilibrium position, the restoring force of the spring is then $k(x + s)$. Assuming that there are no retarding forces acting on the system and assuming that the mass vibrates free of other external forces—**free motion**—we can equate Newton's second law with the net, or resultant, force of the restoring force and the weight:

$$m\frac{d^2x}{dt^2} = -k(s + x) + mg = -kx + \underbrace{mg - ks}_{\text{zero}} = -kx. \qquad (1)$$

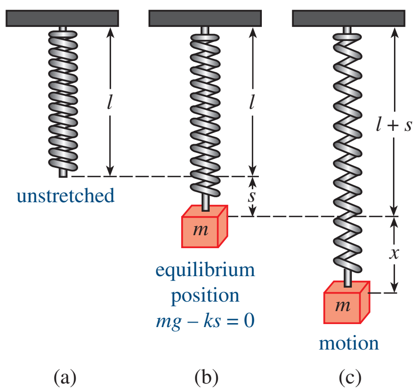


FIGURE 3.8.1 Spring/mass system

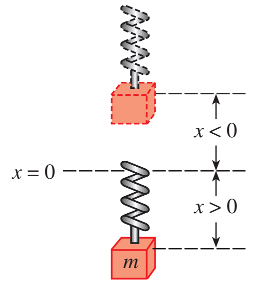


FIGURE 3.8.2 Positive direction is below equilibrium position

The negative sign in (1) indicates that the restoring force of the spring acts opposite to the direction of motion. Furthermore, we can adopt the convention that displacements measured *below* the equilibrium position are positive. See **FIGURE 3.8.2**.

■ DE of Free Undamped Motion By dividing (1) by the mass m we obtain the second-order differential equation $d^2x/dt^2 + (k/m)x = 0$ or

$$\frac{d^2x}{dt^2} + \omega^2 x = 0, \tag{2}$$

where $\omega^2 = k/m$. Equation (2) is said to describe **simple harmonic motion** or **free undamped motion**. Two obvious initial conditions associated with (2) are $x(0) = x_0$, the amount of initial displacement, and $x'(0) = x_1$, the initial velocity of the mass. For example, if $x_0 > 0$, $x_1 < 0$, the mass starts from a point *below* the equilibrium position with an imparted *upward* velocity. When $x_1 = 0$ the mass is said to be released from *rest*. For example, if $x_0 < 0$, $x_1 = 0$, the mass is released from rest from a point $|x_0|$ units *above* the equilibrium position.

■ Solution and Equation of Motion To solve equation (2) we note that the solutions of the auxiliary equation $m^2 + \omega^2 = 0$ are the complex numbers $m_1 = \omega i$, $m_2 = -\omega i$. Thus from (8) of Section 3.3 we find the general solution of (2) to be

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t. \tag{3}$$

The **period** of free vibrations described by (3) is $T = 2\pi/\omega$, and the **frequency** is $f = 1/T = \omega/2\pi$. For example, for $x(t) = 2 \cos 3t - 4 \sin 3t$ the period is $2\pi/3$ and the frequency is $3/2\pi$. The former number means that the graph of $x(t)$ repeats every $2\pi/3$ units; the latter number means that there are three cycles of the graph every 2π units or, equivalently, that the mass undergoes $3/2\pi$ complete vibrations per unit time. In addition, it can be shown that the period $2\pi/\omega$ is the time interval between two successive maxima of $x(t)$. Keep in mind that a maximum of $x(t)$ is a positive displacement corresponding to the mass's attaining a maximum distance *below* the equilibrium position, whereas a minimum of $x(t)$ is a negative displacement corresponding to the mass's attaining a maximum height *above* the equilibrium position. We refer to either case as an **extreme displacement** of the mass. Finally, when the initial conditions are used to determine the constants c_1 and c_2 in (3), we say that the resulting particular solution or response is the **equation of motion**.

EXAMPLE 1 Free Undamped Motion

A mass weighing 2 pounds stretches a spring 6 inches. At $t = 0$ the mass is released from a point 8 inches below the equilibrium position with an upward velocity of $\frac{4}{3}$ ft/s. Determine the equation of free motion.

Solution Because we are using the engineering system of units, the measurements given in terms of inches must be converted into feet: 6 in. $= \frac{1}{2}$ ft; 8 in. $= \frac{2}{3}$ ft. In addition, we must convert the units of weight given in pounds into units of mass. From $m = W/g$ we have $m = \frac{2}{32} = \frac{1}{16}$ slug. Also, from Hooke's law, $2 = k(\frac{1}{2})$ implies that the spring constant is $k = 4$ lb/ft. Hence (1) gives

$$\frac{1}{16}\frac{d^2x}{dt^2} = -4x \quad \text{or} \quad \frac{d^2x}{dt^2} + 64x = 0.$$

The initial displacement and initial velocity are $x(0) = \frac{2}{3}$, $x'(0) = -\frac{4}{3}$, where the negative sign in the last condition is a consequence of the fact that the mass is given an initial velocity in the negative, or upward, direction.

Now $\omega^2 = 64$ or $\omega = 8$, so that the general solution of the differential equation is

$$x(t) = c_1 \cos 8t + c_2 \sin 8t. \tag{4}$$

Applying the initial conditions to $x(t)$ and $x'(t)$ gives $c_1 = \frac{2}{3}$ and $c_2 = -\frac{1}{6}$. Thus the equation of motion is

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t. \tag{5}$$ ≡

Alternative Form of $x(t)$ When $c_1 \neq 0$ and $c_2 \neq 0$, the actual **amplitude** A of free vibrations is not obvious from inspection of equation (3). For example, although the mass in Example 1 is initially displaced $\frac{2}{3}$ foot beyond the equilibrium position, the amplitude of vibrations is a number larger than $\frac{2}{3}$. Hence it is often convenient to convert a solution of form (3) to the simpler form

$$x(t) = A\sin(\omega t + \phi), \tag{6}$$

where $A = \sqrt{c_1^2 + c_2^2}$ and ϕ is a **phase angle** defined by

$$\left.\begin{aligned} \sin\phi &= \frac{c_1}{A} \\ \cos\phi &= \frac{c_2}{A} \end{aligned}\right\} \tan\phi = \frac{c_1}{c_2}. \tag{7}$$

To verify this we expand (6) by the addition formula for the sine function:

$$A\sin\omega t\cos\phi + A\cos\omega t\sin\phi = (A\sin\phi)\cos\omega t + (A\cos\phi)\sin\omega t. \tag{8}$$

It follows from **FIGURE 3.8.3** that if ϕ is defined by

$$\sin\phi = \frac{c_1}{\sqrt{c_1^2 + c_2^2}} = \frac{c_1}{A}, \quad \cos\phi = \frac{c_2}{\sqrt{c_1^2 + c_2^2}} = \frac{c_2}{A},$$

then (8) becomes

$$A\frac{c_1}{A}\cos\omega t + A\frac{c_2}{A}\sin\omega t = c_1\cos\omega t + c_2\sin\omega t = x(t).$$

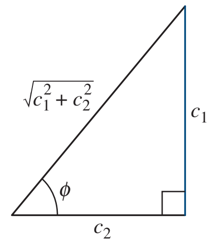


FIGURE 3.8.3 A relationship between $c_1 > 0$, $c_2 > 0$ and phase angle ϕ

EXAMPLE 2 Alternative Form of Solution (5)

In view of the foregoing discussion, we can write the solution (5), $x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t$, in the alternative form $x(t) = A\sin(8t + \phi)$. Computation of the amplitude is straightforward, $A = \sqrt{(\frac{2}{3})^2 + (\frac{-1}{6})^2} = \sqrt{\frac{17}{36}} \approx 0.69$ ft, but some care should be exercised when computing the phase angle ϕ defined by (7). With $c_1 = \frac{2}{3}$ and $c_2 = -\frac{1}{6}$ we find $\tan\phi = -4$, and a calculator then gives $\tan^{-1}(-4) = -1.326$ rad. This is *not* the phase angle, since $\tan^{-1}(-4)$ is located in the *fourth quadrant* and therefore contradicts the fact that $\sin\phi > 0$ and $\cos\phi < 0$ because $c_1 > 0$ and $c_2 < 0$. Hence we must take ϕ to be the *second-quadrant* angle $\phi = \pi + (-1.326) = 1.816$ rad. Thus we have

Be careful in the computation of the phase angle ϕ.

$$x(t) = \frac{\sqrt{17}}{6}\sin(8t + 1.816). \tag{9}$$

The period of this function is $T = 2\pi/8 = \pi/4$. ≡

FIGURE 3.8.4(a) illustrates the mass in Example 2 going through approximately two complete cycles of motion. Reading left to right, the first five positions marked with black dots correspond to the initial position of the mass below the equilibrium position ($x = \frac{2}{3}$), the mass passing through the equilibrium position for the first time heading upward ($x = 0$), the mass at its extreme displacement above the equilibrium position ($x = -\sqrt{17}/6$), the mass at the equilibrium position for the second time heading downward ($x = 0$), and the mass at its extreme displacement below

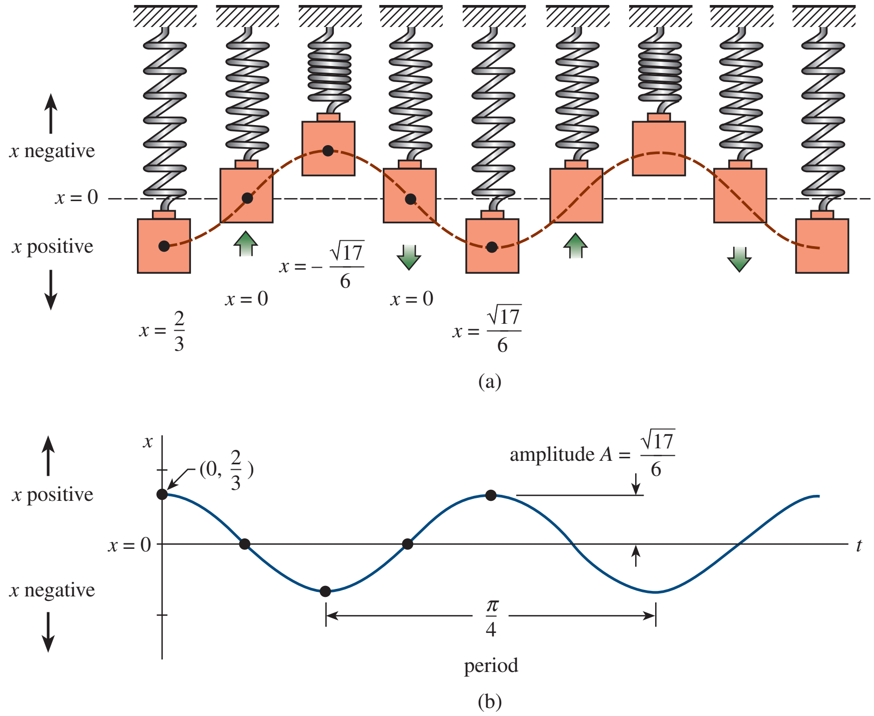


FIGURE 3.8.4 Simple harmonic motion

the equilibrium position ($x = \sqrt{17}/6$). The dots on the graph of (9) given in Figure 3.8.4(b) also agree with the five positions just given. Note, however, that in Figure 3.8.4(b) the positive direction in the *tx*-plane is the usual upward direction and so is opposite to the positive direction indicated in Figure 3.8.4(a). Hence the blue graph representing the motion of the mass in Figure 3.8.4(b) is the mirror image through the *t*-axis of the red dashed curve in Figure 3.8.4(a).

Form (6) is very useful, since it is easy to find values of time for which the graph of $x(t)$ crosses the positive *t*-axis (the line $x = 0$). We observe that $\sin(\omega t + \phi) = 0$ when $\omega t + \phi = n\pi$, where n is a nonnegative integer.

■ Systems with Variable Spring Constants In the model discussed above, we assumed an ideal world, a world in which the physical characteristics of the spring do not change over time. In the nonideal world, however, it seems reasonable to expect that when a spring/mass system is in motion for a long period the spring would weaken; in other words, the "spring constant" would vary, or, more specifically, decay with time. In one model for the **aging spring**, the spring constant k in (1), is replaced by the decreasing function $K(t) = ke^{-\alpha t}$, $k > 0$, $\alpha > 0$. The linear differential equation $mx'' + ke^{-\alpha t}x = 0$ cannot be solved by the methods considered in this chapter. Nevertheless, we can obtain two linearly independent solutions using the methods in Chapter 5. See Problem 15 in Exercises 3.8, Example 4 in Section 5.3, and Problems 33 and 39 in Exercises 5.3.

When a spring/mass system is subjected to an environment in which the temperature is rapidly decreasing, it might make sense to replace the constant k with $K(t) = kt$, $k > 0$, a function that increases with time. The resulting model, $mx'' + ktx = 0$, is a form of **Airy's differential equation**. Like the equation for an aging spring, Airy's equation can be solved by the methods of Chapter 5. See Problem 16 in Exercises 3.8, Example 2 in Section 5.1, and Problems 34, 35, and 40 in Exercises 5.3.

3.8.2 Spring/Mass Systems: Free Damped Motion

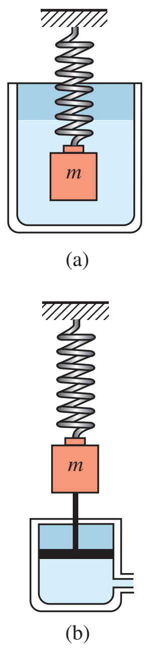


FIGURE 3.8.5 Damping devices

The concept of free harmonic motion is somewhat unrealistic, since the motion described by equation (1) assumes that there are no retarding forces acting on the moving mass. Unless the mass is suspended in a perfect vacuum, there will be at least a resisting force due to the surrounding medium. As **FIGURE 3.8.5** shows, the mass could be suspended in a viscous medium or connected to a dashpot damping device.

■ **DE of Free Damped Motion** In the study of mechanics, damping forces acting on a body are considered to be proportional to a power of the instantaneous velocity. In particular, we shall assume throughout the subsequent discussion that this force is given by a constant multiple of dx/dt. When no other external forces are impressed on the system, it follows from Newton's second law that

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt}, \tag{10}$$

where β is a positive *damping constant* and the negative sign is a consequence of the fact that the damping force acts in a direction opposite to the motion.

Dividing (10) by the mass m, we find the differential equation of **free damped motion** is $d^2x/dt^2 + (\beta/m)dx/dt + (k/m)x = 0$ or

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0, \tag{11}$$

where
$$2\lambda = \frac{\beta}{m}, \quad \omega^2 = \frac{k}{m}. \tag{12}$$

The symbol 2λ is used only for algebraic convenience, since the auxiliary equation is $m^2 + 2\lambda m + \omega^2 = 0$ and the corresponding roots are then

$$m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}, \quad m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}.$$

We can now distinguish three possible cases depending on the algebraic sign of $\lambda^2 - \omega^2$. Since each solution contains the *damping factor* $e^{-\lambda t}$, $\lambda > 0$, the displacements of the mass become negligible over a long period of time.

***Case I*: $\lambda^2 - \omega^2 > 0$** In this situation the system is said to be **overdamped** because the damping coefficient β is large when compared to the spring constant k. The corresponding solution of (11) is $x(t) = c_1e^{m_1t} + c_2e^{m_2t}$ or

$$x(t) = e^{-\lambda t}\left(c_1e^{\sqrt{\lambda^2-\omega^2}\,t} + c_2e^{-\sqrt{\lambda^2-\omega^2}\,t}\right). \tag{13}$$

Equation 13 represents a smooth and nonoscillatory motion. **FIGURE 3.8.6** shows two possible graphs of $x(t)$.

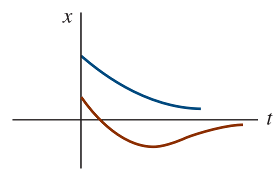


FIGURE 3.8.6 Motion of an overdamped system

***Case II*: $\lambda^2 - \omega^2 = 0$** The system is said to be **critically damped** because any slight decrease in the damping force would result in oscillatory motion. The general solution of (11) is $x(t) = c_1e^{m_1t} + c_2te^{m_1t}$ or

$$x(t) = e^{-\lambda t}(c_1 + c_2t). \tag{14}$$

Some graphs of typical motion are given in **FIGURE 3.8.7**. Notice that the motion is quite similar to that of an overdamped system. It is also apparent from (14) that the mass can pass through the equilibrium position at most one time.

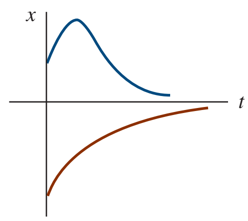


FIGURE 3.8.7 Motion of an critically damped system

***Case III*: $\lambda^2 - \omega^2 < 0$** In this case the system is said to be **underdamped** because the damping coefficient is small compared to the spring constant. The roots m_1 and m_2 are now complex:

$$m_1 = -\lambda + \sqrt{\omega^2 - \lambda^2}\,i, \quad m_2 = -\lambda - \sqrt{\omega^2 - \lambda^2}\,i.$$

Thus the general solution of equation (11) is

$$x(t) = e^{-\lambda t}\left(c_1\cos\sqrt{\omega^2 - \lambda^2}\,t + c_2\sin\sqrt{\omega^2 - \lambda^2}\,t\right). \tag{15}$$

As indicated in **FIGURE 3.8.8**, the motion described by (15) is oscillatory, but because of the coefficient $e^{-\lambda t}$, the amplitudes of vibration $\to 0$ as $t \to \infty$.

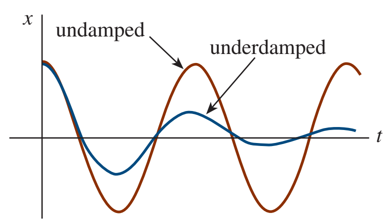


FIGURE 3.8.8 Motion of an underdamped system

EXAMPLE 3 Overdamped Motion

It is readily verified that the solution of the initial-value problem

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0, \quad x(0) = 1, \quad x'(0) = 1$$

is

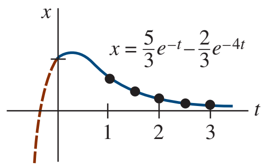
$$x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t}. \tag{16}$$

The problem can be interpreted as representing the overdamped motion of a mass on a spring. The mass starts from a position 1 unit *below* the equilibrium position with a *downward* velocity of 1 ft/s.

To graph $x(t)$ we find the value of t for which the function has an extremum; that is, the value of time for which the first derivative (velocity) is zero. Differentiating (16) gives $x'(t) = -\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t}$ so that $x'(t) = 0$ implies $e^{3t} = \frac{8}{5}$ or $t = \frac{1}{3}\ln\frac{8}{5} = 0.157$. It follows from the first derivative test, as well as our intuition, that $x(0.157) = 1.069$ ft is actually a maximum. In other words, the mass attains an extreme displacement of 1.069 feet below the equilibrium position.

We should also check to see whether the graph crosses the t-axis; that is, whether the mass passes through the equilibrium position. This cannot happen in this instance since the equation $x(t) = 0$, or $e^{3t} = \frac{2}{5}$, has the physically irrelevant solution $t = \frac{1}{3}\ln\frac{2}{5} = -0.305$.

The graph of $x(t)$, along with some other pertinent data, is given in **FIGURE 3.8.9**. ≡



(a)

t	$x(t)$
1	0.601
1.5	0.370
2	0.225
2.5	0.137
3	0.083

(b)

FIGURE 3.8.9 Overdamped system in Example 3

EXAMPLE 4 Critically Damped Motion

An 8-pound weight stretches a spring 2 feet. Assuming that a damping force numerically equal to two times the instantaneous velocity acts on the system, determine the equation of motion if the weight is released from the equilibrium position with an upward velocity of 3 ft/s.

Solution From Hooke's law we see that $8 = k(2)$ gives $k = 4$ lb/ft and that $W = mg$ gives $m = \frac{8}{32} = \frac{1}{4}$ slug. The differential equation of motion is then

$$\frac{1}{4}\frac{d^2x}{dt^2} = -4x - 2\frac{dx}{dt} \quad \text{or} \quad \frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0. \tag{17}$$

The auxiliary equation for (17) is $m^2 + 8m + 16 = (m + 4)^2 = 0$ so that $m_1 = m_2 = -4$. Hence the system is critically damped and

$$x(t) = c_1e^{-4t} + c_2te^{-4t}. \tag{18}$$

Applying the initial conditions $x(0) = 0$ and $x'(0) = -3$, we find, in turn, that $c_1 = 0$ and $c_2 = -3$. Thus the equation of motion is

$$x(t) = -3te^{-4t}. \tag{19}$$

To graph $x(t)$ we proceed as in Example 3. From $x'(t) = -3e^{-4t}(1 - 4t)$ we see that $x'(t) = 0$ when $t = \frac{1}{4}$. The corresponding extreme displacement is $x(\frac{1}{4}) = -3(\frac{1}{4})e^{-1} = -0.276$ ft. As shown in **FIGURE 3.8.10**, we interpret this value to mean that the weight reaches a maximum height of 0.276 foot above the equilibrium position. ≡

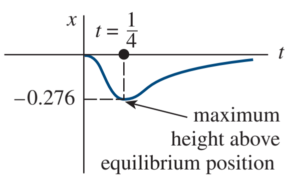


FIGURE 3.8.10 Critically damped system in Example 4

EXAMPLE 5 Underdamped Motion

A 16-pound weight is attached to a 5-foot-long spring. At equilibrium the spring measures 8.2 feet. If the weight is pushed up and released from rest at a point 2 feet above the equilibrium position, find the displacements $x(t)$ if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

Solution The elongation of the spring after the weight is attached is $8.2 - 5 = 3.2$ ft, so it follows from Hooke's law that $16 = k(3.2)$ or $k = 5$ lb/ft. In addition, $m = \frac{16}{32} = \frac{1}{2}$ slug so that the differential equation is given by

$$\frac{1}{2}\frac{d^2x}{dt^2} = -5x - \frac{dx}{dt} \quad \text{or} \quad \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0. \tag{20}$$

Proceeding, we find that the roots of $m^2 + 2m + 10 = 0$ are $m_1 = -1 + 3i$ and $m_2 = -1 - 3i$, which then implies the system is underdamped and

$$x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t). \tag{21}$$

Finally, the initial conditions $x(0) = -2$ and $x'(0) = 0$ yield $c_1 = -2$ and $c_2 = -\frac{2}{3}$, so the equation of motion is

$$x(t) = e^{-t}\left(-2\cos 3t - \frac{2}{3}\sin 3t\right). \tag{22}$$ ≡

■ Alternative Form of $x(t)$ In a manner identical to the procedure used on page 145, we can write any solution

$$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\omega^2 - \lambda^2}\,t + c_2 \sin\sqrt{\omega^2 - \lambda^2}\,t)$$

in the alternative form

$$x(t) = Ae^{-\lambda t}\sin(\sqrt{\omega^2 - \lambda^2}\,t + \phi), \tag{23}$$

where $A = \sqrt{c_1^2 + c_2^2}$ and the phase angle ϕ is determined from the equations

$$\sin\phi = \frac{c_1}{A}, \quad \cos\phi = \frac{c_2}{A}, \quad \tan\phi = \frac{c_1}{c_2}.$$

The coefficient $Ae^{-\lambda t}$ is sometimes called the **damped amplitude** of vibrations. Because (23) is not a periodic function, the number $2\pi/\sqrt{\omega^2 - \lambda^2}$ is called the **quasi period** and $\sqrt{\omega^2 - \lambda^2}/2\pi$ is the **quasi frequency**. The quasi period is the time interval between two successive maxima of $x(t)$. You should verify, for the equation of motion in Example 5, that $A = 2\sqrt{10}/3$ and $\phi = 4.391$. Therefore an equivalent form of (22) is

$$x(t) = \frac{2\sqrt{10}}{3}e^{-t}\sin(3t + 4.391).$$

3.8.3 Spring/Mass Systems: Driven Motion

■ DE of Driven Motion with Damping Suppose we now take into consideration an external force $f(t)$ acting on a vibrating mass on a spring. For example, $f(t)$ could represent a driving force causing an oscillatory vertical motion of the support of the spring. See **FIGURE 3.8.11**. The inclusion of $f(t)$ in the formulation of Newton's second law gives the differential equation of **driven** or **forced motion**:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t). \tag{24}$$

Dividing (24) by m gives

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t), \tag{25}$$

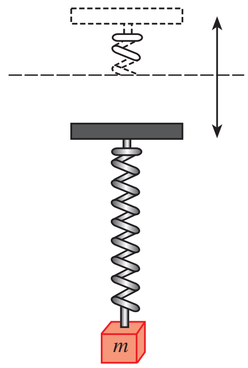


FIGURE 3.8.11 Oscillatory vertical motion of the support

where $F(t) = f(t)/m$ and, as in the preceding section, $2\lambda = \beta/m$, $\omega^2 = k/m$. To solve the latter nonhomogeneous equation we can use either the method of undetermined coefficients or variation of parameters.

EXAMPLE 6 Interpretation of an Initial-Value Problem

Interpret and solve the initial-value problem

$$\frac{1}{5}\frac{d^2x}{dt^2} + 1.2\frac{dx}{dt} + 2x = 5\cos 4t, \quad x(0) = \frac{1}{2}, \quad x'(0) = 0. \tag{26}$$

Solution We can interpret the problem to represent a vibrational system consisting of a mass ($m = \frac{1}{5}$ slug or kilogram) attached to a spring ($k = 2$ lb/ft or N/m). The mass is released from rest $\frac{1}{2}$ unit (foot or meter) below the equilibrium position. The motion is damped ($\beta = 1.2$) and is being driven by an external periodic ($T = \pi/2$ s) force beginning at $t = 0$. Intuitively we would expect that even with damping, the system would remain in motion until such time as the forcing function was "turned off," in which case the amplitudes would diminish. However, as the problem is given, $f(t) = 5\cos 4t$ will remain "on" forever.

We first multiply the differential equation in (26) by 5 and solve

$$\frac{dx^2}{dt^2} + 6\frac{dx}{dt} + 10x = 0$$

by the usual methods. Since $m_1 = -3 + i$, $m_2 = -3 - i$, it follows that

$$x_c(t) = e^{-3t}(c_1 \cos t + c_2 \sin t).$$

Using the method of undetermined coefficients, we assume a particular solution of the form $x_p(t) = A\cos 4t + B\sin 4t$. Differentiating $x_p(t)$ and substituting into the DE gives

$$x_p'' + 6x_p' + 10x_p = (-6A + 24B)\cos 4t + (-24A - 6B)\sin 4t = 25\cos 4t.$$

The resulting system of equations

$$-6A + 24B = 25, \qquad -24A - 6B = 0$$

yields $A = -\frac{25}{102}$ and $B = \frac{50}{51}$. It follows that

$$x(t) = e^{-3t}(c_1 \cos t + c_2 \sin t) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t. \tag{27}$$

When we set $t = 0$ in the above equation, we obtain $c_1 = \frac{38}{51}$. By differentiating the expression and then setting $t = 0$, we also find that $c_2 = -\frac{86}{51}$. Therefore the equation of motion is

$$x(t) = e^{-3t}\left(\frac{38}{51}\cos t - \frac{86}{51}\sin t\right) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t. \tag{28}$$ ≡

Transient and Steady-State Terms When F is a periodic function, such as $F(t) = F_0 \sin \gamma t$ or $F(t) = F_0 \cos \gamma t$, the general solution of (25) for $\lambda > 0$ is the sum of a nonperiodic function $x_c(t)$ and a periodic function $x_p(t)$. Moreover, $x_c(t)$ dies off as time increases; that is, $\lim_{t\to\infty} x_c(t) = 0$. Thus for a long period of time, the displacements of the mass are closely approximated by the particular solution $x_p(t)$. The complementary function $x_c(t)$ is said to be a **transient term** or **transient solution**, and the function $x_p(t)$, the part of the solution that remains after an interval of time, is called a **steady-state term** or **steady-state solution**. Note therefore that the effect of the initial conditions on a spring/mass system driven by F is transient. In the particular solution (28), $e^{-3t}(\frac{38}{51}\cos t - \frac{86}{51}\sin t)$ is a transient term and $x_p(t) = -\frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$ is a steady-state term. The graphs of these two terms and the solution (28) are given in FIGURES 3.8.12(a) and 3.8.12(b), respectively.

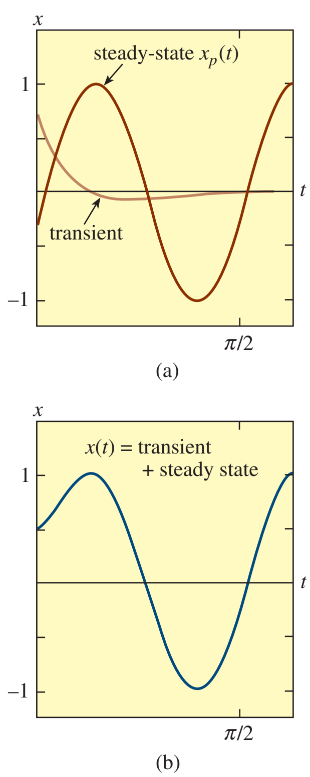


FIGURE 3.8.12 Graph of solution (28) in Example 6

EXAMPLE 7 Transient/Steady-State Solutions

The solution of the initial-value problem

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 4\cos t + 2\sin t, \quad x(0) = 0, \quad x'(0) = x_1,$$

where x_1 is constant, is given by

$$x(t) = \underbrace{(x_1 - 2)e^{-t}\sin t}_{\text{transient}} + \underbrace{2\sin t}_{\text{steady state}}.$$

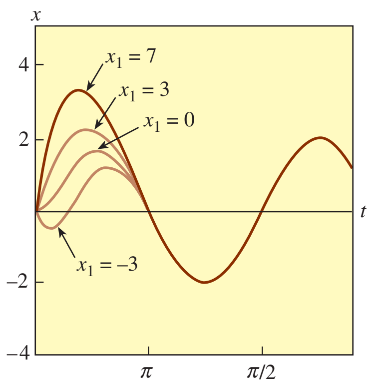


FIGURE 3.8.13 Graphs of solution in Example 7 for various values of x_1

Solution curves for selected values of the initial velocity x_1 are shown in **FIGURE 3.8.13**. The graphs show that the influence of the transient term is negligible for about $t > 3\pi/2$. ≡

DE of Driven Motion Without Damping With a periodic impressed force and no damping force, there is no transient term in the solution of a problem. Also, we shall see that a periodic impressed force with a frequency near or the same as the frequency of free undamped vibrations can cause a severe problem in any oscillatory mechanical system.

EXAMPLE 8 Undamped Forced Motion

Solve the initial-value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin\gamma t, \qquad x(0) = 0, \qquad x'(0) = 0, \tag{29}$$

where F_0 is a constant and $\gamma \neq \omega$.

Solution The complementary function is $x_c(t) = c_1\cos\omega t + c_2\sin\omega t$. To obtain a particular solution we assume $x_p(t) = A\cos\gamma t + B\sin\gamma t$ so that

$$x_p'' + \omega^2 x_p = A(\omega^2 - \gamma^2)\cos\gamma t + B(\omega^2 - \gamma^2)\sin\gamma t = F_0\sin\gamma t.$$

Equating coefficients immediately gives $A = 0$ and $B = F_0/(\omega^2 - \gamma^2)$. Therefore

$$x_p(t) = \frac{F_0}{\omega^2 - \gamma^2}\sin\gamma t.$$

Applying the given initial conditions to the general solution

$$x(t) = c_1\cos\omega t + c_2\sin\omega t + \frac{F_0}{\omega^2 - \gamma^2}\sin\gamma t$$

yields $c_1 = 0$ and $c_2 = -\gamma F_0/\omega(\omega^2 - \gamma^2)$. Thus the solution is

$$x(t) = \frac{F_0}{\omega(\omega^2 - \gamma^2)}(-\gamma\sin\omega t + \omega\sin\gamma t), \qquad \gamma \neq \omega. \tag{30}$$

≡

Pure Resonance Although equation (30) is not defined for $\gamma = \omega$, it is interesting to observe that its limiting value as $\gamma \to \omega$ can be obtained by applying L'Hôpital's rule. This limiting process is analogous to "tuning in" the frequency of the driving force ($\gamma/2\pi$) to the frequency of free vibrations ($\omega/2\pi$). Intuitively we expect that over a length of time we should be able to substantially increase the amplitudes of vibration. For $\gamma = \omega$ we define the solution to be

$$x(t) = \lim_{\gamma\to\omega} F_0 \frac{-\gamma\sin\omega t + \omega\sin\gamma t}{\omega(\omega^2 - \gamma^2)} = F_0 \lim_{\gamma\to\omega} \frac{\frac{d}{d\gamma}(-\gamma\sin\omega t + \omega\sin\gamma t)}{\frac{d}{d\gamma}(\omega^3 - \omega\gamma^2)}$$

$$= F_0 \lim_{\gamma\to\omega} \frac{-\sin\omega t + \omega t\cos\gamma t}{-2\omega\gamma}$$

$$= F_0 \frac{-\sin\omega t + \omega t\cos\omega t}{-2\omega^2}$$

$$= \frac{F_0}{2\omega^2}\sin\omega t - \frac{F_0}{2\omega} t\cos\omega t. \tag{31}$$

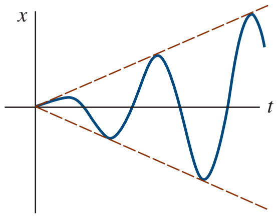


FIGURE 3.8.14 Graph of solution in (31) illustrating pure resonance

As suspected, when $t \to \infty$ the displacements become large; in fact, $|x(t_n)| \to \infty$ when $t_n = n\pi/\omega$, $n = 1, 2, \ldots$. The phenomenon we have just described is known as **pure resonance**. The graph given in **FIGURE 3.8.14** shows typical motion in this case.

In conclusion, it should be noted that there is no actual need to use a limiting process on (30) to obtain the solution for $\gamma = \omega$. Alternatively, equation (31) follows by solving the initial-value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin \omega t, \qquad x(0) = 0, \qquad x'(0) = 0$$

directly by conventional methods.

If the displacements of a spring/mass system were actually described by a function such as (31), the system would necessarily fail. Large oscillations of the mass would eventually force the spring beyond its elastic limit. One might argue too that the resonating model presented in Figure 3.8.14 is completely unrealistic, because it ignores the retarding effects of ever-present damping forces. Although it is true that pure resonance cannot occur when the smallest amount of damping is taken into consideration, large and equally destructive amplitudes of vibration (although bounded as $t \to \infty$) can occur. See Problem 43 in Exercises 3.8.

3.8.4 Series Circuit Analogue

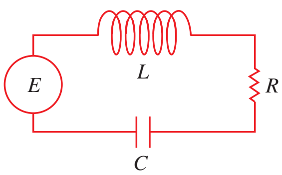


FIGURE 3.8.15 *LRC*-series circuit

■ ***LRC*-Series Circuits** As mentioned in the introduction to this chapter, many different physical systems can be described by a linear second-order differential equation similar to the differential equation of forced motion with damping:

$$m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t). \tag{32}$$

If $i(t)$ denotes current in the ***LRC*-series electrical circuit** shown in **FIGURE 3.8.15**, then the voltage drops across the inductor, resistor, and capacitor are as shown in Figure 1.3.3. By Kirchhoff's second law, the sum of these voltages equals the voltage $E(t)$ impressed on the circuit; that is,

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t). \tag{33}$$

But the charge $q(t)$ on the capacitor is related to the current $i(t)$ by $i = dq/dt$, and so (33) becomes the linear second-order differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t). \tag{34}$$

The nomenclature used in the analysis of circuits is similar to that used to describe spring/mass systems.

If $E(t) = 0$, the **electrical vibrations** of the circuit are said to be **free**. Since the auxiliary equation for (34) is $Lm^2 + Rm + 1/C = 0$, there will be three forms of the solution with $R \neq 0$, depending on the value of the discriminant $R^2 - 4L/C$. We say that the circuit is

	overdamped if	$R^2 - 4L/C > 0$,
	critically damped if	$R^2 - 4L/C = 0$,
and	**underdamped** if	$R^2 - 4L/C < 0$.

In each of these three cases the general solution of (34) contains the factor $e^{-Rt/2L}$, and so $q(t) \to 0$ as $t \to \infty$. In the underdamped case when $q(0) = q_0$, the charge on the capacitor oscillates as it decays; in other words, the capacitor is charging and discharging as $t \to \infty$. When $E(t) = 0$ and $R = 0$, the circuit is said to be undamped and the electrical vibrations do not approach zero as t increases without bound; the response of the circuit is **simple harmonic**.

EXAMPLE 9 Underdamped Series Circuit

Find the charge $q(t)$ on the capacitor in an LRC-series circuit when $L = 0.25$ henry (h), $R = 10$ ohms (Ω), $C = 0.001$ farad (f), $E(t) = 0$ volts (V), $q(0) = q_0$ coulombs (C), and $i(0) = 0$ amperes (A).

Solution Since $1/C = 1000$, equation (34) becomes

$$\frac{1}{4}q'' + 10q' + 1000q = 0 \quad \text{or} \quad q'' + 40q' + 4000q = 0.$$

Solving this homogeneous equation in the usual manner, we find that the circuit is underdamped and $q(t) = e^{-20t}(c_1 \cos 60t + c_2 \sin 60t)$. Applying the initial conditions, we find $c_1 = q_0$ and $c_2 = q_0/3$. Thus $q(t) = q_0 e^{-20t}(\cos 60t + \frac{1}{3}\sin 60t)$. Using (23), we can write the foregoing solution as

$$q(t) = \frac{q_0\sqrt{10}}{3} e^{-20t} \sin(60t + 1.249).$$ ≡

When there is an impressed voltage $E(t)$ on the circuit, the electrical vibrations are said to be **forced**. In the case when $R \neq 0$, the complementary function $q_c(t)$ of (34) is called a **transient solution**. If $E(t)$ is periodic or a constant, then the particular solution $q_p(t)$ of (34) is a **steady-state solution**.

EXAMPLE 10 Steady-State Current

Find the steady-state solution $q_p(t)$ and the **steady-state current** in an LRC-series circuit when the impressed voltage is $E(t) = E_0 \sin \gamma t$.

Solution The steady-state solution $q_p(t)$ is a particular solution of the differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E_0 \sin \gamma t.$$

Using the method of undetermined coefficients, we assume a particular solution of the form $q_p(t) = A \sin \gamma t + B \cos \gamma t$. Substituting this expression into the differential equation, simplifying, and equating coefficients gives

$$A = \frac{E_0\left(L\gamma - \dfrac{1}{C\gamma}\right)}{-\gamma\left(L^2\gamma^2 - \dfrac{2L}{C} + \dfrac{1}{C^2\gamma^2} + R^2\right)}, \quad B = \frac{E_0 R}{-\gamma\left(L^2\gamma^2 - \dfrac{2L}{C} + \dfrac{1}{C^2\gamma^2} + R^2\right)}.$$

It is convenient to express A and B in terms of some new symbols.

$$\text{If } X = L\gamma - \frac{1}{C\gamma}, \quad \text{then} \quad X^2 = L^2\gamma^2 - \frac{2L}{C} + \frac{1}{C^2\gamma^2}.$$

$$\text{If } Z = \sqrt{X^2 + R^2}, \quad \text{then} \quad Z^2 = L^2\gamma^2 - \frac{2L}{C} + \frac{1}{C^2\gamma^2} + R^2.$$

Therefore $A = E_0X/(-\gamma Z^2)$ and $B = E_0R/(-\gamma Z^2)$, so the steady-state charge is

$$q_p(t) = -\frac{E_0X}{\gamma Z^2}\sin\gamma t - \frac{E_0R}{\gamma Z^2}\cos\gamma t.$$

Now the steady-state current is given by $i_p(t) = q_p'(t)$:

$$i_p(t) = \frac{E_0}{Z}\left(\frac{R}{Z}\sin\gamma t - \frac{X}{Z}\cos\gamma t\right). \tag{35}$$ ≡

The quantities $X = L\gamma - 1/(C\gamma)$ and $Z = \sqrt{X^2 + R^2}$ defined in Example 10 are called, respectively, the **reactance** and **impedance** of the circuit. Both the reactance and the impedance are measured in ohms.

3.8 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

3.8.1 Spring/Mass Systems: Free Undamped Motion

1. A mass weighing 4 pounds is attached to a spring whose spring constant is 16 lb/ft. What is the period of simple harmonic motion?

2. A 20-kilogram mass is attached to a spring. If the frequency of simple harmonic motion is $2/\pi$ cycles/s, what is the spring constant k? What is the frequency of simple harmonic motion if the original mass is replaced with an 80-kilogram mass?

3. A mass weighing 24 pounds, attached to the end of a spring, stretches it 4 inches. Initially, the mass is released from rest from a point 3 inches above the equilibrium position. Find the equation of motion.

4. Determine the equation of motion if the mass in Problem 3 is initially released from the equilibrium position with an initial downward velocity of 2 ft/s.

5. A mass weighing 20 pounds stretches a spring 6 inches. The mass is initially released from rest from a point 6 inches below the equilibrium position.

(a) Find the position of the mass at the times $t = \pi/12$, $\pi/8$, $\pi/6$, $\pi/4$, and $9\pi/32$ s.

(b) What is the velocity of the mass when $t = 3\pi/16$ s? In which direction is the mass heading at this instant?

(c) At what times does the mass pass through the equilibrium position?

6. A force of 400 newtons stretches a spring 2 meters. A mass of 50 kilograms is attached to the end of the spring and is initially released from the equilibrium position with an upward velocity of 10 m/s. Find the equation of motion.

7. Another spring whose constant is 20 N/m is suspended from the same rigid support but parallel to the spring/mass system in Problem 6. A mass of 20 kilograms is attached to the second spring, and both masses are initially released from the equilibrium position with an upward velocity of 10 m/s.

(a) Which mass exhibits the greater amplitude of motion?

(b) Which mass is moving faster at $t = \pi/4$ s? At $\pi/2$ s?

(c) At what times are the two masses in the same position? Where are the masses at these times? In which directions are they moving?

8. A mass weighing 32 pounds stretches a spring 2 feet. Determine the amplitude and period of motion if the mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of 2 ft/s. How many complete cycles will the mass have completed at the end of 4π seconds?

9. A mass weighing 8 pounds is attached to a spring. When set in motion, the spring/mass system exhibits simple harmonic motion. Determine the equation of motion if the spring constant is 1 lb/ft and the mass is initially released from a point 6 inches below the equilibrium position with a downward velocity of $\frac{3}{2}$ ft/s. Express the equation of motion in the form given in (6).

10. A mass weighing 10 pounds stretches a spring $\frac{1}{4}$ foot. This mass is removed and replaced with a mass of 1.6 slugs, which is initially released from a point $\frac{1}{3}$ foot above the equilibrium position with a downward velocity of $\frac{5}{4}$ ft/s. Express the equation of motion in the form given in (6). At what times does the mass attain a displacement below the equilibrium position numerically equal to $\frac{1}{2}$ the amplitude?

11. A mass weighing 64 pounds stretches a spring 0.32 foot. The mass is initially released from a point 8 inches above the equilibrium position with a downward velocity of 5 ft/s.

(a) Find the equation of motion.
(b) What are the amplitude and period of motion?
(c) How many complete cycles will the mass have completed at the end of 3π seconds?
(d) At what time does the mass pass through the equilibrium position heading downward for the second time?
(e) At what time does the mass attain its extreme displacement on either side of the equilibrium position?
(f) What is the position of the mass at $t = 3$ s?
(g) What is the instantaneous velocity at $t = 3$ s?
(h) What is the acceleration at $t = 3$ s?
(i) What is the instantaneous velocity at the times when the mass passes through the equilibrium position?
(j) At what times is the mass 5 inches below the equilibrium position?
(k) At what times is the mass 5 inches below the equilibrium position heading in the upward direction?

12. A mass of 1 slug is suspended from a spring whose spring constant is 9 lb/ft. The mass is initially released from a point 1 foot above the equilibrium position with an upward velocity of $\sqrt{3}$ ft/s. Find the times for which the mass is heading downward at a velocity of 3 ft/s.

13. Under some circumstances when two parallel springs, with constants k_1 and k_2, support a single mass, the **effective spring constant** of the system is given by $k = 4k_1k_2/(k_1 + k_2)$. A mass weighing 20 pounds stretches one spring 6 inches and another spring 2 inches. The springs are attached to a common rigid support and then to a metal plate. As shown in **FIGURE 3.8.16**, the mass is attached to the center of the plate in the double-spring arrangement. Determine the effective spring constant of this system. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 2 ft/s.

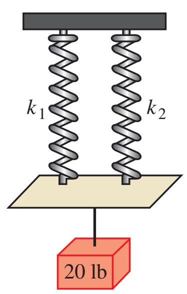


FIGURE 3.8.16 Double-spring system in Problem 13

14. A certain mass stretches one spring $\frac{1}{3}$ foot and another spring $\frac{1}{2}$ foot. The two springs are attached to a common rigid support in the manner indicated in Problem 13 and Figure 3.8.16. The first mass is set aside, a mass weighing 8 pounds is attached to the double-spring arrangement, and the system is set in motion. If the period of motion is $\pi/15$ second, determine how much the first mass weighs.

15. A model of a spring/mass system is $4x'' + e^{-0.1t}x = 0$. By inspection of the differential equation only, discuss the behavior of the system over a long period of time.

16. A model of a spring/mass system is $4x'' + tx = 0$. By inspection of the differential equation only, discuss the behavior of the system over a long period of time.

3.8.2 Spring/Mass Systems: Free Damped Motion

In Problems 17–20, the given figure represents the graph of an equation of motion for a damped spring/mass system. Use the graph to determine

(a) whether the initial displacement is above or below the equilibrium position, and
(b) whether the mass is initially released from rest, heading downward, or heading upward.

17.

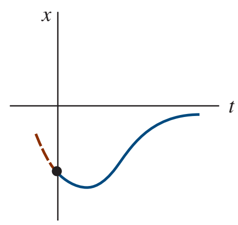


FIGURE 3.8.17 Graph for Problem 17

18.

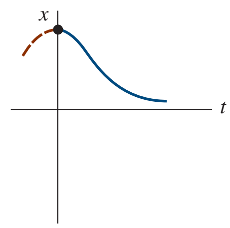


FIGURE 3.8.18 Graph for Problem 18

19.

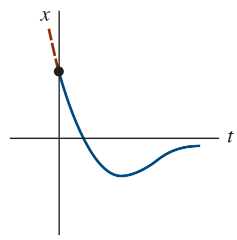


FIGURE 3.8.19 Graph for Problem 19

20.

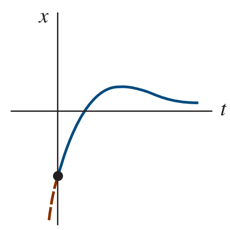


FIGURE 3.8.20 Graph for Problem 20

21. A mass weighing 4 pounds is attached to a spring whose constant is 2 lb/ft. The medium offers a damping force that is numerically equal to the instantaneous velocity. The mass is initially released from a point 1 foot above the equilibrium position with a downward velocity of 8 ft/s. Determine the time at which the mass passes through the equilibrium position. Find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

22. A 4-foot spring measures 8 feet long after a mass weighing 8 pounds is attached to it. The medium through which the mass moves offers damping force numerically equal to $\sqrt{2}$ times the instantaneous velocity. Find the equation of motion if the mass is initially released from the equilibrium position with a downward velocity of 5 ft/s. Find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

23. A 1-kilogram mass is attached to a spring whose constant is 16 N/m, and the entire system is then submerged in a liquid that imparts a damping force numerically equal to 10 times the instantaneous velocity. Determine the equations of motion if

(a) the mass is initially released from rest from a point 1 meter below the equilibrium position, and then
(b) the mass is initially released from a point 1 meter below the equilibrium position with an upward velocity of 12 m/s.

24. In parts (a) and (b) of Problem 23, determine whether the mass passes through the equilibrium position. In each case find the time at which the mass attains its extreme displacement from the equilibrium position. What is the position of the mass at this instant?

25. A force of 2 pounds stretches a spring 1 foot. A mass weighing 3.2 pounds is attached to the spring, and the system is then immersed in a medium that offers a damping force numerically equal to 0.4 times the instantaneous velocity.

(a) Find the equation of motion if the mass is initially released from rest from a point 1 foot above the equilibrium position.

(b) Express the equation of motion in the form given in (23).

(c) Find the first time at which the mass passes through the equilibrium position heading upward.

26. After a mass weighing 10 pounds is attached to a 5-foot spring, the spring measures 7 feet. This mass is removed and replaced with another mass that weighs 8 pounds. The entire system is placed in a medium that offers a damping force numerically equal to the instantaneous velocity.

(a) Find the equation of motion if the mass is initially released from a point $\frac{1}{2}$ foot below the equilibrium position with a downward velocity of 1 ft/s.

(b) Express the equation of motion in the form given in (23).

(c) Find the times at which the mass passes through the equilibrium position heading downward.

(d) Graph the equation of motion.

27. A mass weighing 10 pounds stretches a spring 2 feet. The mass is attached to a dashpot damping device that offers a damping force numerically equal to β ($\beta > 0$) times the instantaneous velocity. Determine the values of the damping constant β so that the subsequent motion is **(a)** overdamped, **(b)** critically damped, and **(c)** underdamped.

28. A mass weighing 24 pounds stretches a spring 4 feet. The subsequent motion takes place in a medium that offers a damping force numerically equal to β ($\beta > 0$) times the instantaneous velocity. If the mass is initially released from the equilibrium position with an upward velocity of 2 ft/s, show that when $\beta > 3\sqrt{2}$ the equation of motion is

$$x(t) = \frac{-3}{\sqrt{\beta^2 - 18}} e^{-2\beta t/3} \sinh \frac{2}{3}\sqrt{\beta^2 - 18}\,t.$$

3.8.3 Spring/Mass Systems: Driven Motion

29. A mass weighing 16 pounds stretches a spring $\frac{8}{3}$ feet. The mass is initially released from rest from a point 2 feet below the equilibrium position, and the subsequent motion takes place in a medium that offers a damping force numerically equal to one-half the instantaneous velocity. Find the equation of motion if the mass is driven by an external force equal to $f(t) = 10 \cos 3t$.

30. A mass of 1 slug is attached to a spring whose constant is 5 lb/ft. Initially the mass is released 1 foot below the equilibrium position with a downward velocity of 5 ft/s, and the subsequent motion takes place in a medium that offers a damping force numerically equal to two times the instantaneous velocity.

(a) Find the equation of motion if the mass is driven by an external force equal to $f(t) = 12 \cos 2t + 3 \sin 2t$.

(b) Graph the transient and steady-state solutions on the same coordinate axes.

(c) Graph the equation of motion.

31. A mass of 1 slug, when attached to a spring, stretches it 2 feet and then comes to rest in the equilibrium position. Starting at $t = 0$, an external force equal to $f(t) = 8 \sin 4t$ is applied to the system. Find the equation of motion if the surrounding medium offers a damping force numerically equal to eight times the instantaneous velocity.

32. In Problem 31 determine the equation of motion if the external force is $f(t) = e^{-t} \sin 4t$. Analyze the displacements for $t \to \infty$.

33. When a mass of 2 kilograms is attached to a spring whose constant is 32 N/m, it comes to rest in the equilibrium position. Starting at $t = 0$, a force equal to $f(t) = 68e^{-2t} \cos 4t$ is applied to the system. Find the equation of motion in the absence of damping.

34. In Problem 33 write the equation of motion in the form $x(t) = A \sin(\omega t + \phi) + Be^{-2t} \sin(4t + \theta)$. What is the amplitude of vibrations after a very long time?

35. A mass m is attached to the end of a spring whose constant is k. After the mass reaches equilibrium, its support begins to oscillate vertically about a horizontal line L according to a formula $h(t)$. The value of h represents the distance in feet measured from L. See **FIGURE 3.8.21**.

(a) Determine the differential equation of motion if the entire system moves through a medium offering a damping force numerically equal to $\beta(dx/dt)$.

(b) Solve the differential equation in part (a) if the spring is stretched 4 feet by a weight of 16 pounds and $\beta = 2$, $h(t) = 5 \cos t$, $x(0) = x'(0) = 0$.

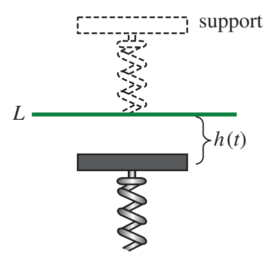


FIGURE 3.8.21 Oscillating support in Problem 35

36. A mass of 100 grams is attached to a spring whose constant is 1600 dynes/cm. After the mass reaches equilibrium, its support oscillates according to the formula $h(t) = \sin 8t$, where h represents displacement from its original position. See Problem 35 and Figure 3.8.21.

(a) In the absence of damping, determine the equation of motion if the mass starts from rest from the equilibrium position.

(b) At what times does the mass pass through the equilibrium position?

(c) At what times does the mass attain its extreme displacements?

(d) What are the maximum and minimum displacements?

(e) Graph the equation of motion.

In Problems 37 and 38, solve the given initial-value problem.

37. $\dfrac{d^2x}{dt^2} + 4x = -5\sin 2t + 3\cos 2t,\ x(0) = -1,\ x'(0) = 1$

38. $\dfrac{d^2x}{dt^2} + 9x = 5\sin 3t,\ x(0) = 2,\ x'(0) = 0$

39. **(a)** Show that the solution of the initial-value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0\cos\gamma t,\ x(0) = 0,\ x'(0) = 0$$

is $$x(t) = \frac{F_0}{\omega^2 - \gamma^2}(\cos\gamma t - \cos\omega t).$$

(b) Evaluate $\displaystyle\lim_{\gamma\to\omega}\frac{F_0}{\omega^2 - \gamma^2}(\cos\gamma t - \cos\omega t)$.

40. Compare the result obtained in part (b) of Problem 39 with the solution obtained using variation of parameters when the external force is $F_0\cos\omega t$.

41. **(a)** Show that $x(t)$ given in part (a) of Problem 39 can be written in the form

$$x(t) = \frac{-2F_0}{\omega^2 - \gamma^2}\sin\frac{1}{2}(\gamma - \omega)t\,\sin\frac{1}{2}(\gamma + \omega)t.$$

(b) If we define $\varepsilon = \frac{1}{2}(\gamma - \omega)$, show that when ε is small, an approximate solution is

$$x(t) = \frac{F_0}{2\varepsilon\gamma}\sin\varepsilon t\,\sin\gamma t.$$

When ε is small the frequency $\gamma/2\pi$ of the impressed force is close to the frequency $\omega/2\pi$ of free vibrations. When this occurs, the motion is as indicated in **FIGURE 3.8.22**. Oscillations of this kind are called **beats** and are due to the fact that the frequency of $\sin\varepsilon t$ is quite small in comparison to the frequency of $\sin\gamma t$. The red dashed curves, or envelope of the graph of $x(t)$, are obtained from the graphs of $\pm(F_0/2\varepsilon\gamma)\sin\varepsilon t$. Use a graphing utility with various values of F_0, ε, and γ to verify the graph in Figure 3.8.22.

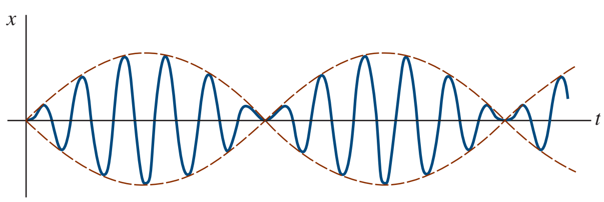


FIGURE 3.8.22 Beats phenomenon in Problem 41

Computer Lab Assignments

42. Can there be beats when a damping force is added to the model in part (a) of Problem 39? Defend your position with graphs obtained either from the explicit solution of the problem

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F_0\cos\gamma t,\quad x(0) = 0,\quad x'(0) = 0$$

or from solution curves obtained using a numerical solver.

43. **(a)** Show that the general solution of

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F_0\sin\gamma t$$

is

$$x(t) = Ae^{-\lambda t}\sin\left(\sqrt{\omega^2 - \lambda^2}\,t + \phi\right) + \frac{F_0}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}\sin(\gamma t + \theta),$$

where $A = \sqrt{c_1^2 + c_2^2}$ and the phase angles ϕ and θ are, respectively, defined by $\sin\phi = c_1/A$, $\cos\phi = c_2/A$ and

$$\sin\theta = \frac{-2\lambda\gamma}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}},$$

$$\cos\theta = \frac{\omega^2 - \gamma^2}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}.$$

(b) The solution in part (a) has the form $x(t) = x_c(t) + x_p(t)$. Inspection shows that $x_c(t)$ is transient, and hence for large values of time, the solution is approximated by $x_p(t) = g(\gamma)\sin(\gamma t + \theta)$, where

$$g(\gamma) = \frac{F_0}{\sqrt{(\omega^2 - \gamma^2)^2 + 4\lambda^2\gamma^2}}.$$

Although the amplitude $g(\gamma)$ of $x_p(t)$ is bounded as $t\to\infty$, show that the maximum oscillations will occur at the value $\gamma_1 = \sqrt{\omega^2 - 2\lambda^2}$. What is the maximum value of g? The number $\sqrt{\omega^2 - 2\lambda^2}/2\pi$ is said to be the **resonance frequency** of the system.

(c) When $F_0 = 2$, $m = 1$, and $k = 4$, g becomes

$$g(\gamma) = \frac{2}{\sqrt{(4 - \gamma^2)^2 + \beta^2\gamma^2}}.$$

Construct a table of the values of γ_1 and $g(\gamma_1)$ corresponding to the damping coefficients $\beta = 2, \beta = 1, \beta = \frac{3}{4}, \beta = \frac{1}{2}$, and $\beta = \frac{1}{4}$. Use a graphing utility to obtain the graphs of g corresponding to these damping coefficients. Use the same coordinate axes. This family of graphs is called the **resonance curve** or **frequency response curve** of the system. What is γ_1 approaching as $\beta\to 0$? What is happening to the resonance curve as $\beta\to 0$?

44. Consider a driven undamped spring/mass system described by the initial-value problem

$$\frac{d^2x}{dt^2} + \omega^2 x = F_0 \sin^n \gamma t, \;\; x(0) = 0, x'(0) = 0.$$

(a) For $n = 2$, discuss why there is a single frequency $\gamma_1/2\pi$ at which the system is in pure resonance.

(b) For $n = 3$, discuss why there are two frequencies $\gamma_1/2\pi$ and $\gamma_2/2\pi$ at which the system is in pure resonance.

(c) Suppose $\omega = 1$ and $F_0 = 1$. Use a numerical solver to obtain the graph of the solution of the initial-value problem for $n = 2$ and $\gamma = \gamma_1$ in part (a). Obtain the graph of the solution of the initial-value problem for $n = 3$ corresponding, in turn, to $\gamma = \gamma_1$ and $\gamma = \gamma_2$ in part (b).

3.8.4 Series Circuit Analogue

45. Find the charge on the capacitor in an *LRC*-series circuit at $t = 0.01$ s when $L = 0.05$ h, $R = 2\ \Omega$, $C = 0.01$ f, $E(t) = 0$ V, $q(0) = 5$ C, and $i(0) = 0$ A. Determine the first time at which the charge on the capacitor is equal to zero.

46. Find the charge on the capacitor in an *LRC*-series circuit when $L = \frac{1}{4}$ h, $R = 20\ \Omega$, $C = \frac{1}{300}$ f, $E(t) = 0$ V, $q(0) = 4$ C, and $i(0) = 0$ A. Is the charge on the capacitor ever equal to zero?

In Problems 47 and 48, find the charge on the capacitor and the current in the given *LRC*-series circuit. Find the maximum charge on the capacitor.

47. $L = \frac{5}{3}$ h, $R = 10\ \Omega$, $C = \frac{1}{30}$ f, $E(t) = 300$ V, $q(0) = 0$ C, $i(0) = 0$ A

48. $L = 1$ h, $R = 100\ \Omega$, $C = 0.0004$ f, $E(t) = 30$ V, $q(0) = 0$ C, $i(0) = 2$ A

49. Find the steady-state charge and the steady-state current in an *LRC*-series circuit when $L = 1$ h, $R = 2\ \Omega$, $C = 0.25$ f, and $E(t) = 50 \cos t$ V.

50. Show that the amplitude of the steady-state current in the *LRC*-series circuit in Example 10 is given by E_0/Z, where Z is the impedance of the circuit.

51. Use Problem 50 to show that the steady-state current in an *LRC*-series circuit when $L = \frac{1}{2}$ h, $R = 20\ \Omega$, $C = 0.001$ f, and $E(t) = 100 \sin 60t$ V, is given by $i_p(t) = 4.160 \sin(60t - 0.588)$.

52. Find the steady-state current in an *LRC*-series circuit when $L = \frac{1}{2}$ h, $R = 20\ \Omega$, $C = 0.001$ f, and $E(t) = 100 \sin 60t + 200 \cos 40t$ V.

53. Find the charge on the capacitor in an *LRC*-series circuit when $L = \frac{1}{2}$ h, $R = 10\ \Omega$, $C = 0.01$ f, $E(t) = 150$ V, $q(0) = 1$ C, and $i(0) = 0$ A. What is the charge on the capacitor after a long time?

54. Show that if L, R, C, and E_0 are constant, then the amplitude of the steady-state current in Example 10 is a maximum when $\gamma = 1/\sqrt{LC}$. What is the maximum amplitude?

55. Show that if L, R, E_0, and γ are constant, then the amplitude of the steady-state current in Example 10 is a maximum when the capacitance is $C = 1/L\gamma^2$.

56. Find the charge on the capacitor and the current in an *LC*-circuit when $L = 0.1$ h, $C = 0.1$ f, $E(t) = 100 \sin \gamma t$ V, $q(0) = 0$ C, and $i(0) = 0$ A.

57. Find the charge on the capacitor and the current in an *LC*-circuit when $E(t) = E_0 \cos \gamma t$ V, $q(0) = q_0$ C, and $i(0) = i_0$ A.

58. In Problem 57, find the current when the circuit is in resonance.

3.9 Linear Models: Boundary-Value Problems

Introduction The preceding section was devoted to dynamic physical systems each described by a mathematical model consisting of a linear second-order differential equation accompanied by prescribed *initial conditions*—that is, side conditions that are specified on the unknown function and its first derivative at a single point. But often the mathematical description of a steady-state phenomenon or a static physical system demands that we solve a linear differential equation subject to *boundary conditions*—that is, conditions specified on the unknown function, or on one of its derivatives, or even on a linear combination of the unknown function and one of its derivatives, at two different points. By and large, the number of specified boundary conditions matches the order of the linear DE. We begin this section with an application of a relatively simple linear fourth-order differential equation associated with four boundary conditions.

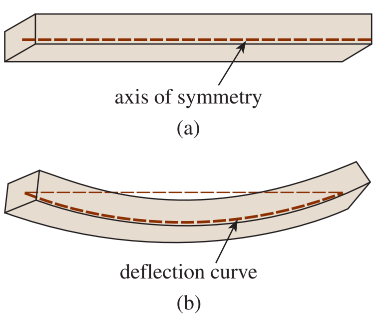


FIGURE 3.9.1 Deflection of a homogeneous beam

Deflection of a Beam Many structures are constructed using girders, or beams, and these beams deflect or distort under their own weight or under the influence of some external force. As we shall now see, this deflection $y(x)$ is governed by a relatively simple linear fourth-order differential equation.

To begin, let us assume that a beam of length L is homogeneous and has uniform cross sections along its length. In the absence of any load on the beam (including its weight), a curve joining the centroids of all its cross sections is a straight line called the **axis of symmetry**. See **FIGURE 3.9.1(a)**.

If a load is applied to the beam in a vertical plane containing the axis of symmetry, the beam, as shown in Figure 3.9.1(b), undergoes a distortion, and the curve connecting the centroids of all cross sections is called the **deflection curve** or **elastic curve**. The deflection curve approximates the shape of the beam. Now suppose that the x-axis coincides with the axis of symmetry and that the deflection $y(x)$, measured from this axis, is positive if downward. In the theory of elasticity it is shown that the bending moment $M(x)$ at a point x along the beam is related to the load per unit length $w(x)$ by the equation

$$\frac{d^2M}{dx^2} = w(x). \tag{1}$$

In addition, the bending moment $M(x)$ is proportional to the curvature κ of the elastic curve

$$M(x) = EI\kappa, \tag{2}$$

where E and I are constants; E is Young's modulus of elasticity of the material of the beam, and I is the moment of inertia of a cross section of the beam (about an axis known as the neutral axis). The product EI is called the **flexural rigidity** of the beam.

Now, from calculus, curvature is given by $\kappa = y''/[1 + (y')^2]^{3/2}$. When the deflection $y(x)$ is small, the slope $y' \approx 0$ and so $[1 + (y')^2]^{3/2} \approx 1$. If we let $\kappa \approx y''$, equation (2) becomes $M = EI\,y''$. The second derivative of this last expression is

$$\frac{d^2M}{dx^2} = EI\frac{d^2}{dx^2}y'' = EI\frac{d^4y}{dx^4}. \tag{3}$$

Using the given result in (1) to replace d^2M/dx^2 in (3), we see that the deflection $y(x)$ satisfies the fourth-order differential equation

$$EI\frac{d^4y}{dx^4} = w(x). \tag{4}$$

Boundary conditions associated with equation (4) depend on how the ends of the beam are supported. A cantilever beam is **embedded** or **clamped** at one end and **free** at the other. A diving board, an outstretched arm, an airplane wing, and a balcony are common examples of such beams, but even trees, flagpoles, skyscrapers, and the George Washington monument can act as cantilever beams, because they are embedded at one end and are subject to the bending force of the wind. For a cantilever beam, the deflection $y(x)$ must satisfy the following two conditions at the embedded end $x = 0$:

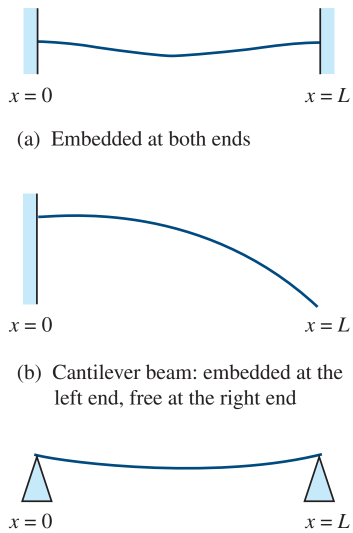
- $y(0) = 0$ since there is no deflection, and
- $y'(0) = 0$ since the deflection curve is tangent to the x-axis (in other words, the slope of the deflection curve is zero at this point).

At $x = L$ the free-end conditions are

- $y''(L) = 0$ since the bending moment is zero, and
- $y'''(L) = 0$ since the shear force is zero.

The function $F(x) = dM/dx = EI\,d^3y/dx^3$ is called the shear force. If an end of a beam is **simply supported** or **hinged** (also called **pin supported**, and **fulcrum supported**), then we must have $y = 0$ and $y'' = 0$ at that end. The following table summarizes the boundary conditions that are associated with (4). See **FIGURE 3.9.2**.

Ends of the Beam	Boundary Conditions
Embedded	$y = 0,\ y' = 0$
Free	$y'' = 0,\ y''' = 0$
Simply supported or hinged	$y = 0,\ y'' = 0$



(a) Embedded at both ends

(b) Cantilever beam: embedded at the left end, free at the right end

(c) Simply supported at both ends

FIGURE 3.9.2 Beams with various end conditions

EXAMPLE 1 An Embedded Beam

A beam of length L is embedded at both ends. Find the deflection of the beam if a constant load w_0 is uniformly distributed along its length—that is, $w(x) = w_0$, $0 < x < L$.

Solution From (4), we see that the deflection $y(x)$ satisfies

$$EI \frac{d^4y}{dx^4} = w_0.$$

Because the beam is embedded at both its left end ($x = 0$) and right end ($x = L$), there is no vertical deflection and the line of deflection is horizontal at these points. Thus the boundary conditions are

$$y(0) = 0, \quad y'(0) = 0, \quad y(L) = 0, \quad y'(L) = 0.$$

We can solve the nonhomogeneous differential equation in the usual manner (find y_c by observing that $m = 0$ is a root of multiplicity four of the auxiliary equation $m^4 = 0$, and then find a particular solution y_p by undetermined coefficients), or we can simply integrate the equation $d^4y/dx^4 = w_0/EI$ four times in succession. Either way, we find the general solution of the equation $y = y_c + y_p$ to be

$$y(x) = c_1 + c_2x + c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4.$$

Now the conditions $y(0) = 0$ and $y'(0) = 0$ give, in turn, $c_1 = 0$ and $c_2 = 0$, whereas the remaining conditions $y(L) = 0$ and $y'(L) = 0$ applied to $y(x) = c_3x^2 + c_4x^3 + \frac{w_0}{24EI}x^4$ yield the simultaneous equations

$$c_3L^2 + c_4L^3 + \frac{w_0}{24EI}L^4 = 0$$

$$3c_3L + 3c_4L^2 + \frac{w_0}{6EI}L^3 = 0.$$

Solving this system gives $c_3 = w_0L^2/24EI$ and $c_4 = -w_0L/12EI$. Thus the deflection is

$$y(x) = \frac{w_0L^2}{24EI}x^2 - \frac{w_0L}{12EI}x^3 + \frac{w_0}{24EI}x^4$$

or $y(x) = \frac{w_0}{24EI}x^2(x - L)^2$. By choosing $w_0 = 24EI$, and $L = 1$, we obtain the graph of the deflection curve in **FIGURE 3.9.3**. ≡

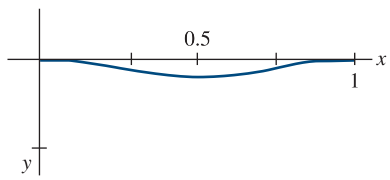


FIGURE 3.9.3 Deflection curve for Example 1

The discussion of the beam not withstanding, a physical system that is described by a two-point boundary-value problem usually involves a second-order differential equation. Hence, for the remainder of the discussion in this section we are concerned with boundary-value problems of the type

$$\textit{Solve:} \quad a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x), \quad a < x < b \tag{5}$$

$$\textit{Subject to:} \quad \begin{aligned} A_1y(a) + B_1y'(a) &= C_1 \\ A_2y(b) + B_2y'(b) &= C_2. \end{aligned} \tag{6}$$

In (5) we assume that the coefficients $a_0(x)$, $a_1(x)$, $a_2(x)$, and $g(x)$ are continuous on the interval $[a, b]$ and that $a_2(x) \neq 0$ for all x in the interval. In (6) we assume that A_1 and B_1 are not both zero and A_2 and B_2 are not both zero. When $g(x) = 0$ for all x in $[a, b]$ and C_1 *and* C_2 are 0, we say that the boundary-value problem is **homogeneous**. Otherwise, we say that the boundary-value problem

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is **nonhomogeneous**. For example, the BVP $y'' + y = 0$, $y(0) = 0$, $y(\pi) = 0$ is homogeneous, whereas the BVP $y'' + y = 1$, $y(0) = 0$, $y(2\pi) = 0$ is nonhomogeneous.

Eigenvalues and Eigenfunctions In applications involving homogeneous boundary-value problems, one or more of the coefficients in the differential equation (5) may depend on a constant parameter λ. As a consequence the solutions $y_1(x)$ and $y_2(x)$ of the homogeneous DE (5) also depend on λ. We often wish to determine those values of the parameter for which the boundary-value problem has nontrivial solutions. The next example illustrates this idea.

EXAMPLE 2 Nontrivial Solutions of a BVP

Solve the homogeneous boundary-value problem

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y(L) = 0.$$

Solution We consider three cases: $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$.

***Case I*:** For $\lambda = 0$, the solution of the DE $y'' = 0$ is $y = c_1x + c_2$. Applying the boundary conditions $y(0) = 0$ and $y(L) = 0$ to this solution yield, in turn, $c_2 = 0$ and $c_1 = 0$. Hence for $\lambda = 0$, the only solution of the boundary-value problem is the trivial solution $y = 0$.

***Case II*:** For $\lambda < 0$, it is convenient to write $\lambda = -\alpha^2$, where $\alpha > 0$. With this new notation the auxiliary equation is $m^2 - \alpha^2 = 0$ and has roots $m_1 = \alpha$ and $m_2 = -\alpha$. Because the interval on which we are working is finite, we choose to write the general solution of $y'' - \alpha^2 y = 0$ in the hyperbolic form $y = c_1 \cosh \alpha x + c_2 \sinh \alpha x$. From y(0) = 0 we see

$$y(0) = c_1 \cosh 0 + c_2 \sinh 0 = c_1 \cdot 1 + c_2 \cdot 0 = c_1$$

implies $c_1 = 0$. Hence $y = c_2 \sinh \alpha x$. The second boundary condition $y(L) = 0$ then requires $c_2 \sinh \alpha L = 0$. When $\alpha \neq 0$, $\sinh \alpha L \neq 0$, and so we are forced to choose $c_2 = 0$. Once again the only solution of the BVP is the trivial solution $y = 0$.

***Case III*:** For $\lambda > 0$ we write $\lambda = \alpha^2$, where $\alpha > 0$. The auxiliary equation $m^2 + \alpha^2 = 0$ now has complex roots $m_1 = i\alpha$ and $m_2 = -i\alpha$, and so the general solution of the DE $y'' + \alpha^2 y = 0$ is $y = c_1 \cos \alpha x + c_2 \sin \alpha x$. As before, $y(0) = 0$ yields $c_1 = 0$ and so $y = c_2 \sin\alpha x$. Then $y(L) = 0$ implies

$$c_2 \sin \alpha L = 0.$$

If $c_2 = 0$, then necessarily $y = 0$. But this time we can require $c_2 \neq 0$ since $\sin \alpha L = 0$ is satisfied whenever αL is an integer multiple of π:

$$\alpha L = n\pi \quad \text{or} \quad \alpha = \frac{n\pi}{L} \quad \text{or} \quad \lambda_n = \alpha_n^2 = \left(\frac{n\pi}{L}\right)^2, \quad n = 1,2,3, \ldots.$$

Therefore for any real nonzero c_2, $y = c_2 \sin(n\pi x/L)$ is a solution of the problem for each n. Since the differential equation is homogeneous, any constant multiple of a solution is also a solution. Thus we may, if desired, simply take $c_2 = 1$. In other words, for each number in the sequence

$$\lambda_1 = \frac{\pi^2}{L^2}, \quad \lambda_2 = \frac{4\pi^2}{L^2}, \quad \lambda_3 = \frac{9\pi^2}{L^2}, \ldots,$$

the *corresponding* function in the sequence

$$y_1 = \sin\frac{\pi}{L}, \quad y_2 = \sin\frac{2\pi}{L}, \quad y_3 = \sin\frac{3\pi}{L}, \ldots,$$

is a nontrivial solution of the original problem. ≡

The numbers $\lambda_n = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$ for which the boundary-value problem in Example 2 has a nontrivial solution are known as **characteristic values** or, more commonly, **eigenvalues**. The solutions depending on these values of λ_n, $y_n = c_2 \sin(n\pi x/L)$ or simply $y_n = \sin(n\pi x/L)$, are called **characteristic functions** or **eigenfunctions**.

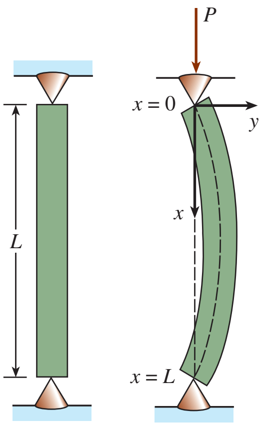

FIGURE 3.9.4 Elastic column buckling under a compressive force

Buckling of a Thin Vertical Column In the eighteenth century Leonhard Euler was one of the first mathematicians to study an eigenvalue problem in analyzing how a thin elastic column buckles under a compressive axial force.

Consider a long slender vertical column of uniform cross section and length L. Let $y(x)$ denote the deflection of the column when a constant vertical compressive force, or load, P is applied to its top, as shown in **FIGURE 3.9.4**. By comparing bending moments at any point along the column we obtain

$$EI\frac{d^2y}{dx^2} = -Py \qquad \text{or} \qquad EI\frac{d^2y}{dx^2} + Py = 0, \tag{7}$$

where E is Young's modulus of elasticity and I is the moment of inertia of a cross section about a vertical line through its centroid.

EXAMPLE 3 The Euler Load

Find the deflection of a thin vertical homogeneous column of length L subjected to a constant axial load P if the column is hinged at both ends.

Solution The boundary-value problem to be solved is

$$EI\frac{d^2y}{dx^2} + Py = 0, \quad y(0) = 0, \quad y(L) = 0.$$

First note that $y = 0$ is a perfectly good solution of this problem. This solution has a simple intuitive interpretation: If the load P is not great enough, there is no deflection. The question then is this: For what values of P will the column bend? In mathematical terms: For what values of P does the given boundary-value problem possess nontrivial solutions?

By writing $\lambda = P/EI$ we see that

$$y'' + \lambda y = 0, \qquad y(0) = 0, \qquad y(L) = 0$$

is identical to the problem in Example 2. From Case III of that discussion we see that the deflection curves are $y_n(x) = c_2 \sin(n\pi x/L)$, corresponding to the eigenvalues $\lambda_n = P_n/EI = n^2\pi^2/L^2$, $n = 1, 2, 3, \ldots$. Physically this means that the column will buckle or deflect only when the compressive force is one of the values $P_n = n^2\pi^2 EI/L^2$, $n = 1, 2, 3, \ldots$. These different forces are called **critical loads**. The deflection curve corresponding to the smallest critical load $P_1 = \pi^2 EI/L^2$, called the **Euler load**, is $y_1(x) = c_2 \sin(\pi x/L)$ and is known as the **first buckling mode**. ≡

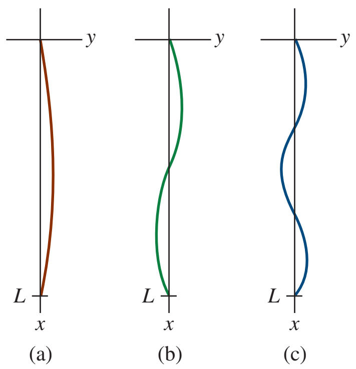

FIGURE 3.9.5 Deflection curves for compressive forces P_1, P_2, P_3

The deflection curves in Example 3 corresponding to $n = 1$, $n = 2$, and $n = 3$ are shown in **FIGURE 3.9.5**. Note that if the original column has some sort of physical restraint put on it at $x = L/2$, then the smallest critical load will be $P_2 = 4\pi^2 EI/L^2$ and the deflection curve will be as shown in Figure 3.9.5(b). If restraints are put on the column at $x = L/3$ and at $x = 2L/3$, then the column will not buckle until the critical load $P_3 = 9\pi^2 EI/L^2$ is applied and the deflection curve will be as shown in Figure 3.9.5(c). See Problem 25 in Exercises 3.9.

Rotating String The simple linear second-order differential equation

$$y'' + \lambda y = 0 \tag{8}$$

occurs again and again as a mathematical model. In Section 3.8 we saw (8) in the forms $d^2x/dt^2 + (k/m)x = 0$ and $d^2q/dt^2 + (1/LC)q = 0$ as models for, respectively, the simple harmonic motion of a spring/mass system and the simple harmonic response of a series circuit. It is apparent when the model for the deflection of a thin column in (7) is written as $d^2y/dx^2 + (P/EI)y = 0$ that it is the same as (8). We encounter the basic equation (8) one more time in this section: as a model that defines the deflection curve or the shape $y(x)$ assumed by a rotating string. The physical situation is analogous to when two persons hold a jump rope and twirl it in a synchronous manner. See **FIGURE 3.9.6** parts (a) and (b).

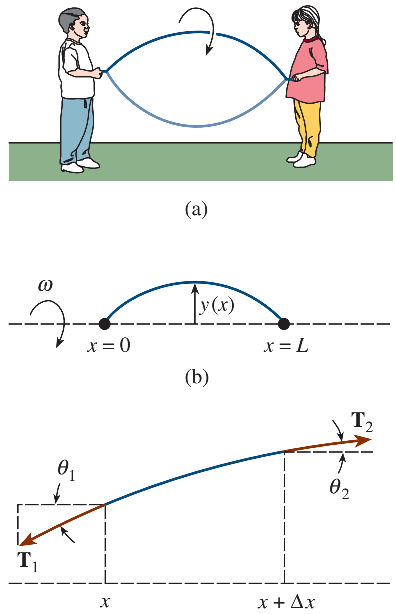

FIGURE 3.9.6 Rotating rope and forces acting on it

Suppose a string of length L with constant linear density ρ (mass per unit length) is stretched along the x-axis and fixed at $x = 0$ and $x = L$. Suppose the string is then rotated about that axis at a constant angular speed ω. Consider a portion of the string on the interval $[x, x + \Delta x]$, where Δx is small. If the magnitude T of the tension $\mathbf{T}$, acting tangential to the string, is constant along the string, then the desired differential equation can be obtained by equating two different formulations of the net force acting on the string on the interval $[x, x + \Delta x]$. First, we see from Figure 3.9.6(c) that the net vertical force is

$$F = T\sin\theta_2 - T\sin\theta_1. \tag{9}$$

When angles θ_1 and θ_2 (measured in radians) are small, we have $\sin\theta_2 \approx \tan\theta_2$ and $\sin\theta_1 \approx \tan\theta_1$. Moreover, since $\tan\theta_2$ and $\tan\theta_1$ are, in turn, slopes of the lines containing the vectors $\mathbf{T}_2$ and $\mathbf{T}_1$, we can also write

$$\tan\theta_2 = y'(x + \Delta x) \quad \text{and} \quad \tan\theta_1 = y'(x).$$

Thus (9) becomes

$$F \approx T\,[\,y'(x + \Delta x) - y'(x)]. \tag{10}$$

Second, we can obtain a different form of this same net force using Newton's second law, $F = ma$. Here the mass of string on the interval is $m = \rho\Delta x$; the centripetal acceleration of a body rotating with angular speed ω in a circle of radius r is $a = r\omega^2$. With Δx small we take $r = y$. Thus the net vertical force is also approximated by

$$F \approx -(\rho\Delta x)y\omega^2, \tag{11}$$

where the minus sign comes from the fact that the acceleration points in the direction opposite to the positive y-direction. Now by equating (10) and (11) we have

difference quotient ↓

$$T[y'(x + \Delta x) - y'(x)] = -(\rho\Delta x)y\omega^2 \qquad \text{or} \qquad T\,\frac{y'(x + \Delta x) - y'(x)}{\Delta x} + \rho\omega^2 y = 0. \tag{12}$$

For Δx close to zero the difference quotient in (12) is approximately the second derivative d^2y/dx^2. Finally we arrive at the model

$$T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0. \tag{13}$$

Since the string is anchored at its ends $x = 0$ and $x = L$, we expect that the solution $y(x)$ of equation (13) should also satisfy the boundary conditions $y(0) = 0$ and $y(L) = 0$.

Remarks

(*i*) We will pursue the subject of eigenvalues and eigenfunctions for linear second-order differential equations in greater detail in Section 12.5.
(*ii*) Eigenvalues are not always easily found as they were in Example 2; you may have to approximate roots of equations such as $\tan x = -x$ or $\cos x \cosh x = 1$. See Problems 34–38 in Exercises 3.9.
(*iii*) Boundary conditions can lead to a homogeneous algebraic system of linear equations where the unknowns are the coefficients c_i in the general solution of the DE. Such a system is always consistent, but in order to possess a nontrivial solution (in the case when the number of equations equals the number of unknowns) we must have the determinant of the coefficients equal to zero. See Problems 19 and 20 in Exercises 3.9.

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3.9 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

Deflection of a Beam

In Problems 1–5, solve equation (4) subject to the appropriate boundary conditions. The beam is of length L, and w_0 is a constant.

1. **(a)** The beam is embedded at its left end and free at its right end, and $w(x) = w_0$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 24EI$ and $L = 1$.

2. **(a)** The beam is simply supported at both ends, and $w(x) = w_0$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 24EI$ and $L = 1$.

3. **(a)** The beam is embedded at its left end and simply supported at its right end, and $w(x) = w_0$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 48EI$ and $L = 1$.

4. **(a)** The beam is embedded at its left end and simply supported at its right end, and $w(x) = w_0 \sin(\pi x/L)$, $0 < x < L$.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 2\pi^3 EI$ and $L = 1$.

(c) Use a root-finding application of a CAS (or a graphic calculator) to approximate the point in the graph in part (b) at which the maximum deflection occurs. What is the maximum deflection?

5. **(a)** The beam is simply supported at both ends, and $w(x) = w_0 x$, $0 < x <$ L.

(b) Use a graphing utility to graph the deflection curve when $w_0 = 36EI$ and $L = 1$.

(c) Use a root-finding application of a CAS (or a graphic calculator) to approximate the point in the graph in part (b) at which the maximum deflection occurs. What is the maximum deflection?

6. **(a)** Find the maximum deflection of the cantilever beam in Problem 1.

(b) How does the maximum deflection of a beam that is half as long compare with the value in part (a)?

(c) Find the maximum deflection of the simply supported beam in Problem 2.

(d) How does the maximum deflection of the simply supported beam in part (c) compare with the value of maximum deflection of the embedded beam in Example 1?

7. A cantilever beam of length L is embedded at its right end, and a horizontal tensile force of P pounds is applied to its free left end. When the origin is taken at its free end, as shown in **FIGURE 3.9.7**, the deflection $y(x)$ of the beam can be shown to satisfy the differential equation

$$EIy'' = Py - w(x)\frac{x}{2}.$$

Find the deflection of the cantilever beam if $w(x) = w_0 x$, $0 < x < L$, and $y(0) = 0$, $y'(L) = 0$.

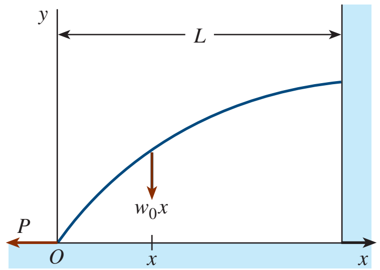

FIGURE 3.9.7 Deflection of cantilever beam in Problem 7

8. When a compressive instead of a tensile force is applied at the free end of the beam in Problem 7, the differential equation of the deflection is

$$EIy'' = -Py - w(x)\frac{x}{2}.$$

Solve this equation if $w(x) = w_0 x$, $0 < x < L$, and $y(0) = 0$, $y'(L) = 0$.

Eigenvalues and Eigenfunctions

In Problems 9–18, find the eigenvalues and eigenfunctions for the given boundary-value problem.

9. $y'' + \lambda y = 0,\ y(0) = 0,\ y(\pi) = 0$

10. $y'' + \lambda y = 0,\ y'(0) = 0,\ y(\pi/4) = 0$

11. $y'' + \lambda y = 0,\ y'(0) = 0,\ y(L) = 0$

12. $y'' + \lambda y = 0,\ y(0) = 0,\ y'(\pi/2) = 0$

13. $y'' + \lambda y = 0,\ y'(0) = 0,\ y(\pi) = 0$

14. $y'' + \lambda y = 0,\ y(-\pi) = 0,\ y(\pi) = 0$

15. $y'' + 2y' + (\lambda + 1)y = 0,\ y(0) = 0,\ y(5) = 0$

16. $y'' + (\lambda + 1)y = 0,\ y'(0) = 0,\ y'(1) = 0$

17. $x^2y'' + xy' + \lambda y = 0,\ y(1) = 0,\ y(e^{\pi}) = 0$

18. $x^2y'' + xy' + \lambda y = 0,\ y'(e^{-1}) = 0,\ y(1) = 0$

In Problems 19 and 20, find the eigenvalues and eigenfunctions for the given boundary-value problem. Consider only the case $\lambda = \alpha^4$, $\alpha > 0$.

19. $y^{(4)} - \lambda y = 0,\quad y(0) = 0,\ y''(0) = 0,\ y(1) = 0,$
$y''(1) = 0$

20. $y^{(4)} - \lambda y = 0,\quad y'(0) = 0,\ y'''(0) = 0,\ y(\pi) = 0,$
$y''(\pi) = 0$

Buckling of a Thin Column

21. Consider Figure 3.9.5. Where should physical restraints be placed on the column if we want the critical load to be P_4? Sketch the deflection curve corresponding to this load.

22. The critical loads of thin columns depend on the end conditions of the column. The value of the Euler load P_1 in Example 3 was derived under the assumption that the column was hinged at both ends. Suppose that a thin vertical homogeneous column is embedded at its base ($x = 0$) and free at its top ($x = L$) and that a constant axial load P is applied to its free end. This load either causes a small deflection δ as shown in **FIGURE 3.9.8** or does not cause such a deflection. In either case the differential equation for the deflection $y(x)$ is

$$EI\frac{d^2y}{dx^2} + Py = P\delta.$$

(a) What is the predicted deflection when $\delta = 0$?

(b) When $\delta \neq 0$, show that the Euler load for this column is one-fourth of the Euler load for the hinged column in Example 3.

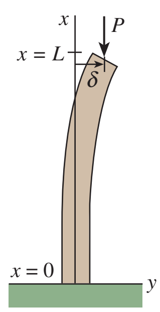

FIGURE 3.9.8 Deflection of vertical column in Problem 22

23. As was mentioned in Problem 22, the differential equation (7) that governs the deflection $y(x)$ of a thin elastic column subject to a constant compressive axial force P is valid only when the ends of the column are hinged. In general, the differential equation governing the deflection of the column is given by

$$\frac{d^2}{dx^2}\left(EI\frac{d^2y}{dx^2}\right) + P\frac{d^2y}{dx^2} = 0.$$

Assume that the column is uniform (EI is a constant) and that the ends of the column are hinged. Show that the solution of this fourth-order differential equation subject to the boundary conditions $y(0) = 0, y''(0) = 0, y(L) = 0, y''(L) = 0$ is equivalent to the analysis in Example 3.

24. Suppose that a uniform thin elastic column is hinged at the end $x = 0$ and embedded at the end $x = L$.

(a) Use the fourth-order differential equation given in Problem 23 to find the eigenvalues λ_n, the critical loads P_n, the Euler load P_1, and the deflections $y_n(x)$.

(b) Use a graphing utility to graph the first buckling mode.

Rotating String

25. Consider the boundary-value problem introduced in the construction of the mathematical model for the shape of a rotating string:

$$T\frac{d^2y}{dx^2} + \rho\omega^2 y = 0, \quad y(0) = 0, \quad y(L) = 0.$$

For constant T and ρ, define the critical speeds of angular rotation ω_n as the values of ω for which the boundary-value problem has nontrivial solutions. Find the critical speeds ω_n and the corresponding deflections $y_n(x)$.

26. When the magnitude of tension T is not constant, then a model for the deflection curve or shape $y(x)$ assumed by a rotating string is given by

$$\frac{d}{dx}\left[T(x)\frac{dy}{dx}\right] + \rho\omega^2 y = 0.$$

Suppose that $1 < x < e$ and that $T(x) = x^2$.

(a) If $y(1) = 0$, $y(e) = 0$, and $\rho\omega^2 > 0.25$, show that the critical speeds of angular rotation are

$$\omega_n = \tfrac{1}{2}\sqrt{(4n^2\pi^2 + 1)/\rho}$$

and the corresponding deflections are

$$y_n(x) = c_2 x^{-1/2}\sin(n\pi\ln x),\ n = 1, 2, 3, \ldots.$$

(b) Use a graphing utility to graph the deflection curves on the interval $[1, e]$ for $n = 1, 2, 3$. Choose $c_2 = 1$.

Miscellaneous Boundary-Value Problems

27. Temperature in a Sphere Consider two concentric spheres of radius $r = a$ and $r = b$, $a < b$. See **FIGURE 3.9.9**. The temperature $u(r)$ in the region between the spheres is determined from the boundary-value problem

$$r\frac{d^2u}{dr^2} + 2\frac{du}{dr} = 0, \quad u(a) = u_0, \quad u(b) = u_1,$$

where u_0 and u_1 are constants. Solve for $u(r)$.

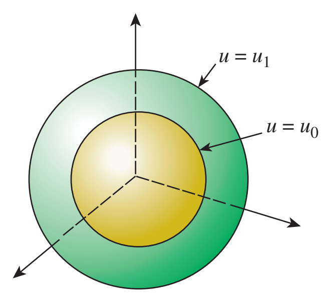

FIGURE 3.9.9 Concentric spheres in Problem 27

28. Temperature in a Ring The temperature $u(r)$ in the circular ring shown in FIGURE 3.9.10 is determined from the boundary-value problem

$$r\frac{d^2u}{dr^2} + \frac{du}{dr} = 0, \quad u(a) = u_0, \quad u(b) = u_1,$$

where u_0 and u_1 are constants. Show that

$$u(r) = \frac{u_0 \ln(r/b) - u_1 \ln(r/a)}{\ln(a/b)}.$$

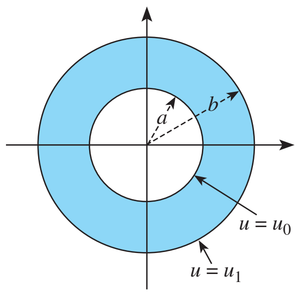

FIGURE 3.9.10 Circular ring in Problem 28

Discussion Problems

29. Simple Harmonic Motion The model $mx'' + kx = 0$ for simple harmonic motion, discussed in Section 3.8, can be related to Example 2 of this section.

Consider a free undamped spring/mass system for which the spring constant is, say, $k = 10$ lb/ft. Determine those masses m_n that can be attached to the spring so that when each mass is released at the equilibrium position at $t = 0$ with a nonzero velocity v_0, it will then pass through the equilibrium position at $t = 1$ second. How many times will each mass m_n pass through the equilibrium position in the time interval $0 < t < 1$?

30. Damped Motion Assume that the model for the spring/mass system in Problem 29 is replaced by $mx'' + 2x' + kx = 0$. In other words, the system is free but is subjected to damping numerically equal to two times the instantaneous velocity. With the same initial conditions and spring constant as in Problem 29, investigate whether a mass m can be found that will pass through the equilibrium position at $t = 1$ second.

In Problems 31 and 32, determine whether it is possible to find values y_0 and y_1 (Problem 31) and values of $L > 0$ (Problem 32) so that the given boundary-value problem has **(a)** precisely one nontrivial solution, **(b)** more than one solution, **(c)** no solution, and **(d)** the trivial solution.

31. $y'' + 16y = 0, \; y(0) = y_0, \; y(\pi/2) = y_1$

32. $y'' + 16y = 0, \; y(0) = 1, \; y(L) = 1$

33. Consider the boundary-value problem

$$y'' + \lambda y = 0, \; y(-\pi) = y(\pi), \; y'(-\pi) = y'(\pi).$$

(a) The type of boundary conditions specified are called **periodic boundary conditions**. Give a geometric interpretation of these conditions.

(b) Find the eigenvalues and eigenfunctions of the problem.

(c) Use a graphing utility to graph some of the eigenfunctions. Verify your geometric interpretation of the boundary conditions given in part (a).

34. Show that the eigenvalues and eigenfunctions of the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(1) + y'(1) = 0$$

are $\lambda_n = \alpha_n^2$ and $y_n = \sin \alpha_n x$, respectively, where α_n, $n = 1, 2, 3, \ldots$ are the consecutive positive roots of the equation $\tan \alpha = -\alpha$.

Computer Lab Assignments

35. Use a CAS to plot graphs to convince yourself that the equation $\tan \alpha = -\alpha$ in Problem 34 has an infinite number of roots. Explain why the negative roots of the equation can be ignored. Explain why $\lambda = 0$ is not an eigenvalue even though $\alpha = 0$ is an obvious solution of the equation $\tan \alpha = -\alpha$.

36. Use a root-finding application of a CAS to approximate the first four eigenvalues λ_1, λ_2, λ_3, and λ_4 for the BVP in Problem 34.

In Problems 37 and 38, find the eigenvalues and eigenfunctions of the given boundary-value problem. Use a CAS to approximate the first four eigenvalues λ_1, λ_2, λ_3, and λ_4.

37. $y'' + \lambda y = 0, \; y(0) = 0, \; y(1) - \frac{1}{2}y'(1) = 0$

38. $y^{(4)} - \lambda y = 0, \; y(0) = 0, \; y'(0) = 0, \; y(1) = 0, \; y'(1) = 0$
[*Hint*: Consider only $\lambda = \alpha^4$, $\alpha > 0$.]

3.10 Green's Functions

Introduction We have seen in Section 3.8 that the linear second-order differential equation

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

plays an important role in applications. In the mathematical analysis of physical systems it is often desirable to express the response or output $y(x)$ of (1) subject to either initial conditions or boundary conditions directly in terms of the forcing function or input $g(x)$. In this manner the response of the system can quickly be analyzed for different forcing functions.

To see how this is done we start by examining solutions of initial-value problems in which the DE (1) has been put into the standard form

$$y'' + P(x)y' + Q(x)y = f(x) \tag{2}$$

by dividing the equation by the lead coefficient $a_2(x)$. We also assume throughout this section that the coefficient functions $P(x)$, $Q(x)$, and $f(x)$ are continuous on some common interval I.

3.10.1 Initial-Value Problems

Three Initial-Value Problems We will see as the discussion unfolds that the solution of the second-order initial-value problem

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = y_0, \quad y'(x_0) = y_1 \tag{3}$$

can be expressed as the superposition of two solutions: the solution y_h of the associated homogeneous DE with nonhomogeneous initial conditions

$$y'' + P(x)y' + Q(x)y = 0, \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \tag{4}$$

Here at least one of the numbers y_0 or y_1 is assumed to be nonzero. If both y_0 and y_1 are 0, then the solution of the IVP is $y = 0$.

and the solution y_p of the nonhomogeneous DE with homogeneous (that is, zero) initial conditions

$$y'' + P(x)y' + Q(x)y = f(x), \quad y(x_0) = 0, \quad y'(x_0) = 0. \tag{5}$$

As we have seen in the preceding sections of this chapter, in the case where P and Q are constants the solution of the IVP (4) presents no difficulties: We use the methods of Sections 3.3 to find the general solution of the homogeneous DE and then use the given initial conditions to determine the two constants in that solution. So we will focus on the solution of the IVP (5). Because of the zero initial conditions, the solution of (5) could describe a physical system that is initially at rest and so is sometimes called a **rest solution**.

Green's Function If $y_1(x)$ and $y_2(x)$ form a fundamental set of solutions on the interval I of the associated homogeneous form of (2), then a particular solution of the nonhomogeneous equation (2) on the interval I can be found by variation of parameters. Recall from (3) of Section 3.5, the form of this solution is

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x). \tag{6}$$

The variable coefficients $u_1(x)$ and $u_2(x)$ in (6) are defined by (5) of Section 3.5:

$$u_1'(x) = -\frac{y_2(x)f(x)}{W}, \quad u_2'(x) = \frac{y_1(x)f(x)}{W}. \tag{7}$$

The linear independence of $y_1(x)$ and $y_2(x)$ on the interval I guarantees that the Wronskian $W = W(y_1(x), y_2(x)) \neq 0$ for all x in I. If x and x_0 are numbers in I, then integrating the derivatives in (7) on the interval $[x_0, x]$ and substituting the results in (6) give

$$\begin{aligned} y_p(x) &= y_1(x)\int_{x_0}^{x}\frac{-y_2(t)f(t)}{W(t)}dt + y_2(x)\int_{x_0}^{x}\frac{y_1(t)f(t)}{W(t)}dt \\ &= \int_{x_0}^{x}\frac{-y_1(x)y_2(t)}{W(t)}f(t)\,dt + \int_{x_0}^{x}\frac{y_1(t)y_2(x)}{W(t)}f(t)\,dt, \end{aligned} \tag{8}$$

Because $y_1(x)$ and $y_2(x)$ are constant with respect to the integration on t, we can move these functions inside the definite integrals.

where $$W(t) = W(y_1(t), y_2(t)) = \begin{vmatrix} y_1(t) & y_2(t) \\ y_1'(t) & y_2'(t) \end{vmatrix}.$$

From the properties of the definite integral, the two integrals in the second line of (8) can be rewritten as a single integral

$$y_p(x) = \int_{x_0}^{x} G(x, t)f(t)dt. \tag{9}$$

The function $G(x, t)$ in (9),

$$G(x, t) = \frac{y_1(t)y_2(x) - y_1(x)y_2(t)}{W(t)}, \tag{10}$$

is called the **Green's function** for the differential equation (2).

Important. Read this paragraph a second time.

Observe that a Green's function (10) depends only on the fundamental solutions, $y_1(x)$ and $y_2(x)$ of the associated homogeneous differential equation for (2) and *not* on the forcing function $f(x)$. Therefore all linear second-order differential equations (2) with the same left-hand side but different forcing functions have the same Green's function. So an alternative title for (10) is the **Green's function for the second-order differential operator** $L = D^2 + P(x)D + Q(x)$.

EXAMPLE 1 Particular Solution

Use (9) and (10) to find a particular solution of $y'' - y = f(x)$.

Solution The solutions of the associated homogeneous equation $y'' - y = 0$ are $y_1 = e^x$, $y_2 = e^{-x}$, and $W(y_1(t), y_2(t)) = -2$. It follows from (10) that the Green's function is

$$G(x, t) = \frac{e^t e^{-x} - e^x e^{-t}}{-2} = \frac{e^{x-t} - e^{-(x-t)}}{2} = \sinh(x - t). \tag{11}$$

Thus from (9), a particular solution of the DE is

$$y_p(x) = \int_{x_0}^{x} \sinh(x - t)f(t)dt. \tag{12}$$ ≡

EXAMPLE 2 General Solutions

Find the general solution of the following nonhomogeneous differential equations.

(a) $y'' - y = 1/x$ (b) $y'' - y = e^{2x}$

Solution From Example 1, both DEs possess the same complementary function $y_c = c_1e^{-x} + c_2e^x$. Moreover, as pointed out in the paragraph preceding Example 1, the Green's function for both differential equations is (11).

(a) With the identifications $f(x) = 1/x$ and $f(t) = 1/t$ we see from (12) that a particular solution of $y'' - y = 1/x$ is $y_p(x) = \int_{x_0}^{x} \frac{\sinh(x - t)}{t} dt$. Thus the general solution $y = y_c + y_p$ of the given DE on any interval $[x_0, x]$ not containing the origin is

$$y = c_1e^x + c_2e^{-x} + \int_{x_0}^{x} \frac{\sinh(x - t)}{t} dt. \tag{13}$$

You should compare this solution with that found in Example 3 of Section 3.5.

(b) With $f(x) = e^{2x}$ in (12), a particular solution of $y'' - y = e^{2x}$ is $y_p(x) = \int_{x_0}^{x} \sinh(x - t)e^{2t}dt$. The general solution $y = y_c + y_p$ is then

$$y = c_1e^x + c_2e^{-x} + \int_{x_0}^{x} \sinh(x - t)e^{2t}dt. \tag{14}$$ ≡

Now consider the special initial-value problem (5) with homogeneous initial conditions. One way of solving the problem when $f(x) \neq 0$ has already been illustrated in Sections 3.4 and 3.5; that is, apply the initial conditions $y(x_0) = 0$, $y'(x_0) = 0$ to the general solution of the nonhomogeneous DE. But there is no actual need to do this because we already have solution of the IVP at hand; it is the function defined in (9).

Theorem 3.10.1 Solution of the IVP in (5)

The function $y_p(x)$ defined in (9) in the solution of the initial-value problem (5).

PROOF

By construction we know that $y_p(x)$ in (9) satisfies the nonhomogeneous DE. Next, because a definite integral has the property $\int_a^a = 0$ we have

$$y_p(x_0) = \int_{x_0}^{x_0} G(x_0, t) f(t)\, dt = 0.$$

Finally, to show that $y_p'(x_0) = 0$ we utilize the Leibniz formula* for the derivative of an integral:

$$y_p'(x) = \overbrace{G(x, x)}^{0 \text{ from } (10)} f(x) + \int_{x_0}^{x} \frac{y_1(t)y_2'(x) - y_1'(x)y_2(t)}{W(t)} f(t)\, dt.$$

Hence,

$$y_p'(x_0) = \int_{x_0}^{x_0} \frac{y_1(t)y_2'(x_0) - y_1'(x_0)y_2(t)}{W(t)} f(t)\, dt = 0.$$ ≡

EXAMPLE 3 Example 2 Revisited

Solve the initial-value problems

(a) $y'' - y = 1/x$, $y(1) = 0$, $y'(1) = 0$ (b) $y'' - y = e^{2x}$, $y(0) = 0$, $y'(0) = 0$.

Solution (a) With $x_0 = 1$ and $f(t) = 1/t$, it follows from (13) of Example 2 and Theorem 3.10.1 that the solution of the initial-value problem is

$$y_p(x) = \int_1^x \frac{\sinh(x - t)}{t} dt,$$

where $[1, x]$, $x > 0$.

(b) Identifying $x_0 = 0$ and $f(t) = e^{2t}$, we see from (14) that the solution of the IVP is

$$y_p(x) = \int_0^x \sinh(x - t) e^{2t} dt. \tag{15}$$ ≡

In Part (b) of Example 3, we can carry out the integration in (15), but bear in mind that x is held constant throughout the integration with respect to t:

$$\begin{aligned} y_p(x) &= \int_0^x \sinh(x - t) e^{2t} dt = \int_0^x \frac{e^{x-t} - e^{-(x-t)}}{2} e^{2t} dt \\ &= \frac{1}{2} e^x \int_0^x e^t dt - \frac{1}{2} e^{-x} \int_0^x e^{3t} dt \\ &= \frac{1}{3} e^{2x} - \frac{1}{2} e^x + \frac{1}{6} e^{-x}. \end{aligned}$$

*This formula, usually discussed in advanced calculus, is given by

$$\frac{d}{dx} \int_{u(x)}^{v(x)} F(x, t)\, dt = F(x, v(x)) v'(x) - F(x, u(x)) u'(x) + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x} F(x, t)\, dt.$$

EXAMPLE 4 Another IVP

Solve the initial-value problem

$$y'' + 4y = x, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution We begin by constructing the Green's function for the given differential equation.

The two linearly independent solutions of $y'' + 4y = 0$ are $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$. From (10), with $W(\cos 2t, \sin 2t) = 2$, we find

$$G(x, t) = \frac{\cos 2t \sin 2x - \cos 2x \sin 2t}{2} = \frac{1}{2}\sin 2(x - t).$$

With the identification $x_0 = 0$, a solution of the given initial-value problem is

$$y_p(x) = \frac{1}{2}\int_0^x t \sin 2(x - t)dt.$$

If we wish to evaluate the integral, we first write

Here we have used the trigonometric identity $\sin(2x - 2t) = \sin 2x \cos 2t - \cos 2x \sin 2t$.

$$y_p(x) = \frac{1}{2}\sin 2x\int_0^x t \cos 2t\, dt - \frac{1}{2}\cos 2x\int_0^x t \sin 2t\, dt$$

and then use integration by parts:

$$y_p(x) = \frac{1}{2}\sin 2x\left[\frac{1}{2}t \sin 2t + \frac{1}{4}\cos 2t\right]_0^x - \frac{1}{2}\cos 2x\left[-\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t\right]_0^x$$

or

$$y_p(x) = \frac{1}{4}x - \frac{1}{8}\sin 2x.$$ ≡

■ **Initial-Value Problems—Continued** We can now make use of Theorem 3.10.1 to find the solution of the initial-value problem posed in (3).

Theorem 3.10.2 Solution of the IVP (3)

If y_h is the solution of the initial-value problem (4) and y_p is the solution (9) of the initial-value problem (5) on the interval I, then

$$y = y_h + y_p \tag{16}$$

is the solution of the initial-value problem (3).

PROOF

Because y_h is a linear combination of the fundamental solutions, it follows from (10) of Section 3.1 that $y = y_h + y_p$ is a solution of the nonhomogeneous DE. Moreover, since y_h satisfies the initial conditions in (4) and y_p satisfies the initial conditions in (5), we have

$$y(x_0) = y_h(x_0) + y_p(x_0) = y_0 + 0 = y_0$$
$$y'(x_0) = y_h'(x_0) + y_p'(x_0) = y_1 + 0 = y_1.$$ ≡

Keeping in mind the absence of a forcing function in (4) and the presence of such a term in (5), we see from (16) that the response $y(x)$ of a physical system described by the initial-value problem (3) can be separated into two different responses:

$$y(x) = \underbrace{y_h(x)}_{\substack{\text{response of system}\\ \text{due to initial conditions}\\ y(x_0) = y_0,\quad y'(x_0) = y_1}} + \underbrace{y_p(x)}_{\substack{\text{response of system}\\ \text{due to the forcing}\\ \text{function } f}} \tag{17}$$

In different symbols, the following initial-value problem represents the pure resonance situation for a vibrating spring/mass system. See page 152.

EXAMPLE 5 Using Theorem 3.10.2

Solve the initial-value problem

$$y'' + 4y = \sin 2x, \quad y(0) = 1, \quad y'(0) = -2.$$

Solution We solve two initial-value problems.

First, we solve $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = -2$. By applying the initial conditions to the general solution $y(x) = c_1 \cos 2x + c_2 \sin 2x$ of the homogeneous DE, we find that $c_1 = 1$ and $c_2 = -1$. Therefore, $y_h(x) = \cos 2x - \sin 2x$.

Next we solve $y'' + 4y = \sin 2x$, $y(0) = 0$, $y'(0) = 0$. Since the left-hand side of the differential equation is the same as the DE in Example 4, the Green's function is the same; namely, $G(x, t) = \frac{1}{2}\sin 2(x - t)$. With $f(t) = \sin 2t$ we see from (9) that the solution of this second problem is $y_p(x) = \frac{1}{2}\int_0^x \sin 2(x - t) \sin 2t\, dt$.

Finally, in view of (16) in Theorem 3.10.2, the solution of the original IVP is

$$y(x) = y_h(x) + y_p(x) = \cos 2x - \sin 2x + \frac{1}{2}\int_0^x \sin 2(x - t) \sin 2t\, dt. \qquad (18)$$

≡

If desired, we can integrate the definite integral in (18) by using the trigonometric identity

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

with $A = 2(x - t)$ and $B = 2t$:

$$\begin{aligned} y_p(x) &= \frac{1}{2}\int_0^x \sin 2(x - t)\sin 2t\, dt \\ &= \frac{1}{4}\int_0^x [\cos(2x - 4t) - \cos 2x]dt \\ &= \frac{1}{4}\left[-\frac{1}{4}\sin(2x - 4t) - t\cos 2x\right]_0^x \\ &= \frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x. \end{aligned} \qquad (19)$$

Hence, the solution (18) can be rewritten as

$$y(x) = y_h(x) + y_p(x) = \cos 2x - \sin 2x + \left(\frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x\right),$$

or

$$y(x) = \cos 2x - \frac{7}{8}\sin 2x - \frac{1}{4}x\cos 2x. \qquad (20)$$

Note that the physical significance indicated in (17) is lost in (20) after combining like terms in the two parts of the solution $y(x) = y_h(x) + y_p(x)$.

The beauty of the solution given in (18) is that we can immediately write down the response of a system if the initial conditions remain the same but the forcing function is changed. For example, if the problem in Example 5 is changed to

$$y'' + 4y = x, \quad y(0) = 1, \quad y'(0) = -2,$$

we simply replace sin $2t$ in the integral in (18) by t and the solution is then

$$y(x) = y_h(x) + y_p(x)$$

$$= \cos 2x - \sin 2x + \frac{1}{2}\int_0^x t \sin 2(x - t)dt \quad \leftarrow \text{see Example 4}$$

$$= \frac{1}{4}x + \cos 2x - \frac{9}{8}\sin 2x.$$

Because the forcing function f is isolated in the particular solution $y_p(x) = \int_{x_0}^{x} G(x, t) f(t)\, dt$, the solution in (16) is useful when f is piecewise defined. The next example illustrates this idea.

EXAMPLE 6 An Initial-Value Problem

Solve the initial-value problem

$$y'' + 4y = f(x), \quad y(0) = 1, y'(0) = -2,$$

when the forcing function f is piecewise defined:

$$f(x) = \begin{cases} 0, & x < 0 \\ \sin 2x, & 0 \le x \le 2\pi \\ 0, & x > 2\pi. \end{cases}$$

Solution From (18), with sin $2t$ replaced by $f(t)$, we can write

$$y(x) = \cos 2x - \sin 2x + \frac{1}{2}\int_0^x \sin 2(x - t) f(t)dt.$$

Because f is defined in three pieces, we consider three cases in the evaluation of the definite integral. For $x < 0$,

$$y_p(x) = \frac{1}{2}\int_0^x \sin 2(x - t) 0\, dt = 0,$$

for $0 \le x \le 2\pi$,

$$y_p(x) = \frac{1}{2}\int_0^x \sin 2(x - t) \sin 2t\, dt \quad \leftarrow \text{using the integration in (19)}$$

$$= \frac{1}{8}\sin 2x - \frac{1}{4}x \cos 2x,$$

and finally for $x > 2\pi$, we can use the integration following Example 5:

$$y_p(x) = \frac{1}{2}\int_0^{2\pi} \sin 2(x - t) \sin 2t\, dt + \frac{1}{2}\int_{2\pi}^{x} \sin 2(x - t)\, 0\, dt$$

$$= \frac{1}{2}\int_0^{2\pi} \sin 2(x - t) \sin 2t\, dt$$

$$= \frac{1}{4}\left[-\frac{1}{4}\sin(2x - 4t) - t \cos 2x\right]_0^{2\pi} \quad \leftarrow \text{using the integration in (19)}$$

$$= -\frac{1}{16}\sin(2x - 8\pi) - \frac{1}{2}\pi \cos 2x + \frac{1}{16}\sin 2x$$

$$= -\frac{1}{2}\pi \cos 2x.$$

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Hence $y_p(x)$ is

$$y_p(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}\sin 2x - \frac{1}{4}x\cos 2x, & 0 \le x \le 2\pi \\ -\frac{1}{2}\pi\cos 2\text{x}, & \text{x} > 2\pi \end{cases}$$

and so

$$y(x) = y_h(x) + y_p(x) = \cos 2x - \sin 2x + y_p(x).$$

Putting all the pieces together we get

$$y(x) = \begin{cases} \cos 2x - \sin 2x, & x < 0 \\ (1 - \frac{1}{4}x)\cos 2x - \frac{7}{8}\sin 2x, & 0 \le x \le 2\pi \\ (1 - \frac{1}{2}\pi)\cos 2x - \sin 2x, & x > 2\pi. \end{cases}$$

The graph $y(x)$ is given in **FIGURE 3.10.1**. ≡

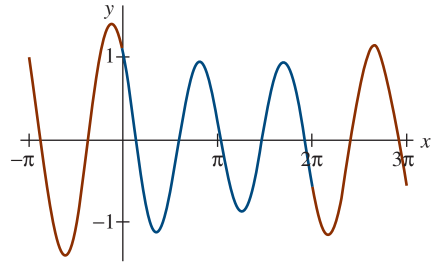


FIGURE 3.10.1 Graph of $y(x)$ in Example 6

We next examine how a boundary-value problem (BVP) can be solved using a different kind of Green's function.

3.10.2 Boundary-Value Problems

In contrast to a second-order IVP in which $y(x)$ and $y'(x)$ are specified at the same point, a BVP for a second-order DE involves conditions on $y(x)$ and $y'(x)$ that are specified at two different points $x = a$ and $x = b$. Conditions such as

$$y(a) = 0, y(b) = 0; \quad y(a) = 0, y'(b) = 0; \quad y'(a) = 0, \quad y'(b) = 0$$

are just special cases of the more general homogeneous boundary conditions

$$A_1 y(a) + B_1 y'(a) = 0 \tag{21}$$

and

$$A_2 y(b) + B_2 y'(b) = 0, \tag{22}$$

where A_1, A_2, B_1, and B_2 are constants. Specifically, our goal is to find an integral solution $y_p(x)$ that is analogous to (9) for nonhomogeneous boundary-value problems of the form

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= f(x), \\ A_1 y(a) + B_1 y'(a) &= 0, \\ A_2 y(b) + B_2 y'(b) &= 0. \end{aligned} \tag{23}$$

In addition to the usual assumptions that $P(x)$, $Q(x)$, and $f(x)$ are continuous on $[a, b]$, we assume that the homogeneous problem

$$\begin{aligned} y'' + P(x)y' + Q(x)y &= 0, \\ A_1 y(a) + B_1 y'(a) &= 0 \\ A_2 y(b) + B_2 y'(b) &= 0, \end{aligned}$$

possesses only the trivial solution $y = 0$. This latter assumption is sufficient to guarantee that a unique solution of (23) exists and is given by an integral $y_p(x) = \int_a^b G(x, t)f(t)\,dt$, where $G(x, t)$ is a Green's function.

The starting point in the construction of $G(x, t)$ is again the variation of parameters formulas (6) and (7).

Another Green's Function Suppose $y_1(x)$ and $y_2(x)$ are linearly independent solutions on $[a, b]$ of the associated homogeneous form of the DE in (23) and that x is a number in the interval $[a, b]$. Unlike the construction of (8) where we started by integrating the derivatives in (7) over the same interval, we now integrate the first equation in (7) on $[b, x]$ and the second equation in (7) on $[a, x]$:

$$u_1(x) = -\int_b^x \frac{y_2(t)f(t)}{W(t)}dt \quad \text{and} \quad u_2(x) = \int_a^x \frac{y_1(t)f(t)}{W(t)}dt. \tag{24}$$

The reason for integrating $u_1'(x)$ and $u_2'(x)$ over different intervals will become clear shortly. From (24), a particular solution $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ of the DE is

$$y_p(x) = y_1(x)\overbrace{\int_x^b}^{\substack{\text{here we used the minus}\\ \text{sign in (24) to reverse}\\ \text{the limits of integration}}} \frac{y_2(t)f(t)}{W(t)}dt + y_2(x)\int_a^x \frac{y_1(t)f(t)}{W(t)}dt$$

or

$$y_p(x) = \int_a^x \frac{y_2(x)y_1(t)}{W(t)}f(t)\,dt + \int_x^b \frac{y_1(x)y_2(t)}{W(t)}f(t)\,dt. \tag{25}$$

The right-hand side of (25) can be written compactly as a single integral

$$y_p(x) = \int_a^b G(x, t)f(t)\,dt, \tag{26}$$

where the function $G(x, t)$ is

$$G(x, t) = \begin{cases} \dfrac{y_1(t)y_2(x)}{W(t)}, & a \le t \le x \\[2ex] \dfrac{y_1(x)y_2(t)}{W(t)}, & x \le t \le b. \end{cases} \tag{27}$$

The piecewise-defined function (27) is called a **Green's function** for the boundary-value problem (23). It can be proved that $G(x, t)$ is a continuous function of x on the interval $[a, b]$.

Now if the solutions $y_1(x)$ and $y_2(x)$ used in the construction of (27) are chosen in such a manner that at $x = a$, $y_1(x)$ satisfies $A_1y_1(a) + B_1y_1'(a) = 0$, and at $x = b$, $y_2(x)$ satisfies $A_2y_2(b) + B_2y_2'(b) = 0$, then, wondrously, $y_p(x)$ defined in (26) satisfies both homogeneous boundary conditions in (23).

To see this we will need

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x) \tag{28}$$

The last line in (29) results from the fact that $y_1(x)u_1'(x) + y_2(x)u_2'(x) = 0$. See the discussion in Section 3.5 following (4).

and

$$\begin{aligned} y_p'(x) &= u_1(x)y_1'(x) + y_1(x)u_1'(x) + u_2(x)y_2'(x) + y_2(x)u_2'(x) \\ &= u_1(x)y_1'(x) + u_2(x)y_2'(x). \end{aligned} \tag{29}$$

Before proceeding, observe in (24) that $u_1(b) = 0$ and $u_2(a) = 0$. In view of the second of these two properties we can show that $y_p(x)$ satisfies (21) whenever $y_1(x)$ satisfies the same boundary condition. From (28) and (29) we have

$$\begin{aligned} A_1y_p(a) + B_1y_p'(a) &= A_1[u_1(a)y_1(a) + \overbrace{u_2(a)}^{0}y_2(a)] + B_1[u_1(a)y_1'(a) + \overbrace{u_2(a)}^{0}y_2'(a)] \\ &= u_1(a)\underbrace{[A_1y_1(a) + B_1y_1'(a)]}_{0 \text{ from (21)}} = 0. \end{aligned}$$

Likewise, $u_1(b) = 0$ implies that whenever $y_2(x)$ satisfies (22) so does $y_p(x)$:

$$A_2 y_p(b) + B_2 y_p'(b) = A_2[\overbrace{u_1(b)}^{0} y_1(b) + u_2(b)y_2(b)] + B_2[\overbrace{u_1(b)}^{0} y_1'(b) + u_2(b)y_2'(b)]$$

$$= u_2(b)\underbrace{[A_2 y_2(b) + B_2 y_2'(b)]}_{0 \text{ from } (22)} = 0.$$

The next theorem summarizes these results.

Theorem 3.10.3 Solution of a BVP

Let $y_1(x)$ and $y_2(x)$ be linearly independent solutions of

$$y'' + P(x)y' + Q(x)y = 0$$

on $[a, b]$, and suppose $y_1(x)$ and $y_2(x)$ satisfy (21) and (22), respectively. Then the function $y_p(x)$ defined in (26) is a solution of the boundary-value problem (23).

EXAMPLE 7 Using Theorem 3.10.3

Solve the boundary-value problem

$$y'' + 4y = 3, \quad y'(0) = 0, \quad y(\pi/2) = 0.$$

The boundary condition $y'(0) = 0$ is a special case of (21) with $a = 0$, $A_1 = 0$, and $B_1 = 1$. The boundary condition $y(\pi/2) = 0$ is a special case of (22) with $b = \pi/2$, $A_2 = 1$, and $B_2 = 0$.

Solution The solutions of the associated homogeneous equation $y'' + 4y = 0$ are $y_1(x) = \cos 2x$ and $y_2(x) = \sin 2x$ and $y_1(x)$ satisfies $y'(0) = 0$, whereas $y_2(x)$ satisfies $y(\pi/2) = 0$. The Wronskian is $W(y_1, y_2) = 2$, and so from (27) we see that the Green's function for the boundary-value problem is

$$G(x, t) = \begin{cases} \frac{1}{2}\cos 2t \sin 2x, & 0 \le t \le x \\ \frac{1}{2}\cos 2x \sin 2t, & x \le t \le \pi/2. \end{cases}$$

It follows from Theorem 3.10.3 that a solution of the BVP is (26) with the identifications $a = 0$, $b = \pi/2$, and $f(t) = 3$:

$$y_p(x) = 3\int_0^{\pi/2} G(x, t)\, dt$$

$$= 3 \cdot \frac{1}{2}\sin 2x \int_0^x \cos 2t\, dt + 3 \cdot \frac{1}{2}\cos 2x \int_x^{\pi/2} \sin 2t\, dt,$$

or after evaluating the definite integrals, $y_p(x) = \frac{3}{4} + \frac{3}{4}\cos 2x$. ≡

Don't infer from the preceding example that the demand that $y_1(x)$ satisfy (21) and $y_2(x)$ satisfy (22) uniquely determines these functions. As we see in the last example, there is a certain arbitrariness in the selection of these functions.

EXAMPLE 8 A Boundary-Value Problem

Solve the boundary-value problem

$$x^2 y'' - 3xy' + 3y = 24x^5, \quad y(1) = 0, \ y(2) = 0.$$

Solution The differential equation is recognized as a Cauchy–Euler DE.

From the auxiliary equation $m(m-1)-3m+3=(m-1)(m-3)=0$ the general solution of the associated homogeneous equation is $y=c_1x+c_2x^3$. Applying $y(1)=0$ to this solution implies $c_1+c_2=0$ or $c_1=-c_2$. By choosing $c_2=-1$ we get $c_1=1$ and $y_1=x-x^3$. On the other hand, $y(2)=0$ applied to the general solution shows $2c_1+8c_2=0$ or $c_1=-4c_2$. The choice $c_2=-1$ now gives $c_1=4$ and so $y_2(x)=4x-x^3$. The Wronskian of these two functions is

$$W(y_1(x), y_2(x)) = \begin{vmatrix} x-x^3 & 4x-x^3 \\ 1-3x^2 & 4-3x^2 \end{vmatrix} = 6x^3.$$

Hence the Green's function for the boundary-value problem is

$$G(x,t) = \begin{cases} \dfrac{(t-t^3)(4x-x^3)}{6t^3}, & 0 \le t \le x \\ \dfrac{(x-x^3)(4t-t^3)}{6t^3}, & x \le t \le 2. \end{cases}$$

In order to identify the correct forcing function f we must write the DE in standard form:

$$y'' - \frac{3}{x}y' + \frac{3}{x^2}y = 24x^3.$$

From this equation we see that $f(t)=24t^3$ and so (26) becomes

$$\begin{aligned} y_p(x) &= 24\int_1^2 G(x,t)t^3\,dt \\ &= 4(4x-x^3)\int_1^x (t-t^3)\,dt + 4(x-x^3)\int_x^2 (4t-t^3)\,dt. \end{aligned}$$

Straightforward definite integration and algebraic simplification yields the solution $y_p(x)=12x-15x^3+3x^5$. ≡

3.10 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

3.10.1 Initial-Value Problems

In Problems 1–6, proceed as in Example 1 to find a particular solution $y_p(x)$ of the given differential equation in the integral form (9).

1. $y''-16y=f(x)$
2. $y''+3y'-10y=f(x)$
3. $y''+2y'+y=f(x)$
4. $4y''-4y'+y=f(x)$
5. $y''+9y=f(x)$
6. $y''-2y'+2y=f(x)$

In Problems 7–12, proceed as in Example 2 to find the general solution of the given differential equation. Use the results obtained in Problems 1–6. Do not evaluate the integral that defines $y_p(x)$.

7. $y''-16y=xe^{-2x}$
8. $y''+3y'-10y=x^2$
9. $y''+2y'+y=e^{-x}$
10. $4y''-4y'+y=\arctan x$
11. $y''+9y=x+\sin x$
12. $y''-2y'+2y=\cos^2 x$

In Problems 13–18, proceed as in Example 3 to find the solution of the given initial-value problem. Evaluate the integral that defines $y_p(x)$.

13. $y''-4y=e^{2x}$, $y(0)=0, y'(0)=0$
14. $y''-y'=1$, $y(0)=0, y'(0)=0$
15. $y''-10y'+25y=e^{5x}$, $y(0)=0, y'(0)=0$
16. $y''+6y'+9y=x$, $y(0)=0, y'(0)=0$
17. $y''+y=\csc x\cot x$, $y(\pi/2)=0, y'(\pi/2)=0$
18. $y''+y=\sec^2 x$, $y(\pi)=0, y'(\pi)=0$

In Problems 19–30, proceed as in Example 5 to find a solution of the given initial-value problem.

19. $y''-4y=e^{2x}$, $y(0)=1, y'(0)=-4$
20. $y''-y'=1$, $y(0)=10, y'(0)=1$
21. $y''-10y'+25y=e^{5x}$, $y(0)=-1, y'(0)=1$
22. $y''+6y'+9y=x$, $y(0)=1, y'(0)=-3$

23. $y'' + y = \csc x \cot x, \quad y(\pi/2) = -\pi/2, y'(\pi/2) = -1$

24. $y'' + y = \sec^2 x, \quad y(\pi) = \frac{1}{2}, y'(\pi) = -1$

25. $y'' + 3y' + 2y = \sin e^x, \quad y(0) = -1, y'(0) = 0$

26. $y'' + 3y' + 2y = \dfrac{1}{1 + e^x}, \quad y(0) = 0, y'(0) = 1$

27. $x^2y'' - 2xy' + 2y = x, \quad y(1) = 2, y'(1) = -1$

28. $x^2y'' - 2xy' + 2y = x \ln x, \quad y(1) = 1, y'(1) = 0$

29. $x^2y'' - 6y = \ln x, \quad y(1) = 1, y'(1) = 3$

30. $x^2y'' - xy' + y = x^2, \quad y(1) = 4, y'(1) = 3$

In Problems 31–34, proceed as in Example 6 to find a solution of the initial-value problem with the given piecewise-defined forcing function.

31. $y'' - y = f(x), \quad y(0) = 8, y'(0) = 2,$

where $\quad f(x) = \begin{cases} -1, & x < 0 \\ \;\;1, & x \geq 0 \end{cases}$

32. $y'' - y = f(x), \quad y(0) = 3, y'(0) = 2,$

where $\quad f(x) = \begin{cases} 0, & x < 0 \\ x, & x \geq 0 \end{cases}$

33. $y'' + y = f(x), \quad y(0) = 1, y'(0) = -1,$

where $\quad f(x) = \begin{cases} 0, & x < 0 \\ 10, & 0 \leq x \leq 3\pi \\ 0, & x > 3\pi \end{cases}$

34. $y'' + y = f(x), \quad y(0) = 0, y'(0) = 1,$

where $\quad f(x) = \begin{cases} 0, & x < 0 \\ \cos x, & 0 \leq x \leq 4\pi \\ 0, & x > 4\pi \end{cases}$

3.10.2 Boundary-Value Problems

In Problems 35 and 36, (**a**) use (25) and (26) to find a solution of the boundary-value problem. (**b**) Verify that function $y_p(x)$ satisfies the differential equations and both boundary conditions.

35. $y'' = f(x), \quad y(0) = 0, y(1) = 0$

36. $y'' = f(x), \quad y(0) = 0, y(1) + y'(1) = 0$

37. In Problem 35 find a solution of the BVP when $f(x) = 1$.

38. In Problem 36 find a solution of the BVP when $f(x) = x$.

In Problems 39–44, proceed as in Examples 7 and 8 to find a solution of the given boundary-value problem.

39. $y'' + y = 1, \quad y(0) = 0, y(1) = 0$

40. $y'' + 9y = 1, \quad y(0) = 0, y'(\pi) = 0$

41. $y'' - 2y' + 2y = e^x, \quad y(0) = 0, y(\pi/2) = 0$

42. $y'' - y' = e^{2x}, \quad y(0) = 0, y(1) = 0$

43. $x^2y'' + xy' = 1, \quad y(e^{-1}) = 0, y(1) = 0$

44. $x^2y'' - 4xy' + 6y = x^4, \quad y(1) - y'(1) = 0, y(3) = 0$

Discussion Problems

45. Suppose the solution of the boundary-value problem

$$y'' + Py' + Qy = f(x), \quad y(a) = 0, y(b) = 0,$$

$a < b$, is given by $y_p(x) = \int_a^b G(x, t) f(t)\,dt$ where $y_1(x)$ and $y_2(x)$ are solutions of the associated homogeneous differential equation chosen in the construction of $G(x, t)$ so that $y_1(a) = 0$ and $y_2(b) = 0$. Prove that the solution of the boundary-value problem with nonhomogeneous DE and boundary conditions,

$$y'' + Py' + Qy = f(x), \quad y(a) = A, y(b) = B$$

is given by

$$y(x) = y_p(x) + \frac{B}{y_1(b)} y_1(x) + \frac{A}{y_2(a)} y_2(x).$$

[*Hint*: In your proof, you will have to show that $y_1(b) \neq 0$ and $y_2(a) \neq 0$. Reread the assumptions following (22).]

46. Use the result in Problem 45 to solve

$$y'' + y = 1, \quad y(0) = 5, y(1) = -10.$$

3.11 Nonlinear Models

Introduction In this section we examine some nonlinear higher-order mathematical models. We are able to solve some of these models using the substitution method introduced on page 139. In some cases where the model cannot be solved, we show how a nonlinear DE can be replaced by a linear DE through a process called *linearization*.

Nonlinear Springs The mathematical model in (1) of Section 3.8 has the form

$$m\frac{d^2x}{dt^2} + F(x) = 0, \tag{1}$$

where $F(x) = kx$. Since x denotes the displacement of the mass from its equilibrium position, $F(x) = kx$ is Hooke's law; that is, the force exerted by the spring that tends to restore the mass to the equilibrium position. A spring acting under a linear restoring force $F(x) = kx$ is naturally referred to as a **linear spring**. But springs are seldom perfectly linear. Depending on how it is constructed and the material used, a spring can range from "mushy" or soft to "stiff" or hard, so that its restorative force may vary from something below to something above that given by the linear law. In the case of free motion, if we assume that a nonaging spring possesses some nonlinear characteristics, then it might be reasonable to assume that the restorative force $F(x)$ of a spring is proportional to, say, the cube of the displacement x of the mass beyond its equilibrium position or that $F(x)$ is a linear combination of powers of the displacement such as that given by the nonlinear function $F(x) = kx + k_1x^3$. A spring whose mathematical model incorporates a nonlinear restorative force, such as

$$m\frac{d^2x}{dt^2} + kx^3 = 0 \quad \text{or} \quad m\frac{d^2x}{dt^2} + kx + k_1x^3 = 0, \tag{2}$$

is called a **nonlinear spring**. In addition, we examined mathematical models in which damping imparted to the motion was proportional to the instantaneous velocity dx/dt, and the restoring force of a spring was given by the linear function $F(x) = kx$. But these were simply assumptions; in more realistic situations damping could be proportional to some power of the instantaneous velocity dx/dt. The nonlinear differential equation

$$m\frac{d^2x}{dt^2} + \beta\left|\frac{dx}{dt}\right|\frac{dx}{dt} + kx = 0 \tag{3}$$

is one model of a free spring/mass system with damping proportional to the square of the velocity. One can then envision other kinds of models: linear damping and nonlinear restoring force, nonlinear damping and nonlinear restoring force, and so on. The point is, nonlinear characteristics of a physical system lead to a mathematical model that is nonlinear.

Notice in (2) that both $F(x) = kx^3$ and $F(x) = kx + k_1x^3$ are odd functions of x. To see why a polynomial function containing only odd powers of x provides a reasonable model for the restoring force, let us express F as a power series centered at the equilibrium position $x = 0$:

$$F(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \cdots.$$

When the displacements x are small, the values of x^n are negligible for n sufficiently large. If we truncate the power series with, say, the fourth term, then

$$F(x) = c_0 + c_1x + c_2x^2 + c_3x^3.$$

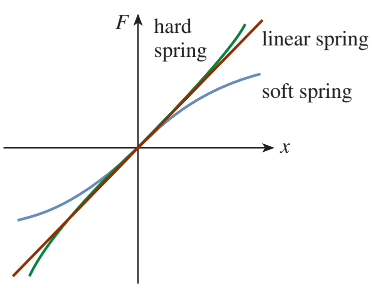


FIGURE 3.11.1 Hard and soft springs

In order for the force at $x > 0$ $(F(x) = c_0 + c_1x + c_2x^2 + c_3x^3)$ and the force at $-x < 0$ $(F(-x) = c_0 - c_1x + c_2x^2 - c_3x^3)$ to have the same magnitude but act in the opposite directions, we must have $F(-x) = -F(x)$. Since this means F is an odd function, we must have $c_0 = 0$ and $c_2 = 0$, and so $F(x) = c_1x + c_3x^3$. Had we used only the first two terms in the series, the same argument yields the linear function $F(x) = c_1x$. For discussion purposes we shall write $c_1 = k$ and $c_2 = k_1$. A restoring force with mixed powers such as $F(x) = kx + k_1x^2$, and the corresponding vibrations, are said to be unsymmetrical.

■ Hard and Soft Springs Let us take a closer look at the equation in (1) in the case where the restoring force is given by $F(x) = kx + k_1x^3$, $k > 0$. The spring is said to be **hard** if $k_1 > 0$ and **soft** if $k_1 < 0$. Graphs of three types of restoring forces are illustrated in **FIGURE 3.11.1**. The next example illustrates these two special cases of the differential equation $m\,d^2x/dt^2 + kx + k_1x^3 = 0$, $m > 0$, $k > 0$.

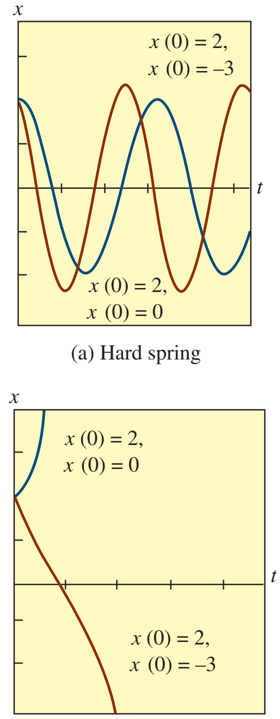
EXAMPLE 1 Comparison of Hard and Soft Springs

The differential equations

$$\frac{d^2x}{dt^2} + x + x^3 = 0 \tag{4}$$

and $$\frac{d^2x}{dt^2} + x - x^3 = 0 \tag{5}$$

are special cases of (2) and are models of a hard spring and soft spring, respectively. **FIGURE 3.11.2(a)** shows two solutions of (4) and Figure 3.11.2(b) shows two solutions of (5) obtained from a numerical solver. The curves shown in red are solutions satisfying the initial conditions $x(0) = 2$, $x'(0) = -3$; the two curves in blue are solutions satisfying $x(0) = 2$, $x'(0) = 0$. These solution curves certainly suggest that the motion of a mass on the hard spring is oscillatory, whereas motion of a mass on the soft spring is not oscillatory. But we must be careful about drawing conclusions based on a couple of solution curves. A more complete picture of the nature of the solutions of both of these equations can be obtained from the qualitative analysis discussed in Chapter 11. ≡



(a) Hard spring

(b) Soft spring

FIGURE 3.11.2 Numerical solution curves

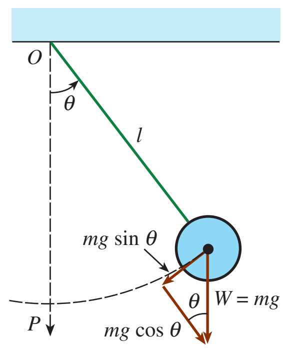


FIGURE 3.11.3 Simple pendulum

■ **Nonlinear Pendulum** Any object that swings back and forth is called a **physical pendulum**. The **simple pendulum** is a special case of the physical pendulum and consists of a rod of length l to which a mass m is attached at one end. In describing the motion of a simple pendulum in a vertical plane, we make the simplifying assumptions that the mass of the rod is negligible and that no external damping or driving forces act on the system. The displacement angle θ of the pendulum, measured from the vertical as shown in **FIGURE 3.11.3**, is considered positive when measured to the right of OP and negative to the left of OP. Now recall that the arc s of a circle of radius l is related to the central angle θ by the formula $s = l\theta$. Hence angular acceleration is

$$a = \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}.$$

From Newton's second law we then have

$$F = ma = ml\frac{d^2\theta}{dt^2}.$$

From Figure 3.11.3 we see that the magnitude of the tangential component of the force due to the weight W is $mg \sin\theta$. In direction this force is $-mg\sin\theta$, since it points to the left for $\theta > 0$ and to the right for $\theta < 0$. We equate the two different versions of the tangential force to obtain $ml\, d^2\theta/dt^2 = -mg\sin\theta$ or

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0. \tag{6}$$

■ **Linearization** Because of the presence of $\sin\theta$, the model in (6) is nonlinear. In an attempt to understand the behavior of the solutions of nonlinear higher-order differential equations, one sometimes tries to simplify the problem by replacing nonlinear terms by certain approximations. For example, the Maclaurin series for $\sin\theta$ is given by

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots,$$

and so if we use the approximation $\sin\theta \approx \theta - \theta^3/6$, equation (6) becomes $d^2\theta/dt^2 + (g/l)\theta + (g/6l)\theta^3 = 0$. Observe that this last equation is the same as the second nonlinear equation in (2) with $m = 1$, $k = g/l$, and $k_1 = -g/6l$. However, if we assume that the displacements θ are small enough to justify using the replacement $\sin\theta \approx \theta$, then (6) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0. \tag{7}$$

See Problem 24 in Exercises 3.11. If we set $\omega^2 = g/l$, we recognize (7) as the differential equation (2) of Section 3.8 that is a model for the free undamped vibrations of a linear spring/mass system. In other words, (7) is again the basic linear equation $y'' + \lambda y = 0$ discussed on page 161 of Section 3.9. As a consequence, we say that equation (7) is a **linearization** of equation (6). Since the general solution of (7) is $\theta(t) = c_1\cos\omega t + c_2\sin\omega t$, this linearization suggests that for initial conditions amenable to small oscillations the motion of the pendulum described by (6) will be periodic.

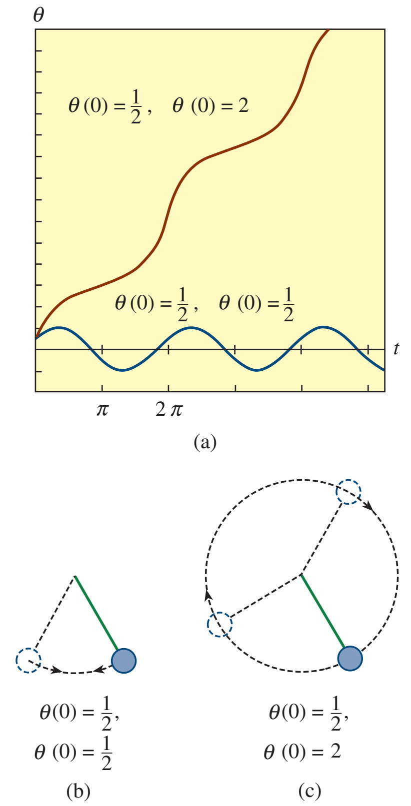


FIGURE 3.11.4 Numerical solution curves in (a); oscillating pendulum in (b); whirling pendulum in (c) in Example 2

EXAMPLE 2 Two Initial-Value Problems

The graphs in **FIGURE 3.11.4(a)** were obtained with the aid of a numerical solver and represent solution curves of equation (6) when $\omega^2 = 1$. The blue curve depicts the solution of (6) that satisfies the initial conditions $\theta(0) = \frac{1}{2}$, $\theta'(0) = \frac{1}{2}$ whereas the red curve is the solution of (6) that satisfies $\theta(0) = \frac{1}{2}$, $\theta'(0) = 2$. The blue curve represents a periodic solution—the pendulum oscillating back and forth as shown in Figure 3.11.4(b) with an apparent amplitude $A \le 1$. The red curve shows that θ increases without bound as time increases—the pendulum, starting from the same initial displacement, is given an initial velocity of magnitude great enough to send it over the top; in other words, the pendulum is whirling about its pivot as shown in Figure 3.11.4(c). In the absence of damping the motion in each case is continued indefinitely. ≡

Telephone Wires

The first-order differential equation

$$\frac{dy}{dx} = \frac{W}{T_1}$$

is equation (17) of Section 1.3. This differential equation, established with the aid of Figure 1.3.8 on page 23, serves as a mathematical model for the shape of a flexible cable suspended between two vertical supports when the cable is carrying a vertical load. In Exercises 2.2, you may have solved this simple DE under the assumption that the vertical load carried by the cables of a suspension bridge was the weight of a horizontal roadbed distributed evenly along the x-axis. With $W = \rho w$, ρ the weight per unit length of the roadbed, the shape of each cable between the vertical supports turned out to be parabolic. We are now in a position to determine the shape of a uniform flexible cable hanging under its own weight, such as a wire strung between two telephone posts. The vertical load is now the wire itself, and so if ρ is the linear density of the wire (measured, say, in lb/ft) and s is the length of the segment P_1P_2 in Figure 1.3.8, then $W = \rho s$. Hence,

$$\frac{dy}{dx} = \frac{\rho s}{T_1}. \tag{8}$$

Since the arc length between points P_1 and P_2 is given by

$$s = \int_0^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx, \tag{9}$$

it follows from the Fundamental Theorem of Calculus that the derivative of (9) is

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{10}$$

Differentiating (8) with respect to x and using (10) lead to the second-order equation

$$\frac{d^2y}{dx^2} = \frac{\rho}{T_1}\frac{ds}{dx} \quad \text{or} \quad \frac{d^2y}{dx^2} = \frac{\rho}{T_1}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}. \tag{11}$$

In the example that follows, we solve (11) and show that the curve assumed by the suspended cable is a **catenary**. Before proceeding, observe that the nonlinear second-order differential equation (11) is one of those equations having the form $F(x, y', y'') = 0$ discussed in Section 3.7. Recall, we have a chance of solving an equation of this type by reducing the order of the equation by means of the substitution $u = y'$.

EXAMPLE 3 An Initial-Value Problem

From the position of the y-axis in Figure 1.3.8 it is apparent that initial conditions associated with the second differential equation in (11) are $y(0) = a$ and $y'(0) = 0$. If we substitute $u = y'$, the last equation in (11) becomes $\dfrac{du}{dx} = \dfrac{\rho}{T_1}\sqrt{1 + u^2}$. Separating variables,

$$\int \frac{du}{\sqrt{1+u^2}} = \frac{\rho}{T_1}\int dx \qquad \text{gives} \qquad \sinh^{-1}u = \frac{\rho}{T_1}x + c_1.$$

Now, $y'(0) = 0$ is equivalent to $u(0) = 0$. Since $\sinh^{-1} 0 = 0$, we find $c_1 = 0$ and so $u = \sinh(\rho x/T_1)$. Finally, by integrating both sides of

$$\frac{dy}{dx} = \sinh\frac{\rho}{T_1}x \qquad \text{we get} \qquad y = \frac{T_1}{\rho}\cosh\frac{\rho}{T_1}x + c_2.$$

Using $y(0) = a$, $\cosh 0 = 1$, the last equation implies that $c_2 = a - T_1/\rho$. Thus we see that the shape of the hanging wire is given by $y = (T_1/\rho)\cosh(\rho x/T_1) + a - T_1/\rho$. ≡

In Example 3, had we been clever enough at the start to choose $a = T_1/\rho$, then the solution of the problem would have been simply the hyperbolic cosine $y = (T_1/\rho)\cosh(\rho x/T_1)$.

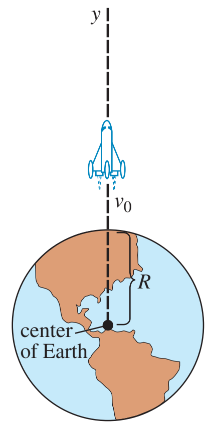


FIGURE 3.11.5 Distance to rocket is large compared to R

Rocket Motion In Section 1.3 we saw that the differential equation of a free-falling body of mass m near the surface of the Earth is given by

$$m\frac{d^2s}{dt^2} = -mg \qquad \text{or simply} \qquad \frac{d^2s}{dt^2} = -g,$$

where s represents the distance from the surface of the Earth to the object and the positive direction is considered to be upward. In other words, the underlying assumption here is that the distance s to the object is small when compared with the radius R of the Earth; put yet another way, the distance y from the center of the Earth to the object is approximately the same as R. If, on the other hand, the distance y to an object, such as a rocket or a space probe, is large compared to R, then we combine Newton's second law of motion and his universal law of gravitation to derive a differential equation in the variable y.

Suppose a rocket is launched vertically upward from the ground as shown in **FIGURE 3.11.5**. If the positive direction is upward and air resistance is ignored, then the differential equation of motion after fuel burnout is

$$m\frac{d^2y}{dt^2} = -k\frac{Mm}{y^2} \qquad \text{or} \qquad \frac{d^2y}{dt^2} = -k\frac{M}{y^2}, \tag{12}$$

where k is a constant of proportionality, y is the distance from the center of the Earth to the rocket, M is the mass of the Earth, and m is the mass of the rocket. To determine the constant k, we use the fact that when $y = R$, $kMm/R^2 = mg$ or $k = gR^2/M$. Thus the last equation in (12) becomes

$$\frac{d^2y}{dt^2} = -g\frac{R^2}{y^2}. \tag{13}$$

See Problem 14 in Exercises 3.11.

Variable Mass Notice in the preceding discussion that we described the motion of the rocket after it has burned all its fuel, when presumably its mass m is constant. Of course during its powered ascent, the total mass of the rocket varies as its fuel is being expended. The second law of motion, as originally advanced by Newton, states that when a body of mass m moves through a force field with velocity v, the time rate of change of the momentum mv of the body is equal to applied or net force F acting on the body:

$$F = \frac{d}{dt}(mv). \tag{14}$$

If m is constant, then (14) yields the more familiar form $F = m\,dv/dt = ma$, where a is acceleration. We use the form of Newton's second law given in (14) in the next example, in which the mass m of the body is variable.

EXAMPLE 4 Chain Pulled Upward by a Constant Force

A uniform 10-foot-long chain is coiled loosely on the ground. One end of the chain is pulled vertically upward by means of a constant force of 5 lb. The chain weighs 1 lb per foot. Determine the height of the end above ground level at time t. See Figure 1.3.18 and Problem 21 in Exercises 1.3.

Solution Let us suppose that $x = x(t)$ denotes the height of the end of the chain in the air at time t, $v = dx/dt$, and that the positive direction is upward. For that portion of the chain in the air at time t we have the following variable quantities:

weight: $W = (x \text{ ft}) \cdot (1 \text{ lb/ft}) = x,$
mass: $m = W/g = x/32,$
net force: $F = 5 - W = 5 - x.$

Thus from (14) we have

Product Rule ↓

$$\frac{d}{dt}\left(\frac{x}{32}v\right) = 5 - x \quad \text{or} \quad x\frac{dv}{dt} + v\frac{dx}{dt} = 160 - 32x. \tag{15}$$

Since $v = dx/dt$ the last equation becomes

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + 32x = 160. \tag{16}$$

The nonlinear second-order differential equation (16) has the form $F(x, x', x'') = 0$, which is the second of the two forms considered in Section 3.7 that can possibly be solved by reduction of order. In order to solve (16), we revert back to (15) and use $v = x'$ along with the Chain Rule. From $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$ the second equation in (15) can be rewritten as

$$xv\frac{dv}{dx} + v^2 = 160 - 32x. \tag{17}$$

On inspection (17) might appear intractable, since it cannot be characterized as any of the first-order equations that were solved in Chapter 2. However, by rewriting (17) in differential form $M(x, v)dx + N(x, v)dv = 0$, we observe that the nonexact equation

$$(v^2 + 32x - 160)\,dx + xv\,dv = 0 \tag{18}$$

can be transformed into an exact equation by multiplying it by an *integrating factor*.* When (18) is multiplied by $\mu(x) = x$, the resulting equation is exact (verify). If we identify $\partial f/\partial x = xv^2 + 32x^2 - 160x$, $\partial f/\partial v = x^2v$, and then proceed as in Section 2.4, we arrive at

$$\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 80x^2 = c_1. \tag{19}$$

From the initial condition $x(0) = 0$ it follows that $c_1 = 0$. Now by solving $\frac{1}{2}x^2v^2 + \frac{32}{3}x^3 - 80x^2 = 0$ for $v = dx/dt > 0$ we get another differential equation,

$$\frac{dx}{dt} = \sqrt{160 - \frac{64}{3}x},$$

which can be solved by separation of variables. You should verify that

$$-\frac{3}{32}\left(160 - \frac{64}{3}x\right)^{1/2} = t + c_2. \tag{20}$$

This time the initial condition $x(0) = 0$ implies $c_2 = -3\sqrt{10}/8$. Finally, by squaring both sides of (20) and solving for x we arrive at the desired result,

$$x(t) = \frac{15}{2} - \frac{15}{2}\left(1 - \frac{4\sqrt{10}}{15}t\right)^2. \tag{21}$$ ≡

See Problem 15 in Exercises 3.11.

*See page 59 in section 2.4.

3.11 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

To the Instructor

In addition to Problems 24 and 25, all or portions of Problems 1–6, 8–13, 15, 17, and 23 could serve as *Computer Lab Assignments.*

Nonlinear Springs

In Problems 1–4, the given differential equation is a model of an undamped spring/mass system in which the restoring force $F(x)$ in (1) is nonlinear. For each equation use a numerical solver to plot the solution curves satisfying the given initial conditions. If the solutions appear to be periodic, use the solution curve to estimate the period T of oscillations.

1. $\dfrac{d^2x}{dt^2} + x^3 = 0,$

$x(0) = 1,\ x'(0) = 1;\ x(0) = \frac{1}{2},\ x'(0) = -1$

2. $\dfrac{d^2x}{dt^2} + 4x - 16x^3 = 0,$

$x(0) = 1,\ x'(0) = 1;\ x(0) = -2,\ x'(0) = 2$

3. $\dfrac{d^2x}{dt^2} + 2x - x^2 = 0,$

$x(0) = 1, x'(0) = 1;\ x(0) = \frac{3}{2},\ x'(0) = -1$

4. $\dfrac{d^2x}{dt^2} + xe^{0.01x} = 0,$

$x(0) = 1, x'(0) = 1;\ x(0) = 3,\ x'(0) = -1$

5. In Problem 3, suppose the mass is released from the initial position $x(0) = 1$ with an initial velocity $x'(0) = x_1$. Use a numerical solver to estimate the smallest value of $|x_1|$ at which the motion of the mass is nonperiodic.

6. In Problem 3, suppose the mass is released from an initial position $x(0) = x_0$ with the initial velocity $x'(0) = 1$. Use a numerical solver to estimate an interval $a \le x_0 \le b$ for which the motion is oscillatory.

7. Find a linearization of the differential equation in Problem 4.

8. Consider the model of an undamped nonlinear spring/mass system given by $x'' + 8x - 6x^3 + x^5 = 0$. Use a numerical solver to discuss the nature of the oscillations of the system corresponding to the initial conditions:

$x(0) = 1, x'(0) = 1;\qquad x(0) = -2, x'(0) = \frac{1}{2};$

$x(0) = \sqrt{2}, x'(0) = 1;\qquad x(0) = 2, x'(0) = \frac{1}{2};$

$x(0) = 2, x'(0) = 0;\qquad x(0) = -\sqrt{2}, x'(0) = -1.$

In Problems 9 and 10, the given differential equation is a model of a damped nonlinear spring/mass system. Predict the behavior of each system as $t \to \infty$. For each equation use a numerical solver to obtain the solution curves satisfying the given initial conditions.

9. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x + x^3 = 0,$

$x(0) = -3,\quad x'(0) = 4;\quad x(0) = 0,\quad x'(0) = -8$

10. $\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x - x^3 = 0,$

$x(0) = 0,\quad x'(0) = \frac{3}{2};\quad x(0) = -1,\quad x'(0) = 1$

11. The model $mx'' + kx + k_1x^3 = F_0 \cos \omega t$ of an undamped periodically driven spring/mass system is called **Duffing's differential equation**. Consider the initial-value problem $x'' + x + k_1x^3 = 5 \cos t, x(0) = 1, x'(0) = 0$. Use a numerical solver to investigate the behavior of the system for values of $k_1 > 0$ ranging from $k_1 = 0.01$ to $k_1 = 100$. State your conclusions.

12. **(a)** Find values of $k_1 < 0$ for which the system in Problem 11 is oscillatory.

(b) Consider the initial-value problem

$$x'' + x + k_1x^3 = \cos \tfrac{3}{2}t,\ x(0) = 0, x'(0) = 0.$$

Find values for $k_1 < 0$ for which the system is oscillatory.

Nonlinear Pendulum

13. Consider the model of the free damped nonlinear pendulum given by

$$\frac{d^2\theta}{dt^2} + 2\lambda\frac{d\theta}{dt} + \omega^2\sin\theta = 0.$$

Use a numerical solver to investigate whether the motion in the two cases $\lambda^2 - \omega^2 > 0$ and $\lambda^2 - \omega^2 < 0$ corresponds, respectively, to the overdamped and underdamped cases discussed in Section 3.8 for spring/mass systems. Choose appropriate initial conditions and values of λ and ω.

Rocket Motion

14. **(a)** Use the substitution $v = dy/dt$ to solve (13) for v in terms of y. Assume that the velocity of the rocket at burnout is $v = v_0$ and that $y \approx R$ at that instant; show that the approximate value of the constant c of integration is $c = -gR + \frac{1}{2}v_0^2$.

(b) Use the solution for v in part (a) to show that the escape velocity of the rocket is given by $v_0 = \sqrt{2gR}$. [*Hint*: Take $y \to \infty$ and assume $v > 0$ for all time t.]

(c) The result in part (b) holds for any body in the solar system. Use the values $g = 32$ ft/s^2 and $R = 4000$ mi to show that the escape velocity from the Earth is (approximately) $v_0 = 25{,}000$ mi/h.

(d) Find the escape velocity from the Moon if the acceleration of gravity is $0.165g$ and $R = 1080$ mi.

Variable Mass

15. **(a)** In Example 4, how much of the chain would you intuitively expect the constant 5-pound force to be able to lift?

(b) What is the initial velocity of the chain?

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(c) Why is the time interval corresponding to $x(t) \geq 0$ not the interval I of definition of the solution (21)? Determine the interval I. How much chain is actually lifted? Explain any difference between this answer and your prediction in part (a).

(d) Why would you expect $x(t)$ to be a periodic solution?

16. A uniform chain of length L, measured in feet, is held vertically so that the lower end just touches the floor. The chain weighs 2 lb/ft. The upper end that is held is released from rest at $t = 0$ and the chain falls straight down. See Figure 1.3.19. As we saw in Problem 22 in Exercises 1.3, if $x(t)$ denotes the length of the chain on the floor at time t, air resistance is ignored, and the positive direction is taken to be downward, then

$$(L - x)\frac{d^2x}{dt^2} - \left(\frac{dx}{dt}\right)^2 = Lg.$$

(a) Solve for v in terms of x. Solve for x in terms of t. Express v in terms of t.

(b) Determine how long it takes for the chain to fall completely to the ground.

(c) What velocity does the model in part (a) predict for the upper end of the chain as it hits the ground?

17. A portion of a uniform chain of length 8 feet is loosely coiled around a peg at the edge of a high horizontal platform, and the remaining portion of the chain hangs at rest over the edge of the platform. Suppose that the length of the overhang is 3 feet and that the chain weighs 2 lb/ft. Starting at $t = 0$, the weight of the overhanging portion causes the chain on the table to uncoil smoothly and to fall to the floor.

(a) Ignore any resistive forces and assume that the positive direction is downward. If $x(t)$ denotes the length of the chain overhanging the platform at time $t > 0$ and $v = dx/dt$, find a differential equation that relates v to x.

(b) Proceed as in Example 4 and solve for v in terms of x by finding an appropriate integrating factor.

(c) Express time t in terms of x. Use a CAS as an aid in determining the time it takes for a 7-foot segment of chain to uncoil completely—that is, fall from the platform.

18. A portion of a uniform chain of length 8 feet lies stretched out on a high horizontal platform, and the remaining portion of the chain hangs over the edge of the platform as shown in **FIGURE 3.11.6**. Suppose the length of the overhang is 3 feet and that the chain weighs 2 lb/ft. The end of the chain on the platform is held until at $t = 0$ it is released from rest, and the chain begins to slide off the platform because of the weight of the overhanging portion.

(a) Ignore any resistive forces and assume that the positive direction is downward. If $x(t)$ denotes the length of the chain overhanging the platform at time $t > 0$ and $v = dx/dt$, show that v is related to x by the differential equation $v\dfrac{dv}{dx} = 4x$.

(b) Solve for v in terms of x. Solve for x in terms of t. Express v in terms of t.

(c) Approximate the time it takes for the rest of the chain to slide off the platform. Find the velocity at which the end of the chain leaves the edge of the platform.

(d) Suppose the chain is L feet long and weighs a total of W pounds. If the overhang at $t = 0$ is x_0 feet, show that the velocity at which the end of the chain leaves the edge of the platform is $v(L) = \sqrt{\dfrac{g}{L}(L^2 - x_0^2)}$.

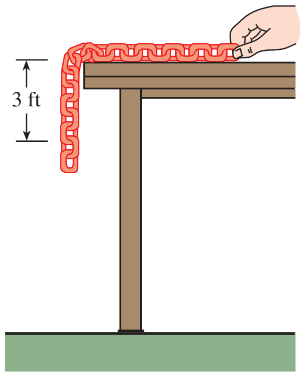


FIGURE 3.11.6 Sliding chain in Problem 18

Miscellaneous Mathematical Models

19. Pursuit Curve In a naval exercise, a ship S_1 is pursued by a submarine S_2, as shown in **FIGURE 3.11.7**. Ship S_1 departs point (0, 0) at $t = 0$ and proceeds along a straight-line course (the y-axis) at a constant speed v_1. The submarine S_2 keeps ship S_1 in visual contact, indicated by the straight dashed line L in the figure, while traveling at a constant speed v_2 along a curve C. Assume that S_2 starts at the point $(a, 0)$, $a > 0$, at $t = 0$ and that L is tangent to C. Determine a mathematical model that describes the curve C. Find an explicit solution of the differential equation. For convenience, define $r = v_1/v_2$. Determine whether the paths of S_1 and S_2 will ever intersect by considering the cases $r > 1$, $r < 1$, and $r = 1$.

[*Hint*: $\dfrac{dt}{dx} = \dfrac{dt}{ds}\dfrac{ds}{dx}$, where s is arc length measured along C.]

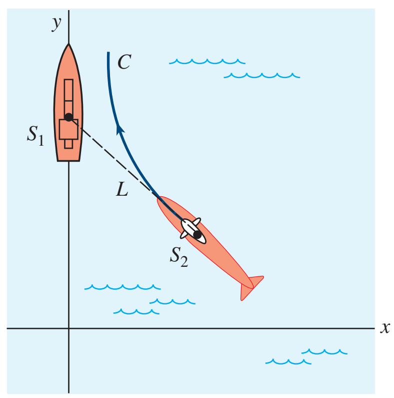


FIGURE 3.11.7 Pursuit curve in Problem 19

20. Pursuit Curve In another naval exercise, a destroyer S_1 pursues a submerged submarine S_2. Suppose that S_1 at (9, 0) on the x-axis detects S_2 at (0, 0) and that S_2 simultaneously detects S_1. The captain of the destroyer S_1 assumes that the submarine will take immediate evasive action and conjectures that its likely new course is the straight line indicated in **FIGURE 3.11.8**.

When S_1 is at (3, 0) it changes from its straight-line course toward the origin to a pursuit curve C. Assume that the speed of the destroyer is, at all times, a constant 30 mi/h and the submarine's speed is a constant 15 mi/h.

(a) Explain why the captain waits until S_1 reaches (3, 0) before ordering a course change to C.

(b) Using polar coordinates, find an equation $r = f(\theta)$ for the curve C.

(c) Let T denote the time, measured from the initial detection, at which the destroyer intercepts the submarine. Find an upper bound for T.

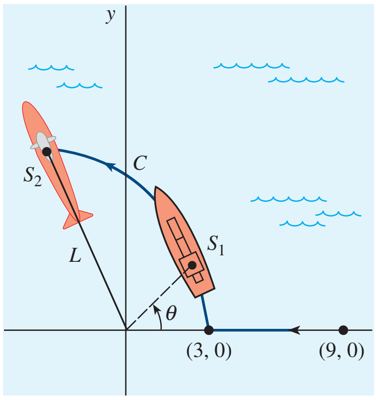


FIGURE 3.11.8 Pursuit curve in Problem 20

Discussion Problems

21. Discuss why the damping term in equation (3) is written as

$$\beta\left|\frac{dx}{dt}\right|\frac{dx}{dt} \quad \text{instead of} \quad \beta\left(\frac{dx}{dt}\right)^2.$$

22. **(a)** Experiment with a calculator to find an interval $0 \le \theta < \theta_1$, where θ is measured in radians, for which you think $\sin\theta \approx \theta$ is a fairly good estimate. Then use a graphing utility to plot the graphs of $y = x$ and $y = \sin x$ on the same coordinate axes for $0 \le x \le \pi/2$. Do the graphs confirm your observations with the calculator?

(b) Use a numerical solver to plot the solutions curves of the initial-value problems

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \quad \theta(0) = \theta_0, \theta'(0) = 0$$

and $$\frac{d^2\theta}{dt^2} + \theta = 0, \quad \theta(0) = \theta_0, \theta'(0) = 0$$

for several values of θ_0 in the interval $0 \le \theta < \theta_1$ found in part (a). Then plot solution curves of the initial-value problems for several values of θ_0 for which $\theta_0 > \theta_1$.

23. **(a)** Consider the nonlinear pendulum whose oscillations are defined by (6). Use a numerical solver as an aid to determine whether a pendulum of length l will oscillate faster on the Earth or on the Moon. Use the same initial conditions, but choose these initial conditions so that the pendulum oscillates back and forth.

(b) For which location in part (a) does the pendulum have greater amplitude?

(c) Are the conclusions in parts (a) and (b) the same when the linear model (7) is used?

Computer Lab Assignments

24. Consider the initial-value problem

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \quad \theta(0) = \frac{\pi}{12}, \quad \theta'(0) = -\frac{1}{3}$$

for the nonlinear pendulum. Since we cannot solve the differential equation, we can find no explicit solution of this problem. But suppose we wish to determine the first time $t_1 > 0$ for which the pendulum in Figure 3.11.3 starting from its initial position to the right, reaches the position OP—that is, find the first positive root of $\theta(t) = 0$. In this problem and the next we examine several ways to proceed.

(a) Approximate t_1 by solving the linear problem $d^2\theta/dt^2 + \theta = 0, \theta(0) = \pi/12, \theta'(0) = -\frac{1}{3}$.

(b) Use the method illustrated in Example 3 of Section 3.7 to find the first four nonzero terms of a Taylor series solution $\theta(t)$ centered at 0 for the nonlinear initial-value problem. Give the exact values of all coefficients.

(c) Use the first two terms of the Taylor series in part (b) to approximate t_1.

(d) Use the first three terms of the Taylor series in part (b) to approximate t_1.

(e) Use a root-finding application of a CAS (or a graphing calculator) and the first four terms of the Taylor series in part (b) to approximate t_1.

(f) In this part of the problem you are led through the commands in *Mathematica* that enable you to approximate the root t_1. The procedure is easily modified so that any root of $\theta(t) = 0$ can be approximated. (*If you do not have* Mathematica, *adapt the given procedure by finding the corresponding syntax for the CAS you have on hand.*) Precisely reproduce and then, in turn, execute each line in the given sequence of commands.

```
sol = NDSolve[{y″[t] + Sin[y[t]] = = 0,
    y[0] = = Pi/12, y′[0] == −1/3} ,
    y, { t, 0, 5} ]//Flatten
solution = y[t]/.sol
Clear[y]
y[t_]: = Evaluate[solution]
y[t]
gr1 = Plot[y[t], { t, 0, 5} ]
root = FindRoot[y[t] = = 0, { t, 1} ]
```

(g) Appropriately modify the syntax in part (f) and find the next two positive roots of $\theta(t) = 0$.

25. Consider a pendulum that is released from rest from an initial displacement of θ_0 radians. Solving the linear model (7) subject to the initial conditions $\theta(0) = \theta_0, \theta'(0) = 0$ gives $\theta(t) = \theta_0 \cos\sqrt{g/l}\,t$. The period of oscillations predicted by this model is given by the familiar formula $T = 2\pi/\sqrt{g/l} = 2\pi\sqrt{l/g}$. The interesting thing about this formula for T is that it does not depend on the magnitude of the initial displacement θ_0. In other words, the linear model predicts that the time that it would take the pendulum to swing from an initial displacement of, say, $\theta_0 = \pi/2$ (= 90°) to $-\pi/2$ and back again would be exactly the same time to cycle from, say, $\theta_0 = \pi/360$ (= 0.5°) to $-\pi/360$. This is intuitively unreasonable; the actual period must depend on θ_0.

If we assume that $g = 32$ ft/s^2 and $l = 32$ ft, then the period of oscillation of the linear model is $T = 2\pi$ s. Let us compare this last number with the period predicted by the nonlinear model when $\theta_0 = \pi/4$. Using a numerical solver that is capable of generating hard data, approximate the solution of

$$\frac{d^2\theta}{dt^2} + \sin\theta = 0, \quad \theta(0) = \frac{\pi}{4}, \quad \theta'(0) = 0$$

for $0 \le t \le 2$. As in Problem 24, if t_1 denotes the first time the pendulum reaches the position OP in Figure 3.11.3, then the period of the nonlinear pendulum is $4t_1$. Here is another way of solving the equation $\theta(t) = 0$. Experiment with small step sizes and advance the time starting at $t = 0$ and ending at $t = 2$. From your hard data, observe the time t_1 when $\theta(t)$ changes, for the first time, from positive to negative. Use the value t_1 to determine the true value of the period of the nonlinear pendulum. Compute the percentage relative error in the period estimated by $T = 2\pi$.

3.12 Solving Systems of Linear Equations

Introduction We conclude this chapter as we did in Chapter 2 with systems of differential equations. But unlike Section 2.9, we will actually solve systems in the discussion that follows.

Coupled Systems/Coupled DEs In Section 2.9 we briefly examined some mathematical models that were systems of linear and nonlinear first-order ODEs. In Section 3.8 we saw that the mathematical model describing the displacement of a mass on a single spring, current in a series circuit, and charge on a capacitor in a series circuit consisted of a *single* differential equation. When physical systems are coupled—for example, when two or more mixing tanks are connected, when two or more spring/mass systems are attached, or when circuits are joined to form a network—the mathematical model of the system usually consists of a set of coupled differential equations; in other words, a system of differential equations.

We did not attempt to solve any of the systems considered in Section 2.9. The same remarks made in Sections 3.7 and 3.11 pertain as well to systems of nonlinear ODEs; that is, it is nearly impossible to solve such systems analytically. However, *linear* systems with constant coefficients can be solved. The method that we shall examine in this section for solving linear systems with constant coefficients simply uncouples the system into distinct linear ODEs in each dependent variable. Thus, this section gives you an opportunity to practice what you learned earlier in the chapter.

Before proceeding, let us continue in the same vein as Section 3.8 by considering a spring/mass system, but this time we derive a mathematical model that describes the vertical displacements of two masses in a coupled spring/mass system.

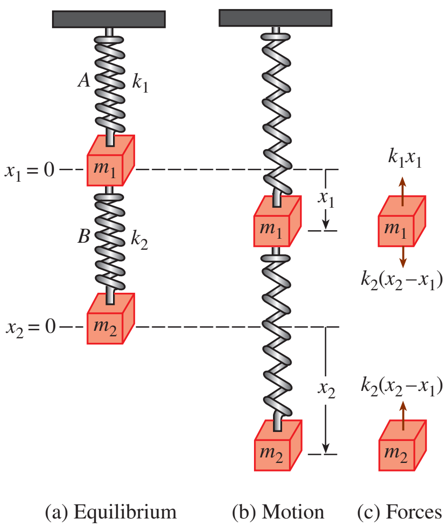


FIGURE 3.12.1 Coupled spring/mass systems

Coupled Spring/Mass System Suppose two masses m_1 and m_2 are connected to two springs A and B of negligible mass having spring constants k_1 and k_2, respectively. As shown in **FIGURE 3.12.1(a)**, spring A is attached to a rigid support and spring B is attached to the bottom of mass m_1. Let $x_1(t)$ and $x_2(t)$ denote the vertical displacements of the masses from their equilibrium positions. When the system is in motion, Figure 3.12.1(b), spring B is subject to both an elongation and a compression; hence its net elongation is $x_2 - x_1$. Therefore it follows from Hooke's law that springs A and B exert forces $-k_1x_1$ and $k_2(x_2 - x_1)$, respectively, on m_1. If no damping is present and no external force is impressed on the system, then the net force on m_1 is $-k_1x_1 + k_2(x_2 - x_1)$. By Newton's second law we can write

$$m_1\frac{d^2x_1}{dt^2} = -k_1x_1 + k_2(x_2 - x_1).$$

Similarly, the net force exerted on mass m_2 is due solely to the net elongation of spring B; that is, $-k_2(x_2 - x_1)$. Hence we have

$$m_2 \frac{d^2x_2}{dt^2} = -k_2(x_2 - x_1).$$

In other words, the motion of the coupled system is represented by the system of linear second-order equations

$$\begin{aligned} m_1x_1'' &= -k_1x_1 + k_2(x_2 - x_1) \\ m_2x_2'' &= -k_2(x_2 - x_1). \end{aligned} \tag{1}$$

After we have illustrated the main idea of this section, we will return to system (1).

Systematic Elimination The method of **systematic elimination** for solving systems of linear equations with constant coefficients is based on the algebraic principle of elimination of variables. The analogue of *multiplying* an algebraic equation by a constant is *operating* on an ODE with some combination of derivatives. The elimination process is expedited by rewriting each equation in a system using differential operator notation. Recall from Section 3.1 that a single linear equation

$$a_ny^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1y' + a_0y = g(t),$$

where the a_i, $i = 0, 1, \ldots, n$ are constants, can be written as

$$(a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0)y = g(t).$$

If an nth-order differential operator $a_nD^n + a_{n-1}D^{n-1} + \cdots + a_1D + a_0$ factors into differential operators of lower order, then the factors commute. Now, for example, to rewrite the system

$$\begin{aligned} x'' + 2x' + y'' &= x + 3y + \sin t \\ x' + y' &= -4x + 2y + e^{-t} \end{aligned}$$

in terms of the operator D, we first bring all terms involving the dependent variables to one side and group the same variables:

$$\begin{aligned} x'' + 2x' - x + y'' - 3y &= \sin t \\ x' - 4x + y' - 2y &= e^{-t} \end{aligned} \quad \text{so that} \quad \begin{aligned} (D^2 + 2D - 1)x + (D^2 - 3)y &= \sin t \\ (D - 4)x + (D - 2)y &= e^{-t}. \end{aligned}$$

Solution of a System A **solution** of a system of differential equations is a set of sufficiently differentiable functions $x = \phi_1(t)$, $y = \phi_2(t)$, $z = \phi_3(t)$, and so on, that satisfies each equation in the system on some common interval I.

Method of Solution Consider the simple system of linear first-order equations

$$\begin{aligned} \frac{dx}{dt} &= 3y \\ \frac{dy}{dt} &= 2x \end{aligned} \quad \text{or, equivalently,} \quad \begin{aligned} Dx - 3y &= 0 \\ 2x - Dy &= 0. \end{aligned} \tag{2}$$

Operating on the first equation in (2) by D while multiplying the second by -3 and then adding eliminates y from the system and gives $D^2x - 6x = 0$. Since the roots of the auxiliary equation of the last DE are $m_1 = \sqrt{6}$ and $m_2 = -\sqrt{6}$, we obtain

$$x(t) = c_1e^{-\sqrt{6}t} + c_2e^{\sqrt{6}t}. \tag{3}$$

Multiplying the first equation in (2) by 2 while operating on the second by D and then subtracting gives the differential equation for y, $D^2y - 6y = 0$. It follows immediately that

$$y(t) = c_3e^{-\sqrt{6}t} + c_4e^{\sqrt{6}t}. \tag{4}$$

This is important. Now, (3) and (4) do not satisfy the system (2) for every choice of c_1, c_2, c_3, and c_4 because the system itself puts a constraint on the number of parameters in a solution that can be chosen arbitrarily. To see this, observe that after substituting $x(t)$ and $y(t)$ into the first equation of the original system, (2) gives, after simplification,

$$(-\sqrt{6}c_1 - 3c_3)e^{-\sqrt{6}t} + (\sqrt{6}c_2 - 3c_4)e^{\sqrt{6}t} = 0.$$

Since the latter expression is to be zero for all values of t, we must have $-\sqrt{6}c_1 - 3c_3 = 0$ and $\sqrt{6}c_2 - 3c_4 = 0$. Thus we can write c_3 and a multiple of c_1 and c_4 as a multiple of c_2:

$$c_3 = -\frac{\sqrt{6}}{3}c_1 \quad \text{and} \quad c_4 = \frac{\sqrt{6}}{3}c_2. \tag{5}$$

Hence we conclude that a solution of the system must be

$$x(t) = c_1e^{-\sqrt{6}t} + c_2e^{\sqrt{6}t}, \quad y(t) = -\frac{\sqrt{6}}{3}c_1e^{-\sqrt{6}t} + \frac{\sqrt{6}}{3}c_2e^{\sqrt{6}t}.$$

You are urged to substitute (3) and (4) into the second equation of (2) and verify that the same relationship (5) holds between the constants.

EXAMPLE 1 Solution by Elimination

Solve

$$\begin{aligned} Dx + (D + 2)y &= 0 \\ (D - 3)x - \quad 2y &= 0. \end{aligned} \tag{6}$$

Solution Operating on the first equation by $D - 3$ and on the second by D and then subtracting eliminates x from the system. It follows that the differential equation for y is

$$[(D - 3)(D + 2) + 2D]y = 0 \quad \text{or} \quad (D^2 + D - 6)y = 0.$$

Since the characteristic equation of this last differential equation is $m^2 + m - 6 = (m - 2)(m + 3) = 0$, we obtain the solution

$$y(t) = c_1e^{2t} + c_2e^{-3t}. \tag{7}$$

Eliminating y in a similar manner yields $(D^2 + D - 6)x = 0$, from which we find

$$x(t) = c_3e^{2t} + c_4e^{-3t}. \tag{8}$$

As we noted in the foregoing discussion, a solution of (6) does not contain four independent constants. Substituting (7) and (8) into the first equation of (6) gives

$$(4c_1 + 2c_3)e^{2t} + (-c_2 - 3c_4)e^{-3t} = 0.$$

From $4c_1 + 2c_3 = 0$ and $-c_2 - 3c_4 = 0$ we get $c_3 = -2c_1$ and $c_4 = -\frac{1}{3}c_2$. Accordingly, a solution of the system is

$$x(t) = -2c_1e^{2t} - \frac{1}{3}c_2e^{-3t}, \quad y(t) = c_1e^{2t} + c_2e^{-3t}.$$ ≡

Since we could just as easily solve for c_3 and c_4 in terms of c_1 and c_2, the solution in Example 1 can be written in the alternative form

$$x(t) = c_3e^{2t} + c_4e^{-3t}, \quad y(t) = -\frac{1}{2}c_3e^{2t} - 3c_4e^{-3t}.$$

It sometimes pays to keep one's eyes open when solving systems. Had we solved for x first, then y could be found, along with the relationship between the constants, by using the last equation in (6). You should verify that substituting $x(t)$ into $y = \frac{1}{2}(Dx - 3x)$ yields $y = -\frac{1}{2}c_3e^{2t} - 3c_4e^{-3t}$.

Watch for a shortcut.

EXAMPLE 2 Solution by Elimination

Solve

$$\begin{aligned} x' - 4x + y'' &= t^2 \\ x' + x + y' &= 0. \end{aligned} \tag{9}$$

Solution First we write the system in differential operator notation:

$$\begin{aligned} (D - 4)x + D^2y &= t^2 \\ (D + 1)x + Dy &= 0. \end{aligned} \tag{10}$$

Then, by eliminating x, we obtain

$$[(D + 1)D^2 - (D - 4)D]y = (D + 1)t^2 - (D - 4)0$$

or

$$(D^3 + 4D)y = t^2 + 2t.$$

Since the roots of the auxiliary equation $m(m^2 + 4) = 0$ are $m_1 = 0$, $m_2 = 2i$, and $m_3 = -2i$, the complementary function is

$$y_c = c_1 + c_2 \cos 2t + c_3 \sin 2t.$$

To determine the particular solution y_p we use undetermined coefficients by assuming $y_p = At^3 + Bt^2 + Ct$. Therefore

$$y_p' = 3At^2 + 2Bt + C, \quad y_p'' = 6At + 2B, \quad y_p''' = 6A,$$
$$y_p''' + 4y_p' = 12At^2 + 8Bt + 6A + 4C = t^2 + 2t.$$

The last equality implies $12A = 1$, $8B = 2$, $6A + 4C = 0$, and hence $A = \frac{1}{12}$, $B = \frac{1}{4}$, $C = -\frac{1}{8}$. Thus

$$y = y_c + y_p = c_1 + c_2 \cos 2t + c_3 \sin 2t + \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t. \tag{11}$$

Eliminating y from the system (9) leads to

$$[(D - 4) - D(D + 1)]x = t^2 \quad \text{or} \quad (D^2 + 4)x = -t^2.$$

It should be obvious that

$$x_c = c_4 \cos 2t + c_5 \sin 2t$$

and that undetermined coefficients can be applied to obtain a particular solution of the form $x_p = At^2 + Bt + C$. In this case the usual differentiations and algebra yield $x_p = -\frac{1}{4}t^2 + \frac{1}{8}$, and so

$$x = x_c + x_p = c_4 \cos 2t + c_5 \sin 2t - \frac{1}{4}t^2 + \frac{1}{8}. \tag{12}$$

Now c_4 and c_5 can be expressed in terms of c_2 and c_3 by substituting (11) and (12) into either equation of (9). By using the second equation, we find, after combining terms,

$$(c_5 - 2c_4 - 2c_2) \sin 2t + (2c_5 + c_4 + 2c_3) \cos 2t = 0$$

so that $c_5 - 2c_4 - 2c_2 = 0$ and $2c_5 + c_4 + 2c_3 = 0$. Solving for c_4 and c_5 in terms of c_2 and c_3 gives $c_4 = -\frac{1}{5}(4c_2 + 2c_3)$ and $c_5 = \frac{1}{5}(2c_2 - 4c_3)$. Finally, a solution of (9) is found to be

$$x(t) = -\frac{1}{5}(4c_2 + 2c_3) \cos 2t + \frac{1}{5}(2c_2 - 4c_3) \sin 2t - \frac{1}{4}t^2 + \frac{1}{8},$$

$$y(t) = c_1 + c_2 \cos 2t + c_3 \sin 2t + \frac{1}{12}t^3 + \frac{1}{4}t^2 - \frac{1}{8}t. \quad \equiv$$

EXAMPLE 3 A Mathematical Model Revisited

In (3) of Section 2.9 we saw that a system of linear first-order differential equations described the number of pounds of salt $x_1(t)$ and $x_2(t)$ of a brine mixture that flows between two tanks. At that time we were not able to solve the system. But now, in terms of differential operators, the system is

$$\begin{aligned} \left(D + \frac{2}{25}\right)x_1 - \qquad \frac{1}{50}x_2 &= 0 \\ -\frac{2}{25}x_1 + \left(D + \frac{2}{25}\right)x_2 &= 0. \end{aligned}$$

Operating on the first equation by $D + \frac{2}{25}$, multiplying the second equation by $\frac{1}{50}$, adding, and then simplifying, give

$$(625D^2 + 100D + 3)x_1 = 0.$$

From the auxiliary equation $625m^2 + 100m + 3 = (25m + 1)(25m + 3) = 0$ we see immediately that

$$x_1(t) = c_1e^{-t/25} + c_2e^{-3t/25}.$$

In like manner we find $(625D^2 + 100D + 3)x_2 = 0$ and so

$$x_2(t) = c_3e^{-t/25} + c_4e^{-3t/25}.$$

Substituting $x_1(t)$ and $x_2(t)$ into, say, the first equation of the system then gives

$$(2c_1 - c_3)e^{-t/25} + (-2c_2 - c_4)e^{-3t/25} = 0.$$

From this last equation we find $c_3 = 2c_1$ and $c_4 = -2c_2$. Thus a solution of the system is

$$x_1(t) = c_1e^{-t/25} + c_2e^{-3t/25}, \qquad x_2(t) = 2c_1e^{-t/25} - 2c_2e^{-3t/25}.$$

In the original discussion we assumed that initial conditions were $x_1(0) = 25$ and $x_2(0) = 0$. Applying these conditions to the solution yields $c_1 + c_2 = 25$ and $2c_1 - 2c_2 = 0$. Solving these equations simultaneously gives $c_1 = c_2 = \frac{25}{2}$. Finally, a solution of the initial-value problem is

$$x_1(t) = \frac{25}{2}e^{-t/25} + \frac{25}{2}e^{-3t/25}, \qquad x_2(t) = 25e^{-t/25} - 25e^{-3t/25}. \quad \equiv$$

In our next example we solve system (1) under the assumption that $k_1 = 6$, $k_2 = 4$, $m_1 = 1$, and $m_2 = 1$.

EXAMPLE 4 A Special Case of System (1)

Solve

$$\begin{aligned} x_1'' + 10x_1 \quad - 4x_2 &= 0 \\ -4x_1 + x_2'' + 4x_2 &= 0 \end{aligned} \tag{13}$$

subject to $x_1(0) = 0$, $x_1'(0) = 1$, $x_2(0) = 0$, $x_2'(0) = -1$.

Solution Using elimination on the equivalent form of the system

$$\begin{aligned} (D^2 + 10)x_1 \quad - 4x_2 &= 0 \\ -4x_1 + (D^2 + 4)x_2 &= 0 \end{aligned}$$

we find that x_1 and x_2 satisfy, respectively,

$$(D^2 + 2)(D^2 + 12)x_1 = 0 \quad \text{and} \quad (D^2 + 2)(D^2 + 12)x_2 = 0.$$

Thus we find

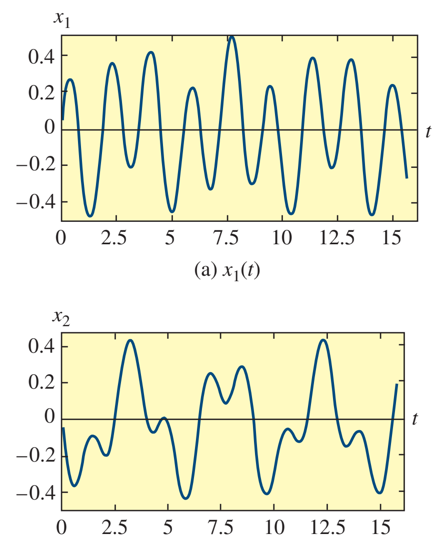
$$\begin{aligned} x_1(t) &= c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t + c_3 \cos 2\sqrt{3}t + c_4 \sin 2\sqrt{3}t \\ x_2(t) &= c_5 \cos \sqrt{2}t + c_6 \sin \sqrt{2}t + c_7 \cos 2\sqrt{3}t + c_8 \sin 2\sqrt{3}t. \end{aligned}$$

Substituting both expressions into the first equation of (13) and simplifying eventually yields $c_5 = 2c_1$, $c_6 = 2c_2$, $c_7 = -\frac{1}{2}c_3$, $c_8 = -\frac{1}{2}c_4$. Thus, a solution of (13) is

$$\begin{aligned} x_1(t) &= c_1 \cos \sqrt{2}t + c_2 \sin \sqrt{2}t + c_3 \cos 2\sqrt{3}t + c_4 \sin 2\sqrt{3}t \\ x_2(t) &= 2c_1 \cos \sqrt{2}t + 2c_2 \sin \sqrt{2}t - \tfrac{1}{2}c_3 \cos 2\sqrt{3}t - \tfrac{1}{2}c_4 \sin 2\sqrt{3}t. \end{aligned}$$

The stipulated initial conditions then imply $c_1 = 0$, $c_2 = -\sqrt{2}/10$, $c_3 = 0$, $c_4 = \sqrt{3}/5$. And so the solution of the initial-value problem is

$$\begin{aligned} x_1(t) &= -\frac{\sqrt{2}}{10}\sin\sqrt{2}t + \frac{\sqrt{3}}{5}\sin 2\sqrt{3}t \\ x_2(t) &= -\frac{\sqrt{2}}{5}\sin\sqrt{2}t - \frac{\sqrt{3}}{10}\sin 2\sqrt{3}t. \end{aligned} \tag{14}$$



(a) $x_1(t)$

(b) $x_2(t)$

FIGURE 3.12.2 Displacements of the two masses in Example 4

The graphs of x_1 and x_2 in **FIGURE 3.12.2** reveal the complicated oscillatory motion of each mass. ≡

We will revisit Example 4 in Section 4.6, where we will solve (13) by means of the Laplace transform.

3.12 Exercises

Answers to selected odd-numbered problems begin on page ANS-000.

In Problems 1–20, solve the given system of differential equations by systematic elimination.

1. $\dfrac{dx}{dt} = 2x - y$

$\dfrac{dy}{dt} = x$

2. $\dfrac{dx}{dt} = 4x + 7y$

$\dfrac{dy}{dt} = x - 2y$

3. $\dfrac{dx}{dt} = -y + t$

$\dfrac{dy}{dt} = x - t$

4. $\dfrac{dx}{dt} - 4y = 1$

$\dfrac{dy}{dt} + x = 2$

5. $(D^2 + 5)x - \quad 2y = 0$

$-2x + (D^2 + 2)y = 0$

6. $(D + 1)x + (D - 1)y = 2$
$3x + (D + 2)y = -1$

7. $\dfrac{d^2x}{dt^2} = 4y + e^t$
$\dfrac{d^2y}{dt^2} = 4x - e^t$

8. $\dfrac{d^2x}{dt^2} + \dfrac{dy}{dt} = -5x$
$\dfrac{dx}{dt} + \dfrac{dy}{dt} = -x + 4y$

9. $Dx + D^2y = e^{3t}$
$(D + 1)x + (D - 1)y = 4e^{3t}$

10. $D^2x - Dy = t$
$(D + 3)x + (D + 3)y = 2$

11. $(D^2 - 1)x - y = 0$
$(D - 1)x + Dy = 0$

12. $(2D^2 - D - 1)x - (2D + 1)y = 1$
$(D - 1)x + Dy = -1$

13. $2\dfrac{dx}{dt} - 5x + \dfrac{dy}{dt} = e^t$
$\dfrac{dx}{dt} - x + \dfrac{dy}{dt} = 5e^t$

14. $\dfrac{dx}{dt} + \dfrac{dy}{dt} = e^t$
$-\dfrac{d^2x}{dt^2} + \dfrac{dx}{dt} + x + y = 0$

15. $(D - 1)x + (D^2 + 1)y = 1$
$(D^2 - 1)x + (D + 1)y = 2$

16. $D^2x - 2(D^2 + D)y = \sin t$
$x + Dy = 0$

17. $Dx = y$
$Dy = z$
$Dz = x$

18. $Dx + z = e^t$
$(D - 1)x + Dy + Dz = 0$
$x + 2y + Dz = e^t$

19. $\dfrac{dx}{dt} = 6y$
$\dfrac{dy}{dt} = x + z$
$\dfrac{dz}{dt} = x + y$

20. $\dfrac{dx}{dt} = -x + z$
$\dfrac{dy}{dt} = -y + z$
$\dfrac{dz}{dt} = -x + y$

In Problems 21 and 22, solve the given initial-value problem.

21. $\dfrac{dx}{dt} = -5x - y$
$\dfrac{dy}{dt} = 4x - y$
$x(1) = 0,\ y(1) = 1$

22. $\dfrac{dx}{dt} = y - 1$
$\dfrac{dy}{dt} = -3x + 2y$
$x(0) = 0,\ y(0) = 0$

≡ Mathematical Models

23. Projectile Motion A projectile shot from a gun has weight $w = mg$ and velocity **v** tangent to its path of motion. Ignoring air resistance and all other forces acting on the projectile except its weight, determine a system of differential equations that describes its path of motion. See **FIGURE 3.12.3**. Solve the system. [*Hint*: Use Newton's second law of motion in the x and y directions.]

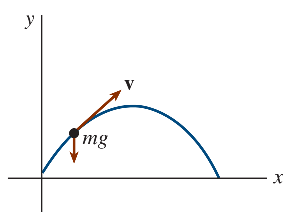


FIGURE 3.12.3 Path of projectile in Problem 23

24. Projectile Motion with Air Resistance Determine a system of differential equations that describes the path of motion in Problem 23 if air resistance is a retarding force **k** (of magnitude k) acting tangent to the path of the projectile but opposite to its motion. See **FIGURE 3.12.4**. Solve the system. [*Hint*: **k** is a multiple of velocity, say c**v**.]

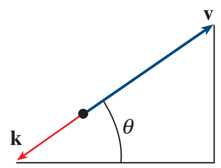


FIGURE 3.12.4 Forces in Problem 24

≡ Computer Lab Assignments

25. Consider the solution $x_1(t)$ and $x_2(t)$ of the initial-value problem given at the end of Example 3. Use a CAS to graph $x_1(t)$ and $x_2(t)$ in the same coordinate plane on the interval [0, 100]. In Example 3, $x_1(t)$ denotes the number of pounds of salt in tank A at time t, and $x_2(t)$ the number of pounds of salt in tank B at time t. See Figure 2.9.1. Use a root-finding application to determine when tank B contains more salt than tank A.

26. **(a)** Reread Problem 8 of Exercises 2.9. In that problem you were asked to show that the system of differential equations

$$\frac{dx_1}{dt} = -\frac{1}{50}x_1$$
$$\frac{dx_2}{dt} = \frac{1}{50}x_1 - \frac{2}{75}x_2$$
$$\frac{dx_3}{dt} = \frac{2}{75}x_2 - \frac{1}{25}x_3$$

is a model for the amounts of salt in the connected mixing tanks A, B, and C shown in Figure 2.9.7. Solve the system subject to $x_1(0) = 15$, $x_2(t) = 10$, $x_3(t) = 5$.

(b) Use a CAS to graph $x_1(t)$, $x_2(t)$, and $x_3(t)$ in the same coordinate plane on the interval [0, 200].

(c) Since only pure water is pumped into tank A, it stands to reason that the salt will eventually be flushed out of all three tanks. Use a root-finding application of a CAS to determine the time when the amount of salt in each tank is less than or equal to 0.5 pounds. When will the amounts of salt $x_1(t)$, $x_2(t)$, and $x_3(t)$ be simultaneously less than or equal to 0.5 pounds?

27. **(a)** Use systematic elimination to solve the system (1) for the coupled spring/mass system when $k_1 = 4$, $k_2 = 2$, $m_1 = 2$, and $m_2 = 1$ and with initial conditions $x_1(0) = 2$, $x_1'(0) = 1$, $x_2(0) = -1$, $x_2'(0) = 1$.

(b) Use a CAS to plot the graphs of $x_1(t)$ and $x_2(t)$ in the tx-plane. What is the fundamental difference in the motions of the masses m_1 and m_2 in this problem and that of the masses illustrated in Figure 3.12.2?

(c) As parametric equations, plot $x_1(t)$ and $x_2(t)$ in the x_1x_2-plane. The curve defined by these parametric equations is called a **Lissajous curve**.

3 in Review

Answers to selected odd-numbered problems begin on page ANS-000.

Answer Problems 1–8 without referring back to the text. Fill in the blank or answer true/false.

1. The only solution of the initial-value problem $y'' + x^2y = 0$, $y(0) = 0$, $y'(0) = 0$ is ______.

2. For the method of undetermined coefficients, the assumed form of the particular solution y_p for $y'' - y = 1 + e^x$ is ______.

3. A constant multiple of a solution of a linear differential equation is also a solution.

4. If f_1 and f_2 are linearly independent functions on an interval I, then their Wronskian $W(f_1, f_2) \neq 0$ for all x in I.

5. If a 10-pound weight stretches a spring 2.5 feet, a 32-pound weight will stretch it _____ feet.

6. The period of simple harmonic motion of an 8-pound weight attached to a spring whose constant is 6.25 lb/ft is _____ seconds.

7. The differential equation describing the motion of a mass attached to a spring is $x'' + 16x = 0$. If the mass is released at $t = 0$ from 1 meter above the equilibrium position with a downward velocity of 3 m/s, the amplitude of vibrations is ______ meters.

8. If simple harmonic motion is described by $x(t) = (\sqrt{2}/2)\sin(2t + \phi)$, the phase angle ϕ is ________ when $x(0) = -\frac{1}{2}$ and $x'(0) = 1$.

9. Give an interval over which $f_1(x) = x^2$ and $f_2(x) = x\,|x|$ are linearly independent. Then give an interval on which f_1 and f_2 are linearly dependent.

10. Without the aid of the Wronskian determine whether the given set of functions is linearly independent or linearly dependent on the indicated interval.

(a) $f_1(x) = \ln x,\ f_2(x) = \ln x^2,\ (0, \infty)$

(b) $f_1(x) = x^n,\ f_2(x) = x^{n+1},\ n = 1, 2, \ldots, (-\infty, \infty)$

(c) $f_1(x) = x,\ f_2(x) = x + 1,\ (-\infty, \infty)$

(d) $f_1(x) = \cos(x + \pi/2),\ f_2(x) = \sin x,\ (-\infty, \infty)$

(e) $f_1(x) = 0,\ f_2(x) = x,\ (-5, 5)$

(f) $f_1(x) = 2,\ f_2(x) = 2x,\ (-\infty, \infty)$

(g) $f_1(x) = x^2,\ f_2(x) = 1 - x^2,\ f_3(x) = 2 + x^2,\ (-\infty, \infty)$

(h) $f_1(x) = xe^{x+1},\ f_2(x) = (4x - 5)e^x,\ f_3(x) = xe^x,\ (-\infty, \infty)$

11. Suppose $m_1 = 3$, $m_2 = -5$, and $m_3 = 1$ are roots of multiplicity one, two, and three, respectively, of an auxiliary equation. Write down the general solution of the corresponding homogeneous linear DE if it is

(a) an equation with constant coefficients,

(b) a Cauchy–Euler equation.

12. Find a Cauchy–Euler differential equation $ax^2y'' + bxy' + cy = 0$, where a, b, and c are real constants, if it is known that

(a) $m_1 = 3$ and $m_2 = -1$ are roots of its auxiliary equation,

(b) $m_1 = i$ is a complex root of its auxiliary equation.

In Problems 13–28, use the procedures developed in this chapter to find the general solution of each differential equation.

13. $y'' - 2y' - 2y = 0$

14. $2y'' + 2y' + 3y = 0$

15. $y''' + 10y'' + 25y' = 0$

16. $2y''' + 9y'' + 12y' + 5y = 0$

17. $3y''' + 10y'' + 15y' + 4y = 0$

18. $2y^{(4)} + 3y''' + 2y'' + 6y' - 4y = 0$

19. $y'' - 3y' + 5y = 4x^3 - 2x$

20. $y'' - 2y' + y = x^2e^x$

21. $y''' - 5y'' + 6y' = 8 + 2\sin x$

22. $y''' - y'' = 6$

23. $y'' - 2y' + 2y = e^x \tan x$

24. $y'' - y = \dfrac{2e^x}{e^x + e^{-x}}$

25. $6x^2y'' + 5xy' - y = 0$

26. $2x^3y''' + 19x^2y'' + 39xy' + 9y = 0$

27. $x^2y'' - 4xy' + 6y = 2x^4 + x^2$

28. $x^2y'' - xy' + y = x^3$

29. Write down the form of the general solution $y = y_c + y_p$ of the given differential equation in the two cases $\omega \neq \alpha$ and $\omega = \alpha$. Do not determine the coefficients in y_p.

(a) $y'' + \omega^2y = \sin \alpha x$

(b) $y'' - \omega^2y = e^{\alpha x}$

30. (a) Given that $y = \sin x$ is a solution of $y^{(4)} + 2y''' + 11y'' + 2y' + 10y = 0$, find the general solution of the DE *without the aid of a calculator or a computer.*

(b) Find a linear second-order differential equation with constant coefficients for which $y_1 = 1$ and $y_2 = e^{-x}$ are solutions of the associated homogeneous equation and $y_p = \frac{1}{2}x^2 - x$ is a particular solution of the nonhomogeneous equation.

31. (a) Write the general solution of the fourth-order DE $y^{(4)} - 2y'' + y = 0$ entirely in terms of hyperbolic functions.

(b) Write down the form of a particular solution of $y^{(4)} - 2y'' + y = \sinh x$.

32. Consider the differential equation $x^2y'' - (x^2 + 2x)y' + (x + 2)y = x^3$. Verify that $y_1 = x$ is one solution of the associated homogeneous equation. Then show that the method of reduction of order discussed in Section 3.2 leads both to a second solution y_2 of the homogeneous equation and to a particular solution y_p of the nonhomogeneous equation. Form the general solution of the DE on the interval $(0, \infty)$.

In Problems 33–38, solve the given differential equation subject to the indicated conditions.

33. $y'' - 2y' + 2y = 0,\ y(\pi/2) = 0,\ y(\pi) = -1$

34. $y'' + 2y' + y = 0,\ y(-1) = 0,\ y'(0) = 0$

35. $y'' - y = x + \sin x,\ y(0) = 2,\ y'(0) = 3$

36. $y'' + y = \sec^3 x,\ y(0) = 1,\ y'(0) = \frac{1}{2}$

37. $y'y'' = 4x,\ y(1) = 5,\ y'(1) = 2$

38. $2y'' = 3y^2,\ y(0) = 1,\ y'(0) = 1$

39. (a) Use a CAS as an aid in finding the roots of the auxiliary equation for $12y^{(4)} + 64y''' + 59y'' - 23y' - 12y = 0$. Give the general solution of the equation.

(b) Solve the DE in part (a) subject to the initial conditions $y(0) = -1, y'(0) = 2, y''(0) = 5, y'''(0) = 0$. Use a CAS as an aid in solving the resulting systems of four equations in four unknowns.

40. Find a member of the family of solutions of

$$xy'' + y' + \sqrt{x} = 0$$

whose graph is tangent to the x-axis at $x = 1$. Use a graphing utility to obtain the solution curve.

In Problems 41–44, use systematic elimination to solve the given system.

41. $\dfrac{dx}{dt} + \dfrac{dy}{dt} = 2x + 2y + 1$

$\dfrac{dx}{dt} + 2\dfrac{dy}{dt} = y + 3$

42. $\dfrac{dx}{dt} = 2x + y + t - 2$

$\dfrac{dy}{dt} = 3x + 4y - 4t$

43. $(D - 2)x \qquad -y = -e^t$

$-3x + (D - 4)y = -7e^t$

44. $(D + 2)x + (D + 1)y = \sin 2t$

$5x + (D + 3)y = \cos 2t$

45. A free undamped spring/mass system oscillates with a period of 3 s. When 8 lb is removed from the spring, the system then has a period of 2 s. What was the weight of the original mass on the spring?

46. A 12-pound weight stretches a spring 2 feet. The weight is released from a point 1 foot below the equilibrium position with an upward velocity of 4 ft/s.

(a) Find the equation describing the resulting simple harmonic motion.

(b) What are the amplitude, period, and frequency of motion?

(c) At what times does the weight return to the point 1 foot below the equilibrium position?

(d) At what times does the weight pass through the equilibrium position moving upward? moving downward?

(e) What is the velocity of the weight at $t = 3\pi/16$ s?

(f) At what times is the velocity zero?

47. A spring with constant $k = 2$ is suspended in a liquid that offers a damping force numerically equal to four times the instantaneous velocity. If a mass m is suspended from the spring, determine the values of m for which the subsequent free motion is nonoscillatory.

48. A 32-pound weight stretches a spring 6 inches. The weight moves through a medium offering a damping force numerically equal to β times the instantaneous velocity. Determine the values of β for which the system will exhibit oscillatory motion.

49. A series circuit contains an inductance of $L = 1$ h, a capacitance of $C = 10^{-4}$ f, and an electromotive force of $E(t) = 100 \sin 50t$ V. Initially the charge q and current i are zero.

(a) Find the equation for the charge at time t.

(b) Find the equation for the current at time t.

(c) Find the times for which the charge on the capacitor is zero.

50. Show that the current $i(t)$ in an LRC-series circuit satisfies the differential equation

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = E'(t),$$

where $E'(t)$ denotes the derivative of $E(t)$.

51. Consider the boundary-value problem

$$y'' + \lambda y = 0, \quad y(0) = y(2\pi), \quad y'(0) = y'(2\pi).$$

Show that except for the case $\lambda = 0$, there are two independent eigenfunctions corresponding to each eigenvalue.

52. A bead is constrained to slide along a frictionless rod of length L. The rod is rotating in a vertical plane with a constant angular velocity ω about a pivot P fixed at the midpoint of the rod, but the design of the pivot allows the bead to move along the entire length of the rod. Let $r(t)$ denote the position of the bead relative to this rotating coordinate system, as shown in **FIGURE 3.R.1**. In order to apply Newton's second law of motion to this rotating frame of reference it is necessary to use the fact that the net force acting on the bead is the sum of the real forces (in this case, the force due to gravity) and the inertial forces (coriolis, transverse, and centrifugal). The mathematics is a little complicated, so we give just the resulting differential equation for r,

$$m\frac{d^2r}{dt^2} = m\omega^2 r - mg\sin(\omega t).$$

(a) Solve the foregoing DE subject to the initial conditions $r(0) = r_0, r'(0) = v_0$.

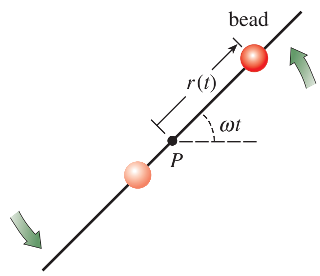


FIGURE 3.R.1 Rotating rod in Problem 52

(b) Determine initial conditions for which the bead exhibits simple harmonic motion. What is the minimum length L of the rod for which it can accommodate simple harmonic motion of the bead?

(c) For initial conditions other than those obtained in part (b), the bead must eventually fly off the rod. Explain using the solution $r(t)$ in part (a).

(d) Suppose $\omega = 1$ rad/s. Use a graphing utility to plot the graph of the solution $r(t)$ for the initial conditions $r(0) = 0$, $r'(0) = v_0$, where v_0 is 0, 10, 15, 16, 16.1, and 17.

(e) Suppose the length of the rod is $L = 40$ ft. For each pair of initial conditions in part (d), use a root-finding application to find the total time that the bead stays on the rod.

53. Suppose a mass m lying on a flat, dry, frictionless surface is attached to the free end of a spring whose constant is k. In **FIGURE 3.R.2(a)** the mass is shown at the equilibrium position $x = 0$; that is, the spring is neither stretched nor compressed. As shown in Figure 3.R.2(b), the displacement $x(t)$ of the mass to the right of the equilibrium position is positive and negative to the left. Derive a differential equation for the free horizontal (sliding) motion of the mass. Discuss the difference between the derivation of this DE and the analysis leading to (1) of Section 3.8.

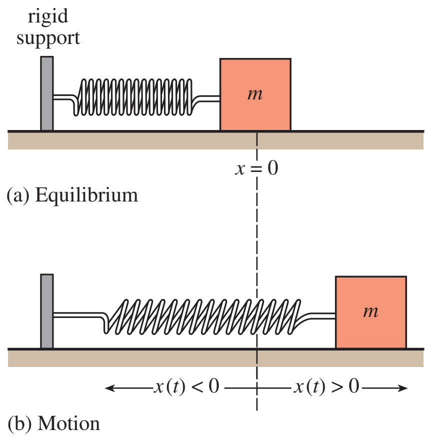


FIGURE 3.R.2 Sliding spring/mass system in Problem 53

54. What is the differential equation of motion in Problem 53 if kinetic friction (but no other damping forces) acts on the sliding mass? [*Hint*: Assume that the magnitude of the force of kinetic friction is $f_k = \mu mg$, where mg is the weight of the mass and the constant $\mu > 0$ is the coefficient of kinetic friction. Then consider two cases: $x' > 0$ and $x' < 0$. Interpret these cases physically.]

In Problems 55 and 56, use a Green's function to solve the given initial-value problem.

55. $y'' + y = \tan x$, $y(0) = 2$, $y'(0) = -5$

56. $x^2y'' - 3xy + 4y = \ln x$, $y(1) = 0$, $y'(1) = 0$

57. Historically, in order to maintain quality control over munitions (bullets) produced by an assembly line, the manufacturer would use a ballistic pendulum to determine the muzzle velocity of a gun; that is, the speed of a bullet as it leaves the barrel. The **ballistic pendulum** (invented in 1742), is simply a plane pendulum consisting of a rod of negligible mass to which a block of wood of mass m_w is attached. The system is set in motion by the impact of a bullet that is moving horizontally at the unknown muzzle velocity v_b; at the time of the impact, $t = 0$, the combined mass is $m_w + m_b$, where m_b is the mass of the bullet embedded in the wood. We have seen in (7) of Section 3.10 that in the case of small oscillations, the angular displacement $\theta(t)$ of a plane pendulum shown in Figure 3.11.3 is given by the linear DE $\theta'' + (g/l)\theta = 0$, where $\theta > 0$ corresponds to motion to the right of vertical. The velocity v_b can be found by measuring the height h of the mass $m_w + m_b$ at the maximum displacement angle θ_{max} shown in **FIGURE 3.R.3**.

Intuitively, the horizontal velocity V of the combined mass $m_w + m_b$ after impact is only a fraction of the velocity v_b of the bullet, that is, $V = \left(\dfrac{m_b}{m_w + m_b}\right)v_b$. Now recall, a distance s traveled by a particle moving along a circular path is related to the radius l and central angle θ by the formula $s = l\theta$. By differentiating the last formula with respect to time t, it follows that the angular velocity ω of the mass and its linear velocity v are related by $v = l\omega$. Thus the initial angular velocity ω_0 at the time t at which the bullet impacts the wood block is related to V by $V = l\omega_0$ or $\omega_0 = \left(\dfrac{m_b}{m_w + m_b}\right)\dfrac{v_b}{l}$.

(a) Solve the initial-value problem

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0,\ \theta(0) = 0,\ \theta'(0) = \omega_0.$$

(b) Use the result from part (a) to show that

$$v_b = \left(\frac{m_w + m_b}{m_b}\right)\sqrt{lg}\,\theta_{max}.$$

(c) Use Figure 3.R.3 to express $\cos\theta_{max}$ in terms of l and h. Then use the first two terms of the Maclaurin series for $\cos\theta$ to express θ_{max} in terms of l and h. Finally, show that v_b is given (approximately) by

$$v_b = \left(\frac{m_w + m_b}{m_b}\right)\sqrt{2gh}.$$

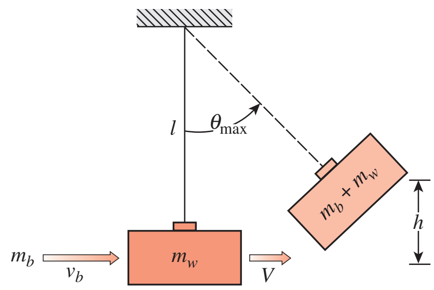


FIGURE 3.R.3 Ballistic pendulum in Problem 57

58. Use the result in Problem 57 to find the muzzle velocity v_b when $m_b = 5\text{g}$, $m_w = 1\text{kg}$, and $h = 6$ cm.