Exercises

4.5.1 In the fox-rabbit model, we saw that the equilibrium point (130, 1200) is technically unstable.

a. Comment on how "unstable" this point is. That is, if the populations start near (130, 1200), do they very quickly move away from this point or not?

b. If a biologist advocates introducing about 20 new foxes and 300 new rabbits to the forest so that the populations are near (130, 1200), would this help stabilize the populations, or would they fluctuate more? Explain.

4.5.2 Consider the parameter "fox death factor."

a. Investigate the sensitivity of the system to the value of this parameter (use initial populations of 110 foxes and 900 rabbits).

b. Can it ever be greater than or equal to 1? Why or why not? (Hint: See equations (4.12) and (4.16))

c. This parameter could be interpreted as the proportion of foxes that survive from one month to the next in the absence of rabbits. Suppose that a disease infects the fox population, decreasing the proportion that survives each month. What effect might this have on the populations?

4.5.3 Consider a model of the form

$$\begin{aligned} F_{n+1} &= aF_n + bF_nR_n \\ R_{n+1} &= cR_n + dF_nR_n \end{aligned}$$

where a, b, c, and d are non–zero constants. Find a formula for the equilibria of the system in terms of a, b, c, and d.

4.5.4 Suppose hunters are allowed to kill m rabbits at the end of each month.

a. Modify the model to take this into account (use the parameters and initial populations shown in Figure 4.23).

b. What effect will this have in the long–term? Would you say the system is sensitive to the parameter m?

c. How many rabbits could they kill each month and still have the populations survive in the long–term?

4.5.5 A biologist is concerned by the large fluctuations in the fox and rabbit populations. To combat this, she advocates the building of rabbit shelters to help increase the survival of baby rabbits, thus increasing the rabbit birth factor. Does our model predict that this would have the intended affect (start with the parameters and initial conditions in Figure 4.23)? Explain your answer.

4.5.6 Consider a forest that contains Foxes and Wolves which compete for the same food resources. If F_n and W_n represent the populations of foxes and wolves, respectively, at the end of month n, a model for their populations is:

$$F_{n+1} = 1.2\,F_n - 0.001\,F_nW_n$$
$$W_{n+1} = 1.3\,W_n - 0.002\,F_nW_n$$

a. Explain the meaning of the parameters 1.2 and 1.3.
b. Why are the parameters -0.001 and -0.002 both negative?
c. Find the equilibria for this system and graphically determine if they are stable or unstable (consider only up to $n = 25$).
d. Is this model sensitive to the initial populations? Why or why not?

4.6 Epidemics

Consider a community of 1,000 people in which three members get sick with the flu. The following week, five new cases of the flu are reported. We are interested in modeling the spread of the disease through the community.

Consider the following assumptions:

1. Nobody enters or leaves the community and no one in the community has contact with anyone outside the community.
2. Each person is either Susceptible (able to get the flu), Infected (currently has the flu and able to spread it), or Removed (already had the flu and is not able to get it again). Initially each person is either susceptible or infected.
3. A susceptible person can get the flu only by contact with an infected person.
4. Once a person gets the flu, he/she cannot get it again.
5. The average duration of the flu is 2 weeks, during which time an infected person can spread the disease to a susceptible person.

The model we are going to build is called an **SIR** model (see, for instance, Allman, Elizabeth Spencer and John A. Rhodes, *Mathematical Models in Biology: An Introduction*, Cambridge University Press, 2004 for more information on this type of model). We begin by dividing the population into three categories: Susceptible, Infected, and Removed, as described in assumption 2. People move between these three categories as illustrated in Figure 4.28.

Let S_n, I_n, and R_n represent the numbers of people that are susceptible, infected, and removed, respectively, at the end of week n. As in previous models, we will begin by modeling the *change* of these variables in terms of difference equations.