Exercises

4.2.1 Consider the linear dynamical system $a_{n+1} = ba_n$.

a. Suppose $b = 1$. Describe the equilibrium values of the system. Are they stable or unstable?
b. Suppose $b \neq 1$. Describe the equilibrium value of the system. For what values of b is this equilibrium value stable? For which is it unstable?

4.2.2 Consider the affine system $a_{n+1} = ba_n + m$ where $b \neq 0$.

a. Analytically (meaning without using Excel) show that $a_n = b^n \left(a_0 - \frac{m}{1-b}\right) + \frac{m}{1-b}$ is the solution to the system.
b. Under what conditions is $\lim_{n\to\infty} = \infty$ (meaning what are the values of m, b, and a_0 for which this limit holds)? Under what conditions is the limit equal to $-\infty$? In these cases, is the equalibrium value $a = \frac{m}{1-b}$ stable or unstable?
c. Under what conditions is $\lim_{n\to\infty} = \frac{m}{1-b}$? In this case, is the equalibrium value $a = \frac{m}{1-b}$ stable or unstable?

4.2.3 Suppose you open a savings account that pays 5% interest compounded yearly with a \$500 initial deposit and make a \$200 deposit at the end of each year.

a. Construct a model of the amount of money in the account at the end of each year and define the variables.
b. Use this model and the solution in Exercise 4.2.2a to find the amount of time it would take to build a value of \$12,000.

4.2.4 Your grandparents have their life savings of \$750,000 in a savings account that pays 6.7% interest compounded yearly. They want to spend all of it before they die. If they plan to live 15 more years, how much should they withdraw at the end of each year to accomplish their goal?

4.2.5 The amount of a drug in a patient's blood stream decreases at the rate of 50% per hour. Suppose an injection is given at the end of each hour which increases the amount of drug in the blood stream by 0.2 units.

a. Formulate a model of the amount of drug in the blood stream at the end of each hour.
b. Find all equilibrium value(s) of your model.
c. Graphically, classify each equilibrium value as stable or unstable.
d. Suppose we give the injection every 3 hours. Describe what happens to the long–term level of the drug in the blood stream.

4.2.6 In the generalized model of the savings account where we withdraw \$5,000 every two years, find a value of a_0 such that $a_2 = a_0$. If the initial deposit is this value, what is the long–term behavior of the system?

4.2.7 Suppose two countries are engaged in an arms race. Further suppose that the two countries have economies of similar strength and they have similar levels of distrust of each other. A simple model for T_n, the total amount of money spent by the two nations, is given by the affine system

$$T_{n+1} = (1 - r + d)\, T_n + c$$

where r is a positive constant that measures the restraint of growth due to the strength (or weakness) of the economies of the countries, d is a positive constant that measures the level of distrust between the countries, and c is a constant. If T_n eventually grows too large, then the countries will not be able to support the arms race and they must either negogiate an end or war breaks out. If T_n approaches a constant level, then the have a "stable" arms race. If T_n eventually decreases to 0, then the race ends.

a. Use the solution in problem 4.2.2 to find the solution to this affine system.
b. If $-1 < (1 - r + d) < 1$, use your solution in part 1 to find $\lim_{n\to\infty} T_n$. What happens to the arms race in this situation? Since $(1 - r + d) < 1$, what relationship is there between d and r?
c. If $T_0 = 199$, $c = -127$, $r = 1/3$, and $d = 1$, use your solution in part 1 to find $\lim_{n\to\infty} T_n$. What happens to the arms race in this situation?
d. Now suppose that $T_0 = 181$, $c = -127$, $r = 1/3$, and $d = 1$. Find $\lim_{n\to\infty} T_n$. What happens to the arms race in this situation?
e. Generalize your findings in the last two parts. Suppose that $(1 - r + d) > 1$ and find a relationship between T_0 and the fraction $\frac{c}{r-d}$ so that the arms race dies out. What is the relationship that leads to war or negotiations?

4.3 Growth of a Bacteria Population

Table 4.2 gives the number of bacteria in a Petri dish, a_n, at the end of each hour n. This data is graphed in Figure 4.10. We want to model a_n in terms of n.

n	0	1	2	3	4	5	6	7	8	9
a_n	10.3	17.2	27	45.3	80.2	125.3	176.2	255.6	330.8	390.4
n	10	11	12	13	14	15	16	17	18	19
a_n	440	520.4	560.4	600.5	610.8	614.5	618.3	619.5	620	621

Table 4.2

When modeling a dynamical system, it is often convenient to think about the way the variable(s) change between time periods. As in Chapter 2, we will consider the change between time periods $\Delta a_n = a_{n+1} - a_n$. The values of this variable for the first 8 values of n are given in Table 4.3.