Exercises

4.4.1 Suppose the rabbit death factor is -0.05 and the other factors are as shown in Figure 4.19. For initial populations of 500 foxes and 200 rabbits, we saw that the populations grow without bound in the long–run. Here we will investigate if this is true for all initial populations.

a. Fix the initial rabbit population at 200. What happens if the initial fox population increases or decreases? How large can the initial fox population be for both species to survive?
b. Fix the initial fox population at 500. What happens if the initial rabbit population increases or decreases? How small can the initial rabbit population be for both species to survive?

4.4.2 Investigate sensitivity to the parameter "rabbit birth factor" using values of the other parameters and the initial populations shown in Figure 4.19 (i.e. increase and decrease the rabbit birth factor and observe what happens to the populations in the long–run). A forest ranger knows that the populations will both die out under the parameters in Figure 4.19. To prevent this, she advocates building rabbit shelters in an attempt to increase the survival rate of baby rabbits. Does this model predict that this would have any effect on the survival of the populations? Why or why not?

4.4.3 With the parameters and the initial populations shown in 4.19 both populations die out. A hunting club claims that if they were allowed to kill (or harvest) a few foxes each month the populations would survive.

a. Modify the model to include this harvesting. Does the model support this claim? If it does, how many foxes could be harvested each month for both species to survive?
b. A conservation group claims that more foxes should be put into the forest each month for the populations to survive. Does your model support this claim?

4.4.4 A biologist models the populations of foxes and rabbits in another forest with the model

$$\begin{aligned} F_{n+1} &= 1.3F_n + 0.4R_n \\ R_{n+1} &= -0.6F_n + 1.05R_n \end{aligned}$$

a. Is this model consistent with the assumptions used in building our model? Why or why not?
b. What does this model suggest about the growth or death of foxes in the absence of rabbits?

4.4.5 Consider a two-dimensional system of the form

$$\begin{aligned} x_{n+1} &= ax_n + by_n \\ y_{n+1} &= cx_n + dy_n \end{aligned}$$

a. If $1 - d - \frac{cb}{1-a} \neq 0$, show that the only equilibrium point is (0, 0).
b. If $1 - d - \frac{cb}{1-a} = 0$, show that any point of the form $(x, x(1-a)/b)$ is an equilibrium point.

4.4.6 Consider the linear two-dimensional discrete dynamical system

$$\begin{aligned} x_{n+1} &= 0.5x_n + 0.4y_n \\ y_{n+1} &= -0.104x_n + 1.1y_n. \end{aligned}$$

a. Numerically verify that the solution to the system is

$$\begin{aligned} x_n &= 10c_1(1.02^n) + 5c_2(0.58^n) \qquad \text{where} \qquad c_1 = \frac{y_0}{11} - \frac{x_0}{55} \\ y_n &= 13c_1(1.02^n) + c_2(0.58^n) \qquad\qquad\qquad\quad c_2 = \frac{13x_0}{55} - \frac{2y_0}{11} \end{aligned}$$

for any initial conditions x_0 and y_0.

b. Using this solution, explain why the origin is a repelling equilibrium point.

4.5 A Nonlinear Predator–Prey Model

Let's consider a similar population of foxes and rabbits along with the same set of assumptions as in section 4.4, but we will model the assumptions differently. We will start with modeling assumptions 2 and 3 the same way:

$$\Delta F_n = F_{n+1} - F_n = -a\,F_n \tag{4.12}$$

$$\Delta R_n = R_{n+1} - R_n = d\,R_n \tag{4.13}$$

where $0 < a,\ d \leq 1$. In section 4.4, the coefficients of proportionality were kept constant. In this section we will model them as increasing or decreasing in the presence of the other population, as we did in modeling the growth of bacteria.

Assumption 4 says that the presence of rabbits increase the rate of growth of foxes, so we have

$$F_{n+1} - F_n = (-a + bR_n)F_n \tag{4.14}$$

where $b \geq 0$. Likewise, assumption 5 says that the presence of foxes decreases the rate of growth of rabbits, so we have

$$R_{n+1} - R_n = (d - cF_n)R_n \tag{4.15}$$

where $c \geq 0$. Rewriting (4.14) and (4.15) we get our model:

$$F_{n+1} = (1 - a)F_n + bR_nF_n \tag{4.16}$$

$$R_{n+1} = -cR_nF_n + (1 + d)R_n \tag{4.17}$$