Exercises

2.3.1 A very simple assumption about the population of rabbits in a forest is that it grows at a rate proportional to size of the population and the rabbits die at a rate proportional to the number of foxes in the forest. If p_n denotes the population of rabbits at time n, and F denotes the (constant) number of foxes, this assumption yields a model for the change in the rabbit population:

$$\Delta p_n = p_{n+1} - p_n = k_1 p_n - k_2 F$$

where k_1, $k_2 \geq 0$.

a. Solve this model for p_{n+1}
b. If $p_0 = 500$, $k_1 = 0.15$, and $k_2 = 0.25$, find the values of F for which the population of rabbits is decreasing, for which it is increasing, and for which it is constant. (**Hint:** If the population is constant, $p_1 = p_0$.)
c. Create a spreadsheet to numerically and graphically support your answer above.

2.3.2 Refer back to Example 2.3.1. We will analyze a refined model. Suppose we assume that Δp_n is proportional to the product of the population and its difference from 621, that is,

$$\Delta p_n = p_{n+1} - p_n = k p_n (621 - p_n)$$

a. Use the data in Table 2.3 to test this assumption by plotting Δp_n vs. $p_n(621 - p_n)$. Use the graph to estimate the constant of proportionality.
b. Use Excel to find the values of the population for days 0 through 20 using this model (use the same initial population as in Table 2.3).
c. Does this model seem more or less reasonable than the original one? Why or why not?

2.3.3 Refer again to Example 2.3.1 and consider the assumption that Δp_n is proportional to p_n^2. Use the data in Table 2.3 to test this assumption. Does this assumption appear to be reasonable? Why or why not?

2.3.4 Answer the following questions about proportionality:

a. If the amount of calories you consume at a meal is proportional to the length of the meal (in minutes), how does the calorie consumption of a meal lasting 5 minutes compare to that of a meal lasting 15 minutes (i.e. do you consume twice as many calories, half as many calories, etc.)?
b. If the surface area of a raindrop is proportional to the square of its radius, how does surface area of a raindrop with radius 2 compare to the area of a raindrop with radius 1?
c. If the volume of a potato is proportional to the cube of its length, how does the volume of potato A compare to potato B if the length of A is three times that of B?
d. If the maximum velocity of a car is inversely proportional to its mass, how much faster will a car go if you cut its mass in half?

2.3.5 A student wishes to determine how the size of a cloud affects the speed of a falling raindrop. She makes the following assumptions:

1. The speed is proportional to the weight of the raindrop.
2. The weight is proportional to the size, or volume, of the raindrop.
3. The size of the raindrop is proportional to the size of the cloud.

Use these assumptions to model the speed in terms of the size of the cloud. Don't forget to define the variables. Does this model seem reasonable? Why or why not?

2.3.6 Prove the following statement: If a is inversely proportional to b and b is directly proportional to c, then a is inversely proportional to c.

2.3.7 Martin wants to determine how the amount of money he has in his wallet will affect his grade in Math Modeling class. Consider the following assumptions:

1. His grade is directly proportional to the amount of time studied.
2. The amount of time studied is directly proportional to the amount of free time he has.
3. The amount of free time he has is inversely proportional to the amount of time he spends going out with his girlfriend.
4. The amount of time he spends going out with his girlfriend is directly proportional to the amount of money he has in his wallet.

Use these assumptions to model his grade in terms of the amount of money in his wallet. Don't forget to define the variables. If he wants a high grade, should he have more or less money in his wallet?

2.3.8 The amount of money A in a savings account with an interest rate of $100\,r\%$ that is *continuously compounded* is

$$A = Pe^{rt}$$

where P is the amount of the initial deposit and t is time from the initial deposit.

a. A man puts \$5000 in an account with an interest rate of 0.4% a month so that $r = 0.004$ and t is measured in months. Find a model for A in terms of t.
b. Suppose at the same time the man opens the account, he goes on a diet and expects to lose 8 pounds a month. If w represents the total amount of weight lost by month t, find a simple model for w in terms of t.
c. Combine the models in part a. and b. to find a model for A in terms of w.
d. At the time the man has lost a total of 26 pounds, find the amount of money in the account.
e. When the account has \$5125, find the total amount of weight lost.

2.3.9 The variable y is said to be *exponentially proportional* to the variable x if there exist non-zero constants k and a such that

$$y = ka^x.$$

a. For the savings account in exercise 2.3.8 part a., show that A is exponentially proportional to t and find the values of k and a.

b. Show that if y is exponentially proportional to x, and x is directly proportional to z, then y is exponentially proportional to z.

2.3.10 When modeling the spread of an infectious disease, the population is often divided into two categories: *susceptible* (those capable of getting the disease) and *infected* (those who have the disease). Let S_n and I_n denote the numbers of susceptible and infected people, respectively, at the beginning of time period n. One simple assumption is that the change in the number of infected people, $\Delta I_n = I_{n+1} - I_n$, is proportional to the product of S_n and I_n. Write down a model for ΔI_n based on this assumption and solve it for I_{n+1}. If we consider the product of S_n and I_n as modeling the "interaction" of suspectible and infected people, do you think the constant of proportionality is positive or negative? Explain.

2.3.11 Consider a forest containing foxes and rabbits where the foxes eat the rabbits. Let F_n and R_n denote the numbers of foxes and rabbits in the forest at the beginning of time period n. Ignoring all factors except the interaction of foxes and rabbits, a simple assumption about the change in the number of rabbits, $\Delta R_n = R_{n+1} - R_n$, is proportional to the product of R_n and F_n. Write down a model for ΔR_n based on this assumption and solve it for R_{n+1}. Do you think the constant of proportionality is positive or negative? Explain.

2.3.12 Translate each of the following descriptions of a physical scenario into a differential equation. Be sure to define the variables.

a. The rate of change of a population at time t is proportional to the population at time t.

b. The rate of change of the volume of a raindrop at time t as it falls through a cloud is proportional to the square of the volume at time t.

c. The velocity at time t of an object moving in a straight line is proportional to the third power of the object's distance from its starting point.

2.3.13 Let $A_1(t)$ and $A_2(t)$ denote the sizes of two opposing armies engaged in battle. One assumption we could make about these variables is that the rate of change of A_1 is proportional to A_2. Set up a differential equation that is consistent with this assumption. Ignoring reinforcements of the armies, is the constant of proportionality positive or negative? Explain.

2.3.14 When a hot cup of coffee is set on a desk, it initially cools very quickly. As its temperature decreases, it does not cool as quickly. This suggests a proportionality relationship. Newtons law of cooling states that the rate at which a hot object (such as a hot cup of coffee) cools is proportional to the difference in the room temperature and the temperature of the object (assuming the room temperature stays constant). Define variables for the temperature of the object and the room temperature and set up a differential equation to model Newton's law of cooling (do not solve the equation).

2.3.15 A simple assumption for the spread of a contagious disease is that the rate at which the number of infected individuals changes is proportional to the product of the total number infected

and the number not yet infected. Assume that initially one resident carries the flu viruses into a dorm with n residents. Let $x(t)$ represent the number of residents that are infected at time t. Set up a differential equation to model the spread of the flu through the dorm (do not solve the equation).

2.3.16 To model the force due to air resistance in Example 2.3.3, we assumed that the force is proportional to velocity. Briefly explain how you might collect data and use the data to determine if this assumption is reasonable.

2.4 Fitting Straight Lines Analytically

As we have seen, modeling with proportionality often requires us to fit a straight line to a set of data. Above, we used a graphical approach which can be rather subjective. In this section we will look at different definitions of a "best fit" line and find formulas for the slope and y-intercept of a best-fit line in terms of the x- and y-coordinates of the data points. This will give an objective approach to fitting a straight line.

The first step is to define criteria for a good-fitting line. The line in the left graph in Figure 2.12 fits the data "better" than the line in the right graph. What's the difference between these two lines? There are probably many ways to answer this question.

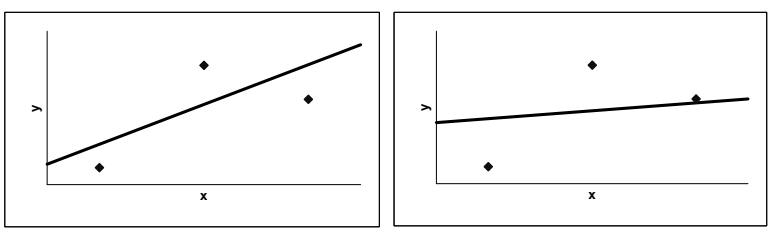

Figure 2.12

We see that in the right graph, the line is very close to the right-most point, but further from the other two points than the line in the left graph. We might say that the line in the left graph is "closer" to the data points in general than the line in the right graph. The idea of minimizing the distance between the line and the points will form the basis of the definition of a best-fit line.

These distances (sometimes called *errors*) are illustrated in Figure 2.13 with the dashed lines. If the coordinates of the points are given by

$$(x_i, y_i) \text{ for } i = 1, 2, \ldots, n$$

and the line is described by the function $f(x) = mx + b$, then the values of the distances are

$$|y_i - f(x_i)| \text{ for } i = 1, 2, \ldots, n$$