Example 5.2.3 Mixing Solutions

Consider a tank that contains 50 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing 50% alcohol is added to the tank at the rate of 2 gallons per minute. At the same time, solution is being drained from the tank at the rate of 3 gallons per minute. Assuming the tank is continuously stirred, find the amount of alcohol in the tank for times between 0 and 50.

Let $y(t)$ represent the amount of alcohol in the tank at time t. Note that alcohol is being added to the tank and drained out. This suggests a basic model:

$$\begin{aligned}\frac{dy}{dt} &= \text{Rate of change of alcohol (in gallons of alcohol/min)}\\ &= \text{Rate in} - \text{Rate out}\end{aligned}$$

Modeling the Rate in is easy. A 50% alcohol solution is being added to the tank at a constant 2 gallons per minute. Therefore,

$$\text{Rate in} = \left(\frac{0.5 \text{ gallons alcohol}}{1 \text{ gallon solution}}\right)\left(\frac{2 \text{ gallons solution}}{\text{min}}\right) = \frac{1 \text{ gallon alcohol}}{\text{min}}$$

Modeling the Rate out is a little more complicated because the proportion of the solution that is alcohol is changing. Note that the volume of solution in the tank is decreasing at the rate of 1 gallon/min. Therefore, the volume of solution in the tank at time t is simply $50 - t$. The amount of alcohol in the tank at time t is simply y. Therefore, similar to the model for Rate in, we have

$$\text{Rate out} = \left(\frac{y \text{ gallons alcohol}}{(50-t) \text{ gallons solution}}\right)\left(\frac{3 \text{ gallons solution}}{\text{min}}\right) = \frac{3y}{50-t}\frac{\text{gallons alcohol}}{\text{min}}$$

Thus, the differential equation is

$$\frac{dy}{dt} = 1 - \frac{3y}{50-t}.$$

In general, the differential equation describing $y(t)$ has the form

$$\frac{dy}{dt} = \begin{pmatrix}\text{Rate of sol}\\ \text{flowing in}\end{pmatrix}\begin{pmatrix}\text{Conc of sol}\\ \text{flowing in}\end{pmatrix} - \begin{pmatrix}\text{Rate of sol}\\ \text{flowing out}\end{pmatrix}\left(\frac{y}{\text{Vol of sol in tank}}\right)$$

The concentration of the solution in the tank is then

$$\text{Concentration} = \frac{y}{\text{Vol of sol in tank}}.$$

To approximate the volume of alcohol in the tank and the concentration and view them graphically, follow these steps:

1. Rename a blank worksheet "**Mixture**" and format it as in Figure 5.6. Copy row 5 down to row 504.

	A	B	C	D
1	**h =**	0.1		
2				
3	**Time**	**Vol Solution in Tank**	**Vol Alcohol**	**Concentration**
4	0	50	=5	=C4/B4
5	=A4+$B$1	=50-A5	=C4+$B$1*(1-3*C4/B4)	=C5/B5

Figure 5.6

2. Create a graph as in Figure 5.7. Notice that the amount of alcohol initially increases and then decreases to 0 as time approaches 50.

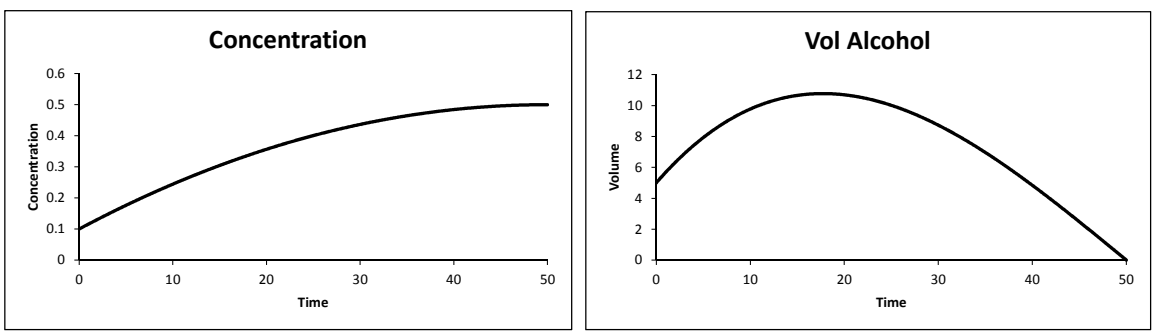

Figure 5.7

Using our results, we can find, for instance, the time at which the concentration in the tank is 0.4. Examining the numerical solution, we see the concentration is approximately 0.4 at time 25 min.

Example 5.2.4 Tank with Valve

Using Euler's method, we can easily solve mixing problems much more complicated than the previous example. Consider the following scenario: A solution of 50% alcohol and 50% water is poured at a constant rage of 1 L/min into a tank initially containing 5 L of pure water. The tank is well-stirred and the solution flowing out of the tank is controlled by a valve which is open for 1 min, closed for 1 min, open for 1 min, and so on. When the valve is open, solution flows out at a rate of 2 L/min. Find the time at which the concentration of alcohol in the tank is 0.4.

To model this scenario, note that the volume of solution in the tank is changing. When the valve is open, the volume decreases by 1 L/min. When the valve is closed, the volume increases by 1 L/min. Thus over a time of h min,

$$\text{Change in volume} = \begin{cases} 1 \cdot h \text{ L} & \text{if the valve is closed} \\ -1 \cdot h \text{ L} & \text{if the valve is open} \end{cases}.$$

If we let $y(t)$ denote the volume of alcohol in the tank at time t, the differential equation describing y is

$$\frac{dy}{dt} = 1 \cdot 0.5 - \begin{pmatrix} 0 \text{ if the valve is closed} \\ 2 \text{ if the valve is open} \end{pmatrix} \left(\frac{y}{\text{Vol of sol in tank}} \right).$$

To solve this differential equation with Euler's method, title a blank worksheet **Tank with Valve** and format it as in Figure 5.8. Copy row 5 down to row 504 then create graphs as in Figure 5.9.

	A	B	C	D	E
1	**h =**	0.1			
2					
3	**Time**	**Valve Open?**	**Vol Solution in Tank**	**Vol Alcohol**	**Concentration**
4	0	=IF(MOD(INT(A4),2)=0,1,0)	5	0	=D4/C4
5	=ROUND(A4+$B$1,1)	=IF(MOD(INT(A5),2)=0,1,0)	=IF(B4=1,C4-$B$1,C4+$B$1)	=D4+$B$1*(0.5-2*D4/C4*B4)	=D5/C5

Figure 5.8

The formulas in column **B** round the time down to the nearest whole number, calculates this rounded time modulus 2 and returns a 1 if this result is 0, meaning the valve is open, and returns 0 otherwise. The **ROUND** function in column **A** rounds off the time to one decimal point. This is necessary because when Excel adds a number repeatedly (as in adding h to the time in each step) it introduces a hidden "rounding" error. Thus when the time is supposed to be a whole number, Excel thinks it is not a whole number. The **ROUND** function corrects this error.

Notice that the volume of alcohol in the tank initially increases but then begins to fluctuate. The concentration initially increases and then levels out near 0.5. This is what we expect because the concentration of solution coming into the tank is 0.5. From the graph and the numerical results, we see that the concentration is about 0.4 near time $t = 7$ min.

Exercises

5.2.1 At time $t = 0$, a yeast culture weighs 0.5 grams. Two hours later, it weighs 2 grams. The maximum weight of the culture is 8 grams.

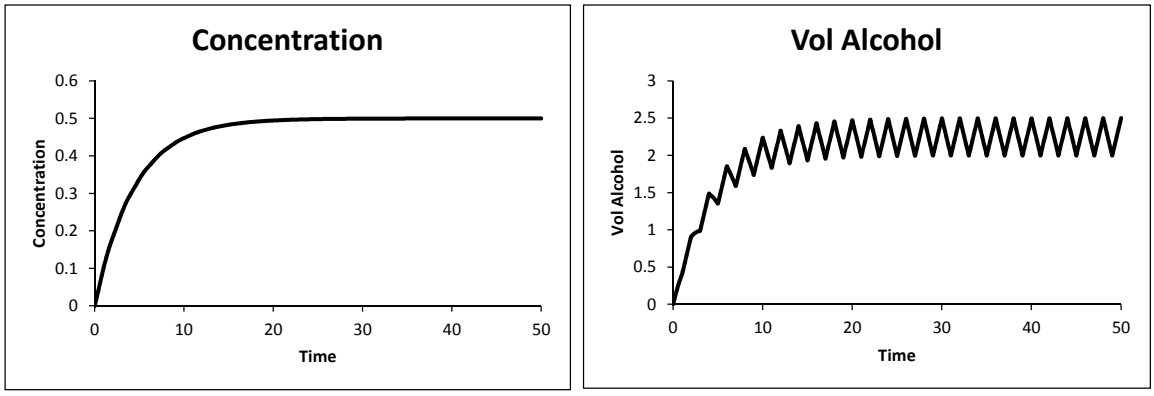

Figure 5.9

a. Create a spreadsheet to model the population using a logistic equation. Use a scroll bar to vary the value of k.

b. Use the scroll bar to find a value of k so that the condition $y(2) = 2$ is satisfied.

c. At what time is the weight increasing most rapidly? Support your answer numerically.

5.2.2 A 100-gallon tank is full of a solution that contains 15 pounds of salt. Starting at time $t = 0$, distilled water is poured into the tank at the rate of 5 gallons per minute. The solution in the tank is continuously well-stirred and is drained out of the tank at the rate of 5 gallons per minute.

a. Let $y(t)$ represent the weight of salt in the tank at time t. Form a differential equation to model dy/dt.

b. Use Euler's method to estimate the amount of salt in the tank at time $t = 10$.

c. At what time is the concentration of salt in the tank 0.1 lb/gal?

d. Find the quantity of the salt in the solution as $t \to \infty$. Is this what you expect?

5.2.3 A solution containing 0.05 kg of salt per L of solution flows at a constant rate of 6 L/min into a tank that initially holds 50 L of pure water. The tank is well-stirred and the solution flows out of the tank at a rate of 6 L/min.

a. Let $y(t)$ represent the mass of salt in the tank at time t. Form a differential equation to model dy/dt.

b. Use Euler's method to estimate the time at which the concentration of salt in the tank is 0.035 kg/L.

c. Now suppose initially there was 0.5 kg of salt in the tank. Find the time at which the concentration is 0.03 kg/L.

5.2.4 A solution containing 0.01 kg of salt per L of solution flows at a constant rate of 5 L/min into a tank that initially holds 15 L of water in which 0.5 kg of salt is dissolved. The tank is well-stirred and the solution flows out of the tank at a rate of 7 L/min.

a. Let $y(t)$ represent the mass of salt in the tank at time t. Form a differential equation to model dy/dt.

b. Use Euler's method to estimate the time at which the amount of salt in the tank is maximized.

c. Estimate the time at which the concentration of salt in the tank is 0.06 kg/L.

5.2.5 A tank initially contains 50 L of pure water. A solution of 50% water and 50% alcohol is poured into the tank at a rate described by $r(t) = 1 + \cos(t)$ L/min. The tank is well-stirred and the solution flows out of the tank at a constant rate of 1 L/min.

a. Use Euler's method to graph the volume of alcohol in the tank over the time interval [0, 250]. (**Hint**: From time t_n to time $t_{n+1} = t_n + h$, the volume in the tank increases by $h \cdot r(t_n)$ L and decreases by $h \cdot 1$ L.)

b. Approximate the time at which there is 17.5 L of alcohol in the tank.

c. Graph the concentration of the alcohol in the tank over the time interval [0, 250]. What is the limit of the concentration as $t \to \infty$? Does this make sense? Explain why or why not.

5.2.6 A tank initially contains 50 L of a solution composed of 90% water and 10% alcohol. A second solution containing 25% alcohol is added to the tank at a constant rate of 1 L/min. The tank is well-stirred and the solution flowing out of the tank is controlled by a valve which is open for 2 min, closed for 2 min, open for 2 min, and so on. When the valve is open, solution flows out at a rate of 3 L/min.

a. Determine the first time at which the volume of solution in the tank is 0 (**Hint**: The valve is open if the time modulus 4 is less than 2.)

b. Use Euler's method to graph the volume of alcohol in the tank from time 0 to the time found in part a.

c. At approximately what time is the volume of alcohol in the tank maximized?

5.2.7 Let $T(t)$ denote the temperature at time t hours from some starting point inside a building that is heated with a furnace. A simple model for T is

$$\frac{dT}{dt} = -k\,(T - M(t)) + k_U(T_D - T)$$

where

- $k > 0$ is a parameter measuring the level of insulation of the building (the more insulation, the lower the value of k),
- $M(t)$ is the temperature outside the building at time t,
- $k_U > 0$ is a parameter measuring the power of the furnace (the more powerful the furnace is, the higher the value of k_U), and
- T_D is the "desired" temperature inside the building.

a. According to this model, is the furnace heating the building faster when T is "close" to T_D or when T is "far" from T_D? Explain why.

b. Let $k = 0.25$, $M(t) = 8°\text{C}$, $k_U = 3$, and $T_D = 21$. If $T(0) = 12$, Use Euler's method to estimate the time at which the temperature reaches 18°C. Does the temperature ever reach the "desired" temperature of 21?

c. Consider the same building as in part a., but suppose the outside temperature is described by $M(t) = 12 + 4\cos(\pi t/12)$. Use Euler's method to estimate the inside temperature at time t. Graph both the inside and outside temperatures over the time interval [0, 72]. Do both temperatures reach their minimum and maximum values at the same time?

d. Consider the same scenario as in part b., but suppose the furnace is turn on between times 0 and 12, then off between 12 and 24, then on between 24 and 36, and so on. Use Euler's method to approximate the graph the temperature over the time interval [0, 72]. Compare this graph to the graph is part b. where the furnace was turned on all the time. Suppose the building is a store that is open only at the time the furnace is turned on. Based on the results of the model, would you recommend leaving the furnace on all the time, or shutting it off when the store is closed? Explain.

5.2.8 In Chapter 2, we modeled the velocity of a free-falling object with air resistance by assuming that the force due to air resistance is proportional to the velocity ($F_A = k\,v$). This yielded the differential equation

$$\frac{dv}{dt} + \frac{k}{m}\,v = g$$

where $g = 9.8\text{m/sec}^2$ and m = mass of the object.

a. Suppose $m = 10\,\text{g}$, $v\,(0) = 0$, and $k = 3$. Use Euler's method to approximate $\lim_{t\to\infty} v\,(t)$.

b. Use a scroll bar to examine what happens to $\lim_{t\to\infty} v\,(t)$ as k changes. Do your results make sense? Why or why not?

c. Add a scroll bar to vary the value of m. What happens to $\lim_{t\to\infty} v\,(t)$ as m changes? Does this make sense?

5.2.9 Consider again the model of the velocity of a free-falling object with air resistance from Chapter 2. Now assume that the force due to air resistance is described by $F_A = (3/n)\,v^n$ where $0 < n < 5$. This assumption yields the differential equation

$$\frac{dv}{dt} + \frac{3}{m \cdot n}\,v^n = g$$

where $g = 9.8\text{m/sec}^2$ and m = mass of the object.

a. Suppose $m = 10\,\text{g}$ and $v\,(0) = 0$. Use Euler's method to approximate $\lim_{t\to\infty} v\,(t)$.

b. Use a scroll bar to examine what happens to $\lim_{t\to\infty} v\,(t)$ for different values of n.

c. Compare the terminal velocity for $n > 1$ to that for $n = 1$. Also compare the amount of time needed to reach the terminal velocity. Repeat for $n < 1$.

d. If we wanted to model the free-fall of an object that is very aerodynamic, would we want to use a small value of n or a large value? Explain.

5.2.10 For a differential equation of the form $dy/dt = F(y)$, Euler's method approximates $y'(t)$ over the interval $[t_n, t_{n+1}]$ with the constant value $F(y_n)$ (see Figure 5.1). That is, $y'(t)$ is approximated using the left end-point of the interval. A better approximation can be found using the average of $y'(t)$ at the left and right end-points. This leads to the *improved* or *modified Euler's method*:

For $n = 0, 1, \ldots,$ let

1. $k_1 = F(y_n)$ (this is an approximation of $y'(t)$ at the left end-point),
2. $u_n = y_n + h \cdot k_1$ (this is a rough approximation of $y(t_{n+1})$, the y-value at the right end-point),
3. $k_2 = F(u_n)$ (this is an approximation of $y'(t)$ at the right end-point),
4. $t_{n+1} = t_n + h$, and
5. $y_{n+1} = y_n + h \cdot \dfrac{k_1 + k_2}{2}$.

Design a spreadsheet that uses the improved Euler's method to approximate the solution to the Newton's law of cooling problem in Example 5.1.1. Calculate the error in the approximation and compare this error to the error from the regular Euler's method. Does the improved Euler's method perform any better?

5.3 Quadratic Population Model

In this section we will model the population of two species with a system of two differential equations. We will not attempt to solve the system, rather we will numerically approximate solutions with Euler's method and analyze the trajectories in the phase–plane.

Euler's method for a system of two differential equations is very similar to that in section 5.2. Consider a two-dimensional system of differential equations with the general form

$$\frac{dx}{dt} = F(x, y), \quad \frac{dy}{dt} = G(x, y)$$

Euler's method for approximating solutions to a system such as this is:

$$\begin{array}{rclrclrcl}
t_1 & = & t_0 + h & x_1 & = & x_0 + hF(x_0, y_0) & y_1 & = & y_0 + hG(x_0, y_0) \\
t_2 & = & t_1 + h & x_2 & = & x_1 + hF(x_1, y_1) & y_2 & = & y_1 + hG(x_1, y_1) \\
 & \vdots & & & \vdots & & & \vdots & \\
t_{n+1} & = & t_n + h & x_{n+1} & = & x_n + hF(x_n, y_n) & y_{n+1} & = & y_n + hG(x_n, y_n)
\end{array}$$