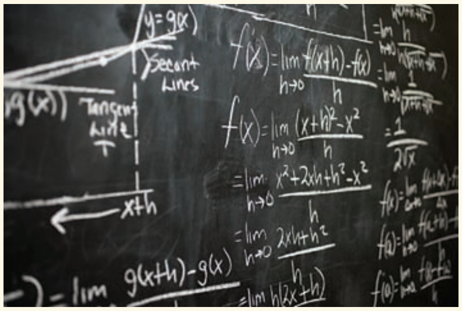
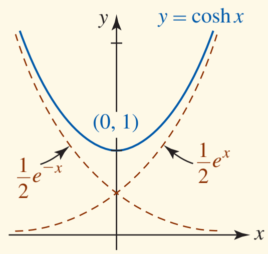
Chapter 3

The Derivative





In This Chapter The word *calculus* is a diminutive form of the Latin word *calx*, which means "stone." In ancient civilizations small stones or pebbles were often used as a means of reckoning. Consequently, the word *calculus* can refer to any systematic method of computation. However, over the last several hundred years the connotation of the word *calculus* has evolved to mean that branch of mathematics concerned with the calculation and application of entities known as derivatives and integrals. Thus, the subject known as **calculus** has been divided into two rather broad but related areas: **differential calculus** and **integral calculus**.

In this chapter we will begin our study of differential calculus.

3.1 The Derivative
3.2 Power and Sum Rules
3.3 Product and Quotient Rules
3.4 Trigonometric Functions
3.5 Chain Rule
3.6 Implicit Differentiation
3.7 Derivatives of Inverse Functions
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Chapter 3 in Review

3.1 The Derivative

Introduction In the last section of Chapter 2 we saw that the tangent line to a graph of a function $y = f(x)$ is the line through a point $(a, f(a))$ with slope given by

Recall, $m_{\tan}$ is also called the slope of the curve at $(a, f(a))$.

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

whenever the limit exists. For many functions it is usually possible to obtain a general formula that gives the value of the slope of a tangent line. This is accomplished by computing

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

for *any* x (for which the limit exists). We then substitute a value of x *after* the limit has been found.

A Definition The limit of the difference quotient in (1) defines a function—a function that is *derived* from the original function $y = f(x)$. This new function is called the **derivative function**, or simply the **derivative**, of f and is denoted by f'.

Definition 3.1.1 Derivative

The **derivative** of a function $y = f(x)$ at x is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{2}$$

whenever the limit exists.

Let us now reconsider Examples 1 and 2 of Section 2.7.

EXAMPLE 1 A Derivative

Find the derivative of $f(x) = x^2 + 2$.

Solution As in the calculation of $m_{\tan}$ in Section 2.7, the process of finding the derivative $f'(x)$ consists of four steps:

(*i*) $f(x+h) = (x+h)^2 + 2 = x^2 + 2xh + h^2 + 2$

(*ii*) $f(x+h) - f(x) = [x^2 + 2xh + h^2 + 2] - x^2 - 2 = h(2x + h)$

(*iii*) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{h(2x+h)}{h} = 2x + h$ ← cancel h's

(*iv*) $\displaystyle\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} [2x + h] = 2x.$

From step (*iv*) we see that the derivative of $f(x) = x^2 + 2$ is $f'(x) = 2x$. ■

Observe that the result $m_{\tan} = 2$ obtained in Example 1 of Section 2.7 is obtained by evaluating the derivative $f'(x) = 2x$ at $x = 1$, that is, $f'(1) = 2$.

EXAMPLE 2 Value of the Derivative

For $f(x) = x^2 + 2$, find $f'(-2)$, $f'(0)$, $f'(\tfrac{1}{2})$, and $f'(1)$. Interpret.

Solution From Example 1 we know that the derivative is $f'(x) = 2x$. Hence,

at $x = -2$, $\begin{cases} f(-2) = 6 & \leftarrow \text{point of tangency is } (-2, 6) \\ f'(-2) = -4 & \leftarrow \text{slope of tangent line at } (-2, 6) \text{ is } m = -4 \end{cases}$

at $x = 0$, $\begin{cases} f(0) = 2 & \leftarrow \text{point of tangency is } (0, 2) \\ f'(0) = 0 & \leftarrow \text{slope of tangent line at } (0, 2) \text{ is } m = 0 \end{cases}$

$$\text{at } x = \tfrac{1}{2}, \quad \begin{cases} f\left(\tfrac{1}{2}\right) = \tfrac{9}{4} & \leftarrow \text{point of tangency is } \left(\tfrac{1}{2}, \tfrac{9}{4}\right) \\ f'\left(\tfrac{1}{2}\right) = 1 & \leftarrow \text{slope of tangent line at } \left(\tfrac{1}{2}, \tfrac{9}{4}\right) \text{ is } m = 1 \end{cases}$$

$$\text{at } x = 1, \quad \begin{cases} f(1) = 3 & \leftarrow \text{point of tangency is } (1, 3) \\ f'(1) = 2. & \leftarrow \text{slope of tangent line at } (1, 3) \text{ is } m = 2 \end{cases}$$

Recall that a horizontal line has 0 slope. So the fact that $f'(0) = 0$ means that the tangent line is horizontal at $(0, 2)$. ■

By the way, if you trace back through the four-step process in Example 1, you will find that the derivative of $g(x) = x^2$ is also $g'(x) = 2x = f'(x)$. This makes intuitive sense; since the graph of $f(x) = x^2 + 2$ is a rigid vertical translation or shift of the graph of $g(x) = x^2$ for a given value of x, the points of tangency change but not the slope of the tangent line at the points. For example, at $x = 3$, $g'(3) = 6 = f'(3)$ but the points of tangency are $(3, g(3)) = (3, 9)$ and $(3, f(3)) = (3, 11)$.

EXAMPLE 3 A Derivative

Find the derivative of $f(x) = x^3$.

Solution To calculate $f(x + h)$, we use the Binomial Theorem.

$$(i)\quad f(x+h) = (x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$(ii)\quad f(x+h) - f(x) = [x^3 + 3x^2h + 3xh^2 + h^3] - x^3 = h(3x^2 + 3xh + h^2)$$

$$(iii)\quad \frac{f(x+h) - f(x)}{h} = \frac{h[3x^2 + 3xh + h^2]}{h} = 3x^2 + 3xh + h^2$$

$$(iv)\quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} [3x^2 + 3xh + h^2] = 3x^2.$$

Recall from algebra that $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Now replace a by x and b by h.

The derivative of $f(x) = x^3$ is $f'(x) = 3x^2$. ■

EXAMPLE 4 Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^3$ at $x = \frac{1}{2}$.

Solution From Example 3 we have two functions $f(x) = x^3$ and $f'(x) = 3x^2$. As we saw in Example 2, when evaluated at the same number $x = \frac{1}{2}$ these functions give different information:

$$f\left(\frac{1}{2}\right) = \left(\frac{1}{2}\right)^3 = \frac{1}{8} \quad \leftarrow \text{point of tangency is } \left(\tfrac{1}{2}, \tfrac{1}{8}\right)$$

$$f'\left(\frac{1}{2}\right) = 3\left(\frac{1}{2}\right)^2 = \frac{3}{4}. \quad \leftarrow \text{slope of tangent at } \left(\tfrac{1}{2}, \tfrac{1}{8}\right) \text{ is } \tfrac{3}{4}$$

Thus, by the point–slope form of a line, an equation of the tangent line is given by

$$y - \frac{1}{8} = \frac{3}{4}\left(x - \frac{1}{2}\right) \quad \text{or} \quad y = \frac{3}{4}x - \frac{1}{4}.$$

The graph of the function and the tangent line are given in FIGURE 3.1.1. ■

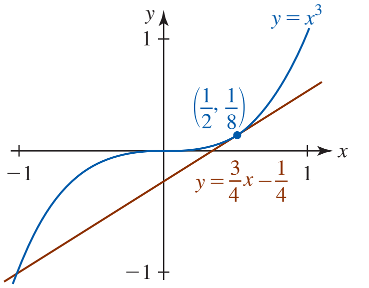


FIGURE 3.1.1 Tangent line in Example 4

EXAMPLE 5 A Derivative

Find the derivative of $f(x) = 1/x$.

Solution In this case you should be able to show that the difference is

$$f(x+h) - f(x) = \frac{1}{x+h} - \frac{1}{x} = \frac{-h}{(x+h)x}. \quad \leftarrow \text{add fractions by using a common denominator}$$

Therefore,

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{-h}{h(x+h)x}$$

$$= \lim_{h \to 0} \frac{-1}{(x+h)x} = \frac{-1}{x^2}.$$

The derivative of $f(x) = 1/x$ is $f'(x) = -1/x^2$. ■

Notation The following is a list of some of the common **notations** used throughout mathematical literature to denote the derivative of a function:

$$f'(x), \quad \frac{dy}{dx}, \quad y', \quad Dy, \quad D_x y.$$

For a function such as $f(x) = x^2$, we write $f'(x) = 2x$; if the same function is written $y = x^2$, we then utilize $dy/dx = 2x$, $y' = 2x$, or $D_x y = 2x$. We will use the first three notations throughout this text. Of course other symbols are used in various applications. Thus, if $z = t^2$, then

$$\frac{dz}{dt} = 2t \qquad \text{or} \qquad z' = 2t.$$

The dy/dx notation has its origin in the derivative form of (3) of Section 2.7. Replacing h by Δx and denoting the difference $f(x + h) - f(x)$ by Δy in (2), the derivative is often defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}. \tag{3}$$

EXAMPLE 6 A Derivative Using (3)

Use (3) to find the derivative of $y = \sqrt{x}$.

Solution In the four-step procedure the important algebra takes place in the third step:

(*i*) $f(x + \Delta x) = \sqrt{x + \Delta x}$

(*ii*) $\Delta y = f(x + \Delta x) - f(x) = \sqrt{x + \Delta x} - \sqrt{x}$

(*iii*)
$$\begin{aligned}\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\ &= \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \quad \leftarrow \text{rationalization of numerator} \\ &= \frac{x + \Delta x - x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\ &= \frac{\Delta x}{\Delta x\left(\sqrt{x + \Delta x} + \sqrt{x}\right)} \\ &= \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}\end{aligned}$$

(*iv*)
$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

The derivative of $y = \sqrt{x}$ is $dy/dx = 1/\left(2\sqrt{x}\right)$. ■

Value of a Derivative The **value** of the derivative at a number a is denoted by the symbols

$$f'(a), \quad \left.\frac{dy}{dx}\right|_{x=a}, \quad y'(a), \quad \left.D_x y\right|_{x=a}.$$

EXAMPLE 7 A Derivative

From Example 6, the value of the derivative of $y = \sqrt{x}$ at, say, $x = 9$ is written

$$\left.\frac{dy}{dx}\right|_{x=9} = \left.\frac{1}{2\sqrt{x}}\right|_{x=9} = \frac{1}{6}.$$

Alternatively, to avoid the clumsy vertical bar we can simply write $y'(9) = \frac{1}{6}$. ■

Differentiation Operators The process of finding or calculating a derivative is called **differentiation**. Thus differentiation is an operation that is performed on a function $y = f(x)$. The operation of differentiation of a function with respect to the variable x is represented by the symbols d/dx and D_x. These symbols are called **differentiation operators**. For instance, the results in Examples 1, 3, and 6 can be expressed, in turn, as

$$\frac{d}{dx}(x^2 + 2) = 2x, \quad \frac{d}{dx}x^3 = 3x^2, \quad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

The symbol

$$\frac{dy}{dx} \quad \text{then means} \quad \frac{d}{dx}y.$$

Differentiability If the limit in (2) exists for a given number x in the domain of f, the function f is said to be **differentiable** at x. If a function f is differentiable at every number x in the open intervals (a, b), $(-\infty, b)$, and (a, ∞), then f is **differentiable on the open interval**. If f is differentiable on $(-\infty, \infty)$, then f is said to be **differentiable everywhere**. A function f is **differentiable on a closed interval** $[a, b]$ when f is differentiable on the open interval (a, b), and

$$\begin{aligned} f'_+(a) &= \lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \\ f'_-(b) &= \lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h} \end{aligned} \tag{4}$$

both exist. The limits in (4) are called **right-hand** and **left-hand derivatives**, respectively. A function is differentiable on $[a, \infty)$ when it is differentiable on (a, ∞) and has a right-hand derivative at a. A similar definition in terms of a left-hand derivative holds for differentiability on $(-\infty, b]$. Moreover, it can be shown:

- *A function f is differentiable at a number c in an interval (a, b) if and only if $f'_+(c) = f'_-(c)$.* (5)

Horizontal Tangents If $y = f(x)$ is continuous at a number a and $f'(a) = 0$, then the tangent line at $(a, f(a))$ is **horizontal**. In Examples 1 and 2 we saw that the value of derivative $f'(x) = 2x$ of the function $f(x) = x^2 + 2$ at $x = 0$ is $f'(0) = 0$. Thus, the tangent line to the graph is horizontal at $(0, f(0))$ or $(0, 0)$. It is left as an exercise (see Problem 7 in Exercises 3.1) to verify by Definition 3.1.1 that the derivative of the continuous function $f(x) = -x^2 + 4x + 1$ is $f'(x) = -2x + 4$. Observe in this latter case that $f'(x) = 0$ when $-2x + 4 = 0$ or $x = 2$. There is a horizontal tangent at the point $(2, f(2)) = (2, 5)$.

Where *f* Fails to be Differentiable A function f fails to have a derivative at $x = a$ if

(*i*) the function f is discontinuous at $x = a$, or
(*ii*) the graph of f has a corner at $(a, f(a))$.

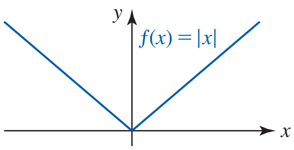
In addition, since the derivative gives slope, f will fail to be differentiable

(*iii*) at a point $(a, f(a))$ at which the tangent line to the graph is vertical.

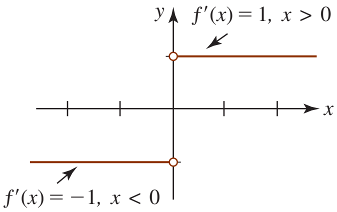
The domain of the derivative f', defined by (2), is the set of numbers x for which the limit exists. Thus the domain of f' is necessarily a subset of the domain of f.

EXAMPLE 8 Differentiability

(a) The function $f(x) = x^2 + 2$ is differentiable for all real numbers x, that is, the domain of $f'(x) = 2x$ is $(-\infty, \infty)$.
(b) Because $f(x) = 1/x$ is discontinuous at $x = 0$, f is not differentiable at $x = 0$ and consequently not differentiable on any interval containing 0. ■



(a) Absolute-value function f



(b) Graph of the derivative f'

FIGURE 3.1.2 Graphs of f and f' in Example 9

EXAMPLE 9 Example 7 of Section 2.7 Revisited

In Example 7 of Section 2.7 we saw that the graph of $f(x) = |x|$ possesses no tangent at the origin (0, 0). Thus $f(x) = |x|$ is not differentiable at $x = 0$. But $f(x) = |x|$ is differentiable on the open intervals $(0, \infty)$ and $(-\infty, 0)$. In Example 5 of Section 2.7, we proved that the derivative of a linear function $f(x) = mx + b$ is $f'(x) = m$. Hence, for $x > 0$ we have $f(x) = |x| = x$ and so $f'(x) = 1$. Also, for $x < 0, f(x) = |x| = -x$ and so $f'(x) = -1$. Since the derivative of f is a piecewise-defined function,

$$f'(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0, \end{cases}$$

we can graph it as we would any function. We see in FIGURE 3.1.2(b) that f' is discontinuous at $x = 0$. ■

In different symbols, what we have shown in Example 9 is that $f'_-(0) = -1$ and $f'_+(0) = 1$. Since $f'_-(0) \neq f'_+(0)$ it follows from (5) that f is not differentiable at 0.

Vertical Tangents Let $y = f(x)$ be continuous at a number a. If $\lim_{x \to a} |f'(x)| = \infty$, then the graph of f is said to have a **vertical tangent** at $(a, f(a))$. The graphs of many functions with rational exponents possess vertical tangents.

In Example 6 of Section 2.7 we mentioned that the graph of $y = x^{1/3}$ possesses a vertical tangent line at (0, 0). We verify this assertion in the next example.

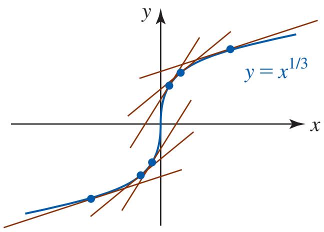


FIGURE 3.1.3 Tangent lines to the graph of the function in Example 10

EXAMPLE 10 Vertical Tangent

It is left as an exercise to prove that the derivative of $f(x) = x^{1/3}$ is given by

$$f'(x) = \frac{1}{3x^{2/3}}.$$

(See Problem 55 in Exercises 3.1.) Although f is continuous at 0, it is clear that f' is not defined at that number. In other words, f is not differentiable at $x = 0$. Moreover, because

$$\lim_{x \to 0^+} f'(x) = \infty \qquad \text{and} \qquad \lim_{x \to 0^-} f'(x) = \infty$$

we have $|f'(x)| \to \infty$ as $x \to 0$. This is sufficient to say that there is a tangent line at $(0, f(0))$ or (0, 0) and that it is vertical. FIGURE 3.1.3 shows that the tangent lines to the graph on either side of the origin become steeper and steeper as $x \to 0$. ■

The graph of a function f can also have a vertical tangent at a point $(a, f(a))$ if f is differentiable only on one side of a, is continuous from the left (right) at a, and either $|f'(x)| \to \infty$ as $x \to a^-$ or $|f'(x)| \to \infty$ as $x \to a^+$.

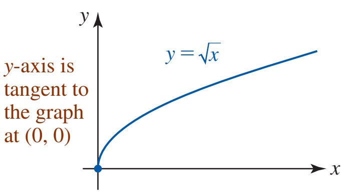


FIGURE 3.1.4 Vertical tangent in Example 11

EXAMPLE 11 One-Sided Vertical Tangent

The function $f(x) = \sqrt{x}$ is not differentiable on the interval $[0, \infty)$ because it is seen from the derivative $f'(x) = 1/(2\sqrt{x})$ that $f'_+(0)$ does not exist. The function $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$ but differentiable on $(0, \infty)$. In addition, because f is continuous at 0 and $\lim_{x \to 0^+} f'(x) = \infty$, there is a one-sided vertical tangent at the origin (0, 0). We see in FIGURE 3.1.4 that the vertical tangent is the y-axis. ■

The functions $f(x) = |x|$ and $f(x) = x^{1/3}$ are continuous everywhere. In particular, both functions are continuous at 0 but neither are differentiable at that number. In other words, continuity at a number a is not sufficient to guarantee that a function is differentiable at a. However, if a function f is differentiable at a, then f must be continuous at that number. We summarize this last fact in the next theorem.

Important ▶

Theorem 3.1.1 Differentiability Implies Continuity

If f is differentiable at a number a, then f is continuous at a.

PROOF To prove continuity of f at a number a it is sufficient to prove that $\lim_{x\to a} f(x) = f(a)$ or equivalently that $\lim_{x\to a}[f(x) - f(a)] = 0$. The hypothesis is that

$$f'(a) = \lim_{h\to 0}\frac{f(a+h) - f(a)}{h}$$

exists. If we let $x = a + h$, then as $h \to 0$ we have $x \to a$. Thus, the foregoing limit is equivalent to

$$f'(a) = \lim_{x\to a}\frac{f(x) - f(a)}{x - a}.$$

Then we can write

$$\begin{aligned}\lim_{x\to a}[f(x) - f(a)] &= \lim_{x\to a}\frac{f(x) - f(a)}{x - a}\cdot(x - a) && \leftarrow \text{multiplication by } \frac{x-a}{x-a} = 1\\ &= \lim_{x\to a}\frac{f(x) - f(a)}{x - a}\cdot\lim_{x\to a}(x - a) && \leftarrow \text{both limits exist}\\ &= f'(a)\cdot 0 = 0.\end{aligned}$$

■

Postscript—A Bit of History It is acknowledged that **Isaac Newton** (1642–1727), an English mathematician and physicist, was the first to set forth many of the basic principles of calculus in unpublished manuscripts on the *method of fluxions*, dated 1665. The word *fluxion* originated from the concept of quantities that "flow"—that is, quantities that change at a certain rate. Newton used the dot notation $\dot{y}$ to represent a fluxion, or as we now know it: the derivative of a function. The symbol $\dot{y}$ never achieved overwhelming popularity among mathematicians and is used today primarily by physicists. For typographical reasons, the so-called "fly-speck notation" has been superseded by the prime notation. Newton attained everlasting fame with the publication of his law of universal gravitation in his monumental treatise *Philosophiae Naturalis Principia Mathematica* in 1687. Newton was also the first to prove, using the calculus and his law of gravitation, Johannes Kepler's three empirical laws of planetary motion and was the first to prove that white light is composed of all colors. Newton was elected to Parliament, was appointed warden of the Royal Mint, and was knighted in 1705. Sir Isaac Newton said about his many accomplishments: "If I have seen farther than others, it is by standing on the shoulders of giants."

Newton

The German mathematician, lawyer, and philosopher **Gottfried Wilhelm Leibniz** (1646–1716) published a short version of his calculus in an article in a periodical journal in 1684. The dy/dx notation for a derivative of a function is due to Leibniz. In fact, it was Leibniz who introduced the word *function* into mathematical literature. But, since it was well known that Newton's manuscripts on the *method of fluxions* dated from 1665, Leibniz was accused of appropriating his ideas from these unpublished works. Fueled by nationalistic prides, a controversy about who was the first to "invent" calculus raged for many years. Historians now agree that both Leibniz and Newton arrived at many of the major premises of calculus independent of each other. Leibniz and Newton are considered the "co-inventors" of the subject.

Leibniz

$\frac{d}{dx}$ **NOTES FROM THE CLASSROOM**

(*i*) In the preceding discussion, we saw that the derivative of a function is itself a function that gives the slope of a tangent line. The derivative is, however, *not* an equation of a tangent line. Also, to say that $y - y_0 = f'(x)\cdot(x - x_0)$ is an equation of the tangent at (x_0, y_0) is incorrect. Remember that $f'(x)$ must be evaluated at x_0 *before* it is used in the point–slope form. If f is differentiable at x_0, then an equation of the tangent line at (x_0, y_0) is $y - y_0 = f'(x_0)\cdot(x - x_0)$.

(*ii*) Although we have emphasized slopes in this section, do not forget the discussion on average rates of change and instantaneous rates of change in Section 2.7. The derivative $f'(x)$ is also the **instantaneous rate of change** of the function $y = f(x)$ with respect to the variable x. More will be said about rates in subsequent sections.

(*iii*) Mathematicians from the seventeenth to the nineteenth centuries believed that a continuous function *usually* possessed a derivative. (We have noted exceptions in this section.) In 1872 the German mathematician Karl Weierstrass conclusively destroyed this tenet by publishing an example of a function that was *everywhere continuous but nowhere differentiable*.

Exercises 3.1

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–20, use (2) of Definition 3.1.1 to find the derivative of the given function.

1. $f(x) = 10$

2. $f(x) = x - 1$

3. $f(x) = -3x + 5$

4. $f(x) = \pi x$

5. $f(x) = 3x^2$

6. $f(x) = -x^2 + 1$

7. $f(x) = -x^2 + 4x + 1$

8. $f(x) = \frac{1}{2}x^2 + 6x - 7$

9. $y = (x + 1)^2$

10. $f(x) = (2x - 5)^2$

11. $f(x) = x^3 + x$

12. $f(x) = 2x^3 + x^2$

13. $y = -x^3 + 15x^2 - x$

14. $y = 3x^4$

15. $y = \dfrac{2}{x + 1}$

16. $y = \dfrac{x}{x - 1}$

17. $y = \dfrac{2x + 3}{x + 4}$

18. $f(x) = \dfrac{1}{x} + \dfrac{1}{x^2}$

19. $f(x) = \dfrac{1}{\sqrt{x}}$

20. $f(x) = \sqrt{2x + 1}$

In Problems 21–24, use (2) of Definition 3.1.1 to find the derivative of the given function. Find an equation of the tangent line to the graph of the function at the indicated value of x.

21. $f(x) = 4x^2 + 7x;\quad x = -1$

22. $f(x) = \frac{1}{3}x^3 + 2x - 4;\quad x = 0$

23. $y = x - \dfrac{1}{x};\quad x = 1$

24. $y = 2x + 1 + \dfrac{6}{x};\quad x = 2$

In Problems 25–28, use (2) of Definition 3.1.1 to find the derivative of the given function. Find point(s) on the graph of the given function where the tangent line is horizontal.

25. $f(x) = x^2 + 8x + 10$

26. $f(x) = x(x - 5)$

27. $f(x) = x^3 - 3x$

28. $f(x) = x^3 - x^2 + 1$

In Problems 29–32, use (2) of Definition 3.1.1 to find the derivative of the given function. Find point(s) on the graph of the given function where the tangent line is parallel to the given line.

29. $f(x) = \frac{1}{2}x^2 - 1;\quad 3x - y = 1$

30. $f(x) = x^2 - x;\quad -2x + y = 0$

31. $f(x) = -x^3 + 4;\quad 12x + y = 4$

32. $f(x) = 6\sqrt{x} + 2;\quad -x + y = 2$

In Problems 33 and 34, show that the given function is not differentiable at the indicated value of x.

33. $f(x) = \begin{cases} -x + 2, & x \le 2 \\ 2x - 4, & x > 2 \end{cases};\quad x = 2$

34. $f(x) = \begin{cases} 3x, & x < 0 \\ -4x, & x \ge 0 \end{cases};\quad x = 0$

In the proof of Theorem 3.1.1 we saw that an alternative formulation of the derivative of a function f at a is given by

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}, \tag{6}$$

whenever the limit exists. In Problems 35–40, use (6) to compute $f'(a)$.

35. $f(x) = 10x^2 - 3$

36. $f(x) = x^2 - 3x - 1$

37. $f(x) = x^3 - 4x^2$

38. $f(x) = x^4$

39. $f(x) = \dfrac{4}{3 - x}$

40. $f(x) = \sqrt{x}$

41. Find an equation of the tangent line shown in red in FIGURE 3.1.5. What are $f(-3)$ and $f'(-3)$?

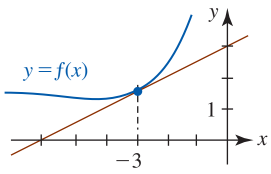


FIGURE 3.1.5 Graph for Problem 41

42. Find an equation of the tangent line shown in red in FIGURE 3.1.6. What is $f'(3)$? What is the y-intercept of the tangent line?

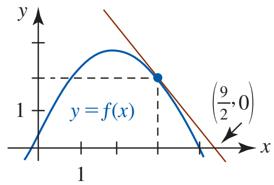


FIGURE 3.1.6 Graph for Problem 42

In Problems 43–48, sketch the graph of f' from the graph of f.

43.

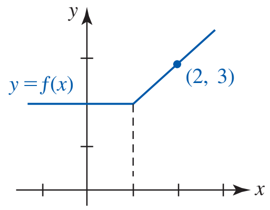


FIGURE 3.1.7 Graph for Problem 43

44.

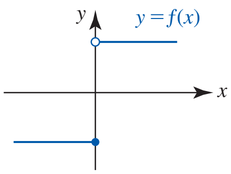


FIGURE 3.1.8 Graph for Problem 44

45.

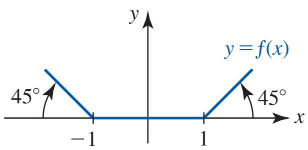


FIGURE 3.1.9 Graph for Problem 45

46.

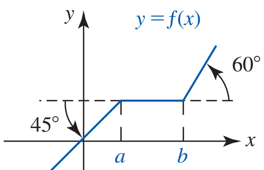


FIGURE 3.1.10 Graph for Problem 46

47.

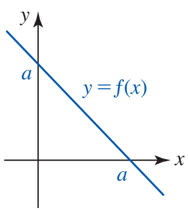


FIGURE 3.1.11 Graph for Problem 47

48.

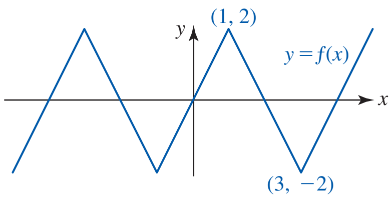
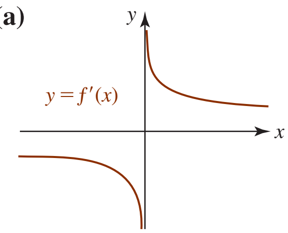


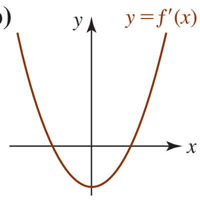
FIGURE 3.1.12 Graph for Problem 48

In Problems 49–54, match the graph of f with a graph of f' from (a)–(f).

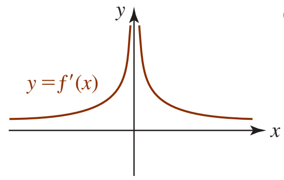
(a)



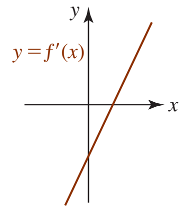
(b)



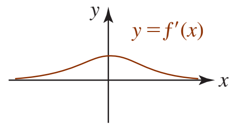
(c)



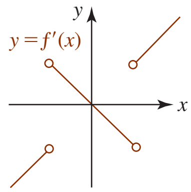
(d)



(e)



(f)



49.

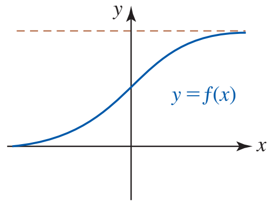


FIGURE 3.1.13 Graph for Problem 49

50.

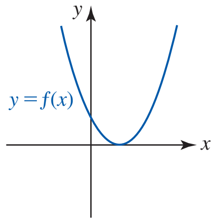


FIGURE 3.1.14 Graph for Problem 50

51.

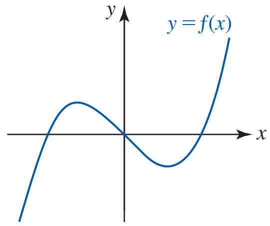


FIGURE 3.1.15 Graph for Problem 51

52.

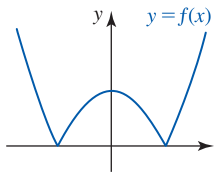


FIGURE 3.1.16 Graph for Problem 52

53.

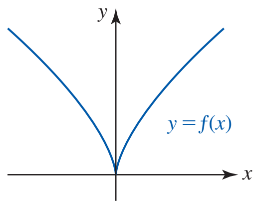


FIGURE 3.1.17 Graph for Problem 53

54.

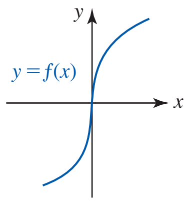


FIGURE 3.1.18 Graph for Problem 54

☰ Think About It

55. Use the alternative definition of the derivative (6) to find the derivative of $f(x) = x^{1/3}$.

[*Hint*: Note that $x - a = (x^{1/3})^3 - (a^{1/3})^3$.]

56. In Examples 10 and 11, we saw, respectively, that the functions $f(x) = x^{1/3}$ and $f(x) = \sqrt{x}$ possessed vertical tangents at the origin $(0, 0)$. Conjecture where the graphs of $y = (x - 4)^{1/3}$ and $y = \sqrt{x + 2}$ may have vertical tangents.

57. Suppose f is differentiable everywhere and has the three properties:

(*i*) $f(x_1 + x_2) = f(x_1)f(x_2)$, (*ii*) $f(0) = 1$, (*iii*) $f'(0) = 1$.

Use (2) of Definition 3.1.1 to show that $f'(x) = f(x)$ for all x.

58. **(a)** Suppose f is an even differentiable function on $(-\infty, \infty)$. On geometric grounds, explain why $f'(-x) = -f'(x)$; that is, f' is an odd function.
(b) Suppose f is an odd differentiable function on $(-\infty, \infty)$. On geometric grounds, explain why $f'(-x) = f'(x)$; that is, f' is an even function.

59. Suppose f is a differentiable function on $[a, b]$ such that $f(a) = 0$ and $f(b) = 0$. By experimenting with graphs discern whether the following statement is true or false: There is some number c in (a, b) such that $f'(c) = 0$.

60. Sketch graphs of various functions f that have the property $f'(x) > 0$ for all x in $[a, b]$. What do these functions have in common?

≡ Calculator/CAS Problem

61. Consider the function $f(x) = x^n + |x|$, where n is a positive integer. Use a calculator or CAS to obtain the graph of f for $n = 1, 2, 3, 4$, and 5. Then use (2) to show that f is not differentiable at $x = 0$ for $n = 1, 2, 3, 4$, and 5. Can you prove this for *any* positive integer n? What is $f'_-(0)$ and $f'_+(0)$ for $n > 1$?

3.2 Power and Sum Rules

Introduction The definition of a derivative

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \tag{1}$$

has the obvious drawback of being rather clumsy and tiresome to apply. To find the derivative of the polynomial function $f(x) = 6x^{100} + 4x^{35}$ using the above definition we would *only* have to juggle 137 terms in the binomial expansions of $(x + h)^{100}$ and $(x + h)^{35}$. There are more efficient ways of computing derivatives of a function than using the definition each time. In this section, and the sections that follow, we will see that there are shortcuts or general **rules** whereby derivatives of functions such as $f(x) = 6x^{100} + 4x^{35}$ can be obtained, literally, with just a flick of a pencil.

In the last section we saw that the derivatives of the power functions

$$f(x) = x^2, \quad f(x) = x^3, \quad f(x) = \frac{1}{x} = x^{-1}, \quad f(x) = \sqrt{x} = x^{1/2}$$

were, in turn,

See Examples 3, 5, and 6 in Section 3.1.

$$f'(x) = 2x, \quad f'(x) = 3x^2, \quad f'(x) = -\frac{1}{x^2} = -x^{-2}, \quad f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}.$$

If the right-hand sides of these four derivatives are written

$$2 \cdot x^{2-1}, \quad 3 \cdot x^{3-1}, \quad (-1) \cdot x^{-1-1}, \quad \frac{1}{2} \cdot x^{\frac{1}{2}-1},$$

we observe that each coefficient (indicated in red) corresponds with the original exponent of x in f and the new exponent of x in f' can be obtained from the old exponent (also indicated in red) by subtracting 1 from it. In other words, the pattern for the derivative of the general power function $f(x) = x^n$ appears to be

bring down exponent as a multiple

$$(\ )x^{(\ )-1}. \tag{2}$$

decrease exponent by 1

Derivative of the Power Function The pattern illustrated in (2) does indeed hold for any real-number exponent n, and we will state it as a formal theorem, but at this point in the course we do not possess the necessary mathematical tools to prove its complete validity. We can, however, readily prove a special case of this power rule; the remaining parts of the proof will be given in the appropriate sections ahead.

Theorem 3.2.1 Power Rule

For any real number n,

$$\frac{d}{dx}x^n = nx^{n-1}. \tag{3}$$

PROOF We present the proof only in the case when n is a positive integer. To compute (1) for $f(x) = x^n$ we use the four-step method:

$$(i)\ f(x+h) = (x+h)^n = \underbrace{x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}_{\text{general Binomial Theorem}}$$

See the *Resource Pages* for a review of the Binomial Theorem.

$$(ii)\ f(x+h) - f(x) = x^n + nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n - x^n$$

$$= nx^{n-1}h + \frac{n(n-1)}{2!}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n$$

$$= h\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}h + \cdots + nxh^{n-2} + h^{n-1}\right]$$

$$(iii)\ \frac{f(x+h)-f(x)}{h} = \frac{h\left[nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}h + \cdots + nxh^{n-2} + h^{n-1}\right]}{h}$$

$$= nx^{n-1} + \frac{n(n-1)}{2!}x^{n-1}h + \cdots + nxh^{n-2} + h^{n-1}$$

$$(iv)\ f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0}\left[nx^{n-1} \underbrace{+ \frac{n(n-1)}{2!}x^{n-1}h + \cdots + nxh^{n-2} + h^{n-1}}_{\text{these terms} \to 0 \text{ as } h \to 0}\right] = nx^{n-1}.$$ ■

EXAMPLE 1 Power Rule

Differentiate

(a) $y = x^7$ **(b)** $y = x$ **(c)** $y = x^{-2/3}$ **(d)** $y = x^{\sqrt{2}}$.

Solution By the Power Rule (3),

(a) with $n = 7$: $\frac{dy}{dx} = 7x^{7-1} = 7x^6$,

(b) with $n = 1$: $\frac{dy}{dx} = 1x^{1-1} = x^0 = 1$,

(c) with $n = -\frac{2}{3}$: $\frac{dy}{dx} = \left(-\frac{2}{3}\right)x^{(-2/3)-1} = -\frac{2}{3}x^{-5/3} = -\frac{2}{3x^{5/3}}$,

(d) with $n = \sqrt{2}$: $\frac{dy}{dx} = \sqrt{2}x^{\sqrt{2}-1}$. ■

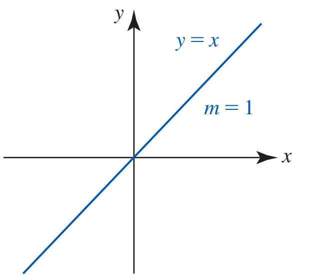


FIGURE 3.2.1 Slope of line $m = 1$ is consistent with $dy/dx = 1$

Observe in part (b) of Example 1 that the result is consistent with the fact that the slope of the line $y = x$ is $m = 1$. See FIGURE 3.2.1.

Theorem 3.2.2 Constant Function Rule

If $f(x) = c$ is a constant function, then $f'(x) = 0$. (4)

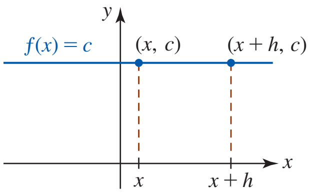


FIGURE 3.2.2 Slope of a horizontal line is 0

PROOF If $f(x) = c$ where c is any real number, then it follows that the difference is $f(x + h) - f(x) = c - c = 0$. Hence from (1),

$$f'(x) = \lim_{h \to 0} \frac{c - c}{h} = \lim_{h \to 0} 0 = 0. \quad \blacksquare$$

Theorem 3.2.2 has an obvious geometric interpretation. As shown in FIGURE 3.2.2, the slope of the horizontal line $y = c$ is, of course, zero. Moreover, Theorem 3.2.2 agrees with (3) in the case when $x \neq 0$ and $n = 0$.

Theorem 3.2.3 Constant Multiple Rule

If c is any constant and f is differentiable at x, then cf is differentiable at x, and

$$\frac{d}{dx} cf(x) = cf'(x). \tag{5}$$

PROOF Let $G(x) = cf(x)$. Then

$$\begin{aligned} G'(x) = \lim_{h \to 0} \frac{G(x + h) - G(x)}{h} &= \lim_{h \to 0} \frac{cf(x + h) - cf(x)}{h} \\ &= \lim_{h \to 0} c \left[\frac{f(x + h) - f(x)}{h} \right] \\ &= c \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = cf'(x). \quad \blacksquare \end{aligned}$$

EXAMPLE 2 A Constant Multiple

Differentiate $y = 5x^4$.

Solution From (3) and (5),

$$\frac{dy}{dx} = 5 \frac{d}{dx} x^4 = 5(4x^3) = 20x^3. \quad \blacksquare$$

Theorem 3.2.4 Sum and Difference Rules

If f and g are functions differentiable at x, then $f + g$ and $f - g$ are differentiable at x, and

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x), \tag{6}$$

$$\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x). \tag{7}$$

PROOF OF (6) Let $G(x) = f(x) + g(x)$. Then

$$\begin{aligned} G'(x) = \lim_{h \to 0} \frac{G(x + h) - G(x)}{h} &= \lim_{h \to 0} \frac{[f(x + h) + g(x + h)] - [f(x) + g(x)]}{h} \\ &= \lim_{h \to 0} \frac{f(x + h) - f(x) + g(x + h) - g(x)}{h} \quad \leftarrow \text{regrouping terms} \\ &= \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \to 0} \frac{g(x + h) - g(x)}{h} \\ &= f'(x) + g'(x). \quad \blacksquare \end{aligned}$$

since limits exist, limit of a sum is → the sum of the limits

Theorem 3.2.4 holds for any finite sum of differentiable functions. For example, if f, g, and h are functions that are differentiable at x, then

$$\frac{d}{dx}[f(x) + g(x) + h(x)] = f'(x) + g'(x) + h'(x).$$

Since $f - g$ can be written as a sum, $f + (-g)$, there is no need to prove (7) since the result follows from a combination of (6) and (5). Hence, we can express Theorem 3.2.4 in words as:

- *The derivative of a sum is the sum of the derivatives.*

Derivative of a Polynomial Because we now know how to differentiate powers of x and constant multiples of those powers we can easily differentiate sums of those constant multiples. The derivative of a polynomial function is particularly easy to obtain. For example, the derivative of the polynomial function $f(x) = 6x^{100} + 4x^{35}$, mentioned in the introduction to this section, is now readily seen to be $f'(x) = 600x^{99} + 140x^{34}$.

EXAMPLE 3 Polynomial with Six Terms

Differentiate $y = 4x^5 - \frac{1}{2}x^4 + 9x^3 + 10x^2 - 13x + 6$.

Solution Using (3), (5), and (6), we obtain

$$\frac{dy}{dx} = 4\frac{d}{dx}x^5 - \frac{1}{2}\frac{d}{dx}x^4 + 9\frac{d}{dx}x^3 + 10\frac{d}{dx}x^2 - 13\frac{d}{dx}x + \frac{d}{dx}6.$$

Since $\frac{d}{dx}6 = 0$ by (4), we obtain

$$\begin{aligned}\frac{dy}{dx} &= 4(5x^4) - \frac{1}{2}(4x^3) + 9(3x^2) + 10(2x) - 13(1) + 0\\ &= 20x^4 - 2x^3 + 27x^2 + 20x - 13.\end{aligned}$$

■

EXAMPLE 4 Tangent Line

Find an equation of a tangent line to the graph of $f(x) = 3x^4 + 2x^3 - 7x$ at the point corresponding to $x = -1$.

Solution From the Sum Rule,

$$f'(x) = 3(4x^3) + 2(3x^2) - 7(1) = 12x^3 + 6x^2 - 7.$$

When evaluated at the same number $x = -1$ the functions f and f' give:

$$f(-1) = 8 \qquad \leftarrow \text{point of tangency is } (-1, 8)$$
$$f'(-1) = -13. \qquad \leftarrow \text{slope of tangent at } (-1, 8) \text{ is } -13$$

The point–slope form gives an equation of the tangent line

$$y - 8 = -13(x - (-1)) \quad \text{or} \quad y = -13x - 5.$$

■

Rewriting a Function In some circumstances, in order to apply a rule of differentiation efficiently it may be necessary to *rewrite* an expression in an alternative form. This alternative form is often the result of some algebraic manipulation or an application of the laws of exponents. For example, we can use (3) to differentiate the following expressions, but first we rewrite them using the laws of exponents

◀ This discussion is worth remembering.

$$\boxed{\frac{4}{x^2},\quad \frac{10}{\sqrt{x}},\quad \sqrt{x^3}} \rightarrow \text{rewrite square roots as powers} \rightarrow \frac{4}{x^2},\quad \frac{10}{x^{1/2}},\quad (x^3)^{1/2},$$

$$\text{then rewrite using negative exponents} \rightarrow 4x^{-2},\quad 10x^{-1/2},\quad x^{3/2},$$

$$\text{the derivative of each term using (3)} \rightarrow \boxed{-8x^{-3},\quad -5x^{-3/2},\quad \frac{3}{2}x^{1/2}}.$$

A function such as $f(x) = (5x + 2)/x^2$ can be rewritten as two fractions

$$f(x) = \frac{5x + 2}{x^2} = \frac{5x}{x^2} + \frac{2}{x^2} = \frac{5}{x} + \frac{2}{x^2} = 5x^{-1} + 2x^{-2}.$$

From the last form of f it is now apparent that the derivative f' is

$$f'(x) = 5(-x^{-2}) + 2(-2x^{-3}) = -\frac{5}{x^2} - \frac{4}{x^3}.$$

EXAMPLE 5 Rewriting the Terms of a Function

Differentiate $y = 4\sqrt{x} + \dfrac{8}{x} - \dfrac{6}{\sqrt[3]{x}} + 10$.

Solution Before differentiating we rewrite the first three terms as powers of x:

$$y = 4x^{1/2} + 8x^{-1} - 6x^{-1/3} + 10.$$

Then
$$\frac{dy}{dx} = 4\frac{d}{dx}x^{1/2} + 8\frac{d}{dx}x^{-1} - 6\frac{d}{dx}x^{-1/3} + \frac{d}{dx}10.$$

By the Power Rule (3) and (4), we obtain

$$\frac{dy}{dx} = 4 \cdot \frac{1}{2}x^{-1/2} + 8 \cdot (-1)x^{-2} - 6 \cdot \left(-\frac{1}{3}\right)x^{-4/3} + 0$$
$$= \frac{2}{\sqrt{x}} - \frac{8}{x^2} + \frac{2}{x^{4/3}}. \quad ■$$

EXAMPLE 6 Horizontal Tangents

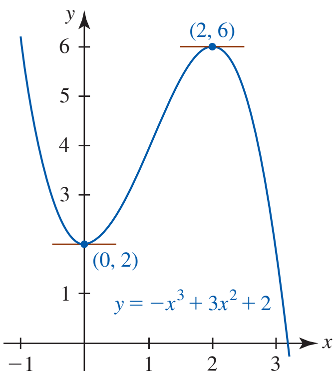


FIGURE 3.2.3 Graph of function in Example 6

Find the points on the graph of $f(x) = -x^3 + 3x^2 + 2$ where the tangent line is horizontal.

Solution At a point $(x, f(x))$ on the graph of f where the tangent is horizontal we must have $f'(x) = 0$. The derivative of f is $f'(x) = -3x^2 + 6x$ and the solutions of $f'(x) = -3x^2 + 6x = 0$ or $-3x(x - 2) = 0$ are $x = 0$ and $x = 2$. The corresponding points are then $(0, f(0)) = (0, 2)$ and $(2, f(2)) = (2, 6)$. See FIGURE 3.2.3. ■

Normal Line A **normal line** at a point P on a graph is one that is perpendicular to the tangent line at P.

EXAMPLE 7 Equation of a Normal Line

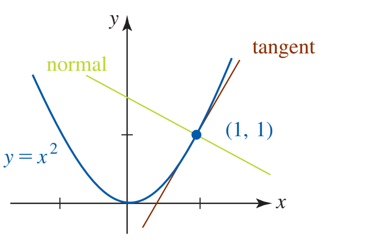


FIGURE 3.2.4 Normal line in Example 7

Find an equation of the normal line to the graph of $y = x^2$ at $x = 1$.

Solution Since $dy/dx = 2x$, we know that $m_{tan} = 2$ at $(1, 1)$. Thus the slope of the normal line shown in green in FIGURE 3.2.4 is the negative reciprocal of the slope of the tangent line, that is, $m = -\frac{1}{2}$. By the point-slope form of a line, an equation of the normal line is then

$$y - 1 = -\frac{1}{2}(x - 1) \quad \text{or} \quad y = -\frac{1}{2}x + \frac{3}{2}. \quad ■$$

EXAMPLE 8 Vertical Tangent

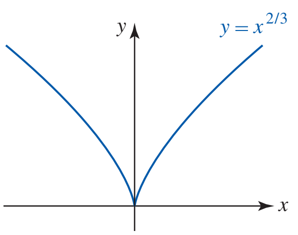


FIGURE 3.2.5 Graph of function in Example 8

For the power function $f(x) = x^{2/3}$ the derivative is

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}.$$

Observe that $\lim_{x\to 0^+} f(x) = \infty$ whereas $\lim_{x\to 0^-} f(x) = -\infty$. Since f is continuous at $x = 0$ and $|f'(x)| \to \infty$ as $x \to 0$, we conclude that the y-axis is a vertical tangent at $(0, 0)$. This fact is apparent from the graph in FIGURE 3.2.5. ■

Cusp The graph of $f(x) = x^{2/3}$ in Example 8 is said to have a **cusp** at the origin. In general, the graph of a function $y = f(x)$ has a cusp at a point $(a, f(a))$ if f is continuous at a, $f'(x)$ has opposite signs on either side of a, and $|f'(x)| \to \infty$ as $x \to a$.

Higher-Order Derivatives We have seen that the derivative $f'(x)$ is a function derived from $y = f(x)$. By differentiation of the first derivative, we obtain yet another function called the **second derivative**, which is denoted by $f''(x)$. In terms of the operation symbol d/dx, we define the second derivative with respect to x as the function obtained by differentiating $y = f(x)$ twice in succession:

$$\frac{d}{dx}\left(\frac{dy}{dx}\right).$$

The second derivative is commonly denoted by the symbols

$$f''(x), \quad y'', \quad \frac{d^2y}{dx^2}, \quad \frac{d^2}{dx^2}f(x), \quad D^2, \quad D_x^2.$$

EXAMPLE 9 Second Derivative

Find the second derivative of $y = \dfrac{1}{x^3}$.

Solution We first simplify the function by rewriting it as $y = x^{-3}$. Then by the Power Rule (3), we have

$$\frac{dy}{dx} = -3x^{-4}.$$

The second derivative follows from differentiating the first derivative

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(-3x^{-4}) = -3(-4x^{-5}) = 12x^{-5} = \frac{12}{x^5}.$$ ■

Assuming that all derivatives exist, we can differentiate a function $y = f(x)$ as many times as we want. The **third derivative** is the derivative of the second derivative; the **fourth derivative** is the derivative of the third derivative; and so on. We denote the third and fourth derivatives by d^3y/dx^3 and d^4y/dx^4 and define them by

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) \quad \text{and} \quad \frac{d^4y}{dx^4} = \frac{d}{dx}\left(\frac{d^3y}{dx^3}\right).$$

In general, if n is a positive integer, then the ***n*th derivative** is defined by

$$\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right).$$

Other notations for the first n derivatives are

$$f'(x), \quad f''(x), \quad f'''(x), \quad f^{(4)}(x), \quad \ldots, \quad f^{(n)}(x),$$
$$y', \quad y'', \quad y''', \quad y^{(4)}, \quad \ldots, \quad y^{(n)},$$
$$\frac{d}{dx}f(x), \quad \frac{d^2}{dx^2}f(x), \quad \frac{d^3}{dx^3}f(x), \quad \frac{d^4}{dx^4}f(x), \quad \ldots, \quad \frac{d^n}{dx^n}f(x),$$
$$D, \quad D^2, \quad D^3, \quad D^4, \quad \ldots, \quad D^n,$$
$$D_x, \quad D_x^2, \quad D_x^3, \quad D_x^4, \quad \ldots, \quad D_x^n.$$

Note that the "prime" notation is used to denote only the first three derivatives; after that we use the superscript $y^{(4)}, y^{(5)}$, and so on. The **value of the *n*th derivative** of a function $y = f(x)$ at a number a is denoted by

$$f^{(n)}(a), \qquad y^{(n)}(a), \qquad \text{and} \qquad \left.\frac{d^ny}{dx^n}\right|_{x=a}.$$

EXAMPLE 10 Fifth Derivative

Find the first five derivatives of $f(x) = 2x^4 - 6x^3 + 7x^2 + 5x$.

Solution We have

$$\begin{aligned} f'(x) &= 8x^3 - 18x^2 + 14x + 5 \\ f''(x) &= 24x^2 - 36x + 14 \\ f'''(x) &= 48x - 36 \\ f^{(4)}(x) &= 48 \\ f^{(5)}(x) &= 0. \end{aligned}$$

■

After reflecting a moment, you should be convinced that the $(n + 1)$st derivative of an nth-degree polynomial function is zero.

$\frac{d}{dx}$ NOTES FROM THE CLASSROOM

(*i*) In the different contexts of science, engineering, and business, functions are often expressed in variables other than x and y. Correspondingly we must adapt the derivative notation to the new symbols. For example,

Function	***Derivative***
$v(t) = 32t$	$v'(t) = \frac{dv}{dt} = 32$
$A(r) = \pi r^2$	$A'(r) = \frac{dA}{dr} = 2\pi r$
$r(\theta) = 4\theta^2 - 3\theta$	$r'(\theta) = \frac{dr}{d\theta} = 8\theta - 3$
$D(p) = 800 - 129p + p^2$	$D'(p) = \frac{dD}{dp} = -129 + 2p.$

(*ii*) You may be wondering what interpretation can be given to the higher-order derivatives. If we think in terms of graphs, then f'' gives the slope of tangent lines to the graph of the function f'; f''' gives the slope of the tangent lines to the graph of f'', and so on. In addition, if f is differentiable, then the first-derivative f' gives the instantaneous rate of change of f. Similarly, if f' is differentiable, then f'' gives the instantaneous rate of change of f'.

Exercises 3.2

Answers to selected odd-numbered problems begin on page ANS-000.

≡ Fundamentals

In Problems 1–8, find dy/dx.

1. $y = -18$

2. $y = \pi^6$

3. $y = x^9$

4. $y = 4x^{12}$

5. $y = 7x^2 - 4x$

6. $y = 6x^3 + 3x^2 - 10$

7. $y = 4\sqrt{x} - \dfrac{6}{\sqrt[3]{x^2}}$

8. $y = \dfrac{x - x^2}{\sqrt{x}}$

In Problems 9–16, find $f'(x)$. Simplify.

9. $f(x) = \frac{1}{5}x^5 - 3x^4 + 9x^2 + 1$

10. $f(x) = -\frac{2}{3}x^6 + 4x^5 - 13x^2 + 8x + 2$

11. $f(x) = x^3(4x^2 - 5x - 6)$

12. $f(x) = \dfrac{2x^5 + 3x^4 - x^3 + 2}{x^2}$

13. $f(x) = x^2(x^2 + 5)^2$ **14.** $f(x) = (x^3 + x^2)^3$

15. $f(x) = \left(4\sqrt{x} + 1\right)^2$ **16.** $f(x) = (9 + x)(9 - x)$

In Problems 17–20, find the derivative of the given function.

17. $h(u) = (4u)^3$ **18.** $p(t) = (2t)^{-4} - (2t^{-1})^2$

19. $g(r) = \dfrac{1}{r} + \dfrac{1}{r^2} + \dfrac{1}{r^3} + \dfrac{1}{r^4}$ **20.** $Q(t) = \dfrac{t^5 + 4t^2 - 3}{6}$

In Problems 21–24, find an equation of the tangent line to the graph of the given function at the indicated value of x.

21. $y = 2x^3 - 1;\quad x = -1$ **22.** $y = -x + \dfrac{8}{x};\quad x = 2$

23. $f(x) = \dfrac{4}{\sqrt{x}} + 2\sqrt{x};\quad x = 4$ **24.** $f(x) = -x^3 + 6x^2;\quad x = 1$

In Problems 25–28, find the point(s) on the graph of the given function at which the tangent line is horizontal.

25. $y = x^2 - 8x + 5$ **26.** $y = \frac{1}{3}x^3 - \frac{1}{2}x^2$

27. $f(x) = x^3 - 3x^2 - 9x + 2$ **28.** $f(x) = x^4 - 4x^3$

In Problems 29–32, find an equation of the normal line to the graph of the given function at the indicated value of x.

29. $y = -x^2 + 1;\quad x = 2$ **30.** $y = x^3;\quad x = 1$

31. $f(x) = \dfrac{1}{3}x^3 - 2x^2;\quad x = 4$ **32.** $f(x) = x^4 - x;\quad x = -1$

In Problems 33–38, find the second derivative of the given function.

33. $y = -x^2 + 3x - 7$ **34.** $y = 15x^2 - 24\sqrt{x}$

35. $y = (-4x + 9)^2$ **36.** $y = 2x^5 + 4x^3 - 6x^2$

37. $f(x) = 10x^{-2}$ **38.** $f(x) = x + \left(\dfrac{2}{x^2}\right)^3$

In Problems 39 and 40, find the indicated higher derivative.

39. $f(x) = 4x^6 + x^5 - x^3;\quad f^{(4)}(x)$

40. $y = x^4 - \dfrac{10}{x};\quad d^5y/dx^5$

In Problems 41 and 42, determine intervals for which $f'(x) > 0$ and intervals for which $f'(x) < 0$.

41. $f(x) = x^2 + 8x - 4$ **42.** $f(x) = x^3 - 3x^2 - 9x$

In Problems 43 and 44, find the point(s) on the graph of f at which $f''(x) = 0$.

43. $f(x) = x^3 + 12x^2 + 20x$ **44.** $f(x) = x^4 - 2x^3$

In Problems 45 and 46, determine intervals for which $f''(x) > 0$ and intervals for which $f''(x) < 0$.

45. $f(x) = (x - 1)^3$ **46.** $f(x) = x^3 + x^2$

An equation containing one or more derivatives of an unknown function $y(x)$ is called a **differential equation**. In Problems 47 and 48, show that the function satisfies the given differential equation.

47. $y = x^{-1} + x^4;\quad x^2y'' - 2xy' - 4y = 0$

48. $y = x + x^3 + 4;\quad x^2y'' - 3xy' + 3y = 12$

49. Find the point on the graph of $f(x) = 2x^2 - 3x + 6$ at which the slope of the tangent line is 5.

50. Find the point on the graph of $f(x) = x^2 - x$ at which the tangent line is $3x - 9y - 4 = 0$.

51. Find the point on the graph of $f(x) = x^2 - x$ at which the slope of the normal line is 2.

52. Find the point on the graph of $f(x) = \frac{1}{4}x^2 - 2x$ at which the tangent line is parallel to the line $3x - 2y + 1 = 0$.

53. Find an equation of the tangent line to the graph of $y = x^3 + 3x^2 - 4x + 1$ at the point where the value of the second derivative is zero.

54. Find an equation of the tangent line to the graph of $y = x^4$ at the point where the value of the third derivative is 12.

≡ Applications

55. The volume V of a sphere of radius r is $V = \frac{4}{3}\pi r^3$. Find the surface area S of the sphere if S is the instantaneous rate of change of the volume with respect to the radius.

56. According to the French physician Jean Louis Poiseuille (1799–1869) the velocity v of blood in an artery with a constant circular cross-section radius R is $v(r) = (P/4\nu l)(R^2 - r^2)$, where P, ν, and l are constants. What is the velocity of blood at the value of r for which $v'(r) = 0$?

57. The potential energy of a spring-mass system when the spring is stretched a distance of x units is $U(x) = \frac{1}{2}kx^2$, where k is the spring constant. The force exerted on the mass is $F = -dU/dx$. Find the force if the spring constant is 30 N/m and the amount of stretch is $\frac{1}{2}$ m.

58. The height s above ground of a projectile at time t is given by

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0,$$

where g, v_0, and s_0 are constants. Find the instantaneous rate of change of s with respect to t at $t = 4$.

≡ Think About It

In Problems 59 and 60, the symbol n represents a positive integer. Find a formula for the given derivative.

59. $\dfrac{d^n}{dx^n}x^n$ **60.** $\dfrac{d^n}{dx^n}\dfrac{1}{x}$

61. From the graphs of f and g in FIGURE 3.2.6, determine which function is the derivative of the other. Explain your choice in words.

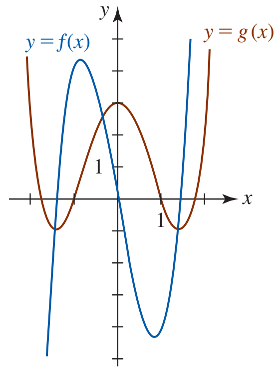


FIGURE 3.2.6 Graphs for Problem 61

62. From the graph of the function $y = f(x)$ given in FIGURE 3.2.7, sketch the graph of f'.

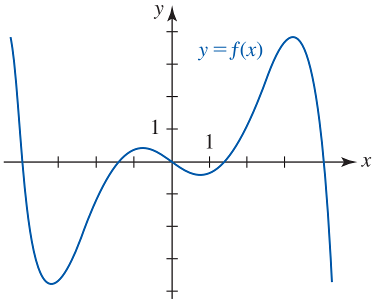


FIGURE 3.2.7 Graph for Problem 62

63. Find a quadratic function $f(x) = ax^2 + bx + c$ such that $f(-1) = -11$, $f'(-1) = 7$, and $f''(-1) = -4$.

64. The graphs of $y = f(x)$ and $y = g(x)$ are said to be **orthogonal** if the tangent lines to each graph are perpendicular at each point of intersection. Show that the graphs of $y = \frac{1}{8}x^2$ and $y = -\frac{1}{4}x^2 + 3$ are orthogonal.

65. Find the values of b and c so that the graph of $f(x) = x^2 + bx$ possesses the tangent line $y = 2x + c$ at $x = -3$.

66. Find an equation of the line(s) that passes through $\left(\frac{3}{2}, 1\right)$ and is tangent to the graph of $f(x) = x^2 + 2x + 2$.

67. Find the point(s) on the graph of $f(x) = x^2 - 5$ such that the tangent line at the point(s) has x-intercept $(-3, 0)$.

68. Find the point(s) on the graph of $f(x) = x^2$ such that the tangent line at the point(s) has y-intercept $(0, -2)$.

69. Explain why the graph of $f(x) = \frac{1}{5}x^5 + \frac{1}{3}x^3$ has no tangent line with slope -1.

70. Find coefficients A and B so that the function $y = Ax^2 + Bx$ satisfies the differential equation $2y'' + 3y' = x - 1$.

71. Find values of a and b such that the slope of the tangent to the graph of $f(x) = ax^2 + bx$ at $(1, 4)$ is -5.

72. Find the slopes of all the normal lines to the graph of $f(x) = x^2$ that pass through the point $(2, 4)$. [*Hint*: Draw a figure and note that *at* $(2, 4)$ there is only one normal line.]

73. Find a point on the graph of $f(x) = x^2 + x$ and a point on the graph of $g(x) = 2x^2 + 4x + 1$ at which the tangent lines are parallel.

74. Find a point on the graph of $f(x) = 3x^5 + 5x^3 + 2x$ at which the tangent has the least possible slope.

75. Find conditions on the coefficients a, b, and c so that the graph of the polynomial function

$$f(x) = ax^3 + bx^2 + cx + d$$

has exactly one horizontal tangent. Exactly two horizontal tangents. No horizontal tangents.

76. Let f be a differentiable function. If $f'(x) > 0$ for all x in the interval (a, b), sketch possible graphs of f on the interval. Describe in words the behavior of the graph of f on the interval. Repeat if $f'(x) < 0$ for all x in the interval (a, b).

77. Suppose f is a differentiable function such that $f'(x) - f(x) = 0$. Find $f^{(100)}(x)$.

78. The graphs of $y = x^2$ and $y = -x^2 + 2x - 3$ given in FIGURE 3.2.8 show that there are two lines L_1 and L_2 that are simultaneously tangent to both graphs. Find the points of tangency on both graphs. Find an equation of each tangent line.

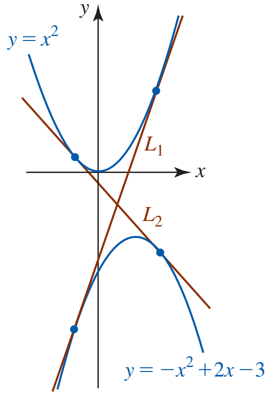


FIGURE 3.2.8 Graphs for Problem 78

Calculator/CAS Problems

79. **(a)** Use a calculator or CAS to obtain the graph of $f(x) = x^4 - 4x^3 - 2x^2 + 12x - 2$.
(b) Evaluate $f''(x)$ at $x = -2$, $x = -1$, $x = 0$, $x = 1$, $x = 2$, $x = 3$, and $x = 4$.
(c) From the data in part (b), do you see any relationship between the shape of the graph of f and the algebraic signs of f''?

80. Use a calculator or CAS to obtain the graph of the given functions. By inspection of the graphs indicate where each function may not be differentiable. Find $f'(x)$ at all points where f is differentiable.

(a) $f(x) = |x^2 - 2x|$ **(b)** $f(x) = |x^3 - 1|$

3.3 Product and Quotient Rules

Introduction So far we know that the derivative of a constant function and a power of x are, in turn:

$$\frac{d}{dx}c = 0 \quad \text{and} \quad \frac{d}{dx}x^n = nx^{n-1}. \tag{1}$$

We also know that for differentiable functions f and g:

$$\frac{d}{dx}cf(x) = cf'(x) \quad \text{and} \quad \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x). \tag{2}$$

Although the results in (1) and (2) allow us to quickly differentiate many algebraic functions (such as polynomials) neither (1) nor (2) are of immediate help in finding derivatives of functions such as $y = x^4\sqrt{x^2+4}$ or $y = x/(2x+1)$. We need additional rules for differentiating products fg and quotients f/g.

Product Rule The rules of differentiation and the derivatives of functions ultimately stem from the definition of the derivative. The Sum Rule in (2), derived in the preceding section, follows from this definition and the fact that the limit of a sum is the sum of the limits whenever the limits exist. We also know that when the limits exist, the limit of a product is the product of the limits. Arguing by analogy, it would then seem plausible that the derivative of a product of two functions is the product of the derivatives. Regrettably, the Product Rule stated next is *not* that simple.

Theorem 3.3.1 Product Rule

If f and g are functions differentiable at x, then fg is differentiable at x, and

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x). \tag{3}$$

PROOF Let $G(x) = f(x)g(x)$. Then by the definition of the derivative along with some algebraic manipulation:

$$\begin{aligned} G'(x) &= \lim_{h\to 0}\frac{G(x+h)-G(x)}{h} = \lim_{h\to 0}\frac{f(x+h)g(x+h)-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\frac{f(x+h)g(x+h)\overbrace{-f(x+h)g(x)+f(x+h)g(x)}^{\text{zero}}-f(x)g(x)}{h} \\ &= \lim_{h\to 0}\left[f(x+h)\frac{g(x+h)-g(x)}{h}+g(x)\frac{f(x+h)-f(x)}{h}\right] \\ &= \lim_{h\to 0}f(x+h)\cdot\lim_{h\to 0}\frac{g(x+h)-g(x)}{h}+\lim_{h\to 0}g(x)\cdot\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}. \end{aligned}$$

Because f is differentiable at x, it is continuous there and so $\lim_{h\to 0}f(x+h) = f(x)$. Furthermore, $\lim_{h\to 0}g(x) = g(x)$. Hence the last equation becomes

$$G'(x) = f(x)g'(x) + g(x)f'(x). \qquad \blacksquare$$

The Product Rule is best memorized in words:

- *The first function times the derivative of the second plus the second function times the derivative of the first.*

EXAMPLE 1 Product Rule

Differentiate $y = (x^3 - 2x^2 + 3)(7x^2 - 4x)$.

Solution From the Product Rule (3),

$$\begin{aligned} \frac{dy}{dx} &= \overbrace{(x^3-2x^2+3)}^{\text{first}}\cdot\overbrace{\frac{d}{dx}(7x^2-4x)}^{\text{derivative of second}}+\overbrace{(7x^2-4x)}^{\text{second}}\cdot\overbrace{\frac{d}{dx}(x^3-2x^2+3)}^{\text{derivative of first}} \\ &= (x^3-2x^2+3)(14x-4)+(7x^2-4x)(3x^2-4x) \\ &= 35x^4-72x^3+24x^2+42x-12. \end{aligned}$$

Alternative Solution The two terms in the given function could be multiplied out to obtain a fifth-degree polynomial. The derivative can then be gotten using the Sum Rule. ■

EXAMPLE 2 Tangent Line

Find an equation of the tangent line to the graph of $y = (1 + \sqrt{x})(x - 2)$ at $x = 4$.

Solution Before taking the derivative we rewrite $\sqrt{x}$ as $x^{1/2}$. Then from the Product Rule (3)

$$\begin{aligned}\frac{dy}{dx} &= (1 + x^{1/2})\frac{d}{dx}(x - 2) + (x - 2)\frac{d}{dx}(1 + x^{1/2})\\ &= (1 + x^{1/2}) \cdot 1 + (x - 2) \cdot \frac{1}{2}x^{-1/2}\\ &= \frac{3x + 2\sqrt{x} - 2}{2\sqrt{x}}.\end{aligned}$$

Evaluating the given function and its derivative at $x = 4$ gives:

$$y(4) = (1 + \sqrt{4})(4 - 2) = 6 \quad \leftarrow \text{point of tangency is } (4, 6)$$

$$\left.\frac{dy}{dx}\right|_{x=4} = \frac{12 + 2\sqrt{4} - 2}{2\sqrt{4}} = \frac{7}{2}. \quad \leftarrow \text{slope of the tangent at } (4, 6) \text{ is } \tfrac{7}{2}$$

By the point–slope form, the tangent line is

$$y - 6 = \frac{7}{2}(x - 4) \qquad \text{or} \qquad y = \frac{7}{2}x - 8.$$ ■

Although (3) is stated for only the product of two functions, it can be applied to functions with a greater number of factors. The idea is to group two (or more) functions and treat this grouping as one function. The next example illustrates the technique.

EXAMPLE 3 Product of Three Functions

Differentiate $y = (4x + 1)(2x^2 - x)(x^3 - 8x)$.

Solution We identify the first two factors as the "first function":

$$\frac{dy}{dx} = \overbrace{(4x + 1)(2x^2 - x)}^{\text{first}}\overbrace{\frac{d}{dx}(x^3 - 8x)}^{\text{derivative of second}} + \overbrace{(x^3 - 8x)}^{\text{second}}\overbrace{\frac{d}{dx}(4x + 1)(2x^2 - x)}^{\text{derivative of first}}.$$

Notice that to find the derivative of the first function, we must apply the Product Rule a second time:

$$\begin{aligned}\frac{dy}{dx} &= (4x + 1)(2x^2 - x) \cdot (3x^2 - 8) + (x^3 - 8x) \cdot \overbrace{[(4x + 1)(4x - 1) + (2x^2 - x) \cdot 4]}^{\text{Product Rule again}}\\ &= (4x + 1)(2x^2 - x)(3x^2 - 8) + (x^3 - 8x)(16x^2 - 1) + 4(x^3 - 8x)(2x^2 - x).\end{aligned}$$ ■

Quotient Rule

The derivative of the quotient of two functions f and g is given next.

Theorem 3.3.2 Quotient Rule

If f and g are functions differentiable at x and $g(x) \neq 0$, then f/g is differentiable at x, and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}. \tag{4}$$

PROOF Let $G(x) = f(x)/g(x)$. Then

$$G'(x) = \lim_{h\to 0}\frac{G(x+h) - G(x)}{h} = \lim_{h\to 0}\frac{\dfrac{f(x+h)}{g(x+h)} - \dfrac{f(x)}{g(x)}}{h}$$

$$= \lim_{h\to 0}\frac{g(x)f(x+h) - f(x)g(x+h)}{hg(x+h)g(x)}$$

$$= \lim_{h\to 0}\frac{g(x)f(x+h) \overbrace{- g(x)f(x) + g(x)f(x)}^{\text{zero}} - f(x)g(x+h)}{hg(x+h)g(x)}$$

$$= \lim_{h\to 0}\frac{g(x)\dfrac{f(x+h) - f(x)}{h} - f(x)\dfrac{g(x+h) - g(x)}{h}}{g(x+h)g(x)}$$

$$= \frac{\lim\limits_{h\to 0} g(x)\cdot\lim\limits_{h\to 0}\dfrac{f(x+h) - f(x)}{h} - \lim\limits_{h\to 0} f(x)\cdot\lim\limits_{h\to 0}\dfrac{g(x+h) - g(x)}{h}}{\lim\limits_{h\to 0} g(x+h)\cdot\lim\limits_{h\to 0} g(x)}.$$

Since all limits are assumed to exist, the last line is the same as

$$G'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}.$$ ■

In words, the Quotient Rule starts with the denominator:

- *The denominator times the derivative of the numerator minus the numerator times the derivative of the denominator all divided by the denominator squared.*

EXAMPLE 4 Quotient Rule

Differentiate $y = \dfrac{3x^2 - 1}{2x^3 + 5x^2 + 7}$.

Solution From the Quotient Rule (4),

$$\frac{dy}{dx} = \frac{\overbrace{(2x^3 + 5x^2 + 7)}^{\text{denominator}}\cdot\overbrace{\frac{d}{dx}(3x^2 - 1)}^{\text{derivative of numerator}} - \overbrace{(3x^2 - 1)}^{\text{numerator}}\cdot\overbrace{\frac{d}{dx}(2x^3 + 5x^2 + 7)}^{\text{derivative of denominator}}}{\underbrace{(2x^3 + 5x^2 + 7)^2}_{\text{square of denominator}}}$$

$$= \frac{(2x^3 + 5x^2 + 7)\cdot 6x - (3x^2 - 1)\cdot(6x^2 + 10x)}{(2x^3 + 5x^2 + 7)^2} \quad \leftarrow \text{multiply out numerator}$$

$$= \frac{-6x^4 + 6x^2 + 52x}{(2x^3 + 5x^2 + 7)^2}.$$ ■

EXAMPLE 5 Quotient and Product Rule

Find the points on the graph of $y = \dfrac{(x^2 + 1)(2x^2 + 1)}{3x^2 + 1}$ where the tangent line is horizontal.

Solution We begin with the Quotient Rule and then use the Product Rule when differentiating the numerator:

Product Rule here

$$\frac{dy}{dx} = \frac{(3x^2+1)\cdot \overbrace{\frac{d}{dx}[(x^2+1)(2x^2+1)]} - (x^2+1)(2x^2+1)\cdot \frac{d}{dx}(3x^2+1)}{(3x^2+1)^2}$$

$$= \frac{(3x^2+1)[(x^2+1)4x + (2x^2+1)2x] - (x^2+1)(2x^2+1)6x}{(3x^2+1)^2} \quad \leftarrow \text{multiply out numerator}$$

$$= \frac{12x^5 + 8x^3}{(3x^2+1)^2}.$$

At a point where the tangent line is horizontal we must have $dy/dx = 0$. The derivative just found can only be 0 when the numerator satisfies

Of course, values of x that make the numerator zero must *not* simultaneously make the denominator zero.

$$12x^5 + 8x^3 = 0 \qquad \text{or} \qquad x^3(12x^2+8) = 0. \tag{5}$$

In (5) because $12x^2 + 8 \neq 0$ for all real numbers x, we must have $x = 0$. Substituting this number into the function gives $y(0) = 1$. The tangent line is horizontal at the y-intercept $(0, 1)$. ■

Postscript—Power Rule Revisited Remember in Section 3.2 we stated that the Power Rule, $(d/dx)x^n = nx^{n-1}$, is valid for all real number exponents n. We are now in a position to prove the rule when the exponent is a negative integer $-m$. Since, by definition, $x^{-m} = 1/x^m$, where m is a positive integer, we can obtain the derivative of x^{-m} by the Quotient Rule and the laws of exponents:

$$\frac{d}{dx}x^{-m} = \frac{d}{dx}\left(\frac{1}{x^m}\right) = \frac{x^m\cdot\frac{d}{dx}1 - 1\cdot\frac{d}{dx}x^m}{(x^m)^2} = -\frac{mx^{m-1}}{x^{2m}} = -mx^{-m-1}.$$

(subtract exponents)

$\frac{d}{dx}$ NOTES FROM THE CLASSROOM

(*i*) The Product and Quotient Rules will usually lead to expressions that demand simplification. If your answer to a problem does not look like the one in the text answer section, you may not have performed sufficient simplifications. Do not be content to simply carry through the mechanics of the various rules of differentiation; it is always a good idea to practice your algebraic skills.

(*ii*) The Quotient Rule is sometimes used when it is not required. Although we could use the Quotient Rule to differentiate functions such as

$$y = \frac{x^5}{6} \qquad \text{and} \qquad y = \frac{10}{x^3},$$

it is simpler (and faster) to rewrite the functions as $y = \frac{1}{6}x^5$ and $y = 10x^{-3}$ and then use the Constant Multiple and Power Rules:

$$\frac{dy}{dx} = \frac{1}{6}\frac{d}{dx}x^5 = \frac{5}{6}x^4 \qquad \text{and} \qquad \frac{dy}{dx} = 10\frac{d}{dx}x^{-3} = -30x^{-4}.$$

Exercises 3.3

Answers to selected odd-numbered problems begin on page ANS-000.

≡ Fundamentals

In Problems 1–10, find dy/dx.

1. $y = (x^2 - 7)(x^3 + 4x + 2)$

2. $y = (7x + 1)(x^4 - x^3 - 9x)$

3. $y = \left(4\sqrt{x} + \frac{1}{x}\right)\left(2x - \frac{6}{\sqrt[3]{x}}\right)$

4. $y = \left(x^2 - \frac{1}{x^2}\right)\left(x^3 + \frac{1}{x^3}\right)$

5. $y = \dfrac{10}{x^2 + 1}$ **6.** $y = \dfrac{5}{4x - 3}$

7. $y = \dfrac{3x + 1}{2x - 5}$ **8.** $y = \dfrac{2 - 3x}{7 - x}$

9. $y = (6x - 1)^2$ **10.** $y = (x^4 + 5x)^2$

In Problems 11–20, find $f'(x)$.

11. $f(x) = \left(\dfrac{1}{x} - \dfrac{4}{x^3}\right)(x^3 - 5x - 1)$

12. $f(x) = (x^2 - 1)\left(x^2 - 10x + \dfrac{2}{x^2}\right)$

13. $f(x) = \dfrac{x^2}{2x^2 + x + 1}$ **14.** $f(x) = \dfrac{x^2 - 10x + 2}{x(x^2 - 1)}$

15. $f(x) = (x + 1)(2x + 1)(3x + 1)$

16. $f(x) = (x^2 + 1)(x^3 - x)(3x^4 + 2x - 1)$

17. $f(x) = \dfrac{(2x + 1)(x - 5)}{3x + 2}$ **18.** $f(x) = \dfrac{x^5}{(x^2 + 1)(x^3 + 4)}$

19. $f(x) = (x^2 - 2x - 1)\left(\dfrac{x + 1}{x + 3}\right)$

20. $f(x) = (x + 1)\left(x + 1 - \dfrac{1}{x + 2}\right)$

In Problems 21–24, find an equation of the tangent line to the graph of the given function at the indicated value of x.

21. $y = \dfrac{x}{x - 1}$; $x = \dfrac{1}{2}$ **22.** $y = \dfrac{5x}{x^2 + 1}$; $x = 2$

23. $y = (2\sqrt{x} + x)(-2x^2 + 5x - 1)$; $x = 1$

24. $y = (2x^2 - 4)(x^3 + 5x + 3)$; $x = 0$

In Problems 25–28, find the point(s) on the graph of the given function at which the tangent line is horizontal.

25. $y = (x^2 - 4)(x^2 - 6)$ **26.** $y = x(x - 1)^2$

27. $y = \dfrac{x^2}{x^4 + 1}$ **28.** $y = \dfrac{1}{x^2 - 6x}$

In Problems 29 and 30, find the point(s) on the graph of the given function at which the tangent line has the indicated slope.

29. $y = \dfrac{x + 3}{x + 1}$; $m = -\dfrac{1}{8}$

30. $y = (x + 1)(2x + 5)$; $m = -3$

In Problems 31 and 32, find the point(s) on the graph of the given function at which the tangent line has the indicated property.

31. $y = \dfrac{x + 4}{x + 5}$; perpendicular to $y = -x$

32. $y = \dfrac{x}{x + 1}$; parallel to $y = \dfrac{1}{4}x - 1$

33. Find the value of k such that the tangent line to the graph of $f(x) = (k + x)/x^2$ has slope 5 at $x = 2$.

34. Show that the tangent to the graph of $f(x) = (x^2 + 14)/(x^2 + 9)$ at $x = 1$ is perpendicular to the tangent to the graph of $g(x) = (1 + x^2)(1 + 2x)$ at $x = 1$.

In Problems 35–40, f and g are differentiable functions. Find $F'(1)$ if $f(1) = 2$, $f'(1) = -3$, and $g(1) = 6$, $g'(1) = 2$.

35. $F(x) = 2f(x)g(x)$ **36.** $F(x) = x^2 f(x)g(x)$

37. $F(x) = \dfrac{2g(x)}{3f(x)}$ **38.** $F(x) = \dfrac{1 + 2f(x)}{x - g(x)}$

39. $F(x) = \left(\dfrac{4}{x} + f(x)\right)g(x)$ **40.** $F(x) = \dfrac{xf(x)}{g(x)}$

41. Suppose $F(x) = \sqrt{x}\, f(x)$, where f is a differentiable function. Find $F''(4)$ if $f(4) = -16$, $f'(4) = 2$, and $f''(4) = 3$.

42. Suppose $F(x) = xf(x) + xg(x)$, where f and g are differentiable functions. Find $F''(0)$ if $f'(0) = -1$ and $g'(0) = 6$.

43. Suppose $F(x) = f(x)/x$, where f is a differentiable function. Find $F''(x)$.

44. Suppose $F(x) = x^3 f(x)$, where f is a differentiable function. Find $F'''(x)$.

In Problems 45–48, determine intervals for which $f'(x) > 0$ and intervals for which $f'(x) < 0$.

45. $f(x) = \dfrac{5}{x^2 - 2x}$ **46.** $f(x) = \dfrac{x^2 + 3}{x + 1}$

47. $f(x) = (-2x + 6)(4x + 7)$

48. $f(x) = (x - 2)(4x^2 + 8x + 4)$

≡ Applications

49. The Law of Universal Gravitation states that the force F between two bodies of masses m_1 and m_2 separated by a distance r is $F = km_1m_2/r^2$, where k is constant. What is the instantaneous rate of change of F with respect to r when $r = \frac{1}{2}$ km?

50. The potential energy U between two atoms in a diatomic molecule is given by $U(x) = q_1/x^{12} - q_2/x^6$, where q_1 and q_2 are positive constants and x is the distance between the atoms. The force between the atoms is defined as $F(x) = -U'(x)$. Show that $F\left(\sqrt[6]{2q_1/q_2}\right) = 0$.

51. The **van der Waals equation of state** for an ideal gas is

$$\left(P + \frac{a}{V^2}\right)(V - b) = RT,$$

where P is pressure, V is volume per mole, R is the universal gas constant, T is temperature, and a and b are constants depending on the gas. Find dP/dV in the case where T is constant.

52. For a convex lens, the focal length f is related to the object distance p and the image distance q by the **lens equation**

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}.$$

Find the instantaneous rate of change of q with respect to p in the case where f is constant. Explain the significance of the negative sign in your answer. What happens to q as p increases?

Think About It

53. **(a)** Graph the rational function $f(x) = \dfrac{2}{x^2 + 1}$.

(b) Find all the points on the graph of f such that the normal lines pass through the origin.

54. Suppose $y = f(x)$ is a differentiable function.

(a) Find dy/dx for $y = [f(x)]^2$.

(b) Find dy/dx for $y = [f(x)]^3$.

(c) Conjecture a rule for finding the derivative of $y = [f(x)]^n$, where n is a positive integer.

(d) Use your conjecture in part (c) to find the derivative of $y = (x^2 + 2x - 6)^{500}$.

55. Suppose $y_1(x)$ satisfies the differential equation $y' + P(x)y = 0$, where P is a known function. Show that $y = u(x)y_1(x)$ satisfies the differential equation

$$y' + P(x)y = f(x)$$

whenever $u(x)$ satisfies $du/dx = f(x)/y_1(x)$.

3.4 Trigonometric Functions

Introduction In this section we develop the derivatives of the six trigonometric functions. Once we have found the derivatives of $\sin x$ and $\cos x$ we can determine the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$ using the Quotient Rule found in the preceding section. We will see immediately that the derivative of $\sin x$ utilizes the following two limit results

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0 \tag{1}$$

found in Section 2.4.

Derivatives of Sine and Cosine To find the derivative of $f(x) = \sin x$ we use the basic definition of the derivative

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} \tag{2}$$

and the four-step process introduced in Sections 2.7 and 3.1. In the first step we use the sum formula for the sine function,

$$\sin(x_1 + x_2) = \sin x_1 \cos x_2 + \cos x_1 \sin x_2, \tag{3}$$

but with x and h playing the parts of the symbols x_1 and x_2.

(i) $f(x+h) = \sin(x+h) = \sin x \cos h + \cos x \sin h$ ← from (3)

(ii) $f(x+h) - f(x) = \sin x \cos h + \cos x \sin h - \sin x$ ← factor $\sin x$ from first and third terms

$= \sin x(\cos h - 1) + \cos x \sin h$

As we see in the next line, we cannot cancel the h's in the difference quotient but we can rewrite the expression to make use of the limit results in (1).

(iii)
$$\frac{f(x+h) - f(x)}{h} = \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} = \sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h}$$

(iv) In this line, the symbol h plays the part of the symbol x in (1):

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \sin x \cdot \lim_{h\to 0}\frac{\cos h - 1}{h} + \cos x \cdot \lim_{h\to 0}\frac{\sin h}{h}.$$

From the limit results in (1), the last line is the same as

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

Hence,

$$\frac{d}{dx}\sin x = \cos x. \tag{4}$$

In a similar manner it can be shown that

$$\frac{d}{dx}\cos x = -\sin x. \tag{5}$$

See Problem 50 in Exercises 3.4.

EXAMPLE 1 Equation of a Tangent Line

Find an equation of the tangent line to the graph of $f(x) = \sin x$ at $x = 4\pi/3$.

Solution From (4) the derivative of $f(x) = \sin x$ is $f'(x) = \cos x$. When evaluated at the same number $x = 4\pi/3$ these functions give:

$$f\left(\frac{4\pi}{3}\right) = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2} \quad \leftarrow \text{point of tangency is } \left(\tfrac{4\pi}{3}, -\tfrac{\sqrt{3}}{2}\right)$$

$$f'\left(\frac{4\pi}{3}\right) = \cos\frac{4\pi}{3} = -\frac{1}{2}. \quad \leftarrow \text{slope of tangent at } \left(\tfrac{4\pi}{3}, -\tfrac{\sqrt{3}}{2}\right) \text{ is } -\tfrac{1}{2}$$

From the point–slope form of a line, an equation of the tangent line is

$$y + \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(x - \frac{4\pi}{3}\right) \quad \text{or} \quad y = -\frac{1}{2}x + \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

The tangent line is shown in red in FIGURE 3.4.1. ■

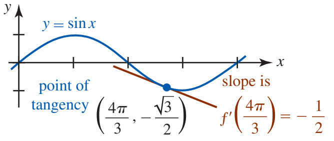


FIGURE 3.4.1 Tangent line in Example 1

Other Trigonometric Functions The results in (4) and (5) can be used in conjunction with the rules of differentiation to find the derivatives of the tangent, cotangent, secant, and cosecant functions.

To differentiate $\tan x = \sin x/\cos x$, we use the Quotient Rule:

$$\frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \frac{d}{dx}\sin x - \sin x \frac{d}{dx}\cos x}{(\cos x)^2}$$

$$= \frac{\cos x\,(\cos x) - \sin x\,(-\sin x)}{(\cos x)^2} = \frac{\overbrace{\cos^2 x + \sin^2 x}^{\text{this equals 1}}}{\cos^2 x}.$$

Using the fundamental Pythagorean identity $\sin^2 x + \cos^2 x = 1$ and the fact that $1/\cos^2 x = (1/\cos x)^2 = \sec^2 x$, the last equation simplifies to

$$\frac{d}{dx}\tan x = \sec^2 x. \tag{6}$$

The derivative formula for the cotangent

$$\frac{d}{dx}\cot x = -\csc^2 x \tag{7}$$

is obtained in an analogous fashion and left as an exercise. See Problem 51 in Exercises 3.4.

Now $\sec x = 1/\cos x$. Therefore, we can use the Quotient Rule again to find the derivative of the secant function:

$$\frac{d}{dx}\frac{1}{\cos x} = \frac{\cos x \frac{d}{dx}1 - 1 \cdot \frac{d}{dx}\cos x}{(\cos x)^2}$$

$$= \frac{0 - (-\sin x)}{(\cos x)^2} = \frac{\sin x}{\cos^2 x}. \tag{8}$$

By writing $$\frac{\sin x}{\cos^2 x} = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x$$

we can express (8) as

$$\frac{d}{dx}\sec x = \sec x\tan x. \tag{9}$$

The final result also follows immediately from the Quotient Rule:

$$\frac{d}{dx}\csc x = -\csc x\cot x. \tag{10}$$

See Problem 52 in Exercises 3.4.

EXAMPLE 2 Product Rule

Differentiate $y = x^2\sin x$.

Solution The Product Rule along with (4) gives

$$\begin{aligned}\frac{dy}{dx} &= x^2\frac{d}{dx}\sin x + \sin x\frac{d}{dx}x^2\\ &= x^2\cos x + 2x\sin x.\end{aligned}$$

■

EXAMPLE 3 Product Rule

Differentiate $y = \cos^2 x$.

Solution One way of differentiating this function is to recognize it as a product: $y = (\cos x)(\cos x)$. Then by the Product Rule and (5),

$$\begin{aligned}\frac{dy}{dx} &= \cos x\frac{d}{dx}\cos x + \cos x\frac{d}{dx}\cos x\\ &= \cos x(-\sin x) + (\cos x)(-\sin x)\\ &= -2\sin x\cos x.\end{aligned}$$

In the next section we will see that there is an alternative procedure for differentiating a power of a function. ■

EXAMPLE 4 Quotient Rule

Differentiate $y = \dfrac{\sin x}{2 + \sec x}$.

Solution By the Quotient Rule, (4), and (9),

$$\begin{aligned}\frac{dy}{dx} &= \frac{(2+\sec x)\dfrac{d}{dx}\sin x - \sin x\dfrac{d}{dx}(2+\sec x)}{(2+\sec x)^2}\\ &= \frac{(2+\sec x)\cos x - \sin x(\sec x\tan x)}{(2+\sec x)^2}\\ &= \frac{1+2\cos x-\tan^2 x}{(2+\sec x)^2}.\end{aligned}$$

← $\sec x\cos x = 1$ and $\sin x(\sec x\tan x) = \sin^2 x/\cos^2 x$

■

EXAMPLE 5 Second Derivative

Find the second derivative of $f(x) = \sec x$.

Solution From (9) the first derivative is

$$f'(x) = \sec x\tan x.$$

To obtain the second derivative we must now use the Product Rule along with (6) and (9):

$$\begin{aligned}f''(x) &= \sec x\frac{d}{dx}\tan x + \tan x\frac{d}{dx}\sec x\\ &= \sec x(\sec^2 x) + \tan x(\sec x\tan x)\\ &= \sec^3 x + \sec x\tan^2 x.\end{aligned}$$

■

For future reference we summarize the derivative formulas introduced in this section.

Theorem 3.4.1 Derivatives of Trigonometric Functions

The derivatives of the six trigonometric functions are

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x, \tag{11}$$

$$\frac{d}{dx}\tan x = \sec^2 x, \qquad \frac{d}{dx}\cot x = -\csc^2 x, \tag{12}$$

$$\frac{d}{dx}\sec x = \sec x\tan x, \qquad \frac{d}{dx}\csc x = -\csc x\cot x. \tag{13}$$

$\frac{d}{dx}$ NOTES FROM THE CLASSROOM

When working the problems in Exercises 3.4 you may not get the same answer as given in the answer section in the back of this book. This is because there are so many trigonometric identities that answers can often be expressed in a more compact form. For example, the answer in Example 3:

$$\frac{dy}{dx} = -2\sin x\cos x \quad \text{is the same as} \quad \frac{dy}{dx} = -\sin 2x$$

by the double-angle formula for the sine function. Try to resolve any differences between your answer and the given answer.

Exercises 3.4

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–12, find dy/dx.

1. $y = x^2 - \cos x$
2. $y = 4x^3 + x + 5\sin x$
3. $y = 1 + 7\sin x - \tan x$
4. $y = 3\cos x - 5\cot x$
5. $y = x\sin x$
6. $y = (4\sqrt{x} - 3\sqrt[3]{x})\cos x$
7. $y = (x^3 - 2)\tan x$
8. $y = \cos x\cot x$
9. $y = (x^2 + \sin x)\sec x$
10. $y = \csc x\tan x$
11. $y = \cos^2 x + \sin^2 x$
12. $y = x^3\cos x - x^3\sin x$

In Problems 13–22, find $f'(x)$.

13. $f(x) = (\csc x)^{-1}$
14. $f(x) = \dfrac{2}{\cos x\cot x}$
15. $f(x) = \dfrac{\cot x}{x + 1}$
16. $f(x) = \dfrac{x^2 - 6x}{1 + \cos x}$
17. $f(x) = \dfrac{x^2}{1 + 2\tan x}$
18. $f(x) = \dfrac{2 + \sin x}{x}$
19. $f(x) = \dfrac{\sin x}{1 + \cos x}$
20. $f(x) = \dfrac{1 + \csc x}{1 + \sec x}$
21. $f(x) = x^4\sin x\tan x$
22. $f(x) = \dfrac{1 + \sin x}{x\cos x}$

In Problems 23–26, find an equation of the tangent line to the graph of the given function at the indicated value of x.

23. $f(x) = \cos x; \quad x = \pi/3$
24. $f(x) = \tan x; \quad x = \pi$
25. $f(x) = \sec x; \quad x = \pi/6$
26. $f(x) = \csc x; \quad x = \pi/2$

In Problems 27–30, consider the graph of the given function on the interval $[0, 2\pi]$. Find the x-coordinates of the point(s) on the graph of the function where the tangent line is horizontal.

27. $f(x) = x + 2\cos x$
28. $f(x) = \dfrac{\sin x}{2 - \cos x}$
29. $f(x) = \dfrac{1}{x + \cos x}$
30. $f(x) = \sin x + \cos x$

In Problems 31–34, find an equation of the normal line to the graph of the given function at the indicated value of x.

31. $f(x) = \sin x; \quad x = 4\pi/3$
32. $f(x) = \tan^2 x; \quad x = \pi/4$
33. $f(x) = x\cos x; \quad x = \pi$
34. $f(x) = \dfrac{x}{1 + \sin x}; \quad x = \pi/2$

In Problems 35 and 36, find the derivative of the given function by first using an appropriate trigonometric identity.

35. $f(x) = \sin 2x$
36. $f(x) = \cos^2\dfrac{x}{2}$

In Problems 37–42, find the second derivative of the given function.

37. $f(x) = x\sin x$
38. $f(x) = 3x - x^2\cos x$
39. $f(x) = \dfrac{\sin x}{x}$
40. $f(x) = \dfrac{1}{1 + \cos x}$
41. $y = \csc x$
42. $y = \tan x$

In Problems 43 and 44, C_1 and C_2 are arbitrary real constants. Show that the function satisfies the given differential equation.

43. $y = C_1 \cos x + C_2 \sin x - \frac{1}{2} x \cos x; \quad y'' + y = \sin x$

44. $y = C_1 \frac{\cos x}{\sqrt{x}} + C_2 \frac{\sin x}{\sqrt{x}}; \quad x^2 y'' + xy' + \left(x^2 - \frac{1}{4}\right) y = 0$

Applications

45. When the angle of elevation of the sun is θ, a telephone pole 40 ft high casts a shadow of length s as shown in FIGURE 3.4.2. Find the rate of change of s with respect to θ when $\theta = \pi/3$ radians. Explain the significance of the minus sign in the answer.

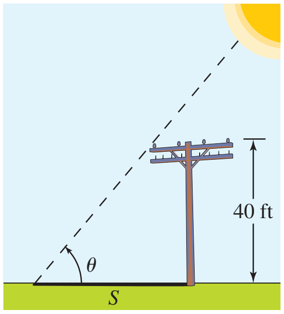


FIGURE 3.4.2 Shadow in Problem 45

46. The two ends of a 10-ft board are attached to perpendicular rails, as shown in FIGURE 3.4.3, so that point P is free to move vertically and point R is free to move horizontally.

(a) Express the area A of triangle PQR as a function of the indicated angle θ.

(b) Find the rate of change of A with respect to θ.

(c) Initially the board rests flat on the horizontal rail. Suppose point R is then moved in the direction of point Q, thereby forcing point P to move up the vertical rail. Initially the area of the triangle is $0\ (\theta = 0)$, but then it increases for a while as θ increases and then decreases as R approaches Q. When the board is vertical, the area of the triangle is again $0\ (\theta = \pi/2)$. Graph the derivative $dA/d\theta$. Interpret this graph to find values of θ for which A is increasing and values of θ for which A is decreasing. Now verify your interpretation of the graph of the derivative by graphing $A(\theta)$.

(d) Use the graphs in part (c) to find the value of θ for which the area of the triangle is the greatest.

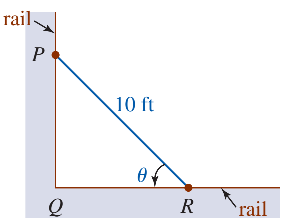


FIGURE 3.4.3 Board in Problem 46

Think About It

47. (a) Find all positive integers n such that

$$\frac{d^n}{dx^n} \sin x = \sin x; \quad \frac{d^n}{dx^n} \cos x = \cos x;$$
$$\frac{d^n}{dx^n} \cos x = \sin x; \quad \frac{d^n}{dx^n} \sin x = \cos x.$$

(b) Use the results in part (a) as an aid in finding

$$\frac{d^{21}}{dx^{21}} \sin x, \quad \frac{d^{30}}{dx^{30}} \sin x, \quad \frac{d^{40}}{dx^{40}} \cos x, \quad \text{and} \quad \frac{d^{67}}{dx^{67}} \cos x.$$

48. Find two distinct points P_1 and P_2 on the graph of $y = \cos x$ so that the tangent line at P_1 is perpendicular to the tangent line at P_2.

49. Find two distinct points P_1 and P_2 on the graph of $y = \sin x$ so that the tangent line at P_1 is parallel to the tangent line at P_2.

50. Use (1), (2), and the sum formula for the cosine to show that

$$\frac{d}{dx} \cos x = -\sin x.$$

51. Use (4) and (5) and the Quotient Rule to show that

$$\frac{d}{dx} \cot x = -\csc^2 x.$$

52. Use (4) and the Quotient Rule to show that

$$\frac{d}{dx} \csc x = -\csc x \cot x.$$

Calculator/CAS Problems

In Problems 53 and 54, use a calculator or CAS to obtain the graph of the given function. By inspection of the graph indicate where the function may not be differentiable.

53. $f(x) = 0.5(\sin x + |\sin x|)$ **54.** $f(x) = |x + \sin x|$

55. As shown in FIGURE 3.4.4, a boy pulls a sled on which his little sister is seated. If the sled and girl weigh a total of 70 lb, and if the coefficient of sliding friction of snow-covered ground is 0.2, then the magnitude F of the force (measured in pounds) required to move the sled is

$$F = \frac{70(0.2)}{0.2 \sin\theta + \cos\theta},$$

where θ is the angle the tow rope makes with the horizontal.

(a) Use a calculator or CAS to obtain the graph of F on the interval $[-1, 1]$.

(b) Find the derivative $dF/d\theta$.

(c) Find the angle (in radians) for which $dF/d\theta = 0$.

(d) Find the value of F corresponding to the angle found in part (c).

(e) Use the graph in part (a) as an aid in interpreting the numbers found in parts (c) and (d).

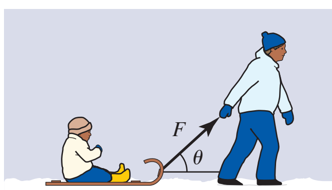


FIGURE 3.4.4 Sled in Problem 55

3.5 Chain Rule

Introduction As discussed in Section 3.2, the Power Rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

is valid for all real number exponents n. In this section we see that a similar rule holds for the derivative of a power of a function $y = [g(x)]^n$. Before stating the formal result, let us consider an example when n is a positive integer.

Suppose we wish to differentiate

$$y = (x^5 + 1)^2. \tag{1}$$

By writing (1) as $y = (x^5 + 1)\cdot(x^5 + 1)$, we can find the derivative using the Product Rule:

$$\begin{aligned}\frac{d}{dx}(x^5 + 1)^2 &= (x^5 + 1)\cdot\frac{d}{dx}(x^5 + 1) + (x^5 + 1)\cdot\frac{d}{dx}(x^5 + 1)\\ &= (x^5 + 1)\cdot 5x^4 + (x^5 + 1)\cdot 5x^4\\ &= 2(x^5 + 1)\cdot 5x^4. \end{aligned}\tag{2}$$

Similarly, to differentiate the function $y = (x^5 + 1)^3$, we can write it as $y = (x^5 + 1)^2\cdot(x^5 + 1)$ and use the Product Rule and the result given in (2):

$$\begin{aligned}\frac{d}{dx}(x^5 + 1)^3 &= \frac{d}{dx}(x^5 + 1)^2\cdot(x^5 + 1)\\ &= (x^5 + 1)^2\cdot\frac{d}{dx}(x^5 + 1) + (x^5 + 1)\cdot\overbrace{\frac{d}{dx}(x^5 + 1)^2}^{\text{we know this from (2)}}\\ &= (x^5 + 1)^2\cdot 5x^4 + (x^5 + 1)\cdot 2(x^5 + 1)\cdot 5x^4\\ &= 3(x^5 + 1)^2\cdot 5x^4. \end{aligned}\tag{3}$$

In like manner, by writing $y = (x^5 + 1)^4$ as $y = (x^5 + 1)^3\cdot(x^5 + 1)$ we can readily show by the Product Rule and (3) that

$$\frac{d}{dx}(x^5 + 1)^4 = 4(x^5 + 1)^3\cdot 5x^4. \tag{4}$$

Power Rule for Functions Inspection of (2), (3), and (4) reveals a pattern for differentiating a power of a function g. For example, in (4) we see

bring down exponent as a multiple ↓ ↓ derivative of function inside parentheses

$$4(x^5 + 1)^3\cdot 5x^4$$

↑ decrease exponent by 1

For emphasis, if we denote a differentiable function by [ ], it appears that

$$\frac{d}{dx}[\ ]^n = n[\ ]^{n-1}\frac{d}{dx}[\ ].$$

The foregoing discussion suggests the result stated in the next theorem.

Theorem 3.5.1 Power Rule for Functions

If n is any real number and $u = g(x)$ is differentiable at x, then

$$\frac{d}{dx}[g(x)]^n = n[g(x)]^{n-1}\cdot g'(x), \tag{5}$$

or equivalently,

$$\frac{d}{dx}u^n = nu^{n-1}\cdot\frac{du}{dx}. \tag{6}$$

Theorem 3.5.1 is itself a special case of a more general theorem, called the **Chain Rule**, which will be presented after we consider some examples of this new power rule.

EXAMPLE 1 Power Rule for Functions

Differentiate $y = (4x^3 + 3x + 1)^7$.

Solution With the identification that $u = g(x) = 4x^3 + 3x + 1$, we see from (6) that

$$\frac{dy}{dx} = \overbrace{7}^{n}\overbrace{(4x^3 + 3x + 1)^6}^{u^{n-1}} \cdot \overbrace{\frac{d}{dx}(4x^3 + 3x + 1)}^{du/dx} = 7(4x^3 + 3x + 1)^6(12x^2 + 3).$$ ∎

EXAMPLE 2 Power Rule for Functions

To differentiate $y = 1/(x^2 + 1)$, we could, of course, use the Quotient Rule. However, by rewriting the function as $y = (x^2 + 1)^{-1}$, it is also possible to use the Power Rule for Functions with $n = -1$:

$$\frac{dy}{dx} = (-1)(x^2 + 1)^{-2} \cdot \frac{d}{dx}(x^2 + 1) = (-1)(x^2 + 1)^{-2}\, 2x = \frac{-2x}{(x^2 + 1)^2}.$$ ∎

EXAMPLE 3 Power Rule for Functions

Differentiate $y = \dfrac{1}{(7x^5 - x^4 + 2)^{10}}$.

Solution Write the given function as $y = (7x^5 - x^4 + 2)^{-10}$. Identify $u = 7x^5 - x^4 + 2$, $n = -10$ and use the Power Rule (6):

$$\frac{dy}{dx} = -10(7x^5 - x^4 + 2)^{-11} \cdot \frac{d}{dx}(7x^5 - x^4 + 2) = \frac{-10(35x^4 - 4x^3)}{(7x^5 - x^4 + 2)^{11}}.$$ ∎

EXAMPLE 4 Power Rule for Functions

Differentiate $y = \tan^3 x$.

Solution For emphasis, we first rewrite the function as $y = (\tan x)^3$ and then use (6) with $u = \tan x$ and $n = 3$:

$$\frac{dy}{dx} = 3(\tan x)^2 \cdot \frac{d}{dx}\tan x.$$

Recall from (6) of Section 3.4 that $(d/dx)\tan x = \sec^2 x$. Hence,

$$\frac{dy}{dx} = 3\tan^2 x \sec^2 x.$$ ∎

EXAMPLE 5 Quotient Rule then Power Rule

Differentiate $y = \dfrac{(x^2 - 1)^3}{(5x + 1)^8}$.

Solution We start with the Quotient Rule followed by two applications of the Power Rule for Functions:

Power Rule for Functions

$$\frac{dy}{dx} = \frac{(5x + 1)^8 \cdot \frac{d}{dx}(x^2 - 1)^3 - (x^2 - 1)^3 \cdot \frac{d}{dx}(5x + 1)^8}{(5x + 1)^{16}}$$

$$= \frac{(5x + 1)^8 \cdot 3(x^2 - 1)^2 \cdot 2x - (x^2 - 1)^3 \cdot 8(5x + 1)^7 \cdot 5}{(5x + 1)^{16}}$$

$$= \frac{6x(5x+1)^8(x^2-1)^2 - 40(5x+1)^7(x^2-1)^3}{(5x+1)^{16}}$$
$$= \frac{(x^2-1)^2(-10x^2+6x+40)}{(5x+1)^9}.$$ ■

EXAMPLE 6 Power Rule then Quotient Rule

Differentiate $y = \sqrt{\dfrac{2x-3}{8x+1}}$.

Solution By rewriting the function as

$$y = \left(\frac{2x-3}{8x+1}\right)^{1/2} \quad \text{we can identify} \quad u = \frac{2x-3}{8x+1}$$

and $n = \frac{1}{2}$. Thus in order to compute du/dx in (6) we must use the Quotient Rule:

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{2x-3}{8x+1}\right)^{-1/2} \cdot \frac{d}{dx}\left(\frac{2x-3}{8x+1}\right)$$
$$= \frac{1}{2}\left(\frac{2x-3}{8x+1}\right)^{-1/2} \cdot \frac{(8x+1)\cdot 2 - (2x-3)\cdot 8}{(8x+1)^2}$$
$$= \frac{1}{2}\left(\frac{2x-3}{8x+1}\right)^{-1/2} \cdot \frac{26}{(8x+1)^2}.$$

Finally, we simplify using the laws of exponents:

$$\frac{dy}{dx} = \frac{13}{(2x-3)^{1/2}(8x+1)^{3/2}}.$$ ■

Chain Rule A power of a function can be written as a composite function. If we identify $f(x) = x^n$ and $u = g(x)$, then $f(u) = f(g(x)) = [g(x)]^n$. The Chain Rule gives us a way of differentiating any composition $f \circ g$ of two differentiable functions f and g.

Theorem 3.5.2 Chain Rule

If the function f is differentiable at $u = g(x)$, and the function g is differentiable at x, then the composition $y = (f \circ g)(x) = f(g(x))$ is differentiable at x and

$$\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x) \tag{7}$$

or equivalently,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}. \tag{8}$$

PROOF FOR $\Delta u \neq 0$ In this partial proof it is convenient to use the form of the definition of the derivative given in (3) of Section 3.1. For $\Delta x \neq 0$,

$$\Delta u = g(x + \Delta x) - g(x) \tag{9}$$

or $g(x + \Delta x) = g(x) + \Delta u = u + \Delta u$. In addition,

$$\Delta y = f(u + \Delta u) - f(u) = f(g(x + \Delta x)) - f(g(x)).$$

When x and $x + \Delta x$ are in some open interval for which $\Delta u \neq 0$, we can write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}.$$

Since g is assumed to be differentiable, it is continuous. Consequently, as $\Delta x \to 0$, $g(x + \Delta x) \to g(x)$, and so from (9) we see that $\Delta u \to 0$. Thus,

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \left(\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u}\right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right)$$

$$= \left(\lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u}\right) \cdot \left(\lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x}\right). \quad \leftarrow \text{note that } \Delta u \to 0 \text{ in the first term}$$

From the definition of the derivative, (3) of Section 3.1, it follows that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$ ■

The assumption that $\Delta u \neq 0$ on some interval does not hold true for every differentiable function g. Although the result given in (7) remains valid when $\Delta u = 0$, the preceding proof does not.

It might help in the understanding of the derivative of a composition $y = f(g(x))$ to think of f as the *outside function* and $u = g(x)$ as the *inside function*. The derivative of $y = f(g(x)) = f(u)$ is then the *product of the derivative of the outside function* (evaluated at the inside function u) *and the derivative of the inside function* (evaluated at x):

derivative of outside function ↓

$$\frac{d}{dx} f(u) = f'(u) \cdot u'. \tag{10}$$

↑ derivative of inside function

The result in (10) is written in various ways. Since $y = f(u)$, we have $f'(u) = dy/du$, and of course $u' = du/dx$. The product of the derivatives in (10) is the same as (8). On the other hand, if we replace the symbols u and u' in (10) by $g(x)$ and $g'(x)$ we obtain (7).

Proof of the Power Rule for Functions As noted previously, a power of a function can be written as a composition of $(f \circ g)(x)$ where the outside function is $y = f(x) = x^n$ and the inside function is $u = g(x)$. The derivative of the inside function $y = f(u) = u^n$ is $\dfrac{dy}{dx} = nu^{n-1}$ and the derivative of the outside function is $\dfrac{du}{dx}$. The product of these derivatives is then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = nu^{n-1}\frac{du}{dx} = n[g(x)]^{n-1}g'(x).$$

This is the Power Rule for Functions given in (5) and (6).

Trigonometric Functions We obtain the derivatives of the trigonometric functions composed with a differentiable function g as another direct consequence of the Chain Rule. For example, if $y = \sin u$, where $u = g(x)$, then the derivative of y with respect to the variable u is

$$\frac{dy}{du} = \cos u.$$

Hence, (8) gives

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \cos u \frac{du}{dx}$$

or equivalently,

$$\frac{d}{dx}\sin[\ ] = \cos[\ ]\frac{d}{dx}[\ ].$$

Similarly, if $y = \tan u$ where $u = g(x)$, then $dy/du = \sec^2 u$ and so

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \sec^2 u \frac{du}{dx}.$$

We summarize the Chain Rule results for the six trigonometric functions.

Theorem 3.5.3 Derivatives of Trigonometric Functions

If $u = g(x)$ is a differentiable function, then

$$\frac{d}{dx}\sin u = \cos u\frac{du}{dx}, \qquad \frac{d}{dx}\cos u = -\sin u\frac{du}{dx}, \tag{11}$$

$$\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}, \qquad \frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}, \tag{12}$$

$$\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx}, \qquad \frac{d}{dx}\csc u = -\csc u\cot u\frac{du}{dx}. \tag{13}$$

EXAMPLE 7 Chain Rule

Differentiate $y = \cos 4x$.

Solution The function is $\cos u$ with $u = 4x$. From the second formula in (11) of Theorem 3.5.3 the derivative is

$$\frac{dy}{dx} = \overbrace{-\sin 4x}^{\frac{dy}{du}} \cdot \overbrace{\frac{d}{dx}4x}^{\frac{du}{dx}} = -4\sin 4x.$$

■

EXAMPLE 8 Chain Rule

Differentiate $y = \tan(6x^2 + 1)$.

Solution The function is $\tan u$ with $u = 6x^2 + 1$. From the first formula in (12) of Theorem 3.5.3 the derivative is

$$\frac{dy}{dx} = \overbrace{\sec^2(6x^2 + 1)}^{\sec^2 u} \cdot \overbrace{\frac{d}{dx}(6x^2 + 1)}^{\frac{du}{dx}} = 12x\sec^2(6x^2 + 1).$$

■

EXAMPLE 9 Product, Power, and Chain Rule

Differentiate $y = (9x^3 + 1)^2\sin 5x$.

Solution We first use the Product Rule:

$$\frac{dy}{dx} = (9x^3 + 1)^2 \cdot \frac{d}{dx}\sin 5x + \sin 5x \cdot \frac{d}{dx}(9x^3 + 1)^2$$

followed by the Power Rule (6) and the first formula in (11) of Theorem 3.5.3,

$$\begin{aligned}\frac{dy}{dx} &= (9x^3 + 1)^2 \cdot \underbrace{\cos 5x \cdot \frac{d}{dx}5x}_{\text{from (11)}} + \sin 5x \cdot \underbrace{2(9x^3 + 1) \cdot \frac{d}{dx}(9x^3 + 1)}_{\text{from (6)}} \\ &= (9x^3 + 1)^2 \cdot 5\cos 5x + \sin 5x \cdot 2(9x^3 + 1) \cdot 27x^2 \\ &= (9x^3 + 1)(45x^3\cos 5x + 5\cos 5x + 54x^2\sin 5x).\end{aligned}$$

■

In Sections 3.2 and 3.3 we saw that even though the Sum and Product Rules were stated in terms of two functions f and g, they were applicable to any finite number of differentiable functions. So too, the Chain Rule is stated for the composition of two functions f and g but we can apply it to the composition of three (or more) differentiable functions. In the case of three functions f, g, and h, (7) becomes

$$\begin{aligned}\frac{d}{dx}f(g(h(x))) &= f'(g(h(x))) \cdot \frac{d}{dx}g(h(x)) \\ &= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x).\end{aligned}$$

EXAMPLE 10 Repeated Use of the Chain Rule

Differentiate $y = \cos^4(7x^3 + 6x - 1)$.

Solution For emphasis we first rewrite the given function as $y = [\cos(7x^3 + 6x - 1)]^4$. Observe that this function is the composition $(f \circ g \circ h)(x) = f(g(h(x)))$ where $f(x) = x^4$, $g(x) = \cos x$, and $h(x) = 7x^3 + 6x - 1$. We first apply the Chain Rule in the form of the Power Rule (6) followed by the second formula in (11):

$$\frac{dy}{dx} = 4[\cos(7x^3 + 6x - 1)]^3 \cdot \frac{d}{dx}\cos(7x^3 + 6x - 1) \qquad \leftarrow \text{first Chain Rule: differentiate the power}$$

$$= 4\cos^3(7x^3 + 6x - 1) \cdot \left[-\sin(7x^3 + 6x - 1) \cdot \frac{d}{dx}(7x^3 + 6x - 1)\right] \qquad \leftarrow \text{second Chain Rule: differentiate the cosine}$$

$$= -4(21x^2 + 6)\cos^3(7x^3 + 6x - 1)\sin(7x^3 + 6x - 1).$$

In the final example, the given function is a composition of four functions.

EXAMPLE 11 Repeated Use of the Chain Rule

Differentiate $y = \sin\left(\tan\sqrt{3x^2 + 4}\right)$.

Solution The function is $f(g(h(k(x))))$, where $f(x) = \sin x$, $g(x) = \tan x$, $h(x) = \sqrt{x}$, and $k(x) = 3x^2 + 4$. In this case we apply the Chain Rule three times in succession

$$\frac{dy}{dx} = \cos\left(\tan\sqrt{3x^2 + 4}\right) \cdot \frac{d}{dx}\tan\sqrt{3x^2 + 4} \qquad \leftarrow \text{first Chain Rule: differentiate the sine}$$

$$= \cos\left(\tan\sqrt{3x^2 + 4}\right) \cdot \sec^2\sqrt{3x^2 + 4} \cdot \frac{d}{dx}\sqrt{3x^2 + 4} \qquad \leftarrow \text{second Chain Rule: differentiate the tangent}$$

$$= \cos\left(\tan\sqrt{3x^2 + 4}\right) \cdot \sec^2\sqrt{3x^2 + 4} \cdot \frac{d}{dx}(3x^2 + 4)^{1/2} \qquad \leftarrow \text{rewrite power}$$

$$= \cos\left(\tan\sqrt{3x^2 + 4}\right) \cdot \sec^2\sqrt{3x^2 + 4} \cdot \frac{1}{2}(3x^2 + 4)^{-1/2} \cdot \frac{d}{dx}(3x^2 + 4) \qquad \leftarrow \text{third Chain Rule: differentiate the power}$$

$$= \cos\left(\tan\sqrt{3x^2 + 4}\right) \cdot \sec^2\sqrt{3x^2 + 4} \cdot \frac{1}{2}(3x^2 + 4)^{-1/2} \cdot 6x \qquad \leftarrow \text{simplify}$$

$$= \frac{3x\cos\left(\tan\sqrt{3x^2 + 4}\right) \cdot \sec^2\sqrt{3x^2 + 4}}{\sqrt{3x^2 + 4}}.$$

You should, of course, become so adept at applying the Chain Rule that you will not have to give a moment's thought as to the number of functions involved in the actual composition.

$\frac{d}{dx}$ NOTES FROM THE CLASSROOM

(*i*) Probably the most common mistake is to forget to carry out the second half of the Chain Rule, namely the derivative of the inside function. This is the du/dx part in

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

For instance, the derivative of $y = (1 - x)^{57}$ is not $dy/dx = 57(1 - x)^{56}$ since $57(1 - x)^{56}$ is only the dy/du part. It might help to consistently use the operation symbol d/dx:

$$\frac{d}{dx}(1 - x)^{57} = 57(1 - x)^{56} \cdot \frac{d}{dx}(1 - x) = 57(1 - x)^{56} \cdot (-1).$$

(*ii*) A less common but probably a worse mistake than the first is to differentiate inside the given function. A student wrote on an examination paper that the derivative of $y = \cos(x^2 + 1)$ was $dy/dx = -\sin(2x)$; that is, the derivative of the cosine is the negative of the sine and the derivative of $x^2 + 1$ is $2x$. Both observations are correct, but how they are put together is incorrect. Bear in mind that the derivative of the inside function is a multiple of the derivative of the outside function. Again, it might help to use the operation symbol d/dx. The correct derivative of $y = \cos(x^2 + 1)$ is the product of two derivatives.

$$\frac{dy}{dx} = -\sin(x^2 + 1) \cdot \frac{d}{dx}(x^2 + 1) = -2x\sin(x^2 + 1).$$

Exercises 3.5

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–20, find dy/dx.

1. $y = (-5x)^{30}$
2. $y = (3/x)^{14}$
3. $y = (2x^2 + x)^{200}$
4. $y = \left(x - \dfrac{1}{x^2}\right)^5$
5. $y = \dfrac{1}{(x^3 - 2x^2 + 7)^4}$
6. $y = \dfrac{10}{\sqrt{x^2 - 4x + 1}}$
7. $y = (3x - 1)^4(-2x + 9)^5$
8. $y = x^4(x^2 + 1)^6$
9. $y = \sin\sqrt{2x}$
10. $y = \sec x^2$
11. $y = \sqrt{\dfrac{x^2 - 1}{x^2 + 1}}$
12. $y = \dfrac{3x - 4}{(5x + 2)^3}$
13. $y = [x + (x^2 - 4)^3]^{10}$
14. $y = \left[\dfrac{1}{(x^3 - x + 1)^2}\right]^4$
15. $y = x(x^{-1} + x^{-2} + x^{-3})^{-4}$
16. $y = (2x + 1)^3\sqrt{3x^2 - 2x}$
17. $y = \sin(\pi x + 1)$
18. $y = -2\cos(-3x + 7)$
19. $y = \sin^3 5x$
20. $y = 4\cos^2\sqrt{x}$

In Problems 21–38, find $f'(x)$.

21. $f(x) = x^3\cos x^3$
22. $f(x) = \dfrac{\sin 5x}{\cos 6x}$
23. $f(x) = (2 + x\sin 3x)^{10}$
24. $f(x) = \dfrac{(1 - \cos 4x)^2}{(1 + \sin 5x)^3}$
25. $f(x) = \tan(1/x)$
26. $f(x) = x\cot(5/x^2)$
27. $f(x) = \sin 2x \cos 3x$
28. $f(x) = \sin^2 2x \cos^3 3x$
29. $f(x) = (\sec 4x + \tan 2x)^5$
30. $f(x) = \csc^2 2x - \csc 2x^2$
31. $f(x) = \sin(\sin 2x)$
32. $f(x) = \tan\left(\cos\dfrac{x}{2}\right)$
33. $f(x) = \cos\left(\sin\sqrt{2x + 5}\right)$
34. $f(x) = \tan(\tan x)$
35. $f(x) = \sin^3(4x^2 - 1)$
36. $f(x) = \sec(\tan^2 x^4)$
37. $f(x) = (1 + (1 + (1 + x^3)^4)^5)^6$
38. $f(x) = \left[x^2 - \left(1 + \dfrac{1}{x}\right)^{-4}\right]^2$

In Problems 39–42, find the slope of the tangent line to the graph of the given function at the indicated value of x.

39. $y = (x^2 + 2)^3$; $x = -1$
40. $y = \dfrac{1}{(3x + 1)^2}$; $x = 0$
41. $y = \sin 3x + 4x\cos 5x$; $x = \pi$
42. $y = 50x - \tan^3 2x$; $x = \pi/6$

In Problems 43–46, find an equation of the tangent line to the graph of the given function at the indicated value of x.

43. $y = \left(\dfrac{x}{x + 1}\right)^2$; $x = -\dfrac{1}{2}$
44. $y = x^2(x - 1)^3$; $x = 2$
45. $y = \tan 3x$; $x = \pi/4$
46. $y = (-1 + \cos 4x)^3$; $x = \pi/8$

In Problems 47 and 48, find an equation of the normal line to the graph of the given function at the indicated value of x.

47. $y = \sin\left(\dfrac{\pi}{6x}\right)\cos(\pi x^2)$; $x = \dfrac{1}{2}$
48. $y = \sin^3\dfrac{x}{3}$; $x = \pi$

In Problems 49–52, find the indicated derivative.

49. $f(x) = \sin\pi x$; $f'''(x)$
50. $y = \cos(2x + 1)$; d^5y/dx^5
51. $y = x\sin 5x$; d^3y/dx^3
52. $f(x) = \cos x^2$; $f''(x)$
53. Find the point(s) on the graph of $f(x) = x/(x^2 + 1)^2$ where the tangent line is horizontal. Does the graph of f have any vertical tangents?
54. Determine the values of t at which the instantaneous rate of change of $g(t) = \sin t + \frac{1}{2}\cos 2t$ is zero.
55. If $f(x) = \cos(x/3)$, what is the slope of the tangent line to the graph of f' at $x = 2\pi$?
56. If $f(x) = (1 - x)^4$, what is the slope of the tangent line to the graph of f'' at $x = 2$?

Applications

57. The function $R = (v_0^2/g)\sin 2\theta$ gives the range of a projectile fired at an angle θ from the horizontal with an initial velocity v_0. If v_0 and g are constants, find those values of θ at which $dR/d\theta = 0$.

58. The volume of a spherical balloon of radius r is $V = \frac{4}{3}\pi r^3$. The radius is a function of time t and increases at a constant rate of 5 in/min. What is the instantaneous rate of change of V with respect to t?

59. Suppose a spherical balloon is being filled at a constant rate $dV/dt = 10\ \text{in}^3/\text{min}$. At what rate is its radius increasing when $r = 2$ in?

60. Consider a mass on a spring shown in FIGURE 3.5.1. In the absence of damping forces, the displacement (or directed distance) of the mass measured from a position called the **equilibrium position** is given by the function

$$x(t) = x_0\cos\omega t + \frac{v_0}{\omega}\sin\omega t,$$

where $\omega = \sqrt{k/m}$, k is the spring constant (an indicator of the stiffness of the spring), m is the mass (measured in slugs or kilograms), y_0 is the initial displacement of the mass (measured above or below the equilibrium position), v_0 is the initial velocity of the mass, and t is time measured in seconds.

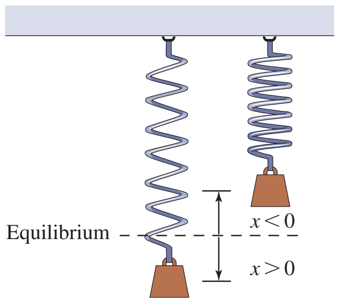


FIGURE 3.5.1 Mass on a spring in Problem 60

(a) Verify that $x(t)$ satisfies the differential equation

$$\frac{d^2x}{dt^2} + \omega^2 x = 0.$$

(b) Verify that $x(t)$ satisfies the initial conditions $x(0) = x_0$ and $x'(0) = v_0$.

Think About It

61. Let F be a differentiable function. What is $\frac{d}{dx}F(3x)$?

62. Let G be a differentiable function. What is $\frac{d}{dx}[G(-x^2)]^2$?

63. Suppose $\frac{d}{du}f(u) = \frac{1}{u}$. What is $\frac{d}{dx}f(-10x + 7)$?

64. Suppose $\frac{d}{dx}f(x) = \frac{1}{1 + x^2}$. What is $\frac{d}{dx}f(x^3)$?

In Problems 65 and 66, the symbol n represents a positive integer. Find a formula for the given derivative.

65. $\frac{d^n}{dx^n}(1 + 2x)^{-1}$

66. $\frac{d^n}{dx^n}\sqrt{1 + 2x}$

67. Suppose $g(t) = h(f(t))$, where $f(1) = 3$, $f'(1) = 6$, and $h'(3) = -2$. What is $g'(1)$?

68. Suppose $g(1) = 2, g'(1) = 3, g''(1) - 1, f'(2) = 4$, and $f''(2) = 3$. What is $\left.\frac{d^2}{dx^2}f(g(x))\right|_{x=1}$?

69. Given that f is an odd differentiable function, use the Chain Rule to show that f' is an even function.

70. Given that f is an even differentiable function, use the Chain Rule to show that f' is an odd function.

3.6 Implicit Differentiation

Introduction The graphs of many equations that we study in mathematics are not the graphs of functions. For example, the equation

$$x^2 + y^2 = 4 \tag{1}$$

describes a circle of radius 2 centered at the origin. Equation (1) is not a function, since for any choice of x satisfying $-2 < x < 2$ there corresponds two values of y. See FIGURE 3.6.1(a). Nevertheless, graphs of equations such as (1) can possess tangent lines at various points (x, y). Equation (1) defines *at least* two functions f and g on the interval $[-2, 2]$. Graphically, the obvious functions are the top half and the bottom half of the circle. To obtain formulas for these functions we solve $x^2 + y^2 = 4$ for y in terms of x:

$$y = f(x) = \sqrt{4 - x^2}, \quad \leftarrow \text{upper semicircle} \tag{2}$$

and

$$y = g(x) = -\sqrt{4 - x^2}. \quad \leftarrow \text{lower semicircle} \tag{3}$$

See Figures 3.6.1(b) and (c). We can now find slopes of tangent lines for $-2 < x < 2$ by differentiating (2) and (3) by the Power Rule for Functions.

In this section we will see how the derivative dy/dx can be obtained for (1), as well as for more complicated equations $F(x, y) = 0$, without the necessity of solving the equation for the variable y.

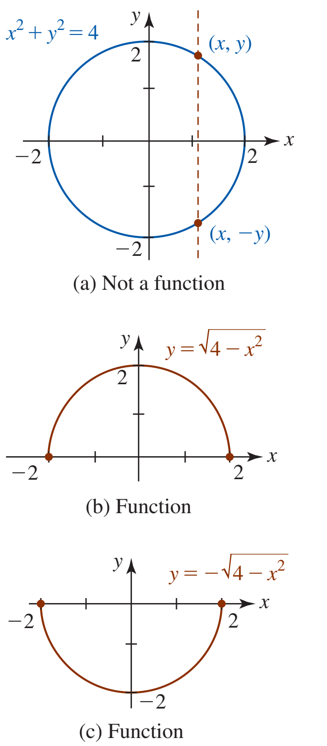


FIGURE 3.6.1 Equation $x^2 + y^2 = 4$ determines at least two functions

Explicit and Implicit Functions A function in which the dependent variable is expressed solely in terms of the independent variable x, namely, $y = f(x)$, is said to be an **explicit function**. For example, $y = \frac{1}{2}x^3 - 1$ is an explicit function. On the other hand, an equivalent equation $2y - x^3 + 2 = 0$ is said to define the function **implicitly**, or y is an **implicit function** of x. We have just seen that the equation $x^2 + y^2 = 4$ defines the two functions $f(x) = \sqrt{4 - x^2}$ and $g(x) = -\sqrt{4 - x^2}$ implicitly.

In general, if an equation $F(x, y) = 0$ defines a function f implicitly on some interval, then $F(x, f(x)) = 0$ is an identity on the interval. The graph of f is a portion or an arc (or all) of the graph of the equation $F(x, y) = 0$. In the case of the functions in (2) and (3), note that both equations

$$x^2 + [f(x)]^2 = 4 \quad \text{and} \quad x^2 + [g(x)]^2 = 4$$

are identities on the interval $[-2, 2]$.

The graph of the equation $x^3 + y^3 = 3xy$ shown in FIGURE 3.6.2(a) is a famous curve called the **Folium of Descartes**. With the aid of a CAS such as *Mathematica* or *Maple*, one of the implicit functions defined by $x^3 + y^3 = 3xy$ is found to be

$$y = \frac{2x}{\sqrt[3]{-4x^3 + 4\sqrt{x^6 - 4x^3}}} + \frac{1}{2}\sqrt[3]{-4x^3 + 4\sqrt{x^6 - 4x^3}}. \tag{4}$$

The graph of this function is the red arc shown in Figure 3.6.2(b). The graph of another implicit function defined by $x^3 + y^3 = 3xy$ is given in Figure 3.6.2(c).

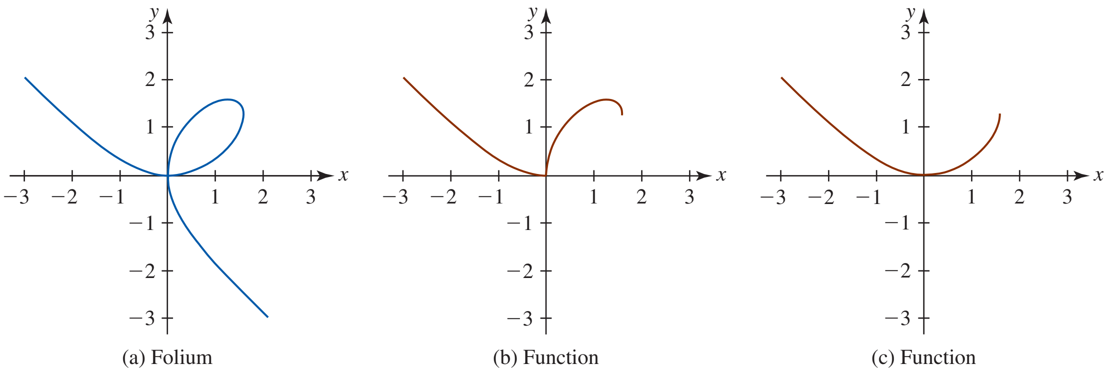


FIGURE 3.6.2 The portions of the graph in (a) that are shown in red in (b) and (c) are graphs of two implicit functions of x

Implicit Differentiation Do not jump to the conclusion from the preceding discussion that we can always solve an equation $F(x, y) = 0$ for an implicit function of x as we did in (2), (3), and (4). For example, solving an equation such as

$$x^4 + x^2y^3 - y^5 = 2x + y \tag{5}$$

Although we cannot solve certain equations for an explicit function, it still may be possible to graph the equation with the aid of a CAS. We can then *see* the functions as we did in Figure 3.6.2.

for y in terms of x is more than an exercise in challenging algebra or a lesson in the use of the correct syntax of a CAS. It is *impossible*! Yet (5) may determine several implicit functions on a suitably restricted interval of the x-axis. Nevertheless, we *can* determine the derivative dy/dx by a process known as **implicit differentiation**. This process consists of differentiating both sides of an equation with respect to x, using the rules of differentiation, and then solving for dy/dx. Since we think of y as being determined by the given equation as a differentiable function of x, the Chain Rule, in the form of the Power Rule for Functions, gives the useful result

$$\frac{d}{dx}y^n = ny^{n-1}\frac{dy}{dx}, \tag{6}$$

where n is any real number. For example,

$$\frac{d}{dx}x^2 = 2x \qquad \text{whereas} \qquad \frac{d}{dx}y^2 = 2y\frac{dy}{dx}.$$

Similarly, if y is a function of x, then by the Product Rule,

$$\frac{d}{dx}xy = x\frac{d}{dx}y + y\frac{d}{dx}x = x\frac{dy}{dx} + y,$$

and by the Chain Rule,

$$\frac{d}{dx}\sin 5y = \cos 5y \cdot \frac{d}{dx}5y = 5\cos 5y\frac{dy}{dx}.$$

Guidelines for Implicit Differentiation

(*i*) Differentiate both sides of the equation with respect to x. Use the rules of differentiation and treat y as a differentiable function of x. For powers of the symbol y use (6).
(*ii*) Collect all terms involving dy/dx on the left-hand side of the differentiated equation. Move all other terms to the right-hand side of the equation.
(*iii*) Factor dy/dx from all terms containing this term. Then solve for dy/dx.

In the following examples we shall assume that the given equation determines at least one differentiable implicit function.

EXAMPLE 1 Using Implicit Differentiation

Find dy/dx if $x^2 + y^2 = 4$.

Solution We differentiate both sides of the equation and then utilize (6):

use Power Rule (6) here ↓

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = \frac{d}{dx}4$$

$$2x + 2y\frac{dy}{dx} = 0.$$

Solving for the derivative yields

$$\frac{dy}{dx} = -\frac{x}{y}. \qquad (7)$$ ■

As illustrated in (7) of Example 1, implicit differentiation usually yields a derivative that depends on both variables x and y. In our introductory discussion we saw that the equation $x^2 + y^2 = 4$ defines two differentiable implicit functions on the open interval $-2 < x < 2$. The symbolism $dy/dx = -x/y$ represents the derivative of either function on the interval. Note that this derivative clearly indicates that functions (2) and (3) are not differentiable at $x = -2$ and $x = 2$ since $y = 0$ for these values of x. In general, implicit differentiation yields the derivative of any differentiable implicit function defined by an equation $F(x, y) = 0$.

EXAMPLE 2 Slope of a Tangent Line

Find the slopes of the tangent lines to the graph of $x^2 + y^2 = 4$ at the points corresponding to $x = 1$.

Solution Substituting $x = 1$ into the given equation gives $y^2 = 3$ or $y = \pm\sqrt{3}$. Hence, there are tangent lines at $(1, \sqrt{3})$ and $(1, -\sqrt{3})$. Although $(1, \sqrt{3})$ and $(1, -\sqrt{3})$ are points on the graphs of two different implicit functions, indicated by the different colors in FIGURE 3.6.3, (7) of Example 1 gives the correct slope at each point for $(-2, 2)$. We have

$$\left.\frac{dy}{dx}\right|_{(1,\sqrt{3})} = -\frac{1}{\sqrt{3}} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{(1,-\sqrt{3})} = -\frac{1}{-\sqrt{3}} = \frac{1}{\sqrt{3}}.$$ ■

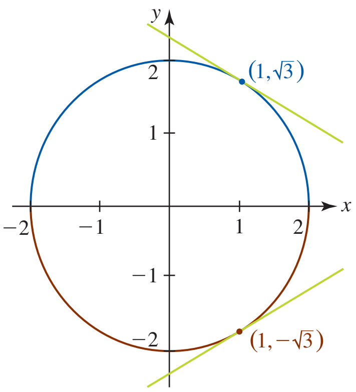


FIGURE 3.6.3 Tangent lines in Example 2 are shown in green

EXAMPLE 3 Using Implicit Differentiation

Find dy/dx if $x^4 + x^2y^3 - y^5 = 2x + 1$.

Solution In this case, we use (6) and the Product Rule:

Product Rule here ↓ Power Rule (6) here ↓

$$\frac{d}{dx}x^4 + \frac{d}{dx}x^2y^3 - \frac{d}{dx}y^5 = \frac{d}{dx}2x + \frac{d}{dx}1$$

$$4x^3 + x^2 \cdot 3y^2\frac{dy}{dx} + 2xy^3 - 5y^4\frac{dy}{dx} = 2 \leftarrow \text{factor } dy/dx \text{ from second and fourth terms}$$

$$(3x^2y^2 - 5y^4)\frac{dy}{dx} = 2 - 4x^3 - 2xy^3$$

$$\frac{dy}{dx} = \frac{2 - 4x^3 - 2xy^3}{3x^2y^2 - 5y^4}.$$ ■

Higher Derivatives

Through implicit differentiation we determine dy/dx. By differentiating dy/dx with respect to x we obtain the second derivative d^2y/dx^2. If the first derivative contains y, then d^2y/dx^2 will again contain the symbol dy/dx; we can eliminate that quantity by substituting its known value. The next example illustrates the method.

EXAMPLE 4 Second Derivative

Find d^2y/dx^2 if $x^2 + y^2 = 4$.

Solution From Example 1, we already know that the first derivative is $dy/dx = -x/y$. The second derivative is the derivative of dy/dx, and so by the Quotient Rule:

substituting for dy/dx

$$\frac{d^2y}{dx^2} = -\frac{d}{dx}\left(\frac{x}{y}\right) = -\frac{y \cdot 1 - x \cdot \frac{dy}{dx}}{y^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2} = -\frac{y^2 + x^2}{y^3}.$$

Noting that $x^2 + y^2 = 4$ permits us to rewrite the second derivative as

$$\frac{d^2y}{dx^2} = -\frac{4}{y^3}.$$ ■

EXAMPLE 5 Chain and Product Rules

Find dy/dx if $\sin y = y\cos 2x$.

Solution From the Chain Rule and Product Rule we obtain

$$\frac{d}{dx}\sin y = \frac{d}{dx}y\cos 2x$$

$$\cos y \cdot \frac{dy}{dx} = y(-\sin 2x \cdot 2) + \cos 2x \cdot \frac{dy}{dx}$$

$$(\cos y - \cos 2x)\frac{dy}{dx} = -2y\sin 2x$$

$$\frac{dy}{dx} = -\frac{2y\sin 2x}{\cos y - \cos 2x}.$$ ■

Postscript—Power Rule Revisited So far we have proved the Power Rule $(d/dx)x^n = nx^{n-1}$ for all integer exponents n. Implicit differentiation provides a way of proving this rule when the exponent is a rational number p/q, where p and q are integers and $q \neq 0$. In the case $n = p/q$, the function

$$y = x^{p/q} \qquad \text{gives} \qquad y^q = x^p.$$

Now for $y \neq 0$, implicit differentiation

$$\frac{d}{dx}y^q = \frac{d}{dx}x^p \qquad \text{yields} \qquad qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

Solving the last equation for dy/dx and simplifying by the laws of exponents gives

$$\frac{dy}{dx} = \frac{p}{q}\frac{x^{p-1}}{y^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{(x^{p/q})^{q-1}} = \frac{p}{q}\frac{x^{p-1}}{x^{p-p/q}} = \frac{p}{q}x^{p/q-1}.$$

Examination of the last result shows that it is (3) of Section 3.2 with $n = p/q$.

Exercises 3.6

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–4, assume that y is a differentiable function of x. Find the indicated derivative.

1. $\frac{d}{dx}x^2y^4$

2. $\frac{d}{dx}\frac{x^2}{y^2}$

3. $\frac{d}{dx}\cos y^2$

4. $\frac{d}{dx}y\sin 3y$

In Problems 5–24, assume that the given equation defines at least one differentiable implicit function. Use implicit differentiation to find dy/dx.

5. $y^2 - 2y = x$

6. $4x^2 + y^2 = 8$

7. $xy^2 - x^2 + 4 = 0$

8. $(y - 1)^2 = 4(x + 2)$

9. $3y + \cos y = x^2$

10. $y^3 - 2y + 3x^3 = 4x + 1$

11. $x^3y^2 = 2x^2 + y^2$

12. $x^5 - 6xy^3 + y^4 = 1$

13. $(x^2 + y^2)^6 = x^3 - y^3$

14. $y = (x - y)^2$

15. $y^{-3}x^6 + y^6x^{-3} = 2x + 1$

16. $y^4 - y^2 = 10x - 3$

17. $(x - 1)^2 + (y + 4)^2 = 25$

18. $\frac{x + y}{x - y} = x$

19. $y^2 = \frac{x - 1}{x + 2}$

20. $\frac{x}{y^2} + \frac{y^2}{x} = 5$

21. $xy = \sin(x + y)$

22. $x + y = \cos(xy)$

23. $x = \sec y$

24. $x\sin y - y\cos x = 1$

In Problems 25 and 26, use implicit differentiation to find the indicated derivative.

25. $r^2 = \sin 2\theta$; $dr/d\theta$

26. $\pi r^2 h = 100$; dh/dr

In Problems 27 and 28, find dy/dx at the indicated point.

27. $xy^2 + 4y^3 + 3x = 0$; $(1, -1)$

28. $y = \sin xy$; $(\pi/2, 1)$

In Problems 29 and 30, find dy/dx at the points that correspond to the indicated number.

29. $2y^2 + 2xy - 1 = 0$; $x = \frac{1}{2}$

30. $y^3 + 2x^2 = 11y$; $y = 1$

In Problems 31–34, find an equation of the tangent line at the indicated point or number.

31. $x^4 + y^3 = 24$; $(-2, 2)$

32. $\frac{1}{x} + \frac{1}{y} = 1$; $x = 3$

33. $\tan y = x$; $y = \pi/4$

34. $3y + \cos y = x^2$; $(1, 0)$

In Problems 35 and 36, find the point(s) on the graph of the given equation where the tangent line is horizontal.

35. $x^2 - xy + y^2 = 3$

36. $y^2 = x^2 - 4x + 7$

37. Find the point(s) on the graph of $x^2 + y^2 = 25$ at which the slope of the tangent is $\frac{1}{2}$.

38. Find the point where the tangent lines to the graph of $x^2 + y^2 = 25$ at $(-3, 4)$ and $(-3, -4)$ intersect.

39. Find the point(s) on the graph of $y^3 = x^2$ at which the tangent line is perpendicular to the line $y + 3x - 5 = 0$.

40. Find the point(s) on the graph of $x^2 - xy + y^2 = 27$ at which the tangent line is parallel to the line $y = 5$.

In Problems 41–48, find d^2y/dx^2.

41. $4y^3 = 6x^2 + 1$

42. $xy^4 = 5$

43. $x^2 - y^2 = 25$

44. $x^2 + 4y^2 = 16$

45. $x + y = \sin y$

46. $y^2 - x^2 = \tan 2x$

47. $x^2 + 2xy - y^2 = 1$

48. $x^3 + y^3 = 27$

In Problems 49–52, first use implicit differentiation to find dy/dx. Then solve for y explicitly in terms of x and differentiate. Show that the two answers are equivalent.

49. $x^2 - y^2 = x$

50. $4x^2 + y^2 = 1$

51. $x^3y = x + 1$

52. $y\sin x = x - 2y$

In Problems 53–56, determine an implicit function from the given equation such that its graph is the blue curve in the figure.

53. $(y - 1)^2 = x - 2$

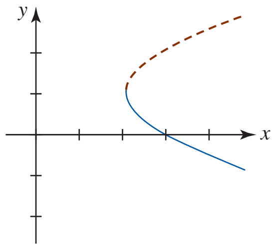


FIGURE 3.6.4 Graph for Problem 53

54. $x^2 + xy + y^2 = 4$

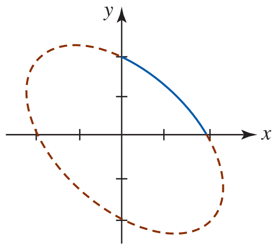


FIGURE 3.6.5 Graph for Problem 54

55. $x^2 + y^2 = 4$

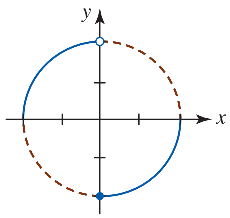


FIGURE 3.6.6 Graph for Problem 55

56. $y^2 = x^2(2 - x)$

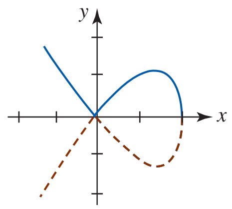


FIGURE 3.6.7 Graph for Problem 56

In Problems 57 and 58, assume that both x and y are differentiable functions of a variable t. Find dy/dt in terms of x, y, and dx/dt.

57. $x^2 + y^2 = 25$

58. $x^2 + xy + y^2 - y = 9$

59. The graph of the equation $x^3 + y^3 = 3xy$ is the Folium of Descartes given in Figure 3.6.2(a).

(a) Find an equation of the tangent line at the point in the first quadrant where the Folium intersects the graph of $y = x$.

(b) Find the point in the first quadrant at which the tangent line is horizontal.

60. The graph of $(x^2 + y^2)^2 = 4(x^2 - y^2)$ shown in FIGURE 3.6.8 is called a **lemniscate**.

(a) Find the points on the graph that correspond to $x = 1$.

(b) Find an equation of the tangent line to the graph at each point found in part (a).

(c) Find the points on the graph at which the tangent is horizontal.

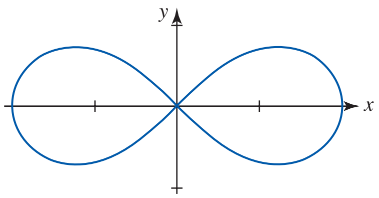


FIGURE 3.6.8 Lemniscate in Problem 60

In Problems 61 and 62, show that the graphs of the given equations are orthogonal at the indicated point of intersection. See Problem 64 in Exercises 3.2.

61. $y^2 = x^3, \quad 2x^2 + 3y^2 = 5; \quad (1, 1)$

62. $y^3 + 3x^2y = 13, \quad 2x^2 - 2y^2 = 3x; \quad (2, 1)$

If all the curves of one family of curves $G(x, y) = c_1$, c_1 a constant, intersect orthogonally all the curves of another family $H(x, y) = c_2$, c_2 a constant, then the families are said to be **orthogonal trajectories** of each other. In Problems 63 and 64, show that the families of curves are orthogonal trajectories of each other. Sketch the two families of curves.

63. $x^2 - y^2 = c_1, \quad xy = c_2$

64. $x^2 + y^2 = c_1, \quad y = c_2x$

☰ Applications

65. A woman drives toward a freeway sign as shown in FIGURE 3.6.9. Let θ be her viewing angle of the sign and let x be her distance (measured in feet) to that sign.

(a) If her eye level is 4 ft from the surface of the road, show that

$$\tan\theta = \frac{4x}{x^2 + 252}.$$

(b) Find the rate at which θ changes with respect to x.

(c) At what distance is the rate in part (b) equal to zero?

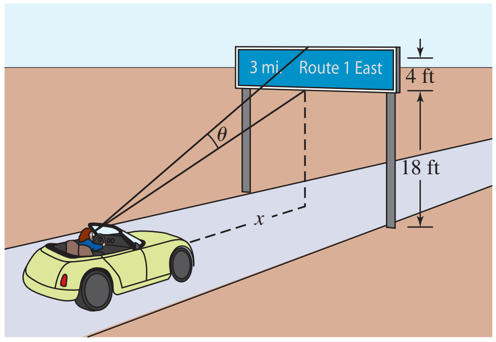


FIGURE 3.6.9 Car in Problem 65

66. A jet fighter "loops the loop" in a circle of radius 1 km as shown in FIGURE 3.6.10. Suppose a rectangular coordinate system is chosen so that the origin is at the center of the circular loop. The aircraft releases a missile that flies on a straight-line path that is tangent to the circle and hits a target on the ground whose coordinates are $(2, -2)$.

(a) Determine the point on the circle where the missile was released.

(b) If a missile is released at the point $\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ on the circle, at what point does it hit the ground?

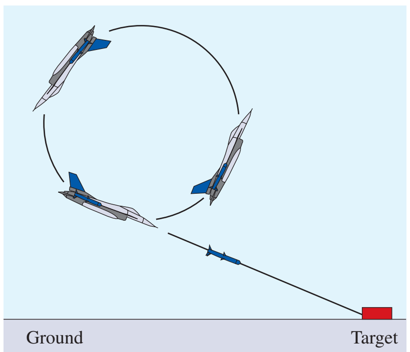


FIGURE 3.6.10 Jet fighter in Problem 66

Think About It

67. The angle θ $(0 < \theta < \pi)$ between two curves is defined to be the angle between their tangent lines at the point P of intersection. If m_1 and m_2 are the slopes of the tangent lines at P, it can be shown that $\tan\theta = (m_1 - m_2)/(1 + m_1 m_2)$. Determine the angle between the graphs of $x^2 + y^2 + 4y = 6$ and $x^2 + 2x + y^2 = 4$ at (1, 1).

68. Show that an equation of the tangent line to the ellipse $x^2/a^2 + y^2/b^2 = 1$ at the point (x_0, y_0) is given by

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} = 1.$$

69. Consider the equation $x^2 + y^2 = 4$. Make up another implicit function $h(x)$ defined by this equation for $-2 \le x \le 2$ different from the ones given in (2), (3), and Problem 55.

70. For $-1 < x < 1$ and $-\pi/2 < y < \pi/2$, the equation $x = \sin y$ defines a differentiable implicit function.

(a) Find dy/dx in terms of y.
(b) Find dy/dx in terms of x.

3.7 Derivatives of Inverse Functions

Introduction In Section 1.5 we saw that the graphs of a one-to-one function f and its inverse f^{-1} are **reflections** of each other in the line $y = x$. As a consequence, if (a, b) is a point on the graph of f, then (b, a) is a point on the graph of f^{-1}. In this section we will also see that the slopes of tangent lines to the graph of a differentiable function f are related to the slopes of tangents to the graph of f^{-1}.

We begin with two theorems about the continuity of f and f^{-1}.

Continuity of f^{-1} Although we state the next two theorems without proof, their plausibility follows from the fact that the graph of f^{-1} is a reflection of the graph of f in the line $y = x$.

Theorem 3.7.1 Continuity of an Inverse Function

Let f be a continuous one-to-one function on its domain X. Then f^{-1} is continuous on its domain.

Increasing–Decreasing Functions Suppose $y = f(x)$ is a function defined on an interval I, and that x_1 and x_2 are any two numbers in the interval such that $x_1 < x_2$. Then from Section 1.3 and Figure 1.3.4 recall that f is said to be

- **increasing** on the interval if $f(x_1) < f(x_2)$, and (1)
- **decreasing** on the interval if $f(x_1) > f(x_2)$. (2)

The next two theorems establish a link between the notions of increasing/decreasing and the existence of an inverse function.

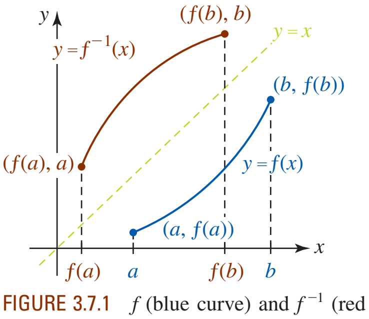


FIGURE 3.7.1 f (blue curve) and f^{-1} (red curve) are continuous and increasing

Theorem 3.7.2 Existence of an Inverse Function

Let f be a continuous function and increasing on an interval $[a, b]$. Then f^{-1} exists and is continuous and increasing on $[f(a), f(b)]$.

Theorem 3.7.2 also holds when the word *increasing* is replaced with the word *decreasing* and the interval in the conclusion is replaced by $[f(b), f(a)]$. See FIGURE 3.7.1. In addition, we can conclude from Theorem 3.7.2 that if f is continuous and increasing on an interval $(-\infty, \infty)$, then f^{-1} exists and is continuous and increasing on its domain. Inspection of Figures 1.3.4 and 3.7.1 also shows that if f in Theorem 3.7.2 is a differentiable function on (a, b), then:

f increasing and differentiable means the tangent lines have positive slope. ▶

- f is increasing on the interval $[a, b]$ if $f'(x) > 0$ on (a, b), and
- f is decreasing on the interval $[a, b]$ if $f'(x) < 0$ on (a, b).

We will prove these statements in the next chapter.

Theorem 3.7.3 Existence of an Inverse Function

Suppose f is a differentiable function on an open interval (a, b). If either $f'(x) > 0$ on the interval or $f'(x) < 0$ on the interval, then f is one-to-one. Moreover, f^{-1} is differentiable for all x in the range of f.

EXAMPLE 1 Existence of an Inverse

Prove that $f(x) = 5x^3 + 8x - 9$ has an inverse.

Solution Since f is a polynomial function it is differentiable everywhere, that is, f is differentiable on the interval $(-\infty, \infty)$. Also, $f'(x) = 15x^2 + 8 > 0$ for all x implies that f is increasing on $(-\infty, \infty)$. It follows from Theorem 3.7.3 that f is one-to-one and hence f^{-1} exists. ■

Derivative of f^{-1} If f is differentiable on an interval I and is one-to-one on that interval, then for a in I the point (a, b) on the graph of f and the point (b, a) on the graph of f^{-1} are mirror images of each other in the line $y = x$. As we see next, the slopes of the tangent lines at (a, b) and (b, a) are also related.

EXAMPLE 2 Derivative of an Inverse

In Example 5 of Section 1.5 we showed that the inverse of the one-to-one function $f(x) = x^2 + 1, x \geq 0$ is $f^{-1}(x) = \sqrt{x-1}$. At $x = 2$,

$$f(2) = 5 \qquad \text{and} \qquad f^{-1}(5) = 2.$$

Now from

$$f'(x) = 2x \qquad \text{and} \qquad (f^{-1})'(x) = \frac{1}{2\sqrt{x-1}}$$

we see $f'(2) = 4$ and $(f^{-1})'(5) = \frac{1}{4}$. This shows that the slope of the tangent to the graph of f at $(2, 5)$ and the slope of the tangent to the graph of f^{-1} at $(5, 2)$ are reciprocals:

$$(f^{-1})'(5) = \frac{1}{f'(2)} \qquad \text{or} \qquad (f^{-1})'(5) = \frac{1}{f'(f^{-1}(5))}.$$

See FIGURE 3.7.2. ■

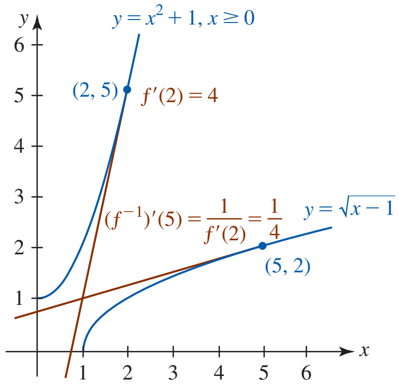


FIGURE 3.7.2 Tangent lines in Example 2

The next theorem shows that the result in Example 2 is no coincidence.

Theorem 3.7.4 Derivative of an Inverse Function

Suppose that f is differentiable on an interval I and $f'(x)$ is never zero on I. If f has an inverse f^{-1} on I, then f^{-1} is differentiable at a number x and

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}. \tag{3}$$

PROOF As we have seen in (5) of Section 1.5, $f(f^{-1}(x)) = x$ for every x in the domain of f^{-1}. By implicit differentiation and the Chain Rule,

$$\frac{d}{dx}f(f^{-1}(x)) = \frac{d}{dx}x \qquad \text{or} \qquad f'(f^{-1}(x)) \cdot \frac{d}{dx}f^{-1}(x) = 1.$$

Solving the last equation for $\frac{d}{dx}f^{-1}(x)$ gives (3). ■

Equation (3) clearly shows that to find the derivative function for f^{-1} we must know $f^{-1}(x)$ explicitly. For a one-to-one function $y = f(x)$ solving the equation $x = f(y)$ for y is sometimes difficult and often impossible. In this case it is convenient to rewrite (3) using different notation. Again by implicit differentiation

$$\frac{d}{dx}x = \frac{d}{dx}f(y) \qquad \text{gives} \qquad 1 = f'(y)\cdot\frac{dy}{dx}.$$

Solving the last equation for dy/dx and writing $dx/dy = f'(y)$ yields

$$\frac{dy}{dx} = \frac{1}{dx/dy}. \tag{4}$$

If (a, b) is a known point on the graph of f, the result in (4) enables us to evaluate the derivative of f^{-1} at (b, a) without an equation that defines $f^{-1}(x)$.

EXAMPLE 3 Derivative of an Inverse

It was pointed out in Example 1 that the polynomial function $f(x) = 5x^3 + 8x - 9$ is differentiable on $(-\infty, \infty)$ and hence continuous on the interval. Since the end behavior of f is that of the single-term polynomial function $y = 5x^3$ we can conclude that the range of f is also $(-\infty, \infty)$. Moreover, since $f'(x) = 15x^2 + 8 > 0$ for all x, f is increasing on its domain $(-\infty, \infty)$. Hence by Theorem 3.7.3, f has a differentiable inverse f^{-1} with domain $(-\infty, \infty)$. By interchanging x and y, the inverse is defined by the equation $x = 5y^3 + 8y - 9$, but solving this equation for y in terms of x is difficult (it requires the cubic formula). Nevertheless, using $dx/dy = 15y^2 + 8$, the derivative of the inverse function is given by (4):

$$\frac{dy}{dx} = \frac{1}{15y^2 + 8}. \tag{5}$$

For example, since $f(1) = 4$ we know that $f^{-1}(4) = 1$. Thus, the slope of the tangent line to the graph of f^{-1} at (4, 1) is given by (5):

$$\left.\frac{dy}{dx}\right|_{x=4} = \left.\frac{1}{15y^2 + 8}\right|_{y=1} = \frac{1}{23}.$$ ■

Read this paragraph a second time.

In Example 3, the derivative of the inverse function can also be obtained directly from $x = 5y^3 + 8y - 9$ using implicit differentiation:

$$\frac{d}{dx}x = \frac{d}{dx}(5y^3 + 8y - 9) \qquad \text{gives} \qquad 1 = 15y^2\frac{dy}{dx} + 8\frac{dy}{dx}.$$

Solving the last equation for dy/dx gives (5). As a consequence of this observation implicit differentiation can be used to find the derivative of an inverse function with minimum effort. In the discussion that follows we will find the derivatives of the inverse trigonometric functions.

Derivatives of Inverse Trigonometric Functions A review of Figures 1.5.15 and 1.5.17(a) reveals that the inverse tangent and inverse cotangent are differentiable for all x. However, the remaining four inverse trigonometric functions are not differentiable at either $x = -1$ or $x = 1$. We shall confine our attention to the derivations of the derivative formulas for the inverse sine, inverse tangent, and inverse secant and leave the others as exercises.

Inverse Sine: $y = \sin^{-1}x$ if and only if $x = \sin y$, where $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$. Therefore, implicit differentiation

$$\frac{d}{dx}x = \frac{d}{dx}\sin y \qquad \text{gives} \qquad 1 = \cos y\cdot\frac{dy}{dx}$$

and so

$$\frac{dy}{dx} = \frac{1}{\cos y}. \tag{6}$$

For the given restriction on the variable y, $\cos y \ge 0$ and so $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$. By substituting this quantity in (6), we have shown that

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1 - x^2}}. \tag{7}$$

As predicted, note that (7) is not defined at $x = -1$ and $x = 1$. The inverse sine or arcsine function is differentiable on the open interval $(-1, 1)$.

Inverse Tangent: $y = \tan^{-1}x$ if and only if $x = \tan y$, where $-\infty < x < \infty$ and $-\pi/2 < y < \pi/2$. Thus,

$$\frac{d}{dx}x = \frac{d}{dx}\tan y \qquad \text{gives} \qquad 1 = \sec^2 y \cdot \frac{dy}{dx}$$

or

$$\frac{dy}{dx} = \frac{1}{\sec^2 y}. \tag{8}$$

In view of the identity $\sec^2 y = 1 + \tan^2 y = 1 + x^2$, (8) becomes

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1 + x^2}. \tag{9}$$

Inverse Secant: For $|x| > 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$,

$$y = \sec^{-1}x \qquad \text{if and only if} \qquad x = \sec y.$$

Differentiating the last equation implicitly gives

$$\frac{dy}{dx} = \frac{1}{\sec y \tan y}. \tag{10}$$

In view of the restrictions on y, we have $\tan y = \pm\sqrt{\sec^2 y - 1} = \pm\sqrt{x^2 - 1}$, $|x| > 1$. Hence, (10) becomes

$$\frac{d}{dx}\sec^{-1}x = \pm\frac{1}{x\sqrt{x^2 - 1}}. \tag{11}$$

We can get rid of the $\pm$ sign in (11) by observing in Figure 1.5.17(b) that the slope of the tangent line to the graph of $y = \sec^{-1}x$ is positive for $x < -1$ and positive for $x > 1$. Thus, (11) is equivalent to

$$\frac{d}{dx}\sec^{-1}x = \begin{cases} -\dfrac{1}{x\sqrt{x^2 - 1}}, & x < -1 \\ \dfrac{1}{x\sqrt{x^2 - 1}}, & x > 1. \end{cases} \tag{12}$$

The result in (12) can be rewritten in a compact form using the absolute value symbol:

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2 - 1}}. \tag{13}$$

The derivative of the composition of an inverse trigonometric function with a differentiable function $u = g(x)$ is obtained from the Chain Rule.

Theorem 3.7.5 Inverse Trigonometric Functions

If $u = g(x)$ is a differentiable function, then

$$\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1 - u^2}}\frac{du}{dx}, \qquad \frac{d}{dx}\cos^{-1}u = \frac{-1}{\sqrt{1 - u^2}}\frac{du}{dx}, \tag{14}$$

$$\frac{d}{dx}\tan^{-1}u = \frac{1}{1 + u^2}\frac{du}{dx}, \qquad \frac{d}{dx}\cot^{-1}u = \frac{-1}{1 + u^2}\frac{du}{dx}, \tag{15}$$

$$\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}, \qquad \frac{d}{dx}\csc^{-1}u = \frac{-1}{|u|\sqrt{u^2 - 1}}\frac{du}{dx}. \tag{16}$$

In the formulas in (14) we must have $|u| < 1$, whereas in the formulas in (16) we must have $|u| > 1$.

EXAMPLE 4 Derivative of Inverse Sine

Differentiate $y = \sin^{-1} 5x$.

Solution With $u = 5x$, we have from the first formula in (14),

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-(5x)^2}} \cdot \frac{d}{dx} 5x = \frac{5}{\sqrt{1-25x^2}}.$$ ■

EXAMPLE 5 Derivative of Inverse Tangent

Differentiate $y = \tan^{-1}\sqrt{2x+1}$.

Solution With $u = \sqrt{2x+1}$, we have from the first formula in (15),

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{1+\left(\sqrt{2x+1}\right)^2} \cdot \frac{d}{dx}(2x+1)^{1/2} \\ &= \frac{1}{1+(2x+1)} \cdot \frac{1}{2}(2x+1)^{-1/2} \cdot 2 \\ &= \frac{1}{(2x+2)\sqrt{2x+1}}.\end{aligned}$$ ■

EXAMPLE 6 Derivative of Inverse Secant

Differentiate $y = \sec^{-1} x^2$.

Solution For $x^2 > 1 > 0$, we have from the first formula in (16),

$$\begin{aligned}\frac{dy}{dx} &= \frac{1}{|x^2|\sqrt{(x^2)^2-1}} \cdot \frac{d}{dx}x^2 \\ &= \frac{2x}{x^2\sqrt{x^4-1}} = \frac{2}{x\sqrt{x^4-1}}.\end{aligned} \tag{17}$$

With the aid of a graphing utility we obtain the graph of $y = \sec^{-1}x^2$ given in FIGURE 3.7.3. Notice that (17) gives positive slope for $x > 1$ and negative slope for $x < -1$. ■

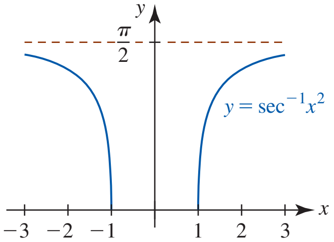


FIGURE 3.7.3 Graph of function in Example 6

EXAMPLE 7 Tangent Line

Find an equation of the tangent line to the graph of $f(x) = x^2\cos^{-1}x$ at $x = -\frac{1}{2}$.

Solution By the Product Rule and the second formula in (14),

$$f'(x) = x^2\left(\frac{-1}{\sqrt{1-x^2}}\right) + 2x\cos^{-1}x.$$

Since $\cos^{-1}\left(-\frac{1}{2}\right) = 2\pi/3$, the two functions f and f' evaluated at $x = -\frac{1}{2}$ give:

$$f\left(-\frac{1}{2}\right) = \frac{\pi}{6} \qquad \leftarrow \text{point of tangency is } \left(-\tfrac{1}{2}, \tfrac{\pi}{6}\right)$$

$$f'\left(-\frac{1}{2}\right) = -\frac{1}{2\sqrt{3}} - \frac{2\pi}{3}. \leftarrow \text{slope of tangent at } \left(-\tfrac{1}{2}, \tfrac{\pi}{6}\right) \text{ is } -\tfrac{1}{2\sqrt{3}} - \tfrac{2\pi}{3}$$

By the point–slope form of a line, the unsimplified equation of the tangent line is

$$y - \frac{\pi}{6} = \left(-\frac{1}{2\sqrt{3}} - \frac{2\pi}{3}\right)\left(x + \frac{1}{2}\right).$$

Since the domain of $\cos^{-1}x$ is the interval $[-1, 1]$ the domain of f is $[-1, 1]$. The corresponding range is $[0, \pi]$. FIGURE 3.7.4 was obtained with the aid of a graphing utility. ■

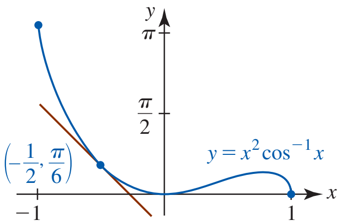


FIGURE 3.7.4 Tangent line in Example 7.

Exercises 3.7 Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–4, without graphing determine whether the given function f has an inverse.

1. $f(x) = 10x^3 + 8x + 12$
2. $f(x) = -7x^5 - 6x^3 - 2x + 17$
3. $f(x) = x^3 + x^2 - 2x$
4. $f(x) = x^4 - 2x^2$

In Problems 5 and 6, use (3) to find the derivative of f^{-1} at the indicated point.

5. $f(x) = 2x^3 + 8;\quad \left(f\left(\tfrac{1}{2}, \tfrac{1}{2}\right)\right)$
6. $f(x) = -x^3 - 3x + 7;\quad (f(-1), -1)$

In Problems 7 and 8, find f^{-1}. Use (3) to find $(f^{-1})'$ and then verify this result by direct differentiation of f^{-1}.

7. $f(x) = \dfrac{2x+1}{x}$
8. $f(x) = (5x + 7)^3$

In Problems 9–12, without finding the inverse, find, at the indicated value of x, the corresponding point on the graph of f^{-1}. Then use (4) to find an equation of the tangent line at this point.

9. $y = \dfrac{1}{3}x^3 + x - 7;\quad x = 3$
10. $y = \dfrac{2x+1}{4x-1};\quad x = 0$
11. $y = (x^5 + 1)^3;\quad x = 1$
12. $y = 8 - 6\sqrt[3]{x+2};\quad x = -3$

In Problems 13–32, find the derivative of the given function.

13. $y = \sin^{-1}(5x - 1)$
14. $y = \cos^{-1}\left(\dfrac{x+1}{3}\right)$
15. $y = 4\cot^{-1}\dfrac{x}{2}$
16. $y = 2x - 10\sec^{-1}5x$
17. $y = 2\sqrt{x}\tan^{-1}\sqrt{x}$
18. $y = (\tan^{-1}x)(\cot^{-1}x)$
19. $y = \dfrac{\sin^{-1}2x}{\cos^{-1}2x}$
20. $y = \dfrac{\sin^{-1}x}{\sin x}$
21. $y = \dfrac{1}{\tan^{-1}x^2}$
22. $y = \dfrac{\sec^{-1}x}{x}$
23. $y = 2\sin^{-1}x + x\cos^{-1}x$
24. $y = \cot^{-1}x - \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$
25. $y = \left(x^2 - 9\tan^{-1}\dfrac{x}{3}\right)^3$
26. $y = \sqrt{x - \cos^{-1}(x+1)}$
27. $F(t) = \arctan\left(\dfrac{t-1}{t+1}\right)$
28. $g(t) = \arccos\sqrt{3t+1}$
29. $f(x) = \arcsin(\cos 4x)$
30. $f(x) = \arctan\left(\dfrac{\sin x}{2}\right)$
31. $f(x) = \tan(\sin^{-1}x^2)$
32. $f(x) = \cos(x\sin^{-1}x)$

In Problems 33 and 34, use implicit differentiation to find dy/dx.

33. $\tan^{-1}y = x^2 + y^2$
34. $\sin^{-1}y - \cos^{-1}x = 1$

In Problems 35 and 36, show that $f'(x) = 0$. Interpret the result.

35. $f(x) = \sin^{-1}x + \cos^{-1}x$
36. $f(x) = \tan^{-1}x + \tan^{-1}(1/x)$.

In Problems 37 and 38, find the slope of the tangent line to the graph of the given function at the indicated value of x.

37. $y = \sin^{-1}\dfrac{x}{2};\quad x = 1$
38. $y = (\cos^{-1}x)^2;\quad x = 1/\sqrt{2}$

In Problems 39 and 40, find an equation of the tangent line to the graph of the given function at the indicated value of x.

39. $f(x) = x\tan^{-1}x;\quad x = 1$
40. $f(x) = \sin^{-1}(x - 1);\quad x = \dfrac{1}{2}$
41. Find the points on the graph of $f(x) = 5 - 2\sin x$, $0 \le x \le 2\pi$, at which the tangent line is parallel to the line $y = \sqrt{3}x + 1$.
42. Find all tangent lines to the graph of $f(x) = \arctan x$ that have slope $\frac{1}{4}$.

Think About It

43. If f and $(f^{-1})'$ are differentiable, use (3) to find a formula for $(f^{-1})''(x)$.

3.8 Exponential Functions

Introduction In Section 1.6 we saw that the exponential function $f(x) = b^x$, $b > 0$, $b \neq 1$, is defined for all real numbers, that is, the domain of f is $(-\infty, \infty)$. Inspection of Figure 1.6.2 shows that f is everywhere continuous. It turns out that an exponential function is also differentiable everywhere. In this section we develop the derivative of $f(x) = b^x$.

Derivative of an Exponential Function To find the derivative of an exponential function $f(x) = b^x$ we will use the definition of the derivative given in (2) of Definition 3.1.1. We first compute the difference quotient

$$\frac{f(x+h) - f(x)}{h} \tag{1}$$

in three steps. For the exponential function $f(x) = b^x$, we have

(i) $f(x+h) = b^{x+h} = b^x b^h$ ← laws of exponents

(ii) $f(x+h) - f(x) = b^{x+h} - b^x = b^x b^h - b^x = b^x(b^h - 1)$ ← laws of exponents and factoring

(iii) $\dfrac{f(x+h) - f(x)}{h} = \dfrac{b^x(b^h - 1)}{h} = b^x \cdot \dfrac{b^h - 1}{h}.$

In the fourth step, the calculus step, we let $h \to 0$ but analogous to the derivatives of $\sin x$ and $\cos x$ in Section 3.4, there is no apparent way of canceling the h in the difference quotient (iii). Nonetheless, the derivative of $f(x) = b^x$ is

$$f'(x) = \lim_{h\to 0} b^x \cdot \frac{b^h - 1}{h}. \tag{2}$$

Because b^x does not depend on the variable h, we can rewrite (2) as

$$f'(x) = b^x \cdot \lim_{h\to 0} \frac{b^h - 1}{h}. \tag{3}$$

Now here are the amazing results. The limit in (3),

$$\lim_{h\to 0} \frac{b^h - 1}{h}, \tag{4}$$

can be shown to exist for every positive base b. However, as one might expect, we will get a different answer for each base b. So for convenience let us denote the expression in (4) by the symbol $m(b)$. The derivative of $f(x) = b^x$ is then

$$f'(x) = b^x m(b). \tag{5}$$

You are asked to approximate the value of $m(b)$ in the four cases $b = 1.5, 2, 3$, and 5 in Problems 57–60 of Exercises 3.8. For example, it can be shown that $m(10) \approx 2.302585\ldots$ and as a consequence if $f(x) = 10^x$, then

$$f'(x) = (2.302585\ldots)10^x. \tag{6}$$

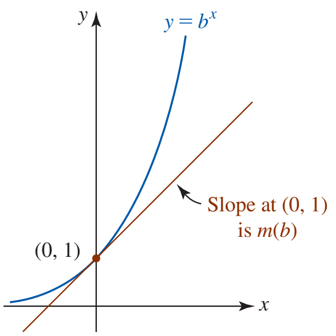


FIGURE 3.8.1 Find a base b so that the slope $m(b)$ of the tangent line at (0, 1) is 1

We can get a better understanding of what $m(b)$ is by evaluating (5) at $x = 0$. Since $b^0 = 1$, we have $f'(0) = m(b)$. In other words, $m(b)$ is the slope of the tangent line to the graph of $f(x) = b^x$ at $x = 0$, that is, at the y-intercept (0, 1). See FIGURE 3.8.1. Given that we have to calculate a different $m(b)$ for each base b, and that $m(b)$ is likely to be an "ugly" number as in (6), over time the following question arose *naturally*:

- *Is there a base b for which* $m(b) = 1$? (7)

Derivative of the Natural Exponential Function To answer the question posed in (7), we must return to the definitions of e given in Section 1.6. Specifically, (4) of Section 1.6,

$$e = \lim_{h\to 0} (1 + h)^{1/h} \tag{8}$$

provides the means for answering the question posed in (7). We know that on an intuitive level, the equality in (8) means that as h gets closer and closer to 0 then $(1 + h)^{1/h}$ can be made arbitrarily close to the number e. Thus for values of h near 0, we have the approximation $(1 + h)^{1/h} \approx e$ and so it follows that $1 + h \approx e^h$. The last expression written in the form

$$\frac{e^h - 1}{h} \approx 1 \tag{9}$$

suggests that

$$\lim_{h\to 0}\frac{e^h - 1}{h} = 1. \tag{10}$$

Since the left-hand side of (10) is $m(e)$ we have the answer to the question posed in (7):

- *The base b for which $m(b) = 1$ is $b = e$.* (11)

In addition, from (3) we have discovered a wonderfully simple result. The derivative of $f(x) = e^x$ is e^x. In summary,

$$\frac{d}{dx}e^x = e^x. \tag{12}$$

The result in (12) is the same as $f'(x) = f(x)$. Moreover, if $c \neq 0$ is a constant, then the only other nonzero function f in calculus whose derivative is equal to itself is $y = ce^x$ since by the Constant Multiple Rule of Section 3.2

$$\frac{dy}{dx} = \frac{d}{dx}ce^x = c\frac{d}{dx}e^x = ce^x = y.$$

Derivative of $f(x) = b^x$—Revisited In the preceding discussion we saw that $m(e) = 1$, but left unanswered the question of whether $m(b)$ has an exact value for each $b > 0$. It has. From the identity $e^{\ln b} = b$, $b > 0$, we can write any exponential function $f(x) = b^x$ in terms of the e base:

$$f(x) = b^x = (e^{\ln b})^x = e^{x(\ln b)}.$$

From the Chain Rule the derivative of b^x is

$$f'(x) = \frac{d}{dx}e^{x(\ln b)} = e^{x(\ln b)}\cdot\frac{d}{dx}x(\ln b) = e^{x(\ln b)}(\ln b).$$

Returning to $b^x = e^{x(\ln b)}$, the preceding line shows that

$$\frac{d}{dx}b^x = b^x(\ln b). \tag{13}$$

Matching the result in (5) with that in (13) we conclude that $m(b) = \ln b$. For example, the derivative of $f(x) = 10^x$ is $f'(x) = 10^x(\ln 10)$. Because $\ln 10 \approx 2.302585$ we see $f'(x) = 10^x(\ln 10)$ is the same as the result in (6).

The Chain Rule forms of the results in (12) and (13) are given next.

Theorem 3.8.1 Derivatives of Exponential Functions

If $u = g(x)$ is a differentiable function, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}, \tag{14}$$

and

$$\frac{d}{dx}b^u = b^u(\ln b)\frac{du}{dx}. \tag{15}$$

EXAMPLE 1 Chain Rule

Differentiate

(a) $y = e^{-x}$ **(b)** $y = e^{1/x^3}$ **(c)** $y = 8^{5x}$.

Solution

(a) With $u = -x$ we have from (14),

$$\frac{dy}{dx} = e^{-x}\cdot\frac{d}{dx}(-x) = e^{-x}(-1) = -e^{-x}.$$

(b) By rewriting $u = 1/x^3$ as $u = x^{-3}$ we have from (14),

$$\frac{dy}{dx} = e^{1/x^3} \cdot \frac{d}{dx}x^{-3} = e^{1/x^3}(-3x^{-4}) = -3\,\frac{e^{1/x^3}}{x^4}.$$

(c) With $u = 5x$ we have from (15),

$$\frac{dy}{dx} = 8^{5x} \cdot (\ln 8) \cdot \frac{d}{dx}5x = 5 \cdot 8^{5x}(\ln 8). \qquad ■$$

EXAMPLE 2 Product and Chain Rule

Find the points on the graph of $y = 3x^2e^{-x^2}$ where the tangent line is horizontal.

Solution We use the Product Rule along with (14):

$$\begin{aligned}\frac{dy}{dx} &= 3x^2 \cdot \frac{d}{dx}e^{-x^2} + e^{-x^2} \cdot \frac{d}{dx}3x^2\\ &= 3x^2(-2xe^{-x^2}) + 6xe^{-x^2}\\ &= e^{-x^2}(-6x^3 + 6x).\end{aligned}$$

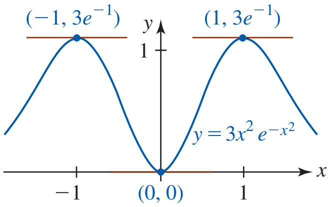


FIGURE 3.8.2 Graph of function in Example 2

Since $e^{-x^2} \neq 0$ for all real numbers x, $\frac{dy}{dx} = 0$ when $-6x^3 + 6x = 0$. Factoring the last equation gives $x(x + 1)(x - 1) = 0$ and so $x = 0$, $x = -1$, and $x = 1$. The corresponding points on the graph of the given function are then $(0, 0)$, $(-1, 3e^{-1})$, and $(1, 3e^{-1})$. The graph of $y = 3x^2e^{-x^2}$ along with the three tangent lines (in red) are shown in FIGURE 3.8.2. ■

In the next example we recall the fact that an exponential statement can be written in an equivalent logarithmic form. In particular, we use (9) of Section 1.6 in the form

$$y = e^x \qquad \text{if and only if} \qquad x = \ln y. \tag{16}$$

EXAMPLE 3 Tangent Line Parallel to a Line

Find the point on the graph of $f(x) = 2e^{-x}$ at which the tangent line is parallel to $y = -4x - 2$.

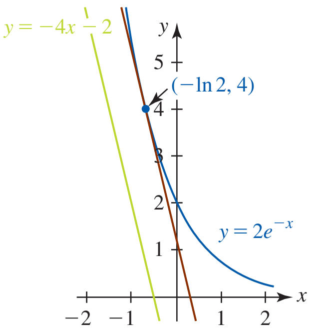


FIGURE 3.8.3 Graph of function and lines in Example 3

Solution Let $(x_0, f(x_0)) = (x_0, 2e^{-x_0})$ be the unknown point on the graph of $f(x) = 2e^{-x}$ where the tangent line is parallel to $y = -4x - 2$. From the derivative $f'(x) = -2e^{-x}$ the slope of the tangent line at this point is then $f'(x_0) = -2e^{-x_0}$. Since $y = -4x - 2$ and the tangent line are parallel at that point, the slopes are equal:

$$f'(x_0) = -4 \qquad \text{or} \qquad -2e^{-x_0} = -4 \qquad \text{or} \qquad e^{-x_0} = 2.$$

From (16) the last equation gives $-x_0 = \ln 2$ or $x_0 = -\ln 2$. Hence, the point is $(-\ln 2, 2e^{\ln 2})$. Since $e^{\ln 2} = 2$, the point is $(-\ln 2, 4)$. In FIGURE 3.8.3 the given line is shown in green and the tangent line in red. ■

$\frac{d}{dx}$ NOTES FROM THE CLASSROOM

The numbers e and π are **transcendental** as well as irrational numbers. A transcendental number is one that is *not* a root of a polynomial equation with integer coefficients. For example, $\sqrt{2}$ is irrational but is not transcendental, since it is a root of the polynomial equation $x^2 - 2 = 0$. The number e was proved to be transcendental by the French mathematician Charles Hermite (1822–1901) in 1873, whereas π was proved to be transcendental nine years later by the German mathematician Ferdinand Lindemann (1852–1939). The latter proof showed conclusively that "squaring a circle" with a rule and a compass was impossible.

Exercises 3.8

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–26, find the derivative of the given function.

1. $y = e^{-x}$
2. $y = e^{2x+3}$
3. $y = e^{\sqrt{x}}$
4. $y = e^{\sin 10x}$
5. $y = 5^{2x}$
6. $y = 10^{-3x^2}$
7. $y = x^3e^{4x}$
8. $y = e^{-x}\sin \pi x$
9. $f(x) = \dfrac{e^{-2x}}{x}$
10. $f(x) = \dfrac{xe^x}{x + e^x}$
11. $y = \sqrt{1 + e^{-5x}}$
12. $y = (e^{2x} - e^{-2x})^{10}$
13. $y = \dfrac{2}{e^{x/2} + e^{-x/2}}$
14. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}}$
15. $y = \dfrac{e^{7x}}{e^{-x}}$
16. $y = e^{2x}e^{3x}e^{4x}$
17. $y = (e^3)^{x-1}$
18. $y = \left(\dfrac{1}{e^x}\right)^{100}$
19. $f(x) = e^{x^{1/3}} + (e^x)^{1/3}$
20. $f(x) = (2x + 1)^3e^{-(1-x)^4}$
21. $f(x) = e^{-x}\tan e^x$
22. $f(x) = \sec e^{2x}$
23. $f(x) = e^{x\sqrt{x^2+1}}$
24. $y = e^{\frac{x+2}{x-2}}$
25. $y = e^{e^{x^2}}$
26. $y = e^x + e^{x+e^{-x}}$

27. Find an equation of the tangent line to the graph of $y = (e^x + 1)^2$ at $x = 0$.

28. Find the slope of the normal line to the graph of $y = (x - 1)e^{-x}$ at $x = 0$.

29. Find the point on the graph of $y = e^x$ at which the tangent line is parallel to $3x - y = 7$.

30. Find the point on the graph of $y = 5x + e^{2x}$ at which the tangent line is parallel to $y = 6x$.

In Problems 31 and 32, find the point(s) on the graph of the given function at which the tangent line is horizontal. Use a graphing utility to obtain the graph of each function.

31. $f(x) = e^{-x}\sin x$
32. $f(x) = (3 - x^2)e^{-x}$

In Problems 33–36, find the indicated higher derivative.

33. $y = e^{x^2}; \quad \dfrac{d^3y}{dx^3}$
34. $y = \dfrac{1}{1 + e^{-x}}; \quad \dfrac{d^2y}{dx^2}$
35. $y = \sin e^{2x}; \quad \dfrac{d^2y}{dx^2}$
36. $y = x^2e^x; \quad \dfrac{d^4y}{dx^4}$

In Problems 37 and 38, C_1 and C_2 are arbitrary real constants. Show that the function satisfies the given differential equation.

37. $y = C_1e^{-3x} + C_2e^{2x}; \quad y'' + y' - 6y = 0$

38. $y = C_1e^{-x}\cos 2x + C_2e^{-x}\sin 2x; \quad y'' + 2y' + 5y = 0$

39. If C and k are real constants, show that the function $y = Ce^{kx}$ satisfies the differential equation $y' = ky$.

40. Use Problem 39 to find a function that satisfies the given conditions.
    (a) $y' = -0.01y$ and $y(0) = 100$
    (b) $\dfrac{dP}{dt} - 0.15P = 0$ and $P(0) = P_0$

In Problems 41–46, use implicit differentiation to find dy/dx.

41. $y = e^{x+y}$
42. $xy = e^y$
43. $y = \cos e^{xy}$
44. $y = e^{(x+y)^2}$
45. $x + y^2 = e^{x/y}$
46. $e^x + e^y = y$

47. **(a)** Sketch the graph of $f(x) = e^{-|x|}$.
    (b) Find $f'(x)$.
    (c) Sketch the graph of f'.
    (d) Is the function differentiable at $x = 0$?

48. **(a)** Show that the function $f(x) = e^{\cos x}$ is periodic with period 2π.
    (b) Find all points on the graph of f where the tangent is horizontal.
    (c) Sketch the graph of f.

Applications

49. The logistic function

$$P(t) = \frac{aP_0}{bP_0 + (a - bP_0)e^{-at}},$$

where a and b are positive constants, often serves as a mathematical model for an expanding but limited population.
    (a) Show that $P(t)$ satisfies the differential equation

$$\frac{dP}{dt} = P(a - bP).$$

    (b) The graph of $P(t)$ is called a **logistic curve** where $P(0) = P_0$ is the initial population. Consider the case when $a = 2, b = 1$, and $P_0 = 1$. Find horizontal asymptotes for the graph of $P(t)$ by determining the limits $\lim_{t\to-\infty} P(t)$ and $\lim_{t\to\infty} P(t)$.
    (c) Graph $P(t)$.
    (d) Find the value(s) of t for which $P''(t) = 0$.

50. The **Jenss mathematical model** (1937) represents one of the most accurate empirically devised formulas for predicting the height h (in centimeters) in terms of age t (in years) for preschool-age children (3 months to 6 years):

$$h(t) = 79.04 + 6.39t - e^{3.26-0.99t}.$$

    (a) What height does this model predict for a 2-year-old?
    (b) How fast is a 2-year-old increasing in height?
    (c) Use a calculator or CAS to obtain the graph of h on the interval $[\frac{1}{4}, 6]$.
    (d) Use the graph in part (c) to estimate the age of a preschool-age child who is 100 cm tall.

Think About It

51. Show that the x-intercept of the tangent line to the graph of $y = e^{-x}$ at $x = x_0$ is one unit to the right of x_0.

52. How is the tangent line to the graph of $y = e^x$ at $x = 0$ related to the tangent line to the graph of $y = e^{-x}$ at $x = 0$?

53. Explain why there is no point on the graph of $y = e^x$ at which the tangent line is parallel to $2x + y = 1$.

54. Find all tangent lines to the graph of $f(x) = e^x$ that pass through the origin.

In Problems 55 and 56, the symbol n represents a positive integer. Find a formula for the given derivative.

55. $\dfrac{d^n}{dx^n}\sqrt{e^x}$ **56.** $\dfrac{d^n}{dx^n}xe^{-x}$

Calculator/CAS Problems

In Problems 57–60, use a calculator to estimate the value $m(b) = \lim_{h\to 0}\dfrac{b^h - 1}{h}$ for $b = 1.5, b = 2, b = 3,$ and $b = 5$ by filling out the given table.

57.

$h \to 0$	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\dfrac{(1.5)^h - 1}{h}$						

58.

$h \to 0$	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\dfrac{2^h - 1}{h}$						

59.

$h \to 0$	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\dfrac{3^h - 1}{h}$						

60.

$h \to 0$	0.1	0.01	0.001	0.0001	0.00001	0.000001
$\dfrac{5^h - 1}{h}$						

61. Use a calculator or CAS to obtain the graph of

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

Show that f is differentiable for all x. Compute $f'(0)$ using the definition of the derivative.

3.9 Logarithmic Functions

Introduction Because the inverse of the exponential function $y = b^x$ is the logarithmic function $y = \log_b x$ we can find the derivative of the latter function in three different ways: (3) of Section 3.7, implicit differentiation, or from the fundamental definition (2) of Section 3.1. We will demonstrate the last two methods.

Derivative of the Natural Logarithm We know from (9) of Section 1.6 that $y = \ln x$ is the same as $x = e^y$. By implicit differentiation, the Chain Rule, and (14) of Section 3.8,

$$\frac{d}{dx}x = \frac{d}{dx}e^y \quad \text{gives} \quad 1 = e^y\frac{dy}{dx}.$$

Therefore
$$\frac{dy}{dx} = \frac{1}{e^y}.$$

Replacing e^y by x, we get the following result:

Like the inverse trigonometric functions, the derivative of the inverse of the natural exponential function is an algebraic function.

$$\frac{d}{dx}\ln x = \frac{1}{x}. \tag{1}$$

Derivative of $f(x) = \log_b x$ In precisely the same manner used to obtain (1), the derivative of $y = \log_b x$ can be gotten by differentiating $x = b^y$ implicitly:

$$\frac{d}{dx}x = \frac{d}{dx}b^y \quad \text{gives} \quad 1 = b^y(\ln b)\frac{dy}{dx}.$$

Therefore
$$\frac{dy}{dx} = \frac{1}{b^y(\ln b)}.$$

Replacing b^y by x gives

$$\frac{d}{dx}\log_b x = \frac{1}{x(\ln b)}. \tag{2}$$

Because $\ln e = 1$, (2) becomes (1) when $b = e$.

EXAMPLE 1 Product Rule

Differentiate $f(x) = x^2 \ln x$.

Solution By the Product Rule and (1) we have

$$f'(x) = x^2 \cdot \frac{d}{dx}\ln x + (\ln x)\cdot\frac{d}{dx}x^2 = x^2\cdot\frac{1}{x} + (\ln x)\cdot 2x$$

or

$$f'(x) = x + 2x\ln x.$$

■

EXAMPLE 2 Slope of a Tangent Line

Find the slope of the tangent to the graph of $y = \log_{10} x$ at $x = 2$.

Solution By (2) the derivative of $y = \log_{10} x$ is

$$\frac{dy}{dx} = \frac{1}{x(\ln 10)}.$$

With the aid of a calculator, the slope of the tangent line at $(2, \log_{10} 2)$ is

$$\left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2\ln 10} \approx 0.2171.$$

■

We summarize the results in (1) and (2) in their Chain Rule forms.

Theorem 3.9.1 Derivatives of Logarithmic Functions

If $u = g(x)$ is a differentiable function, then

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \tag{3}$$

and

$$\frac{d}{dx}\log_b u = \frac{1}{u(\ln b)}\frac{du}{dx}. \tag{4}$$

EXAMPLE 3 Chain Rule

Differentiate

(a) $f(x) = \ln(\cos x)$ and **(b)** $y = \ln(\ln x)$.

Solution

(a) By (3), with $u = \cos x$ we have

$$f'(x) = \frac{1}{\cos x}\cdot\frac{d}{dx}\cos x = \frac{1}{\cos x}\cdot(-\sin x)$$

or

$$f'(x) = -\tan x.$$

(b) Using (3) again, this time with $u = \ln x$, we get

$$\frac{dy}{dx} = \frac{1}{\ln x}\cdot\frac{d}{dx}\ln x = \frac{1}{\ln x}\cdot\frac{1}{x} = \frac{1}{x\ln x}.$$

■

EXAMPLE 4 Chain Rule

Differentiate $f(x) = \ln x^3$.

Solution Because x^3 must be positive it is understood that $x > 0$. Hence by (3), with $u = x^3$ we have

$$f'(x) = \frac{1}{x^3} \cdot \frac{d}{dx} x^3 = \frac{1}{x^3} \cdot (3x^2) = \frac{3}{x}.$$

Alternative Solution: From (*iii*) of the laws of logarithms (Theorem 1.6.1), $\ln N^c = c \ln N$ and so we can rewrite $y = \ln x^3$ as $y = 3 \ln x$ and then differentiate:

$$f(x) = 3 \frac{d}{dx} \ln x = 3 \cdot \frac{1}{x} = \frac{3}{x}.$$ ■

Although the domain of the natural logarithm $y = \ln x$ is the set $(0, \infty)$, the domain of $y = \ln|x|$ extends to the set $(-\infty, 0) \cup (0, \infty)$. For the numbers in this last domain,

$$|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0. \end{cases}$$

Therefore

$$\begin{aligned} &\text{for } x > 0, \quad \frac{d}{dx} \ln x = \frac{1}{x} \\ &\text{for } x < 0, \quad \frac{d}{dx} \ln(-x) = \frac{1}{-x} \cdot (-1) = \frac{1}{x}. \end{aligned} \tag{5}$$

The derivatives in (5) prove that for $x \neq 0$,

$$\frac{d}{dx} \ln|x| = \frac{1}{x}. \tag{6}$$

The result in (6) then generalizes by the Chain Rule. For a differentiable function $u = g(x)$, $u \neq 0$,

$$\frac{d}{dx} \ln|u| = \frac{1}{u} \frac{du}{dx}. \tag{7}$$

EXAMPLE 5 Using (6)

Find the slope of the tangent line to the graph of $y = \ln|x|$ at $x = -2$ and at $x = 2$.

Solution Since (6) gives $dy/dx = 1/x$, we have

$$\left.\frac{dy}{dx}\right|_{x=-2} = -\frac{1}{2} \quad \text{and} \quad \left.\frac{dy}{dx}\right|_{x=2} = \frac{1}{2}. \tag{8}$$

Because $\ln|-2| = \ln 2$, (8) gives, respectively, the slopes of the tangent lines at the points $(-2, \ln 2)$ and $(2, \ln 2)$. Observe in FIGURE 3.9.1 that the graph of $y = \ln|x|$ is symmetric with respect to the y-axis; the tangent lines are shown in red. ■

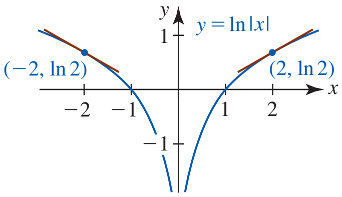


FIGURE 3.9.1 Graphs of tangent lines and function in Example 5

EXAMPLE 6 Using (7)

Differentiate

(a) $y = \ln(2x - 3)$ and **(b)** $y = \ln|2x - 3|$.

Solution

(a) For $2x - 3 > 0$, or $x > \frac{3}{2}$, we have from (3),

$$\frac{dy}{dx} = \frac{1}{2x - 3} \cdot \frac{d}{dx}(2x - 3) = \frac{2}{2x - 3}. \tag{9}$$

(b) For $2x - 3 \neq 0$, or $x \neq \frac{3}{2}$, we have from (7),

$$\frac{dy}{dx} = \frac{1}{2x - 3} \cdot \frac{d}{dx}(2x - 3) = \frac{2}{2x - 3}. \tag{10}$$

Although (9) and (10) *appear* to be equal, they are definitely not the same function. The difference is simply that the domain of the derivative in (9) is the interval $\left(\frac{3}{2}, \infty\right)$, whereas the domain of the derivative in (10) is the set of real numbers except $x = \frac{3}{2}$. ■

EXAMPLE 7 A Distinction

The functions $f(x) = \ln x^4$ and $g(x) = 4\ln x$ are not the same. Since $x^4 > 0$ for all $x \neq 0$, the domain of f is the set of real numbers except $x = 0$. The domain of g is the interval $(0, \infty)$. Thus,

$$f'(x) = \frac{4}{x}, \quad x \neq 0 \qquad \text{whereas} \qquad g'(x) = \frac{4}{x}, \quad x > 0.$$ ■

EXAMPLE 8 Simplifying Before Differentiating

Differentiate $y = \ln \dfrac{x^{1/2}(2x+7)^4}{(3x^2+1)^2}$.

Solution Using the laws of logarithms given in Section 1.6, for $x > 0$ we can rewrite the right-hand side of the given function as

$$\begin{aligned} y &= \ln x^{1/2}(2x+7)^4 - \ln(3x^2+1)^2 && \leftarrow \ln(M/N) = \ln M - \ln N \\ &= \ln x^{1/2} + \ln(2x+7)^4 - \ln(3x^2+1)^2 && \leftarrow \ln(MN) = \ln M + \ln N \\ &= \frac{1}{2}\ln x + 4\ln(2x+7) - 2\ln(3x^2+1) && \leftarrow \ln N^c = c\ln N \end{aligned}$$

so that

$$\frac{dy}{dx} = \frac{1}{2}\cdot\frac{1}{x} + 4\cdot\frac{1}{2x+7}\cdot 2 - 2\cdot\frac{1}{3x^2+1}\cdot 6x$$

or

$$\frac{dy}{dx} = \frac{1}{2x} + \frac{8}{2x+7} - \frac{12x}{3x^2+1}.$$ ■

Logarithmic Differentiation Differentiation of a complicated function $y = f(x)$ that consists of products, quotients, and powers can be simplified by a technique known as **logarithmic differentiation**. The procedure consists of three steps.

Guidelines for Logarithmic Differentiation

(*i*) Take the natural logarithm of both sides of $y = f(x)$. Simplify the right-hand side of $\ln y = \ln f(x)$ as much as possible using the general properties of logarithms.

(*ii*) Differentiate the simplified version of $\ln y = \ln f(x)$ implicitly:

$$\frac{d}{dx}\ln y = \frac{d}{dx}\ln f(x).$$

(*iii*) Since the derivative of the left-hand side is $\dfrac{1}{y}\dfrac{dy}{dx}$, multiply both sides by y and replace y by $f(x)$.

We know how to differentiate any function of the type

$$y = (\text{constant})^{\text{variable}} \qquad \text{and} \qquad y = (\text{variable})^{\text{constant}}.$$

For example,

$$\frac{d}{dx}\pi^x = \pi^x(\ln \pi) \qquad \text{and} \qquad \frac{d}{dx}x^\pi = \pi x^{\pi-1}.$$

There are functions where both the base and the exponent are variable:

$$y = (\text{variable})^{\text{variable}}. \tag{11}$$

For example, $f(x) = (1 + 1/x)^x$ is a function of the type described in (11). Recall, in Section 1.6 we saw that $f(x) = (1 + 1/x)^x$ played an important role in the definition of the number e. Although we will not develop a general formula for the derivative of functions of the type given in (11), we can nonetheless obtain their derivatives through the process of logarithmic differentiation. The next example illustrates the method for finding dy/dx.

EXAMPLE 9 Logarithmic Differentiation

Differentiate $y = x^{\sqrt{x}}, x > 0$.

Solution Taking the natural logarithm of both sides of the given equation and simplifying yields

$$\ln y = \ln x^{\sqrt{x}} = \sqrt{x}\ln x. \qquad \leftarrow \text{property } (iii) \text{ of the laws of logarithms, Section 1.6}$$

Then we differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = \sqrt{x}\cdot\frac{1}{x} + \frac{1}{2}x^{-1/2}\cdot\ln x \qquad \leftarrow \text{Product Rule}$$

$$\frac{dy}{dx} = y\left[\frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}\right] \qquad \leftarrow \text{now replace } y \text{ by } x^{\sqrt{x}}$$

$$= \frac{1}{2}x^{\sqrt{x}-\frac{1}{2}}(2 + \ln x). \qquad \leftarrow \text{common denominator and laws of exponents}$$

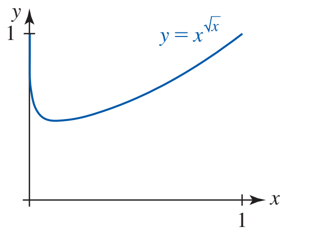


FIGURE 3.9.2 Graph of function in Example 9

We obtained the graph of $y = x^{\sqrt{x}}$ in FIGURE 3.9.2 with the aid of a graphing utility. Note that the graph has a horizontal tangent at the point at which $dy/dx = 0$. Thus, the x-coordinate of the point of horizontal tangency is determined from $2 + \ln x = 0$ or $\ln x = -2$. The last equation gives $x = e^{-2}$. ■

EXAMPLE 10 Logarithmic Differentiation

Find the derivative of $y = \dfrac{\sqrt[3]{x^4 + 6x^2}(8x + 3)^5}{(2x^2 + 7)^{2/3}}$.

Solution Notice that the given function contains no logarithms. As such, we can find dy/dx using the ordinary application of the Quotient, Product, and Power Rules. This procedure, which is tedious, can be avoided by first taking the logarithm of both sides of the given equation, simplifying as we did in Example 9 by the laws of logarithms, and *then* differentiating implicitly. We take the natural logarithm of both sides of the given equation and simplify the right-hand side:

$$\ln y = \ln\frac{\sqrt[3]{x^4 + 6x^2}(8x + 3)^5}{(2x^2 + 7)^{2/3}}$$

$$= \ln\sqrt[3]{x^4 + 6x^2} + \ln(8x + 3)^5 - \ln(2x^2 + 7)^{2/3}$$

$$= \frac{1}{3}\ln(x^4 + 6x^2) + 5\ln(8x + 3) - \frac{2}{3}\ln(2x^2 + 7).$$

Differentiating the last line with respect to x gives

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3}\cdot\frac{1}{x^4 + 6x^2}\cdot(4x^3 + 12x) + 5\cdot\frac{1}{8x + 3}\cdot 8 - \frac{2}{3}\cdot\frac{1}{2x^2 + 7}\cdot 4x$$

$$\frac{dy}{dx} = y\left[\frac{4x^3 + 12x}{3(x^4 + 6x^2)} + \frac{40}{8x + 3} - \frac{8x}{3(2x^2 + 7)}\right] \qquad \leftarrow \text{multiply both sides by } y$$

$$= \frac{\sqrt[3]{x^4 + 6x^2}(8x + 3)^5}{(2x^2 + 7)^{2/3}}\left[\frac{4x^3 + 12x}{3(x^4 + 6x^2)} + \frac{40}{8x + 3} - \frac{8x}{3(2x^2 + 7)}\right]. \qquad \leftarrow \text{replace } y \text{ by the original expression}$$

■

Postscript—Derivative of $f(x) = \log_b x$ Revisited As stated in the introduction to this section we can obtain the derivative of $f(x) = \log_b x$ using the definition of the derivative. From (2) of Section 3.1,

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{\log_b(x+h)-\log_b x}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\log_b\frac{x+h}{x} && \leftarrow \text{algebra and the laws of logarithms} \\ &= \lim_{h\to 0}\frac{1}{h}\log_b\left(1+\frac{h}{x}\right) && \leftarrow \text{division of } x+h \text{ by } x \\ &= \lim_{h\to 0}\frac{1}{x}\cdot\frac{x}{h}\log_b\left(1+\frac{h}{x}\right) && \leftarrow \text{multiplication by } x/x = 1 \\ &= \frac{1}{x}\lim_{h\to 0}\log_b\left(1+\frac{h}{x}\right)^{x/h} && \leftarrow \text{the laws of logarithms} \\ &= \frac{1}{x}\log_b\left[\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{x/h}\right]. && (12) \end{aligned}$$

The last step, taking the limit inside the logarithmic function, is justified by invoking the continuity of the function on $(0, \infty)$ and assuming that the limit inside the brackets exists. If we let $t = h/x$ in the last equation, then since x is fixed, $h \to 0$ implies $t \to 0$. Consequently, we see from (4) of Section 1.6 that

$$\lim_{h\to 0}\left(1+\frac{h}{x}\right)^{x/h} = \lim_{t\to 0}(1+t)^{1/t} = e.$$

Hence the result in (12) shows that,

$$\frac{d}{dx}\log_b x = \frac{1}{x}\log_b e. \qquad (13)$$

Those with sharp eyes and long memories will have noticed that (13) is not the same as (2). The results are equivalent, since by the change of base formula for logarithms $\log_b e = \ln e/\ln b = 1/\ln b$.

When the "natural" choice of $b = e$ is made, (13) becomes (1) since $\log_e e = \ln e = 1$.

Postscript—Power Rule Revisited We are finally in a position to prove the Power Rule $(d/dx)x^n = nx^{n-1}$, (3) of Section 3.2, for all real number exponents n. Our demonstration uses the following fact: For $x > 0$, x^n is defined for all real numbers n. Then in view of the identity $x = e^{\ln x}$ we can write

$$x^n = (e^{\ln x})^n = e^{n\ln x}.$$

Thus,

$$\frac{d}{dx}x^n = \frac{d}{dx}e^{n\ln x} = e^{n\ln x}\frac{d}{dx}(n\ln x) = \frac{n}{x}e^{n\ln x}.$$

Substituting $e^{n\ln x} = x^n$ in the last result completes the proof for $x > 0$,

$$\frac{d}{dx}x^n = \frac{n}{x}x^n = nx^{n-1}.$$

The last derivative formula is also valid for $x < 0$ when $n = p/q$ is a rational number and q is an odd integer.

Exercises 3.9

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–24, find the derivative of the given function.

1. $y = 10\ln x$
2. $y = \ln 10x$
3. $y = \ln x^{1/2}$
4. $y = (\ln x)^{1/2}$
5. $y = \ln(x^4 + 3x^2 + 1)$
6. $y = \ln(x^2 + 1)^{20}$
7. $y = x^2\ln x^3$
8. $y = x - \ln|5x + 1|$
9. $y = \dfrac{\ln x}{x}$
10. $y = x(\ln x)^2$
11. $y = \ln\dfrac{x}{x+1}$
12. $y = \dfrac{\ln 4x}{\ln 2x}$
13. $y = -\ln|\cos x|$
14. $y = \dfrac{1}{3}\ln|\sin 3x|$
15. $y = \dfrac{1}{\ln x}$
16. $y = \ln\dfrac{1}{x}$
17. $f(x) = \ln(x\ln x)$
18. $f(x) = \ln(\ln(\ln x))$
19. $g(x) = \sqrt{\ln\sqrt{x}}$
20. $w(\theta) = \theta\sin(\ln 5\theta)$
21. $H(t) = \ln t^2(3t^2 + 6)$
22. $G(t) = \ln\sqrt{5t+1}\,(t^3 + 4)^6$
23. $f(x) = \ln\dfrac{(x+1)(x+2)}{x+3}$
24. $f(x) = \ln\sqrt{\dfrac{(3x+2)^5}{x^4+7}}$

25. Find an equation of the tangent line to the graph of $y = \ln x$ at $x = 1$.

26. Find an equation of the tangent line to the graph of $y = \ln(x^2 - 3)$ at $x = 2$.

27. Find the slope of the tangent to the graph of $y = \ln(e^{3x} + x)$ at $x = 0$.

28. Find the slope of the tangent to the graph of $y = \ln(xe^{-x^3})$ at $x = 1$.

29. Find the slope of the tangent to the graph of f' at the point where the slope of the tangent to the graph of $f(x) = \ln x^2$ is 4.

30. Determine the point on the graph of $y = \ln 2x$ at which the tangent line is perpendicular to $x + 4y = 1$.

In Problems 31 and 32, find the point(s) on the graph of the given function at which the tangent line is horizontal.

31. $f(x) = \dfrac{\ln x}{x}$ **32.** $f(x) = x^2 \ln x$

In Problems 33–36, find the indicated derivative and simplify as much as possible.

33. $\dfrac{d}{dx}\ln\left(x + \sqrt{x^2 - 1}\right)$ **34.** $\dfrac{d}{dx}\ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right)$

35. $\dfrac{d}{dx}\ln(\sec x + \tan x)$ **36.** $\dfrac{d}{dx}\ln(\csc x - \cot x)$

In Problems 37–40, find the indicated higher derivative.

37. $y = \ln x$; $\dfrac{d^3y}{dx^3}$ **38.** $y = x\ln x$; $\dfrac{d^2y}{dx^2}$

39. $y = (\ln|x|)^2$; $\dfrac{d^2y}{dx^2}$ **40.** $y = \ln(5x - 3)$; $\dfrac{d^4y}{dx^4}$

In Problems 41 and 42, C_1 and C_2 are arbitrary real constants. Show that the function satisfies the given differential equation for $x > 0$.

41. $y = C_1x^{-1/2} + C_2x^{-1/2}\ln x$; $4x^2y'' + 8xy' + y = 0$

42. $y = C_1x^{-1}\cos\left(\sqrt{2}\ln x\right) + C_2x^{-1}\sin\left(\sqrt{2}\ln x\right)$; $x^2y'' + 3xy' + 3y = 0$

In Problems 43–48, use implicit differentiation to find dy/dx.

43. $y^2 = \ln xy$ **44.** $y = \ln(x + y)$

45. $x + y^2 = \ln\dfrac{x}{y}$ **46.** $y = \ln xy^2$

47. $xy = \ln(x^2 + y^2)$ **48.** $x^2 + y^2 = \ln(x + y)^2$

In Problems 49–56, use logarithmic differentiation to find dy/dx.

49. $y = x^{\sin x}$ **50.** $y = (\ln|x|)^x$

51. $y = x(x - 1)^x$ **52.** $y = \dfrac{(x^2 + 1)^x}{x^2}$

53. $y = \dfrac{\sqrt{(2x + 1)(3x + 2)}}{4x + 3}$ **54.** $y = \dfrac{x^{10}\sqrt{x^2 + 5}}{\sqrt[3]{8x^2 + 2}}$

55. $y = \dfrac{(x^3 - 1)^5(x^4 + 3x^3)^4}{(7x + 5)^9}$ **56.** $y = x\sqrt{x + 1}\,\sqrt[3]{x^2 + 2}$

57. Find an equation of the tangent line to the graph of $y = x^{x+2}$ at $x = 1$.

58. Find an equation of the tangent line to the graph of $y = x(\ln x)^x$ at $x = e$.

In Problems 59 and 60, find the point on the graph of the given function at which the tangent line is horizontal. Use a graphing utility to obtain the graph of each function on the interval $[0.01, 1]$.

59. $y = x^x$ **60.** $y = x^{2x}$

≡ Think About It

61. Find the derivatives of
(a) $y = \tan x^x$ **(b)** $y = x^x e^{x^x}$ **(c)** $y = x^{x^x}$.

62. Find d^2y/dx^2 for $y = \sqrt{x^x}$.

63. The function $f(x) = \ln|x|$ is not differentiable only at $x = 0$. The function $g(x) = |\ln x|$ is not differentiable at $x = 0$ and at one other value of $x > 0$. What is it?

64. Find a way to compute $\dfrac{d}{dx}\log_x e$.

≡ Calculator/CAS Problems

65. **(a)** Use a calculator or CAS to obtain the graph of $y = (\sin x)^{\ln x}$ on the interval $(0, 5\pi)$.
(b) Explain why there appears to be no graph on certain intervals. Identify the intervals.

66. **(a)** Use a calculator or CAS to obtain the graph of $y = |\cos x|^{\cos x}$ on the interval $[0, 5\pi]$.
(b) Determine, at least approximately, the values of x in the interval $[0, 5\pi]$ for which the tangent to the graph is horizontal.

67. Use a calculator or CAS to obtain the graph of $f(x) = x^3 - 12\ln x$. Then find the *exact* value of the least value of $f(x)$.

3.10 Hyperbolic Functions

Introduction If you have ever toured the 630-ft-high Gateway Arch in St. Louis, Missouri, you may have asked the question, What is the shape of the arch? and received the rather cryptic reply: the shape of an inverted catenary. The word *catenary* stems from the Latin word *catena* and literally means "a hanging chain" (the Romans used the catena as a dog leash). It can be demonstrated that the shape assumed by a long flexible wire, chain, cable, or rope hanging under its own weight between two points is the shape of the graph of the function

$$f(x) = \frac{k}{2}(e^{cx} + e^{-cx}) \tag{1}$$

for appropriate choices of the constants c and k. The graph of any function of the form given in (1) is called a **catenary**.

The Gateway Arch in St. Louis, MO.

Hyperbolic Functions Combinations such as (1) involving the exponential functions e^x and e^{-x} occur so often in applied mathematics that they warrant special definitions.

Definition 3.10.1 Hyperbolic Sine and Cosine

For any real number x, the **hyperbolic sine** of x is

$$\sinh x = \frac{e^x - e^{-x}}{2} \tag{2}$$

and the **hyperbolic cosine** of x is

$$\cosh x = \frac{e^x + e^{-x}}{2}. \tag{3}$$

Because the domain of each of the exponential functions e^x and e^{-x} is the set of real numbers $(-\infty, \infty)$, the domain of $y = \sinh x$ and of $y = \cosh x$ is $(-\infty, \infty)$. From (2) and (3) of Definition 3.10.1 it is also apparent that

$$\sinh 0 = 0 \quad \text{and} \quad \cosh 0 = 1.$$

The shape of the St. Louis Gateway Arch is based on the mathematical model

$$y = A - B\cosh(Cx/L),$$

where $A = 693.8597$, $B = 68.7672$, $L = 299.2239$, $C = 3.0022$, and x and y are measured in feet. When $x = 0$, we get the approximate height of 630 ft.

Analogous to the trigonometric functions $\tan x$, $\cot x$, $\sec x$, and $\csc x$ that are defined in terms of $\sin x$ and $\cos x$, we define four additional hyperbolic functions in terms of $\sinh x$ and $\cosh x$.

Definition 3.10.2 Other Hyperbolic Functions

For a real number x, the **hyperbolic tangent** of x is

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \tag{4}$$

the **hyperbolic cotangent** of x, $x \neq 0$, is

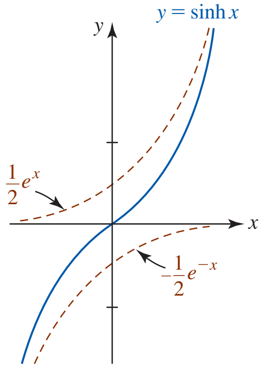
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}, \tag{5}$$

the **hyperbolic secant** of x is

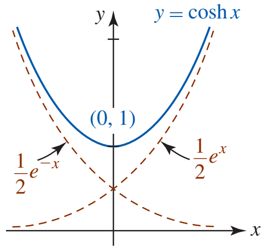
$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}, \tag{6}$$

the **hyperbolic cosecant** of x, $x \neq 0$, is

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}. \tag{7}$$



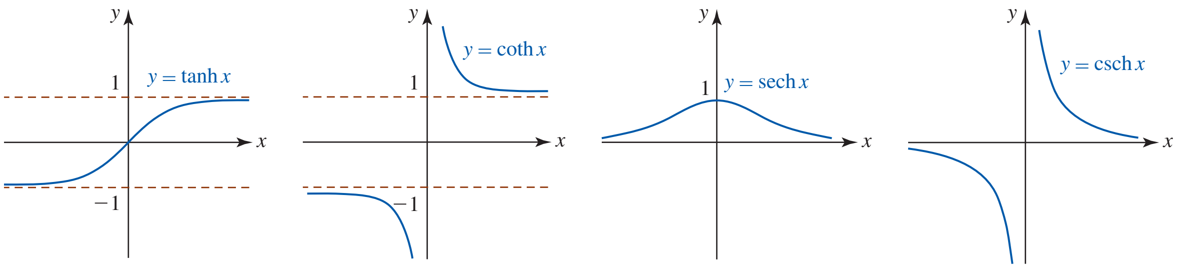
(a) $y = \sinh x$



(b) $y = \cosh x$

FIGURE 3.10.1 Graphs of hyperbolic sine and cosine

Graphs of Hyperbolic Functions The graphs of the hyperbolic sine and hyperbolic cosine are given in FIGURE 3.10.1. Note the similarity of the graph in Figure 3.10.1(b) and the shape of the Gateway Arch in the photo at the beginning of this section. The graphs of the hyperbolic tangent, cotangent, secant, and cosecant are given in FIGURE 3.10.2. Note that $x = 0$ is a vertical asymptote of the graphs of $y = \coth x$ and $y = \operatorname{csch} x$.



(a) $y = \tanh x$ (b) $y = \coth x$ (c) $y = \operatorname{sech} x$ (d) $y = \operatorname{csch} x$

FIGURE 3.10.2 Graphs of the hyperbolic tangent, cotangent, secant, and cosecant

Identities Although the hyperbolic functions are not periodic, they possess many identities that are similar to those for the trigonometric functions. Notice that the graphs in Figure 3.10.1(a) and (b) are symmetric with respect to the origin and the y-axis, respectively. In other words, $y = \sinh x$ is an odd function and $y = \cosh x$ is an even function:

$$\sinh(-x) = -\sinh x, \tag{8}$$

$$\cosh(-x) = \cosh x. \tag{9}$$

In trigonometry a fundamental identity is $\cos^2 x + \sin^2 x = 1$. For hyperbolic functions the analogue of this identity is

$$\cosh^2 x - \sinh^2 x = 1. \tag{10}$$

To prove (10) we resort to (2) and (3) of Definition 3.10.1:

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2 \\ &= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1.\end{aligned}$$

We summarize (8)–(10) along with eleven other identities in the theorem that follows.

Theorem 3.10.1 Hyperbolic Identities

$\sinh(-x) = -\sinh x$	$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$	(11)
$\cosh(-x) = \cosh x$	$\sinh(x - y) = \sinh x \cosh y - \cosh x \sinh y$	(12)
$\tanh(-x) = -\tanh x$	$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$	(13)
$\cosh^2 x - \sinh^2 x = 1$	$\cosh(x - y) = \cosh x \cosh y - \sinh x \sinh y$	(14)
$1 - \tanh^2 x = \operatorname{sech}^2 x$	$\sinh 2x = 2 \sinh x \cosh x$	(15)
$\coth^2 x - 1 = \operatorname{csch}^2 x$	$\cosh 2x = \cosh^2 x + \sinh^2 x$	(16)
$\sinh^2 x = \frac{1}{2}(-1 + \cosh 2x)$	$\cosh^2 x = \frac{1}{2}(1 + \cosh 2x)$	(17)

Derivatives of Hyperbolic Functions The derivatives of the hyperbolic functions follow from (14) of Section 3.8 and the rules of differentiation; for example

$$\frac{d}{dx}\sinh x = \frac{d}{dx}\frac{e^x - e^{-x}}{2} = \frac{1}{2}\left[\frac{d}{dx}e^x - \frac{d}{dx}e^{-x}\right] = \frac{e^x + e^{-x}}{2}.$$

That is,

$$\frac{d}{dx}\sinh x = \cosh x. \tag{18}$$

Similarly, it should be apparent from the definition of the hyperbolic cosine in (3) that

$$\frac{d}{dx}\cosh x = \sinh x. \tag{19}$$

To differentiate, say, the hyperbolic tangent, we use the Quotient Rule and the definition given in (4):

$$\frac{d}{dx}\tanh x = \frac{d}{dx}\frac{\sinh x}{\cosh x}$$

$$= \frac{\cosh x \cdot \frac{d}{dx}\sinh x - \sinh x \cdot \frac{d}{dx}\cosh x}{\cosh^2 x}$$

$$= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \quad \leftarrow \text{this is equal to 1 by (10)}$$

$$= \frac{1}{\cosh^2 x}.$$

In other words,

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x. \tag{20}$$

The derivatives of the six hyperbolic functions in the most general case follow from the Chain Rule.

Theorem 3.10.2 Derivatives of Hyperbolic Functions

If $u = g(x)$ is a differentiable function, then

$$\frac{d}{dx}\sinh u = \cosh u \frac{du}{dx}, \qquad \frac{d}{dx}\cosh u = \sinh u \frac{du}{dx}, \tag{21}$$

$$\frac{d}{dx}\tanh u = \operatorname{sech}^2 u \frac{du}{dx}, \qquad \frac{d}{dx}\coth u = -\operatorname{csch}^2 u \frac{du}{dx}, \tag{22}$$

$$\frac{d}{dx}\operatorname{sech} u = -\operatorname{sech} u \tanh u \frac{du}{dx}, \qquad \frac{d}{dx}\operatorname{csch} u = -\operatorname{csch} u \coth u \frac{du}{dx}. \tag{23}$$

You should take careful note of the slight difference in the results in (21)–(23) and the analogous formulas for the trigonometric functions:

$$\frac{d}{dx}\cos x = -\sin x \qquad \text{whereas} \qquad \frac{d}{dx}\cosh x = \sinh x$$

$$\frac{d}{dx}\sec x = \sec x \tan x \qquad \text{whereas} \qquad \frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x.$$

EXAMPLE 1 Chain Rule

Differentiate

(a) $y = \sinh\sqrt{2x+1}$ **(b)** $y = \coth x^3$.

Solution

(a) From the first result in (21),

$$\frac{dy}{dx} = \cosh\sqrt{2x+1} \cdot \frac{d}{dx}(2x+1)^{1/2}$$

$$= \cosh\sqrt{2x+1}\left(\frac{1}{2}(2x+1)^{-1/2} \cdot 2\right)$$

$$= \frac{\cosh\sqrt{2x+1}}{\sqrt{2x+1}}.$$

(b) From the second result in (22),

$$\frac{dy}{dx} = -\operatorname{csch}^2 x^3 \cdot \frac{d}{dx}x^3 = -\operatorname{csch}^2 x^3 \cdot 3x^2.$$

EXAMPLE 2 Value of a Derivative

Evaluate the derivative of $y = \dfrac{3x}{4 + \cosh 2x}$ at $x = 0$.

Solution From the Quotient Rule,

$$\frac{dy}{dx} = \frac{(4 + \cosh 2x) \cdot 3 - 3x(\sinh 2x \cdot 2)}{(4 + \cosh 2x)^2}.$$

Because $\sinh 0 = 0$ and $\cosh 0 = 1$, we have

$$\left.\frac{dy}{dx}\right|_{x=0} = \frac{15}{25} = \frac{3}{5}.$$

Inverse Hyperbolic Functions Inspection of Figure 3.10.1(a) shows that $y = \sinh x$ is a one-to-one function. That is, for any real number y in the range $(-\infty, \infty)$ of the hyperbolic sine, there corresponds only one real number x in its domain $(-\infty, \infty)$. Hence, $y = \sinh x$ has an inverse function that is written $y = \sinh^{-1}x$. See FIGURE 3.10.3(a). As in our earlier discussion of the inverse trigonometric functions in Section 1.5, this later notation is equivalent to $x = \sinh y$. From Figure 3.10.2(a) it is also seen that $y = \tanh x$ with domain $(-\infty, \infty)$ and range $(-1, 1)$ is also one-to-one and has an inverse $y = \tanh^{-1}x$ with domain $(-1, 1)$ and range $(-\infty, \infty)$. See Figure 3.10.3(c). But from Figures 3.10.1(b) and 3.10.2(c) it is apparent that $y = \cosh x$ and $y = \operatorname{sech} x$ are not one-to-one functions and so do not possess inverse functions unless their domains are suitably restricted. Inspection of Figure 3.10.1(b) shows that when the domain of $y = \cosh x$ is restricted to the interval $[0, \infty)$, the corresponding range is $[1, \infty)$. The inverse function $y = \cosh^{-1}x$ then has domain $[1, \infty)$ and range $[0, \infty)$. See Figure 3.10.3(b). The graphs of all the inverse hyperbolic functions along with their domains and ranges are summarized in Figure 3.10.3.

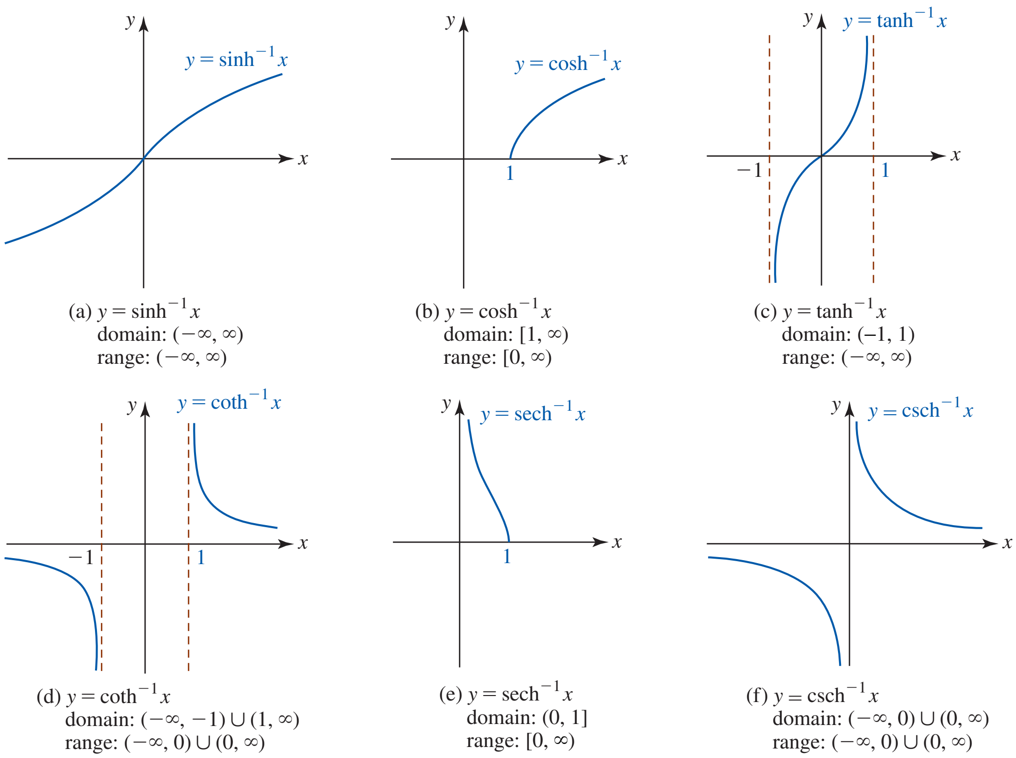


FIGURE 3.10.3 Graphs of the inverses of the hyperbolic sine, cosine, tangent, cotangent, secant, and cosecant

Inverse Hyperbolic Functions as Logarithms Because all the hyperbolic functions are defined in terms of combinations of e^x, it should not come as any surprise to find that the inverse hyperbolic functions can be expressed in terms of the natural logarithm. For example, $y = \sinh^{-1} x$ is equivalent to $x = \sinh y$, so that

$$x = \frac{e^y - e^{-y}}{2} \quad \text{or} \quad 2x = \frac{e^{2y} - 1}{e^y} \quad \text{or} \quad e^{2y} - 2xe^y - 1 = 0.$$

Because the last equation is quadratic in e^y, the quadratic formula gives

$$e^y = \frac{2x \pm \sqrt{4x^2 + 4}}{2} = x \pm \sqrt{x^2 + 1}. \tag{24}$$

Now the solution corresponding to the minus sign in (24) must be rejected since $e^y > 0$ but $x - \sqrt{x^2 + 1} < 0$. Thus, we have

$$e^y = x + \sqrt{x^2 + 1} \quad \text{or} \quad y = \sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Similarly, for $y = \tanh^{-1} x$, $|x| < 1$,

$$x = \tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}}$$

gives

$$e^y(1 - x) = (1 + x)e^{-y}$$

$$e^{2y} = \frac{1 + x}{1 - x}$$

$$2y = \ln\left(\frac{1 + x}{1 - x}\right)$$

or

$$y = \tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right).$$

We have proved two of the results in the next theorem.

Theorem 3.10.3 Logarithmic Identities

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad \cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right), x \geq 1 \tag{25}$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), |x| < 1 \qquad \coth^{-1} x = \frac{1}{2}\ln\left(\frac{x + 1}{x - 1}\right), |x| > 1 \tag{26}$$

$$\operatorname{sech}^{-1} x = \ln\left(\frac{1 + \sqrt{1 - x^2}}{x}\right), 0 < x \leq 1 \qquad \operatorname{csch}^{-1} x = \ln\left(\frac{1}{x} + \frac{\sqrt{1 + x^2}}{|x|}\right), x \neq 0 \tag{27}$$

The foregoing identities are a convenient means for obtaining the numerical values of an inverse hyperbolic function. For example, with the aid of a calculator we see from the first result in (25) in Theorem 3.10.3 that when $x = 4$

$$\sinh^{-1} 4 = \ln\left(4 + \sqrt{17}\right) \approx 2.0947.$$

Derivatives of Inverse Hyperbolic Functions To find the derivative of an inverse hyperbolic function, we can proceed in two different ways. For example, if

$$y = \sinh^{-1} x \quad \text{then} \quad x = \sinh y.$$

Using implicit differentiation, we can write

$$\frac{d}{dx}x = \frac{d}{dx}\sinh y$$

$$1 = \cosh y \frac{dy}{dx}.$$

Hence

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{\sinh^2 y + 1}} = \frac{1}{\sqrt{x^2 + 1}}.$$

The foregoing result can be obtained in an alternative manner. We know from Theorem 3.10.3 that

$$y = \ln\left(x + \sqrt{x^2 + 1}\right).$$

Therefore, from the derivative of the logarithm, we obtain

$$\frac{dy}{dx} = \frac{1}{x + \sqrt{x^2+1}}\left(1 + \frac{1}{2}(x^2+1)^{-1/2} \cdot 2x\right) \quad \leftarrow \text{by (3) of Section 3.9}$$

$$= \frac{1}{x + \sqrt{x^2+1}} \, \frac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}} = \frac{1}{\sqrt{x^2+1}}.$$

We have essentially proved the first entry in (28) in the next theorem.

Theorem 3.10.4 Derivatives of Inverse Hyperbolic Functions

If $u = g(x)$ is a differentiable function, then

$$\frac{d}{dx}\sinh^{-1}u = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}, \qquad \frac{d}{dx}\cosh^{-1}u = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx},\ u > 1, \tag{28}$$

$$\frac{d}{dx}\tanh^{-1}u = \frac{1}{1-u^2}\frac{du}{dx},\quad |u| < 1, \qquad \frac{d}{dx}\coth^{-1}u = \frac{1}{1-u^2}\frac{du}{dx},\ |u| > 1, \tag{29}$$

$$\frac{d}{dx}\operatorname{sech}^{-1}u = \frac{-1}{u\sqrt{1-u^2}}\frac{du}{dx},\ 0 < u < 1, \qquad \frac{d}{dx}\operatorname{csch}^{-1}u = \frac{-1}{|u|\sqrt{1+u^2}}\frac{du}{dx},\ u \neq 0. \tag{30}$$

EXAMPLE 3 Derivative of Inverse Hyperbolic Cosine

Differentiate $y = \cosh^{-1}(x^2 + 5)$.

Solution With $u = x^2 + 5$, we have from the second formula in (28),

$$\frac{dy}{dx} = \frac{1}{\sqrt{(x^2+5)^2 - 1}} \cdot \frac{d}{dx}(x^2+5) = \frac{2x}{\sqrt{x^4 + 10x^2 + 24}}.$$

■

EXAMPLE 4 Derivative of Inverse Hyperbolic Tangent

Differentiate $y = \tanh^{-1}4x$.

Solution With $u = 4x$, we have from the first formula in (29),

$$\frac{dy}{dx} = \frac{1}{1-(4x)^2} \cdot \frac{d}{dx}4x = \frac{4}{1-16x^2}.$$

■

EXAMPLE 5 Product and Chain Rules

Differentiate $y = e^{x^2}\operatorname{sech}^{-1}x$.

Solution By the Product Rule and the first formula in (30), we have

$$\frac{dy}{dx} = e^{x^2}\underbrace{\left(\frac{-1}{x\sqrt{1-x^2}}\right)}_{\text{by first formula in (30)}} + \underbrace{2xe^{x^2}}_{\text{by (14) of Section 3.8}}\operatorname{sech}^{-1}x$$

$$= -\frac{e^{x^2}}{x\sqrt{1-x^2}} + 2xe^{x^2}\operatorname{sech}^{-1}x.$$

■

$\frac{d}{dx}$ NOTES FROM THE CLASSROOM

(*i*) As mentioned in the introduction to this section, the graph of any function of the form $f(x) = k\cosh cx$, k and c constants, is called a **catenary**. The shape assumed by a flexible wire or heavy rope strung between two posts has the basic shape of a graph of a hyperbolic cosine. Furthermore, if two circular rings are held vertically and are not too far apart, then a soap film stretched between the rings will assume a surface having minimum area. The surface is a portion of a **catenoid**, which is the surface obtained by revolving a catenary about the x-axis. See FIGURE 3.10.4.

(*ii*) The similarity between trigonometric and hyperbolic functions extends beyond the derivative formulas and basic identities. If t is an angle measured in radians whose terminal side is OP, then the coordinates of P on a unit *circle* $x^2 + y^2 = 1$ are $(\cos t, \sin t)$. Now, the area of the shaded circular sector shown in FIGURE 3.10.5(a) is $A = \frac{1}{2}t$ and so $t = 2A$. In this manner, the *circular functions* $\cos t$ and $\sin t$ can be considered functions of the area A.

You might already know that the graph of the equation $x^2 - y^2 = 1$ is called a *hyperbola*. Because $\cosh t \geq 1$ and $\cosh^2 t - \sinh^2 t = 1$, it follows that the coordinates of a point P on the right-hand branch of the hyperbola are $(\cosh t, \sinh t)$. Furthermore, it can be shown that the area of the hyperbolic sector in Figure 3.10.5(b) is related to the number t by $t = 2A$. Whence we see the origin of the name *hyperbolic function.*

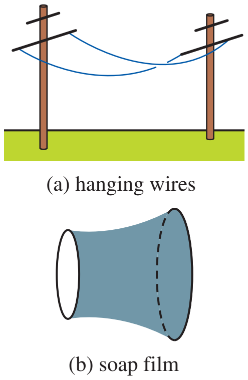


FIGURE 3.10.4 Catenary in (a); catenoid in (b)

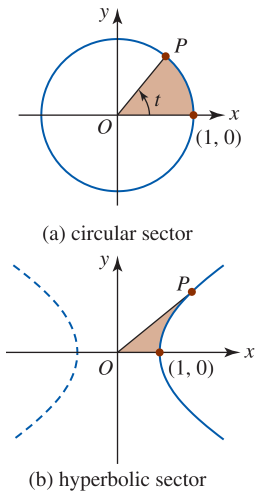


FIGURE 3.10.5 Circle in (a); hyperbola in (b)

Exercises 3.10

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

1. If $\sinh x = -\frac{1}{2}$, find the values of the remaining hyperbolic functions.
2. If $\cosh x = 3$, find the values of the remaining hyperbolic functions.

In Problems 3–26, find the derivative of the given function.

3. $y = \cosh 10x$
4. $y = \operatorname{sech} 8x$
5. $y = \tanh\sqrt{x}$
6. $y = \operatorname{csch}\dfrac{1}{x}$
7. $y = \operatorname{sech}(3x - 1)^2$
8. $y = \sinh e^{x^2}$
9. $y = \coth(\cosh 3x)$
10. $y = \tanh(\sinh x^3)$
11. $y = \sinh 2x \cosh 3x$
12. $y = \operatorname{sech} x \coth 4x$
13. $y = x\cosh x^2$
14. $y = \dfrac{\sinh x}{x}$
15. $y = \sinh^3 x$
16. $y = \cosh^4\sqrt{x}$
17. $f(x) = (x - \cosh x)^{2/3}$
18. $f(x) = \sqrt{4 + \tanh 6x}$
19. $f(x) = \ln(\cosh 4x)$
20. $f(x) = (\ln(\operatorname{sech} x))^2$
21. $f(x) = \dfrac{e^x}{1 + \cosh x}$
22. $f(x) = \dfrac{\ln x}{x^2 + \sinh x}$
23. $F(t) = e^{\sinh t}$
24. $H(t) = e^t e^{\operatorname{csch} t^2}$
25. $g(t) = \dfrac{\sin t}{1 + \sinh 2t}$
26. $w(t) = \dfrac{\tanh t}{(1 + \cosh t)^2}$
27. Find an equation of the tangent line to the graph of $y = \sinh 3x$ at $x = 0$.
28. Find an equation of the tangent line to the graph of $y = \cosh x$ at $x = 1$.

In Problems 29 and 30, find the point(s) on the graph of the given function at which the tangent is horizontal.

29. $f(x) = (x^2 - 2)\cosh x - 2x\sinh x$
30. $f(x) = \cos x \cosh x - \sin x \sinh x$

In Problems 31 and 32, find d^2y/dx^2 for the given function.

31. $y = \tanh x$
32. $y = \operatorname{sech} x$

In Problems 33 and 34, C_1, C_2, C_3, C_4 and k are arbitrary real constants. Show that the function satisfies the given differential equation.

33. $y = C_1\cosh kx + C_2\sinh kx$; $\quad y'' - k^2y = 0$
34. $y = C_1\cos kx + C_2\sin kx + C_3\cosh kx + C_4\sinh kx$; $y^{(4)} - k^4y = 0$

In Problems 35–48, find the derivative of the given function.

35. $y = \sinh^{-1} 3x$

36. $y = \cosh^{-1}\dfrac{x}{2}$

37. $y = \tanh^{-1}(1 - x^2)$

38. $y = \coth^{-1}\dfrac{1}{x}$

39. $y = \coth^{-1}(\csc x)$

40. $y = \sinh^{-1}(\sin x)$

41. $y = x\sinh^{-1} x^3$

42. $y = x^2 \operatorname{csch}^{-1} x$

43. $y = \dfrac{\operatorname{sech}^{-1} x}{x}$

44. $y = \dfrac{\coth^{-1} e^{2x}}{e^{2x}}$

45. $y = \ln(\operatorname{sech}^{-1} x)$

46. $y = x\tanh^{-1} x + \ln\sqrt{1 - x^2}$

47. $y = (\cosh^{-1} 6x)^{1/2}$

48. $y = \dfrac{1}{(\tanh^{-1} 2x)^3}$

≡ Applications

49. **(a)** Assume that k, m, and g are real constants. Show that the function

$$v(t) = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{kg}{m}}\,t\right)$$

satisfies the differential equation $m\dfrac{dv}{dt} = mg - kv^2$.

(b) The function v represents the velocity of a falling mass m when air resistance is taken to be proportional to the square of the instantaneous velocity. Find the limiting or **terminal velocity** $v_{\text{ter}} = \lim_{t\to\infty} v(t)$ of the mass.

(c) Suppose a 80-kg skydiver delays opening the parachute until terminal velocity is attained. Determine the terminal velocity if it is known that $k = 0.25$ kg/m.

50. A woman, W, starting at the origin, moves in the direction of the positive x-axis pulling a boat along the curve C, called a **tractrix**, indicated in FIGURE 3.10.6. The boat, initially located on the y-axis at $(0, a)$, is pulled by a rope of constant length a that is kept taut throughout the motion. An equation of the tractrix is given by

$$x = a\ln\left(\frac{a + \sqrt{a^2 - y^2}}{y}\right) - \sqrt{a^2 - y^2}.$$

(a) Rewrite this equation using a hyperbolic function.

(b) Use implicit differentiation to show that the equation of the tractrix satisfies the differential equation

$$\frac{dy}{dx} = -\frac{y}{\sqrt{a^2 - y^2}}.$$

(c) Interpret geometrically the differential equation in part (b).

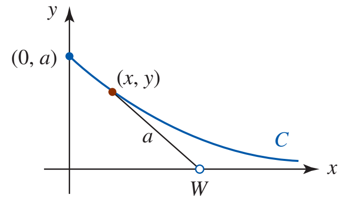


FIGURE 3.10.6 Tractrix in Problem 50

≡ Think About It

In Problems 51 and 52, find the exact numerical value of the given quantity.

51. $\cosh(\ln 4)$

52. $\sinh(\ln 0.5)$

In Problems 53 and 54, express the given quantity as a rational function of x.

53. $\sinh(\ln x)$

54. $\tanh(3\ln x)$

55. Show that for any positive integer n,

$$(\cosh x + \sinh x)^n = \cosh nx + \sinh nx.$$

Chapter 3 in Review

Answers to selected odd-numbered problems begin on page ANS-000.

A. True/False

In Problems 1–20, indicate whether the given statement is true or false.

1. If $y = f(x)$ is continuous at a number a, then there is a tangent line to the graph of f at $(a, f(a))$. _____

2. If f is differentiable at every real number x, then f is continuous everywhere. _____

3. If $y = f(x)$ has a tangent line at $(a, f(a))$, then f is necessarily differentiable at $x = a$. _____

4. The instantaneous rate of change of $y = f(x)$ with respect to x at x_0 is the slope of the tangent line to the graph at $(x_0, f(x_0))$. _____

5. At $x = -1$, the tangent line to the graph of $f(x) = x^3 - 3x^2 - 9x$ is parallel to the line $y = 2$. _____

6. The derivative of a product is the product of the derivatives. _____

7. A polynomial function has a tangent line at every point on its graph. _____

8. For $f(x) = -x^2 + 5x + 1$ an equation of the tangent line is $f'(x) = -2x + 5$._____
9. The function $f(x) = x/(x^2 + 9)$ is differentiable on the interval $[-3, 3]$._____
10. If $f'(x) = g'(x)$, then $f(x) = g(x)$. _____
11. If m is the slope of a tangent line to the graph of $f(x) = \sin x$, then $-1 \leq m \leq 1$. _____
12. For $y = \tan^{-1}x$, $dy/dx > 0$ for all x. _____
13. $\frac{d}{dx}\cos^{-1}x = -\sin^{-1}x$ _____
14. The function $f(x) = x^5 + x^3 + x$ has an inverse. _____
15. If $f'(x) < 0$ on the interval $[2, 8]$, then $f(3) > f(5)$. _____
16. If f is an increasing differentiable function on an interval, then $f'(x)$ is also increasing on the interval. _____
17. The only function for which $f'(x) = f(x)$ is $f(x) = e^x$. _____
18. $\frac{d}{dx}\ln|x| = \frac{1}{|x|}$ _____
19. $\frac{d}{dx}\cosh^2 x = \frac{d}{dx}\sinh^2 x$ _____
20. Every inverse hyperbolic function is a logarithm. _____

B. Fill in the Blanks

In Problems 1–20, fill in the blanks.

1. If $y = f(x)$ is a polynomial function of degree 3, then $\frac{d^4}{dx^4}f(x) =$ ________.
2. The slope of the tangent line to the graph of $y = \ln|x|$ at $x = -\frac{1}{2}$ is ________.
3. The slope of the normal line to the graph of $f(x) = \tan x$ at $x = \pi/3$ is ________.
4. $f(x) = \frac{x^{n+1}}{n+1}$, $n \neq -1$, then $f'(x) =$ ________.
5. An equation of the tangent line to the graph of $y = (x + 3)/(x - 2)$ at $x = 0$ is ________.
6. For $f(x) = 1/(1 - 3x)$ the instantaneous rate of change of f' with respect to x at $x = 0$ is ________.
7. If $f'(4) = 6$ and $g'(4) = 3$, then the slope of the tangent line to the graph of $y = 2f(x) - 5g(x)$ at $x = 4$ is ________.
8. If $f(2) = 1$, $f'(2) = 5$, $g(2) = 2$, and $g'(2) = -3$, then $\frac{d}{dx}\frac{x^2 f(x)}{g(x)}\bigg|_{x=2} =$ ________.
9. If $g(1) = 2$, $g'(1) = 3$, $g''(1) = -1$, $f'(2) = 4$, and $f''(2) = 3$, then $\frac{d^2}{dx^2}f(g(x))\bigg|_{x=1} =$ _________.
10. If $f'(x) = x^2$, then $\frac{d}{dx}f(x^3) =$ ________.
11. If F is a differentiable function, then $\frac{d^2}{dx^2}F(\sin 4x) =$ ________.
12. The function $f(x) = \cot x$ is not differentiable on the interval $[0, \pi]$ because _________.
13. The function

$$f(x) = \begin{cases} ax + b, & x \leq 3 \\ x^2, & x > 3 \end{cases}$$

is differentiable at $x = 3$ when $a =$ ________ and $b =$ _________.
14. If $f'(x) = \sec^2 2x$, then $f(x) =$ _________.
15. The tangent line to the graph of $f(x) = 5 - x + e^{x-1}$ is horizontal at the point _________.

16. $\dfrac{d}{dx}2^x =$ __________.

17. $\dfrac{d}{dx}\log_{10}x =$ __________.

18. If $f(x) = \ln|2x - 4|$, the domain of $f'(x)$ is __________.

19. The graph of $y = \cosh x$ is called a _________.

20. $\cosh^{-1}1 =$ ________.

C. Exercises

In Problems 1–28, find the derivative of the given function.

1. $f(x) = \dfrac{4x^{0.3}}{5x^{0.2}}$

2. $y = \dfrac{1}{x^3 + 4x^2 - 6x + 11}$

3. $F(t) = \left(t + \sqrt{t^2+1}\right)^{10}$

4. $h(\theta) = \theta^{1.5}(\theta^2 + 1)^{0.5}$

5. $y = \sqrt[4]{x^4 + 16}\,\sqrt[3]{x^3 + 8}$

6. $g(u) = \sqrt{\dfrac{6u - 1}{u + 7}}$

7. $y = \dfrac{\cos 4x}{4x + 1}$

8. $y = 10\cot 8x$

9. $f(x) = x^3\sin^2 5x$

10. $y = \tan^2(\cos 2x)$

11. $y = \sin^{-1}\dfrac{3}{x}$

12. $y = \cos x\cos^{-1}x$

13. $y = (\cot^{-1}x)^{-1}$

14. $y = \operatorname{arcsec}(2x - 1)$

15. $y = 2\cos^{-1}x + 2x\sqrt{1 - x^2}$

16. $y = x^2\tan^{-1}\sqrt{x^2 - 1}$

17. $y = xe^{-x} + e^{-x}$

18. $y = (e + e^2)^x$

19. $y = x^7 + 7^x + 7^\pi + e^{7x}$

20. $y = (e^x + 1)^{-e}$

21. $y = \ln\left(x\sqrt{4x - 1}\right)$

22. $y = \left(\ln\cos^2 x\right)^2$

23. $y = \sinh^{-1}(\sin^{-1}x)$

24. $y = (\tan^{-1}x)(\tanh^{-1}x)$

25. $y = xe^{x\cosh^{-1}x}$

26. $y = \sinh^{-1}\sqrt{x^2 - 1}$

27. $y = \sinh e^{x^3}$

28. $y = (\tanh 5x)^{-1}$

In Problems 29–34, find the indicated derivative.

29. $y = (3x + 1)^{5/2};\quad \dfrac{d^3y}{dx^3}$

30. $y = \sin(x^3 - 2x);\quad \dfrac{d^2y}{dx^2}$

31. $s = t^2 + \dfrac{1}{t^2};\quad \dfrac{d^4s}{dt^4}$

32. $W = \dfrac{v - 1}{v + 1};\quad \dfrac{d^3W}{dv^3}$

33. $y = e^{\sin 2x};\quad \dfrac{d^2y}{dx^2}$

34. $f(x) = x^2\ln x;\quad f'''(x)$

35. First use the laws of logarithms to simplify

$$y = \ln\left|\frac{(x+5)^4(2-x)^3}{(x+8)^{10}\sqrt[3]{6x+4}}\right|,$$

and then find dy/dx.

36. Find dy/dx for $y = 5^{x^2}x^{\sin 2x}$.

37. Given that $y = x^3 + x$ is a one-to-one function, find the slope of the tangent line to the graph of the inverse function at $x = 1$.

38. Given that $f(x) = 8/(1 - x^3)$ is a one-to-one function, find f^{-1} and $(f^{-1})'$.

In Problems 39 and 40, find dy/dx.

39. $xy^2 = e^x - e^y$

40. $y = \ln(xy)$

41. Find an equation of a tangent line to the graph of $f(x) = x^3$ that is perpendicular to the line $y = -3x$.

42. Find the point(s) on the graph of $f(x) = \frac{1}{2}x^2 - 5x + 1$ at which
(a) $f''(x) = f(x)$ and **(b)** $f''(x) = f'(x)$.

43. Find equations for the lines through $(0, -9)$ that are tangent to the graph of $y = x^2$.

44. **(a)** Find the x-intercept of the tangent line to the graph of $y = x^2$ at $x = 1$.
(b) Find an equation of the line with the same x-intercept that is perpendicular to the tangent line in part (a).
(c) Find the point(s) where the line in part (b) intersects the graph of $y = x^2$.

45. Find the point on the graph of $f(x) = \sqrt{x}$ at which the tangent line is parallel to the secant line through $(1, f(1))$ and $(9, f(9))$.

46. If $f(x) = (1 + x)/x$, what is the slope of the tangent line to the graph of f'' at $x = 2$?

47. Find the x-coordinates of all points on the graph of $f(x) = 2\cos x + \cos 2x$, $0 \le x \le 2\pi$, at which the tangent line is horizontal.

48. Find the point on the graph of $y = \ln 2x$ such that the tangent line passes through the origin.

49. Suppose a series circuit contains a capacitor and a variable resistor. If the resistance at time t is given by $R = k_1 + k_2 t$, where k_1 and k_2 are positive known constants, then the charge $q(t)$ on the capacitor is given by

$$q(t) = E_0 C + (q_0 - E_0 C)\left(\frac{k_1}{k_1 + k_2 t}\right)^{1/Ck_2},$$

where C is a constant called the **capacitance** and $E(t) = E_0$ is the impressed voltage. Show that $q(t)$ satisfies the initial condition $q(0) = q_0$ and the differential equation

$$(k_1 + k_2 t)\frac{dq}{dt} + \frac{1}{C}q = E_0.$$

50. Assume that C_1 and C_2 are arbitrary real constants. Show that the function

$$y = C_1 x + C_2\left[\frac{x}{2}\ln\left(\frac{x-1}{x+1}\right) - 1\right]$$

satisfies the differential equation

$$(1 - x^2)y'' - 2xy' + 2y = 0.$$

In Problems 51 and 52, C_1, C_2, C_3, and C_4 are arbitrary real constants. Show that the function satisfies the given differential equation.

51. $y = C_1 e^{-x} + C_2 e^x + C_3 x e^{-x} + C_4 x e^x$; $y^{(4)} - 2y'' + y = 0$

52. $y = C_1 \cos x + C_2 \sin x + C_3 x \cos x + C_4 x \sin x$; $y^{(4)} + 2y'' + y = 0$

53. **(a)** Find the points on the graph of $y^3 - y + x^2 - 4 = 0$ corresponding to $x = 2$.
(b) Find the slopes of the tangent lines at the points found in part (a).

54. Sketch the graph of f' from the graph of f given in FIGURE 3.R.1.

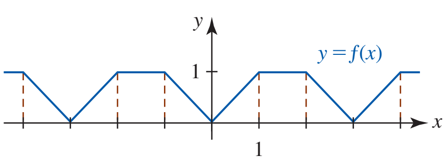


FIGURE 3.R.1 Graph for Problem 54

55. The graph of $x^{2/3} + y^{2/3} = 1$, shown in FIGURE 3.R.2, is called a **hypocycloid**.*
Find equations of the tangent lines to the graph at the points corresponding to $x = \frac{1}{8}$.

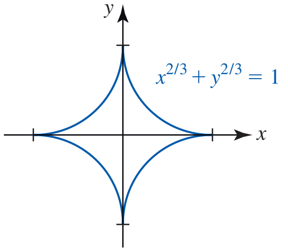


FIGURE 3.R.2 Hypocycloid in Problem 55

56. Find d^2y/dx^2 for the equation in Problem 55.

57. Suppose

$$f(x) = \begin{cases} x^2, & x \le 0 \\ \sqrt{x}, & x > 0. \end{cases}$$

Find $f'(x)$ for $x \neq 0$. Use the definition of the derivative, (2) of Section 3.1, to determine whether $f'(0)$ exists.

*Go to the website **http://mathworld.wolfram.com/Hypocycloid.html** to see various kinds of hypocycloids and their properties.