Preface

This student study guide is designed to accompany the text *A First Course in Complex Analysis with Applications, Second Edition* (Jones and Bartlett Publishers, 2009) by Dennis G. Zill and Patrick D. Shanahan. It consists of seven chapters which correspond to the seven chapters of the text. Each chapter has the following features.

Review Topics

Many sections of the study guide are preceded by a review of selected topics from calculus and differential equations that are required for that section. These reviews provide concise summaries of prerequisite notation, terminology, and concepts. For additional review, students are encouraged to consult appropriate mathematics texts. Two excellent sources that were used repeatedly for the review topics are *Calculus: Early Transcendentals, Fourth Edition* (Jones and Bartlett Publishers, 2010) by Dennis G. Zill and Warren S. Wright and *Advanced Engineering Mathematics, Third Edition* (Jones and Bartlett Publishers, 2006) by Dennis G. Zill and Michael R. Cullen.

Summaries

A summary of every section of the text is provided. The summary reviews all of the key ideas of the section including all terminology, formulas, theorems, and concepts. Figures with two colors are included to aid in geometric understanding.

Exercises

Following the summary, complete solutions are given for every other odd exercise in the section (eg. problems 3, 7, 11, etc.). These are full solutions, supported by figures with two colors, that supply all of the pertinent details of the problem and incorporate the same techniques and writing style used in the text. The solutions also include references to equations, definitions, theorems, and figures in the text. The answer to each problem is typeset in color for easy reference.

Focus on Concepts

The focus on concepts problems from the text consist of conceptual word, proof, and geometrical problems. Since they are often used as topics for classroom discussion or independent study we have included detailed hints rather than full solutions for these problems. As with the exercises, only every other odd problem is included.

Final Note to Students

The most effective way to learn mathematics is to work many, many problems. You should not review a solution in this study guide before first working or, at the very least, attempting to work the problem yourself. Learning advanced mathematical topics takes significant time and effort. It may be quicker to look at the solutions, then try to work problems, but ultimately this approach will not lead to an independent understanding of concepts and problem solving strategies that are required for success.

Chapter 2

Complex Functions and Mappings

2.1 Complex Functions

Review Topic: Functions

function: A function f from a set A to a set B is a rule that assigns to each element $a \in A$ exactly one element $b \in B$. The notation $f : A \to B$ is used to denote a function from A to B.

values: If the element $b \in B$ is assigned to the element $a \in A$ by a function $f : A \to B$, then we call b the image of a under f, or the value of f at a. We write $b = f(a)$ to indicate this assignment.

domain: If $f : A \to B$, then the set A is called the domain of f, written as $\text{Dom}(f)$.

range: If $f : A \to B$, then the set of all images in B is called the range of f, written as $\text{Range}(f)$. Using set notation the range is defined by

$$\text{Range}(f) = \{f(a) \bigm| a \in A\}.$$

one-to-one If $f : A \to B$, then f is called one-to-one if every element in the range of f corresponds to exactly one element in the domain of f. In other words, f is one-to-one if whenever $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$.

onto: If $f : A \to B$, then f is called onto if the range of f is equal to the entire set B. That is, f is onto if for every $b \in B$ there is some $a \in A$ so that $f(a) = b$.

2.1 Summary

complex function: A complex function, or a complex-valued function of a complex variable, is a function whose domain and range are subsets of the set $\mathbf{C}$. That is, if $f : A \to B$ is a complex function, then we must have $A \subseteq \mathbf{C}$ and $B \subseteq \mathbf{C}$.

$w = f(z)$: The notation $w = f(z)$ is used to represent a complex-valued function of a complex variable, while the notation $y = f(x)$ is used to represent a real-valued function of a real variable.

Re(f) and Im(f): Given a complex function $w = f(z)$, if we set $z = x + iy$ and express the function in terms of two real functions as:

$$f(z) = u(x, y) + iv(x, y),$$

then the functions $u(x, y)$ and $v(x, y)$ are called the real and imaginary parts of f, respectively. We use the notation $\text{Re}(f) = u(x, y)$ and $\text{Im}(f) = v(x, y)$.

complex exponential function: The complex exponential function is defined by:

$$e^z = e^x \cos y + ie^x \sin y.$$

polar coordinates: Given a complex function $w = f(z)$, we can replace the complex variable z with its polar form $z = r(\cos\theta + i \sin\theta)$ and express the function in terms of two real functions as:

$$f(z) = u(r, \theta) + iv(r, \theta).$$

We still call $\text{Re}(f) = u(r, \theta)$ and $\text{Im}(f) = v(r, \theta)$ the real and imaginary parts of f, but these real functions are different from those obtained using Cartesian form $x + iy$ of the complex variable z.

Exercises 2.1

3. **(a)** For $z = 1$ we have $|z| = 1$ and $\text{Arg}(z) = 0$. Therefore,

$$f(1) = \log_e 1 + i(0) = 0.$$

(b) For $z = 4i$ we have $|z| = 4$ and $\text{Arg}(z) = \frac{\pi}{2}$. Therefore,

$$f(4i) = \log_e 4 + i\frac{\pi}{2} \approx 1.38629 + 1.57080i.$$

(c) For $z = 1 + i$ we have $|z| = \sqrt{2}$ and $\text{Arg}(z) = \frac{\pi}{4}$. Therefore,

$$f(1 + i) = \log_e \sqrt{2} + i\frac{\pi}{4} = \frac{1}{2}\log_e 2 + i\frac{\pi}{4} \approx 0.34657 + 0.78540i.$$

7. **(a)** For $z = 3$ we have $r = |z| = 3$ and $\theta = \text{Arg}(z) = 0$. Therefore,

$$f(3) = 3 + i\cos^2 0 = 3 + i(1)^2 = 3 + i.$$

(b) For $z = -2i$ we have $r = |z| = 2$ and $\theta = \mathrm{Arg}(z) = -\frac{\pi}{2}$. Therefore,

$$f(-2i) = 2 + i\cos^2\left(-\frac{\pi}{2}\right) = 2 + i(0)^2 = 2.$$

(c) For $z = 2 - i$ we have $r = |z| = \sqrt{5}$. Moreover, from equating the Cartesian and polar forms of the point $z = 2 - i = \sqrt{5}(\cos\theta + i\sin\theta)$ we obtain $\cos\theta = 2/\sqrt{5}$. Therefore,

$$f(2-i) = \sqrt{5} + i\left(\frac{2}{\sqrt{5}}\right)^2 = \sqrt{5} + \frac{4}{5}i \approx 2.23607 + 0.8i.$$

11. If we set $z = x + iy$, then

$$\begin{aligned} f(z) &= (x+iy)^3 - 2(x+iy) + 6 \\ &= x^3 + 3x^2yi + 3xy^2i^2 + y^3i^3 - 2x - 2yi + 6 \\ &= x^3 + 3x^2yi - 3xy^2 - y^3i - 2x - 2yi + 6 \\ &= (x^3 - 3xy^2 - 2x + 6) + (3x^2y - y^3 - 2y)i. \end{aligned}$$

Therefore, $\mathrm{Re}(f) = x^3 - 3xy^2 - 2x + 6$ and $\mathrm{Im}(f) = 3x^2y - y^3 - 2y$.

15. If we set $z = x + iy$, then $2(x+iy) + i = 2x + (2y+1)i$. So, from (3) of Section 2.1 we have:

$$\begin{aligned} e^{2z+i} &= e^{2x+(2y+1)i} \\ &= e^{2x}\cos(2y+1) + ie^{2x}\sin(2y+1). \end{aligned}$$

Therefore, $\mathrm{Re}(f) = e^{2x}\cos(2y+1)$ and $\mathrm{Im}(f) = e^{2x}\sin(2y+1)$.

19. If we set $z = r(\cos\theta + i\sin\theta)$, then

$$\begin{aligned} f(z) &= \left[r(\cos\theta + i\sin\theta)\right]^4 \\ &= r^4(\cos 4\theta + i\sin 4\theta) \qquad \leftarrow \text{see (9) in Section 1.3} \\ &= r^4\cos 4\theta + ir^4\sin 4\theta. \end{aligned}$$

Therefore, $\mathrm{Re}(f) = r^4\cos 4\theta$ and $\mathrm{Im}(f) = r^4\sin 4\theta$.

23. Since $\mathrm{Re}(z)$ and z^2 are defined for all complex numbers, the natural domain of $f(z) = 2\mathrm{Re}(z) - iz^2$ is the set **C** of all complex numbers.

Focus on Concepts

27. **(a)** Does $\arg(z)$ assign one and only one value to z?

(d) See part (a).

31. In order to determine the natural domain consider the questions: For which values of x and y is $\cos(x - y)$ defined? How about $\sin(x - y)$? In order to determine the range, consider $|f(z)|$.

35. Verify that $|e^{-z}| = e^{-x}$, then use this identity to answer the question.

2.2 Complex Functions as Mappings

Review Topic: Parametric Curves

parametric curve: If $f(t)$ and $g(t)$ are real-valued functions of a real variable t, then the set C of all points $(f(t), g(t))$, where $a \leq t \leq b$, is called a parametric curve. The equations $x = f(t)$ and $y = g(t)$ are called parametric equations for C, and the variable t is called the parameter.

eliminating the parameter: Given a set of parametric equations $x = f(t)$, $y = g(t)$ it is sometimes possible to eliminate the parameter t to obtain a single Cartesian equation of the curve. For example, if the parametric equations $x = t^2 - t + 2$, $y = -t^2 + t$ for a curve C are added together we obtain the single equation $x + y = 2$ (so, the curve C defined by these equations is a line).

smooth curves: A parametric curve given by $x = f(t)$, $y = g(t)$, $a \leq t \leq b$ is called smooth if f' and g' are continuous on $[a, b]$ and not simultaneously zero on (a, b). A piecewise smooth curve consists of a finite number of smooth curves joined end to end.

tangent lines: If C is a parametric curve given by $x = x(t)$, $y = y(t)$, $a \leq t \leq b$ and if both $f(t)$ and $g(t)$ are differentiable on (a, b), then the slope of the tangent line to C at $(f(t), g(t))$ is given by

$$\frac{dy}{dx} = \frac{g'(t)}{f'(t)}$$

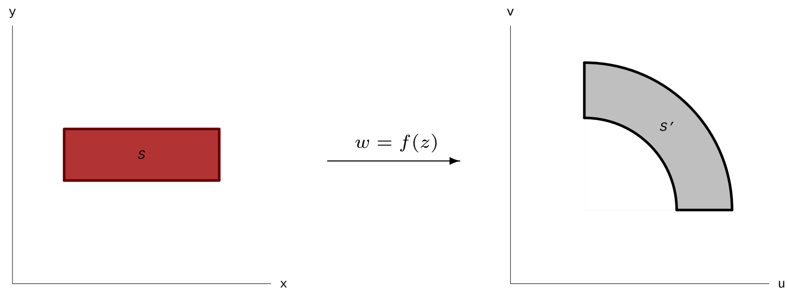
provided $f'(t) \neq 0$.

2.2 Summary

complex mapping: The term complex mapping is used to refer to the correspondence determined by a complex function $w = f(z)$ between points in the z-plane and their images in the w-plane.

image of S under f: If S is a set and $f(z)$ is a complex function, then the image of S under f is the set S' consisting of the images under the mapping $w = f(z)$ of all of the points in S.

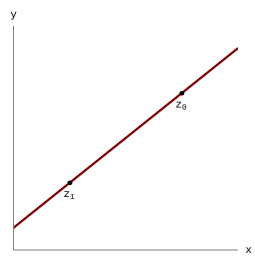
geometric representation: A geometric representation of a complex mapping consists of a set S shown in one copy of the complex plane and the image S' of S under f shown in a second copy of the complex plane. See below.



complex parametric curves: If $x(t)$ and $y(t)$ are real-valued functions of a real parameter t, then the set C consisting of all points $z(t) = x(t) + iy(t)$, $a \le t \le b$ is called a complex parametric curve. The complex-valued function of the real variable t, $z(t) = x(t) + iy(t)$ is called a parametrization of C.

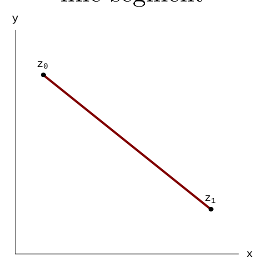
commonly used parametrizations: The following parametrizations are commonly used to help describe complex mappings.

line



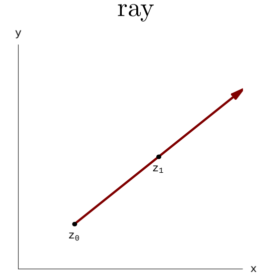
$z(t) = z_0(1-t) + z_1 t$,
$-\infty < t < \infty$

line segment



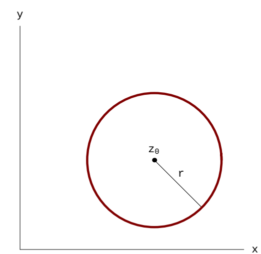
$z(t) = z_0(1-t) + z_1 t$,
$0 \le t \le 1$

ray



$z(t) = z_0(1-t) + z_1 t$,
$0 \le t < \infty$

circle



$z(t) = z_0 + re^{it}$,
$0 \le t \le 2\pi$

image of a parametric curve: If $w = f(z)$ is a complex mapping and if C is a curve parametrized by $z(t)$, $a \le t \le b$, then $w(t) = f(z(t))$, $a \le t \le b$ is a parametrization of the image, C', of C under $w = f(z)$.

Exercises 2.2

3. The half-plane S can be described by the two simultaneous inequalities

$$-\infty < x < \infty \quad \text{and} \quad y > 2. \tag{1}$$

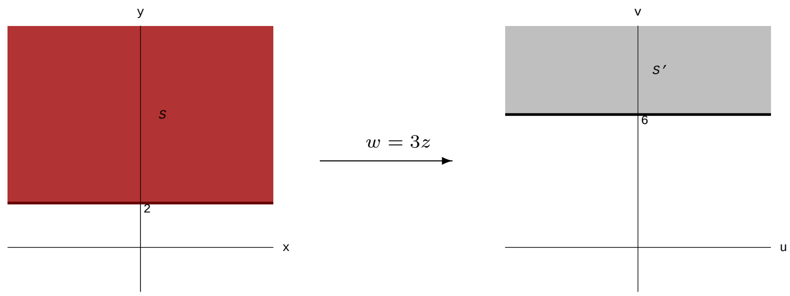
After replacing z by $x + iy$ in the mapping $w = 3z$, we obtain $w = 3(x + iy) = 3x + 3yi$, and so the real and imaginary parts of the mapping $w = 3z$ are

$$u = 3x \quad \text{and} \quad v = 3y. \tag{2}$$

The image S' of S under $w = 3z$ is found by using the equations in (2) to transform the bounds on x and y in (1) into bounds on u and v. If we substitute $x = u/3$ and $y = v/3$ in (1) and simplify, then we find that S' is given by

$$-\infty < u < \infty \quad \text{and} \quad v > 6.$$

Therefore, S' is the half-plane $\text{Im}(w) > 6$. The mapping is depicted below.



7. The half-plane S can be described by the two simultaneous inequalities

$$-\infty < x < \infty \quad \text{and} \quad y \leq 1. \tag{3}$$

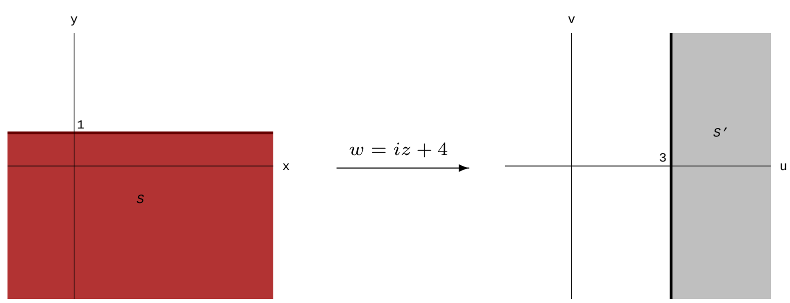
After replacing z by $x + iy$ in the mapping $w = iz + 4$, we obtain $w = i(x + iy) + 4 = (4 - y) + ix$, and so the real and imaginary parts of the mapping $w = iz + 4$ are

$$u = 4 - y \quad \text{and} \quad v = x. \tag{4}$$

The image S' of S under $w = iz + 4$ is found by using the equations in (4) to transform the bounds on x and y in (3) into bounds on u and v. If we substitute $x = v$ and $y = 4 - u$ in (3) and simplify, then we find that S' is given by

$$u \geq 3 \quad \text{and} \quad -\infty < v < \infty.$$

Therefore, S' is the half-plane $\text{Re}(w) \geq 3$. The mapping is depicted below.



11. The line $x = 0$ can be described by

$$x = 0 \quad \text{and} \quad -\infty < y < \infty. \tag{5}$$

From (1) of Section 2.1, the real and imaginary parts of $w = z^2$ are

$$u = x^2 - y^2 \quad \text{and} \quad v = 2xy. \tag{6}$$

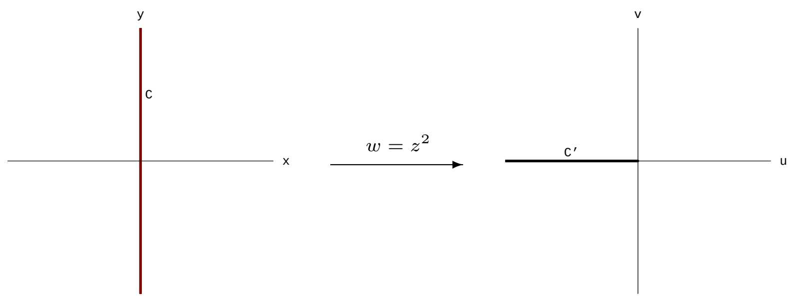
For any point $z = 0 + iy$ on the line $x = 0$, the equations describing the image in (6) become

$$u = -y^2 \quad \text{and} \quad v = 0. \tag{7}$$

Since $-\infty < y < \infty$ from (5), we have $0 \le y^2 < \infty$, and consequently, $-\infty < u \le 0$ from the first equation in (7). Therefore, the image C' of the line $x = 0$ under $w = z^2$ is the ray

$$-\infty < u \le 0 \quad \text{and} \quad v = 0.$$

The mapping is depicted below.



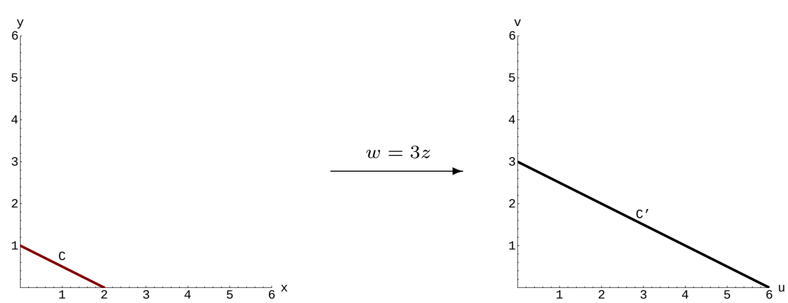
15. (a) With the identifications $z_0 = 2$ and $z_1 = i$ in parametrization (7) of Section 2.2, we see that the parametric curve C is the line segment from 2 to i.

(b) Using $z(t) = 2(1-t) + it$ and $f(z) = 3z$, the image of the parametric curve C is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= f(z(t)) \\ &= 3\,(2(1-t) + it) \\ &= 6(1-t) + 3it. \end{aligned}$$

Therefore, a parametrization of the image C' is $w(t) = 6(1-t) + 3it,\ 0 \le t \le 1$.

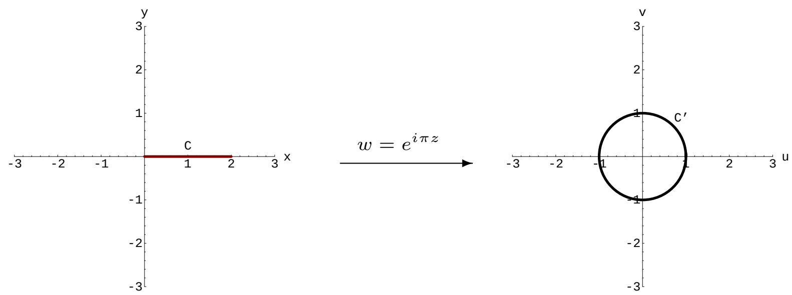
(c) With the identifications $z_0 = 6$ and $z_1 = 3i$ in parametrization (7) of Section 2.2, we see that the parametric curve C' is the line segment from 6 to $3i$. Plots for parts (a) and (c) are shown below.



19. (a) Replacing the parameter t in $z(t)$ with a new parameter $2s$ we obtain a different parametrization of C, $Z(s) = 2s,\ 0 \le s \le 1$. Thus, with the identifications $z_0 = 0$ and $z_1 = 2$ in parametrization (7) of Section 2.2, we see that the parametric curve C is the line segment from 0 to 2.

(b) Using $z(t) = t$ and $f(z) = e^{i\pi z}$, the image of the parametric curve C is given by (11) of Section 2.2, $w(t) = f(z(t) = e^{i\pi t},\ 0 \leq t \leq 2$. Therefore, a parametrization of the image C' is $w(t) = e^{i\pi t},\ 0 \leq t \leq 2$.

(c) Replacing the parameter t in $w(t)$ with a new parameter s/π we obtain a different parametrization of C', $W(s) = e^{is},\ 0 \leq s \leq 2\pi$. Now with the identifications $z_0 = 0$ and $r = 1$ in parametrization (10) of Section 2.2, we see that the parametric curve C' is the unit circle. Plots for parts (a) and (c) are shown below.



23. The circle $|z| = 2$ has center $z_0 = 0$ and radius $r = 2$. Thus, from (10) of Section 2.2, a parametrization of C is $z(t) = 2e^{it},\ 0 \leq t \leq 2\pi$. The image of C under $f(z) = 1/z$ is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= f(z(t)) \\ &= \frac{1}{2e^{it}} \\ &= \frac{1}{2}e^{-it}. \end{aligned}$$

Replacing the parameter t in $w(t)$ with a new parameter $-s$ we obtain a different parametrization of C', $W(s) = \frac{1}{2}e^{is},\ -2\pi \leq s \leq 0$. Now with the identifications $z_0 = 0$ and $r = \frac{1}{2}$ in parametrization (10) of Section 2.2, we see that C' is a circle centeblue at 0 with radius 1/2. That is, C' is the circle $|w| = \frac{1}{2}$.

Focus on Concepts

27. **(a)** Set $z = 1 + iy$ and then write $1/z$ in the form $u + iv$.

(b) Substitute the expressions for u and v from part (a) into the equation $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$ and simplify to confirm that the equation is an identity.

(c) Use the general equation of a circle.

(d) The point $u + iv = 0 + i0$ does satisfy the equation $(u - \frac{1}{2})^2 + v^2 = \frac{1}{4}$. But can this point be given by an appropriate choice of y in the expressions for u and v from part (a)?

31. Consider the geometric relationship between a point $x + iy$ and its conjugate $x - iy$.

2.3 Linear Mappings

Review Topic: Composition

composition: Let f and g be functions. The composition of f and g is the function

$$(f \circ g)(z) = f(g(z)).$$

Notice that in order for $f \circ g$ to be defined at z, the value of $g(z)$ must be in the domain of f. In other words, the domain of $f \circ g$ is the subset of those z in the domain of g for which $g(z)$ is in the domain of f.

order: In general $(f \circ g)(z) \neq (g \circ f)(z)$. For example, if $f(z) = z^2$ and $g(z) = z + i$, then $(f \circ g)(z) = (z + i)^2 = z^2 + 2iz - 1$ but $(g \circ f)(z) = z^2 + i$.

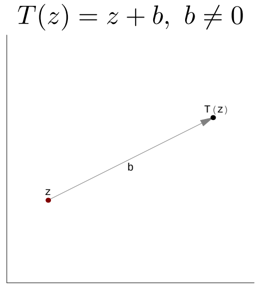
2.3 Summary

complex linear function: A complex linear function is a function of the form $f(z) = az + b$ where a and b are any complex constants.

translation/rotation/magnification: There are three special types of linear functions: a translation, a rotation, and a magnification. These functions are described below.

translation

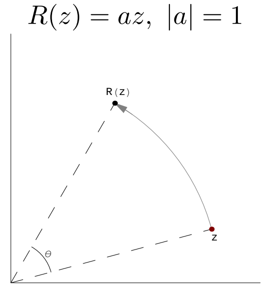
$T(z) = z + b,\ b \neq 0$



translates the point z along the vector b

rotation

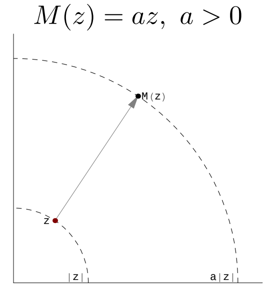
$R(z) = az,\ |a| = 1$



rotates the point z through an angle of $\theta = \text{Arg}(a)$ radians about the origin

magnification

$M(z) = az,\ a > 0$



magnifies the modulus of the point z by a factor of a

linear mapping as a composition: If $f(z) = az + b$ is a complex linear mapping and $a \neq 0$, then we can express f as:

$$f(z) = |a| \left(\frac{a}{|a|} z \right) + b.$$

This means that $f(z)$ is the composition $f(z) = (T \circ M \circ R)(z)$ where $R(z) = \frac{a}{|a|} z$ is rotation by $\text{Arg}(a)$, $M(z) = |a|z$ is magnification by $|a|$, and $T(z) = z + b$ is translation by b.

image of a point under a linear mapping: If $f(z) = az + b$ is a complex linear mapping with $a \neq 0$, and if $w_0 = f(z_0)$ is plotted in the same copy of the complex plane as z_0, then w_0 is the point obtained by

(i) rotating z_0 through angle $\text{Arg}(a)$ about the origin,

(ii) magnifying the result by $|a|$, and

(iii) translating the result by b.

geometry of linear mappings: Because a non-constant linear mapping is a composition of a rotation, a magnification, and a translation, the image of a geometric figure can have a different size, but its basic shape is the same. For example, the image of a circle is a circle, the image of a 5-sided polygon is a 5-sided polygon, etc.

Exercises 2.3

3. (a) Using (6) of Section 2.3, we express the linear mapping $f(z) = 3iz$ as

$$f(z) = |3i| \left(\frac{3i}{|3i|} z \right) = 3(iz) + 0.$$

Thus, the image of the closed disk $|z| \leq 1$ is found by

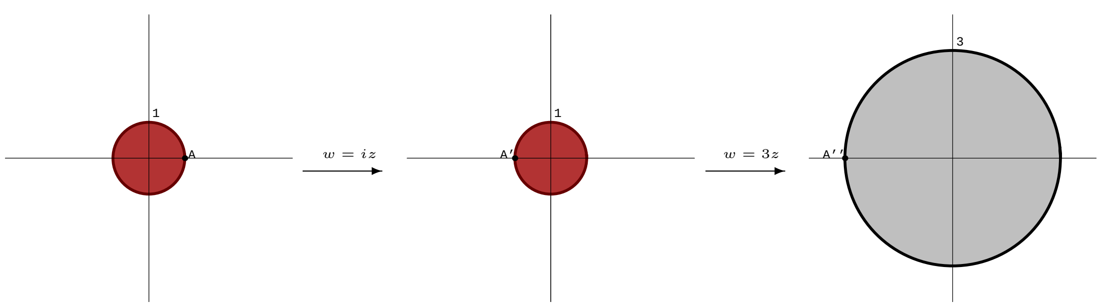
(i) rotating the disk by $\text{Arg}(i) = \pi$,

(ii) magnifying the result by $|3i| = 3$, and

(iii) translating the result by 0 (no translation).

Under the rotation (i), the disk is mapped onto itself and under the magnification (ii) the result is mapped onto a disk centered at the origin with radius 3. Therefore, the image is the closed disk $|w| \leq 3$.

(b)



7. (a) Using (6) of Section 2.3, we express the linear mapping $f(z) = z + 2i$ as

$$f(z) = |1| \left(\frac{1}{|1|} z \right) + 2i = 1(1z) + 2i.$$

Thus, the image of the triangle is found by

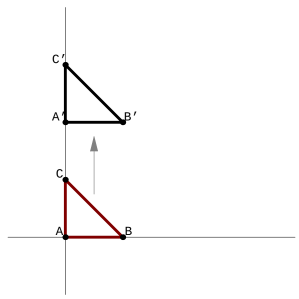
(i) rotating the disk by $\text{Arg}(1) = 0$ (no rotation),

(ii) magnifying the result by $|1| = 1$ (no magnification), and

(iii) translating the result by $2i$.

Under the translation (iii), the vertices 0, 1, and i are mapped to $2i$, $1 + 2i$, and $3i$, respectively. Therefore, the image is the triangle with vertices $2i$, $1 + 2i$, and $3i$.

(b)



11. **(a)** Using (6) of Section 2.3, we express the linear mapping $f(z) = -3z + i$ as

$$f(z) = \big| -3\big| \left(\frac{-3}{|-3|} z \right) + i = 3\big(-1z\big) + i.$$

Thus, the image of the triangle is found by

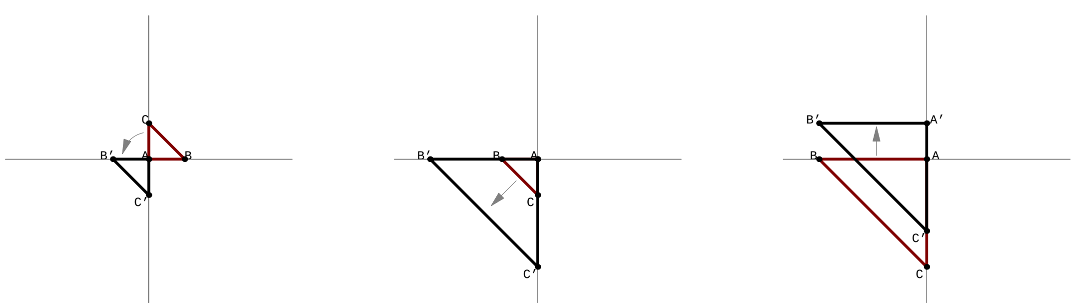
(i) rotating the disk by $\text{Arg}(-1) = \pi$,

(ii) magnifying the result by $|-3| = 3$, and

(iii) translating the result by i.

Under the rotation (i), the vertices 0, 1, and i are mapped to 0, -1, and $-i$, respectively. Under the magnification (ii), these vertices are then mapped to 0, -3, and $-3i$. Finally, these vertices are mapped to i, $-3 + i$, and $-2i$ under the translation (iii). Therefore, the image is the triangle with vertices i, $-3 + i$, and $-2i$.

(b)



rotate by π magnify by 3 translate by i

15. Using (6) of Section 2.3, we express the linear mapping $f(z) = -\frac{1}{2}z + 1 - \sqrt{3}i$ as

$$f(z) = \left|-\frac{1}{2}\right|\left(\frac{-1/2}{|-1/2|}z\right) + 1 - \sqrt{3}i = \frac{1}{2}\big(-1z\big) + 1 - \sqrt{3}i.$$

Thus, the image of the triangle is found by

(i) rotating the disk by $\text{Arg}(-1) = \pi$,

(ii) magnifying the result by $\left|-\frac{1}{2}\right| = \frac{1}{2}$, and

(iii) translating the result by $1 - \sqrt{3}i$.

19. There are many ways to map the imaginary axis onto the line containing i and $1 + 2i$. One approach consists of first rotating the imaginary axis onto the line $y = x$, then translating this line 1 unit in the y-direction. In other words, one solution to the problem is a linear mapping that consists of a rotation clockwise through $\pi/4$ radians followed by a translation by i. This gives $f(z) = e^{-\pi i/4}z + i$.

23. **(a)** Using $z(t) = z_0(1-t) + z_1 t$ and $T(z) = z + b$, the image of the line segment is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= T(z(t)) \\ &= z_0(1-t) + z_1 t + b \\ &= z_0(1-t) + z_1 t + b(1-t) + bt \\ &= (z_0 + b)(1-t) + (z_1 + b)t. \end{aligned}$$

Therefore, a parametrization of the image is $w(t) = (z_0 + b)(1-t) + (z_1 + b)t,\ 0 \le t \le 1$. From parametrization (7) of Section 2.2, we see that the image is a line segment from $z_0 + b$ to $z_1 + b$.

(b) Using $z(t) = z_0(1-t) + z_1 t$ and $R(z) = az,\ |a| = 1$, the image of the line segment is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= R(z(t)) \\ &= a\left(z_0(1-t) + z_1 t\right) \\ &= (az_0)(1-t) + (az_1)t. \end{aligned}$$

Therefore, a parametrization of the image is $w(t) = (az_0)(1-t) + (az_1)t,\ 0 \le t \le 1$. From parametrization (7) of Section 2.2, we see that the image is a line segment from az_0 to az_1. Since $|a| = 1$, the endpoints az_0 and az_1 of the image line segment are obtained by rotating the points z_0 and z_1, respectively, though an angle of $\text{Arg}(a)$ about the origin.

(c) Using $z(t) = z_0(1-t) + z_1 t$ and $M(z) = az,\ a > 0$, the image of the line segment is given by (11) of Section 2.2:

$$\begin{aligned} w(t) &= M(z(t)) \\ &= a\left(z_0(1-t) + z_1 t\right) \\ &= (az_0)(1-t) + (az_1)t. \end{aligned}$$

Therefore, a parametrization of the image is $w(t) = (az_0)(1-t) + (az_1)t,\ 0 \le t \le 1$. From parametrization (7) of Section 2.2, we see that the image is a line segment from az_0 to az_1. Since $a > 0$, the endpoints az_0 and az_1 of the image line segment are obtained by magnifying the points z_0 and z_1, respectively, by a factor of a.

Focus on Concepts

27. **(a)** Given translations $T_1(z) = z + b_1$ and $T_2(z) = z + b_2$, their composition is

$$T_1 \circ T_2(z) = T_1(z + b_2) = (z + b_2) + b_1 = z + (b_1 + b_2).$$

If we set $b = b_1 + b_2$, then the composition can be written as $T_1 \circ T_2(z) = z + b$. Therefore, $T_1 \circ T_2$ is a translation if $b_1 + b_2 \neq 0$ and the identity if $b_1 + b_2 = 0$. Since $T_2 \circ T_1(z) = z + (b_1 + b_2)$, the order of composition does not matter.

(b) Modify the argument in part (a).

(c) Modify the argument in part (a).

31. Let $f(z) = az + b$ for some complex constants a and b. Consider the identity $|z| = |az + b|$ for the values $z = 0$ and $z = 1$. What do the two resulting equations tell you about the coefficients a and b, and, consequently, the linear mapping $w = f(z)$?

35. **(a)** Solve the system of equations:

$$\begin{aligned} az_1 + b &= w_1 \\ az_2 + b &= w_2 \end{aligned}$$

for a and b.

(b) Consider f_1 a rotation and f_2 a magnification. What is the image of $z = 0$?

39. **(a)** Rotate, magnify, and translate the annulus in order to determine its image, then determine the least and greatest distance a point in the image can be from the origin.

(b) Determine the points in the original annulus that map onto the points that are closest and farthest from the origin identified in part (a).

(c) Notice that the closer a point $f(z)$ is to the origin, the farther $1/f(z)$ is from the origin and the farther $f(z)$ is from the origin, the closer $1/f(z)$ is to the origin.

2.4 Special Power Functions

Review Topic: Inverse Functions

one-to-one functions: A function f is one-to-one if whenever $f(a_1) = f(a_2)$, then $a_1 = a_2$. In other words, f is one-to-one if whenever $a_1 \neq a_2$, then $f(a_1) \neq f(a_2)$. Put yet another way, f is one-to-one if every element b in the range of f corresponds to exactly one element a in the domain of f.

inverse function: If f is a one-to-one function with domain A and range B, then the inverse function f^{-1} is the function with domain B and range A defined by $f^{-1}(b) = a$ if $f(a) = b$.

composition properties of inverse functions: If f is a one-to-one function with domain A and range B and if f^{-1} is its inverse, then

$$\begin{aligned} \left(f \circ f^{-1}\right)(b) &= f\left(f^{-1}(b)\right) = b \qquad \text{for all } b \text{ in } B, \text{ and} \\ \left(f^{-1} \circ f\right)(a) &= f^{-1}\left(f(a)\right) = a \qquad \text{for all } a \text{ in } A. \end{aligned}$$

inverse mappings: If a one-to-one function f maps a set S onto a set T, then the inverse function f^{-1} maps the set T onto the set S.

finding inverses algebraically: In order to find the inverse of a one-to-one function f algebraically we solve the equation $f(a) = b$ for the independent variable a, then relabel the independent and dependent variables appropriately. For example, the complex function $f(z) = 5z - 2i$ is one-to-one. If we solve the equation $w = 5z - 2i$ for the independent variable z we obtain $z = (w + 2i)/5$. Relabeling the independent and dependent variables gives the inverse function $f^{-1}(z) = (z + 2i)/5$.

restricting domains: If f is a function that is not one-to-one, then it does not have an inverse function. However, it may be possible to restrict the domain of f to a subset on which f is one-to-one and, consequently, use the restricted domain to determine an inverse function. For example, the real function $f(x) = x^2$ is not one-to-one on its natural domain $(-\infty, \infty)$. The function $f(x) = x^2$ is one-to-one, however, on the restricted domain $[0, \infty)$. On this restricted domain we have the inverse function $f^{-1}(x) = \sqrt{x}$.

2.4 Summary

complex polynomial function: A complex polynomial function is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_1 z + a_0$$

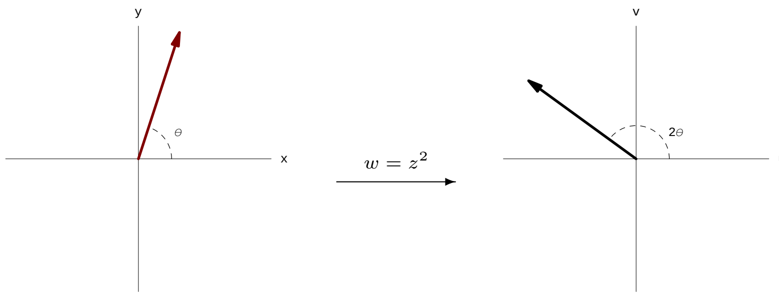
where n is a positive integer and a_n, a_{n-1}, ... a_1, and a_0 are complex constants. If $a_n \neq 0$, then n is called the degree of the polynomial $p(z)$.

complex power functions: A complex power function is a function of the form $f(z) = z^\alpha$ where α is a complex constant.

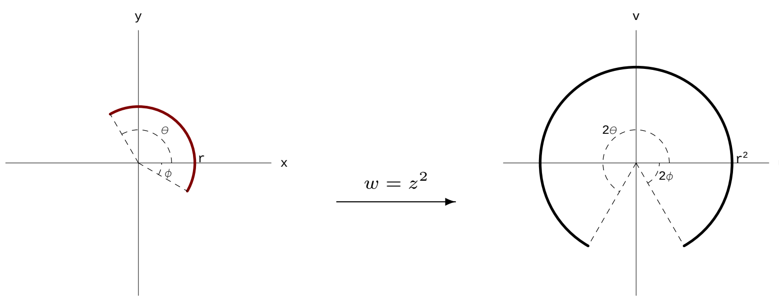
complex squaring function $f(z) = z^2$: The complex squaring function $f(z) = z^2$ can be expressed in terms of it real and imaginary parts using either a Cartesian, polar, or exponential form of the variable z. Each is given below:

$$f(z) = z^2 = \overbrace{(x^2 - y^2) + 2xyi}^{\text{Cartesian}} = \overbrace{r^2(\cos 2\theta + i \sin 2\theta)}^{\text{polar}} = \overbrace{r^2 e^{i2\theta}}^{\text{exponential}}$$

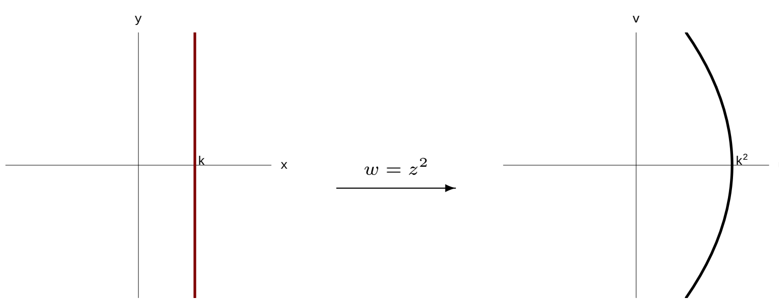
the mapping $w = z^2$: As a mapping, $w = z^2$ squares the modulus of a point and doubles its argument. The following summarizes some properties of the mapping $w = z^2$



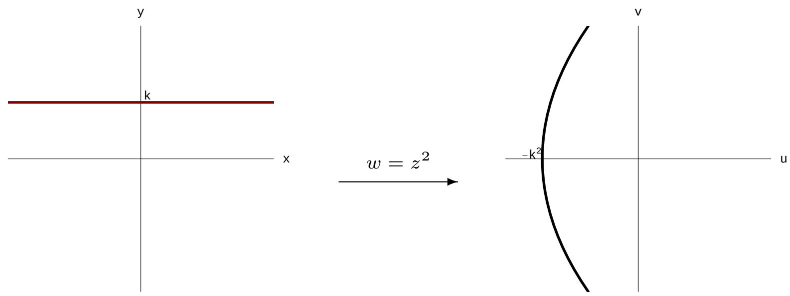
ray $\arg(z) = \theta$ ray $\arg(w) = 2\theta$



circular segment $|z| = r$, $\phi \le \arg(z) \le \theta$ circular segment $|w| = r^2$, $2\phi \le \arg(w) \le 2\theta$

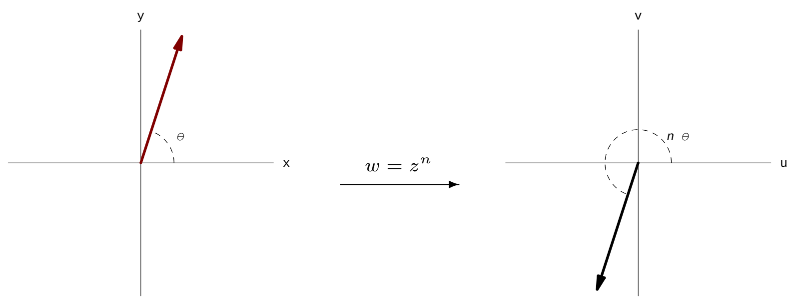


vertical line $\mathrm{Re}(z) = k$ parabola $u = k^2 - v^2/(4k^2)$

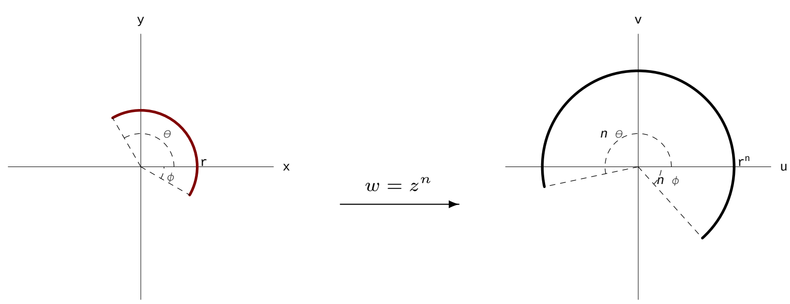


horizontal line $\text{Im}(z) = k$ parabola $u = v^2/(4k^2) - k^2$

the mapping $w = z^n$: As a mapping, $w = z^n = r^n e^{in\theta}$ raises the modulus of a point to the nth power and increases its argument by a factor of n. The following summarizes some properties of the mapping $w = z^n$



ray $\arg(z) = \theta$ ray $\arg(w) = n\theta$



circular segment $|z| = r$, $\phi \le \arg(z) \le \theta$ circular segment $|w| = r^n$, $n\phi \le \arg(w) \le n\theta$

principal square root function $z^{1/2}$: The principal square root function is defined by

$$\begin{aligned} z^{1/2} &= \sqrt{|z|}\, e^{i\text{Arg}(z)/2} \\ &= \sqrt{r} e^{i\theta/2}, \qquad \theta = \text{Arg}(z). \end{aligned}$$

If the domain of $f(z) = z^2$ is restricted to the set $-\pi/2 < \arg(z) \leq \pi/2$, then f is one-to-one and the principal square root function is the inverse of the function $f(z) = z^2$ on this restricted domain.

principal nth root function $z^{1/n}$: For integers $n \geq 2$, the principal nth root function is defined by

$$\begin{aligned} z^{1/n} &= \sqrt[n]{|z|}\, e^{i\operatorname{Arg}(z)/n} \\ &= \sqrt[n]{r} e^{i\theta/n}, \qquad \theta = \operatorname{Arg}(z). \end{aligned}$$

If the domain of $f(z) = z^n$ is restricted to the set $-\pi/n < \arg(z) \leq \pi/n$, then f is one-to-one and the principal square root function is the inverse of the function $f(z) = z^n$ on this restricted domain.

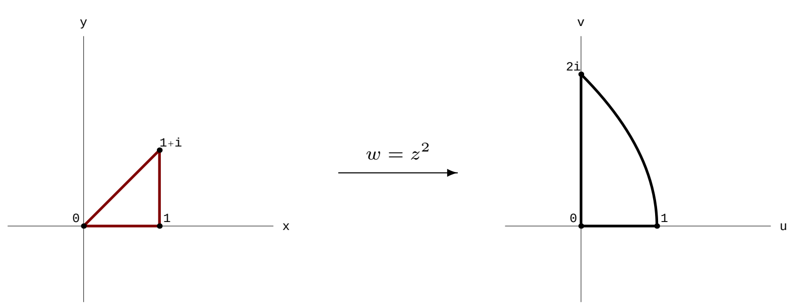
multiple-valued functions: A rule F that assigns a set of one or more complex numbers to each complex number in a subset of $\mathbf{C}$ is called a multiple-valued function. For example, $G(z) = \arg(z)$ is a multiple-valued function that assigns an infinite set of numbers (all differing by a multiple of 2π) to a nonzero complex number z. Multiple-valued functions are not functions (as defined in Section 2.1).

Exercises 2.4.1

3. By identifying $k = 3$ in equation (3) of Section 2.4.1 we see that the image of the line $x = 3$ under the mapping $w = z^2$ is the parabola $u = 9 - v^2/36$.

7. Notice that the positive imaginary axis is the ray $\arg(z) = \pi/2$. From the 2.4 Summary, the image of this ray is a ray making an angle of $2(\pi/2) = \pi$ radians with the positive real axis. Therefore, the image is the ray $\arg(w) = \pi$, the negative real aixs.

11. The triangle with vertices 0, 1, and $1+i$ consists of three line segments. We treat each of these segments separately. The first segment from 0 to 1 lies in the ray $\arg(z) = 0$. So, by the discussion on pages 73-74 of Section 2.4, its image under $w = z^2$ lies in the ray $\arg(w) = 2 \cdot 0 = 0$. Since the endpoints $z = 0$ and $z = 1$ map to $w = 0$ and $w = 1$, respectively, the image is the line segment from 0 to 1. Similarly, the line segment from 0 to $1 + i$ lies in the ray $\arg(z) = \frac{\pi}{4}$ and so its image lies in the ray $\arg(w) = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}$. Since the endpoints $z = 0$ and $z = 1 + i$ map to $w = 0$ and $w = 2i$, respectively, the image is the line segment from 0 to $2i$. The last line segment from 1 to $1 + i$ lies in the vertical line $x = 1$, not a ray emanating from the origin, thus its image will lie in a parabola. Identifying $k = 1$ in equation (3) of Section 2.4.1 we see that the image will lie in the parabola $u = 1 - v^2/4$. Since the endpoints $z = 1$ and $z = 1 + i$ map to $w = 1$ ($v = 0$) and $w = 2i$ ($v = 2$), respectively, the image of this last line segment is the parabolic arc $u = 1 - v^2/4$, $0 \leq v \leq 2$. Therefore, the image consists of the line segment from 0 to 1, line segment from 0 to $2i$, and parabolic arc $u = 1 - v^2/4$, $0 \leq v \leq 2$. The mapping is depicted in the figure below.



15. The function $f(z) = 2z^2 + 1 - i$ can be expressed as a composition $f(z) = (h \circ g)(z)$ of the linear function $h(z) = 2z + 1 - i$ and the squaring function $g(z) = z^2$. Under the squaring function, the ray $\arg(z) = \pi/3$ is mapped onto the ray $\arg(w) = 2\pi/3$. Next we consider the action of the linear function on this image. By (6) of Section 2.3, the linear mapping $h(z) = 2z + 1 - i$ consists of a rotation by $\text{Arg}(2/|2|) = 0$ (no rotation), followed by a magnification by $|2| = 2$, and a translation by $1 - i$. The ray $\arg(w) = 2\pi/3$ is mapped onto itself by the magnification. After the translation by $1 - i$, the image is a ray emanating from $1 - i$ making an angle of $2\pi/3$ with the line $y = 1$. Since 0 is not in the set $\arg(z) = \pi/3$, the point $f(0) = 1 - i$ is not included in the image. Therefore, the image can be described as the ray emanating from $1-i$ and containing $(\sqrt{3}-1)i$ excluding the point $1-i$.

19. The function $f(z) = \frac{1}{4}e^{i\pi/4}z^2$ can be expressed as a composition $f(z) = (h \circ g)(z)$ of the linear function $h(z) = \frac{1}{4}e^{i\pi/4}z$ and the squaring function $g(z) = z^2$. Because the squaring function squares moduli and doubles arguments, the circular arc $|z| = 2$, $0 \le \arg(z) \le \pi/2$ is mapped onto the circular arc $|w| = 4$, $0 \le \arg(w) \le \pi$. Next we consider the action of the linear function on this image. By (6) of Section 2.3, the linear mapping $h(z) = \frac{1}{4}e^{i\pi/4}z$ consists of a rotation by $\pi/4$, followed by a magnification by $1/4$, and a translation by 0 (no translation). Under the rotation, the circular arc $|w| = 4$, $0 \le \arg(w) \le \pi$ maps onto the circular arc $|w| = 4$, $\pi/4 \le \arg(w) \le 5\pi/4$. Finally, under the magnification, this image is mapped onto the circular arc $|w| = 1$, $\pi/4 \le \arg(w) \le 5\pi/4$. Therefore, the image is the circular arc $|w| = 1$, $\pi/4 \le \arg(w) \le 5\pi/4$.

23. **(a)** Since the function $f(z) = z^2$ squares the modulus and doubles the argument, the image of the region $1 \le |z| \le 2$, $\pi/4 \le \arg(z) \le 3\pi/4$ is the region $1 \le |w| \le 4$, $\pi/2 \le \arg(w) \le 3\pi/2$.

(b) Since the function $f(z) = z^3$ cubes the modulus and triples the argument, the image of the region $1 \le |z| \le 2$, $\pi/4 \le \arg(z) \le 3\pi/4$ is the region $1 \le |w| \le 8$, $3\pi/4 \le \arg(w) \le 9\pi/4$.

(c) Since the function $f(z) = z^4$ raises the modulus to the 4th power and quadruples the argument, the image of the region $1 \le |z| \le 2$, $\pi/4 \le \arg(z) \le 3\pi/4$ is the region $1 \le |w| \le 16$, $\pi \le \arg(w) \le 3\pi$. That is, the image is the annulus $1 \le |w| \le 16$.

Exercises 2.4.2

27. From (14) of Section 2.4.2, since $|-1| = 1$ and $\text{Arg}(-1) = \pi$, we have that

$$(-1)^{1/3} = \sqrt[3]{1}e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

31. Since the principal square root function takes the square root of the modulus and halves the argument, the image of the ray $\arg(z) = \pi/4$ is the ray $\arg(w) = \pi/8$.

35. Since the principal square root function takes the square root of the modulus and halves the argument, the image of the circular arc $|z| = 9$, $-\pi/2 \le \arg(z) \le \pi$ is the circular arc $|w| = 3$, $-\pi/4 \le \arg(w) \le \pi/2$.

39. We consider the image of the boundary of the region first. The boundary of the region lies in the union of the ray $\arg(z) = \pi/2$ and the parabola $x = 4 - \frac{1}{16}y^2$. Since the principal square root function takes the square root of the modulus and halves the argument, the image of the ray $\arg(z) = \pi/2$ is the ray $\arg(w) = \pi/4$. In order to determine the image of the parabola we use the fact that $f(z) = z^{1/2}$ is the inverse function of $f(z) = z^2$ defined on the restricted domain $-\pi/2 < \arg(z) \leq \pi/2$. By (3) of Section 2.4.1 with the identification $k^2 = 4$, the image of the lines $x = \pm 2$ map onto the parabola $u = 4 - \frac{1}{16}v^2$ under $w = z^2$. Only the line $x = 2$ is in the restricted domain of $f(z) = z^2$, and so we conclude that the image of the parabola $x = 4 - \frac{1}{16}y^2$ under $w = z^{1/2}$ is the vertical line $u = 2$. Thus, the image is bounded by the ray $\arg(w) = \pi/4$ (or, equivalently, the line $u = v$) and the line $u = 2$. Since the point $-7 + 24i$ is in the region and $(-7 + 24i)^{1/2} = 3 + 4i$, the image of the region must be the set bounded by the lines $u = v$ and $u = 2$ containing the point $3 + 4i$.

Focus on Concepts

43. Use the fact that $\arg(w) = \pi/2$ and $\arg(w) = -3\pi/2$ are the same ray.

47. Consider different subarcs of the circle $|z| = \sqrt{2}$.

51. Remember that the range of the principal nth root function is $-\dfrac{\pi}{n} < \arg(w) \leq \dfrac{\pi}{n}$.

55. The mapping $w = 2iz^2 - i$ is a composition of the mappings $w = z^2$ and $w = 2iz - i$. First, find the image of the quarter disk S under the mapping $w = z^2$. Second, determine the image of this set under the linear mapping $w = 2iz - i$. The point in the image of this composition that is farthest from the origin realizes the maximum modulus M and the point closest to the origin realizes the minimum modulus L.

2.5 Reciprocal Function

2.5 Summary

reciprocal function: The function $f(z) = 1/z$ is called the reciprocal function.

inversion in the unit circle: The mapping

$$g(z) = \frac{1}{z} = \frac{1}{r}e^{i\theta}$$

is called inversion in the unit circle. Under inversion in the unit circle a point z and its image $w = g(z)$ have the same argument but the modulus of z and w are reciprocals of each other. Therefore, if $z_0 \neq 0$, then $w_0 = 1/z_0$ is the unique point on the ray $\arg(z) = \arg(z_0)$ with modulus $|w_0| = 1/|z_0|$.

complex conjugation: The function

$$c(z) = \overline{z} = re^{-i\theta}$$

is called the complex conjugation function. As a mapping, complex conjugation is a reflection across the real axis. Therefore, given z_0, $w_0 = \overline{z_0}$ is the unique point with the same modulus as z_0 and $\arg(w_0) = -\arg(z_0)$.

reciprocal mapping: The complex reciprocal function $f(z) = 1/z$ can be written as the composition

$$f(z) = \frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} = \overline{\left(\frac{1}{r}e^{i\theta}\right)} = c(g(z))$$

where $g(z) = \frac{1}{r}e^{i\theta}$ is inversion in the unit circle and $c(z) = \overline{z}$ is complex conjugation. Therefore, if $z_0 \neq 0$, then $w_0 = 1/z_0$ is the point obtained by

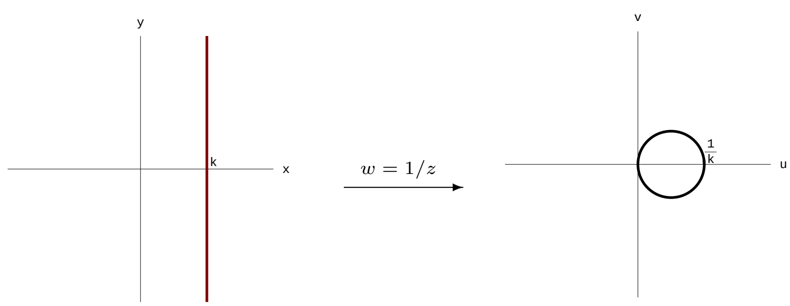
(i) inverting z_0 in the unit circle, then

(ii) reflecting the result across the real axis.

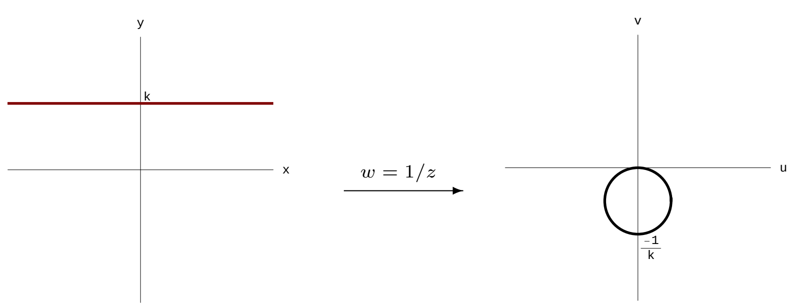
$1/z$ on the extended complex plane: The reciprocal function on the extended complex plane is defined by

$$f(z) = \begin{cases} 1/z, & \text{if } z \neq 0 \text{ or } \infty \\ \infty, & \text{if } z = 0 \\ 0, & \text{if } z = \infty. \end{cases}$$

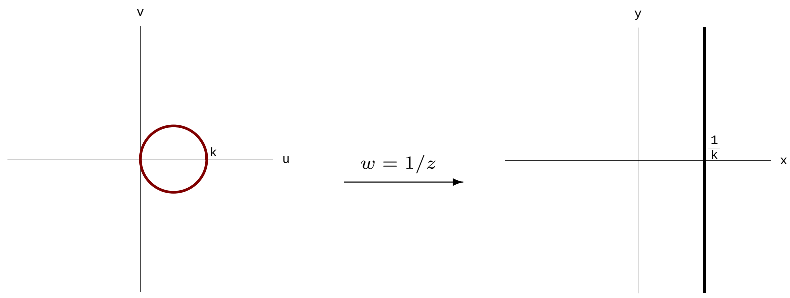
mapping properties of $w = 1/z$: The following summarizes some properties of the mapping $w = 1/z$.



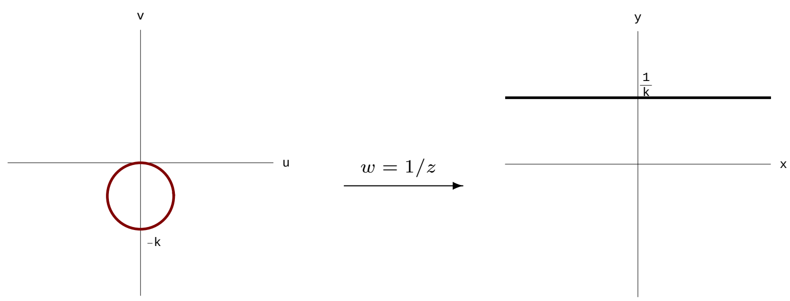
vertical line $x = k$ circle $\left|w - \frac{1}{2k}\right| = \left|\frac{1}{2k}\right|$



horizontal line $y = k$ circle $\left|w + \frac{1}{2k}i\right| = \left|\frac{1}{2k}\right|$



circle $\left|z - \frac{k}{2}\right| = \left|\frac{k}{2}\right|$ vertical line $u = 1/k$



circle $\left|z + \frac{k}{2}i\right| = \left|\frac{k}{2}\right|$ horizontal line $v = 1/k$

Exercises 2.5

3. In order to find the image of the set $|z| = 3$, $-\pi/4 \le \arg(z) \le 3\pi/4$ under the mapping $w = 1/z$, we first invert this set in the unit circle, then reflect the result across the real axis. Under inversion in the unit circle, points with modulus 3 have images with modulus 1/3, and the arguments are unchanged. Hence, the image under inversion in the unit circle is the set $|w| = 1/3$, $-\pi/4 \le \arg(w) \le 3\pi/4$. Reflecting this set across the real axis negates arguments and does not change the modulus. Therefore, the image is the semicircle $|w| = 1/3$, $-3\pi/4 \le \arg(w) \le \pi/4$.

7. In order to find the image of the set $\arg(z) = \pi/4$ under the mapping $w = 1/z$, we first invert this set in the unit circle, then reflect the result across the real axis. If we set $z = re^{i\theta}$, then this set is given by $0 < r$, $\theta = \pi/4$. Under inversion in the unit circle, points with modulus $r > 0$ have images with modulus $1/r > 0$, , and the arguments are unchanged. Hence, the image under inversion in the unit circle is the set $\arg(w) = \pi/4$. Reflecting this set across the real axis negates arguments and does not change the modulus. Therefore, the image is the ray $\arg(w) = -\pi/4$.

11. From the summary of Section 2.5 we have that the circle $\left|z + \frac{k}{2}i\right| = \left|\frac{k}{2}\right|$ is mapped onto the horizontal line $v = 1/k$ by the reciprocal mapping $w = 1/z$ on the extended complex plane. By identifying $k = 2$, we have that the image of the circle $|z + i| = 1$ is the horizontal line $v = 1/2$.

15. From the summary of Section 2.5 we have that the vertical line $x = k$ is mapped onto the circle $\left|w - \frac{1}{2k}\right| = \left|\frac{1}{2k}\right|$ by the reciprocal mapping $w = 1/z$ on the extended complex plane. By making the identifications $k_1 = -2$ and $k_2 = -1$, we have that the boundary lines $x = -2$ and $x = -1$ of the set S are mapped onto the circles $\left|w + \frac{1}{4}\right| = \left|\frac{1}{4}\right|$ and $\left|w + \frac{1}{2}\right| = \left|\frac{1}{2}\right|$, respectively. Since the point $z = -3/2$ is in the set S, the point $w = -2/3$ is in the image of the set. Therefore, the image of S under the reciprocal mapping on the extended complex plane is the set bounded by the circles $\left|w + \frac{1}{4}\right| = \frac{1}{4}$ and $\left|w + \frac{1}{2}\right| = \frac{1}{2}$ and containing the point $w = -2/3$.

19. **(a)** The first function in the composition is $f(z) = 1/z$. Thus, the mapping $w = g(f(z))$ first inverts in the unit circle, then reflects across the real axis. From the discussion on page 65 of Section 2.3, the second function in the composition, $g(z) = 2iz + 1$, rotates through an angle of $\text{Arg}(2i) = \pi/2$ about the origin, magnifies by $|2i| = 2$, and then translates by 1. Therefore, the mapping $w = f(z)$ inverts in the unit circle, reflects across the real axis, rotates through an angle of $\text{Arg}(2i) = \pi/2$ about the origin, magnifies by $|2i| = 2$, and then translates by 1.

(b) Under the reciprocal mapping on the extended complex plane, the vertical line $x = 4$ maps onto the circle $\left|w - \frac{1}{8}\right| = \frac{1}{8}$. Now under the linear mapping $g(z) = 2iz + 1$, the circle is rotated through an angle of $\pi/2$ about the origin, magnified by a factor of 2, and then translated by 1. The rotation maps the circle onto the circle $\left|w - \frac{1}{8}i\right| = \frac{1}{8}$ with the same radius but whose center has been rotated by $\pi/2$. The magnification maps this image onto the circle $\left|w - \frac{1}{4}i\right| = \frac{1}{4}$ whose center and radius have been doubled. Finally, the translation maps this image onto the circle with the same radius, but whose center has been transformed to $w = 1 + \frac{1}{4}i$. Therefore, the image of the line $x = 4$ is the circle $\left|w - 1 - \frac{1}{4}i\right| = \frac{1}{4}$.

(c) Under the reciprocal mapping on the extended complex plane, the circle $|z + 2| = 2$ maps onto the vertical line $u = -\frac{1}{4}$. Now under the linear mapping $g(z) = 2iz + 1$, the line is rotated through an angle of $\pi/2$ about the origin, magnified by a factor of 2, and then translated by 1. The rotation maps the line onto the line $v = -\frac{1}{4}$. The magnification maps this image onto the line $v = -\frac{1}{2}$. Finally, since the translation is along a vector $(b = 1 + 0i)$ in the same direction of this line, the line maps onto itself. Therefore, the image of the circle $|z + 2| = 2$ is the line $v = -\frac{1}{2}$.

Focus on Concepts

23. Modify the procedure used in Example 2 of Section 2.5.

27. **(a)** Notice that L is a generalized circle with $A = 0$ (see (7) in Problem 25 of Section 2.5). By Problem 26, the image of L is the generalized circle given by (8). What must be true about the coefficients in (8) in order for the image to be a line?

(b) Use part (a) and consider the coefficients of the image line.

(c) As in part (a), use (8) to determine an equation of the image circle. Complete the square in order to determine the center and radius.

2.6 Limits and Continuity

Review Topic: Real Limits

limit of a real function: Informally, the limit of f as x tends to x_0 exists and is equal to L, denoted $\lim_{x\to x_0} f(x) = L$, means that values of the real function $f(x)$ can be made arbitrarily close to the real number L if values of x are chosen sufficiently close to, but not equal to, x_0. The precise definition is:

$\lim_{x\to x_0} f(x) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(x) - L| < \varepsilon$ *whenever* $0 < |x - x_0| < \delta$.

left and right-hand limits: The limit of f as x tends to x_0 from the left, denoted by $\lim_{x\to x_0^-} f(x)$, and the limit from the right, denoted $\lim_{x\to x_0^+} f(x)$, are defined by:

$\lim_{x\to x_0^-} f(x) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(x) - L| < \varepsilon$ *whenever* $x_0 - \delta < x < x_0$*; for* $\lim_{x\to x_0^+} f(x) = L$*, we require* $x_0 < x < x_0 + \delta$.

A real limit exists when the left and right-hand limits are equal. That is,

$$\lim_{x\to x_0} f(x) = L \quad \text{if and only if} \quad \lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x) = L.$$

properties of real limits: The following properties allow many real limits to be evaluated in a somewhat mechanical fashion. Suppose that f and g are real functions. If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then

(*i*) $\lim_{x\to x_0} c = c$, where c is a real constant,

(*ii*) $\lim_{x\to x_0} x = x_0$,

(*iii*) $\lim_{x\to x_0} cf(x) = cL$, where c is a real constant,

(*iv*) $\lim_{x\to x_0} (f(x) \pm g(x)) = L \pm M$,

(*v*) $\lim_{x\to x_0} f(x) \cdot g(x) = L \cdot M$,

(*vi*) $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{M}$, provided $M \neq 0$.

continuity of a real function: A real function f is continuous at a point x_0 if $\lim_{x\to x_0} f(x) = f(x_0)$. If follows from the properties of real limits that real polynomial and rational functions are continuous at all points in their domains.

limit of a real multivariable function: The definition of limit for a real multivariable function $F(x, y)$ is similar to that for a real function f.

$\lim_{(x,y)\to(x_0,y_0)} F(x, y) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(x, y) - L| < \varepsilon$ *whenever* $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

continuity of a real multivariable function: A real multivariable function F is continuous at a point (x_0, y_0) if $\lim_{(x,y)\to(x_0,y_0)} F(x, y) = F(x_0, y_0)$. If follows from properties of real multivariable limits that two-variable polynomial and rational functions are continuous at all points in their domains.

2.6 Summary

limit of a complex function: Informally, the limit of f as z tends to z_0 exists and is equal to L, denoted $\lim_{z\to z_0} f(z) = L$, means that values of the complex function $f(z)$ can be made arbitrarily close to the complex number L if values of z are chosen sufficiently close to, but not equal to, z_0. The precise definition is similar to that of a real function:

$\lim_{z\to z_0} f(z) = L$ *if for every* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that* $|f(z) - L| < \varepsilon$ *whenever* $0 < |z - z_0| < \delta$.

In this definition, both ε and δ are real numbers, while z, z_0, $f(z)$, and L are complex numbers.

criterion for the nonexistence of a limit: In the limit of a complex function, z is allowed to approach z_0 from any direction in the complex plane. This provides a means of showing that certain complex limits do not exist: if $f(z)$ approaches two complex numbers $L_1 \neq L_2$ as z approaches z_0 along two different curves, then $\lim_{z\to z_0} f(z)$ does not exist.

real and imaginary parts of a limit: One method that is used to evaluate complex limits is to consider the real multivariable limits of the real and imaginary parts of f. If $f(z) = u(x, y) + iv(x, y)$, $z_0 = x_0 + iy_0$, and $L = u_0 + iv_0$, then

$$\lim_{z\to z_0} f(z) = L \quad \text{if and only if} \quad \lim_{(x,y)\to(x_0,y_0)} u(x, y) = u_0 \;\text{ and }\; \lim_{(x,y)\to(x_0,y_0)} v(x, y) = v_0.$$

properties of complex limits: The following properties allow many complex limits to be evaluated in a somewhat mechanical fashion. Suppose that f and g are complex functions. If $\lim_{z\to z_0} f(z) = L$ and $\lim_{z\to z_0} g(z) = M$, then

(*i*) $\lim_{z\to z_0} c = c$, where c is a complex constant,

(*ii*) $\lim_{z\to z_0} z = z_0$,

(*iii*) $\lim_{z\to z_0} cf(z) = cL$, where c is a complex constant, *(iv)* $\lim_{z\to z_0} (f(z) \pm g(z)) = L \pm M$,

(v) $\lim_{z\to z_0} f(z) \cdot g(z) = L \cdot M$,

(vi) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{L}{M}$, provided $M \neq 0$.

continuity of a complex function: A complex function f is continuous at a point z_0 if $\lim_{z\to z_0} f(z) = f(z_0)$.

criteria for continuity at a point: A complex function f is continuous at a point z_0 if each of the following three conditions hold:

(i) $\lim_{z\to z_0} f(z)$ exists,

(ii) f is defined at z_0, and

(iii) $\lim_{z\to z_0} f(z) = f(z_0)$.

real and imaginary parts of a continuous function: If $f(z) = u(x, y) + iv(x, y)$ and $z_0 = x_0 + iy_0$, then the complex function f is continuous at the point z_0 if and only if both real multivariable functions u and v are continuous at the point (x_0, y_0).

properties of continuous functions: If f and g are continuous at the point z_0, then cf (where c is a complex constant), $f \pm g$, and $f \cdot g$ are continuous functions at the point z_0. If $g(z_0) \neq 0$, then f/g is also continuous at the point z_0. From this it follows that all complex polynomial functions and all complex rational functions are continuous on their domains.

a bounding property for continuous functions: If f is a continuous function defined on a closed and bounded region R, then f is bounded on R. That is, there is a real constant $M > 0$ such that $|f(z)| < M$ for all z in R.

branches of multiple-valued functions: Recall from Section 2.4 that a multiple-valued function F is a rule that assigns a set of one or more complex numbers to each complex number in a subset of **C**. A branch of a multiple-valued function F is a function f_1 that is continuous on some domain and that assigns exactly one of the multiple values of F to each point z in that domain.

branch points and branch cuts: A branch cut for a branch f_1 is a curve that is excluded from the domain of the multiple-valued function F so that f_1 is continuous on the remaining points. A branch point is a point that is on the branch cut of every branch.

limit at infinity: The limit of f as z tends to ∞ exists and is equal to L, denoted $\lim_{z\to\infty} f(z) = L$, means that values of the complex function $f(z)$ can be made arbitrarily close to the complex number L if values of z are chosen so that $|z|$ is sufficiently large. A useful result for evaluating limits at infinity is:

$$\lim_{z\to\infty} f(z) = L \quad \text{if and only if} \quad \lim_{z\to 0} f\left(\frac{1}{z}\right) = L.$$

infinite limit: The limit of f as z tends to z_0 is infinity, denoted $\lim_{z\to z_0} f(z) = \infty$, means that $|f(z)|$ can be made arbitrarily large if values of z are chosen sufficiently close to, but not equal to, z_0. A useful result for evaluating infinite limits is:

$$\lim_{z\to z_0} f(z) = \infty \quad \text{if and only if} \quad \lim_{z\to z_0} \frac{1}{f(z)} = 0.$$

Exercises 2.6.1

3. In order to use Theorem 2.6.1 we need to find the real and imaginary parts of $f(z) = |z|^2 - i\bar{z}$. Setting $z = x + iy$ we obtain

$$\begin{aligned} |z|^2 - i\bar{z} &= |x+iy|^2 - i(\overline{x+iy}) \\ &= x^2 + y^2 - ix - y \\ &= (x^2 + y^2 - y) + ix. \end{aligned}$$

Identifying $x_0 = 1$, $y_0 = -1$, $u(x, y) = x^2 + y^2 - y$, and $v(x, y) = x$, we have

$$\lim_{(x,y)\to(1,-1)} (x^2 + y^2 - y) = 3 \quad \text{and} \quad \lim_{(x,y)\to(1,-1)} x = 1.$$

Therefore, by Theorem 2.6.1, $\lim_{z\to 1-i} \left(|z|^2 - i\bar{z}\right) = 3 - i$.

7. In order to use Theorem 2.6.1 we need to find the real and imaginary parts of $f(z) = \dfrac{e^z - e^{\bar{z}}}{\operatorname{Im}(z)}$. Setting $z = x + iy$ we obtain

$$\begin{aligned} \frac{e^z - e^{\bar{z}}}{\operatorname{Im}(z)} &= \frac{e^{x+iy} - e^{x-iy}}{y} \\ &= \frac{e^x \cos y + ie^x \sin y - e^x \cos y + ie^x \sin y}{y} \\ &= 2e^x \left(\frac{\sin y}{y}\right) i. \end{aligned}$$

Identifying $x_0 = 0$, $y_0 = 0$, $u(x, y) = 0$, and $v(x, y) = 2e^x \left(\dfrac{\sin y}{y}\right)$, we have

$$\lim_{(x,y)\to(0,0)} 0 = 0 \quad \text{and} \quad \lim_{(x,y)\to(0,0)} 2e^x \left(\frac{\sin y}{y}\right) = 2\left(\lim_{x\to 0} e^x\right)\left(\lim_{y\to 0} \frac{\sin y}{y}\right) = 2.$$

In the limit of $v(x, y)$ we made use of the fundamental trigonometric limit $\lim_{y\to 0} \dfrac{\sin y}{y} = 1$. Therefore, by Theorem 2.6.1, $\lim_{z\to 1-i} \left(|z|^2 - i\bar{z}\right) = 2i$.

11. By (15), we have

$$\lim_{z\to e^{i\pi/4}} z = e^{i\pi/4}.$$

Using this limit, Theorem 2.6.2(ii), and Theorem 2.6.2(iv), we obtain

$$\begin{aligned}\lim_{z\to e^{i\pi/4}}\left(z+\frac{1}{z}\right) &= e^{i\pi/4}+\frac{1}{e^{i\pi/4}}\\ &= e^{i\pi/4}+e^{-i\pi/4}\\ &= \cos(\pi/4)+i\sin(\pi/4)+\cos(-\pi/4)+i\sin(-\pi/4)\\ &= \sqrt{2}.\end{aligned}$$

15. By (15), (16), and Theorem 2.6.2(ii), we have

$$\lim_{z\to z_0}(z-z_0)=0.$$

Thus, we cannot apply Theorem 2.6.2(iv) without simplifying the rational function in the limit. Notice that:

$$\begin{aligned}\frac{(az+b)-(az_0+b)}{z-z_0} &= \frac{az+\not b - az_0 - \not b}{z-z_0}\\ &= \frac{a(z-z_0)}{z-z_0}.\end{aligned}$$

Becuase z is not allowed to take on the value z_0 in the limit we can cancel the common factor in the limit:

$$\begin{aligned}\lim_{z\to z_0}\frac{(az+b)-(az_0+b)}{z-z_0} &= \lim_{z\to z_0}\frac{a(z-z_0)}{z-z_0}\\ &= \lim_{z\to z_0} a\\ &= a.\end{aligned}$$

19. **(a)** If z approaches 0 along the real axis, then $z=x+0i$ where the real number x is approaching 0. For this approach we have

$$\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2=\lim_{x\to 0}\left(\frac{x+0i}{x-0i}\right)^2=\lim_{x\to 0}\left(\frac{x}{x}\right)^2=\lim_{x\to 0}(1)^2=1.$$

(b) If z approaches 0 along the imaginary axis, then $z=0+yi$ where the real number y is approaching 0. For this approach we have

$$\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2=\lim_{y\to 0}\left(\frac{0+yi}{0-yi}\right)^2=\lim_{y\to 0}\left(\frac{yi}{-yix}\right)^2=\lim_{y\to 0}(-1)^2=1.$$

(c) No, the two limits in parts (a) and (b) do not imply that the limit is 1. In order for the limit to exist and be equal to one, the function must approach 1 along every possible curve through 0. This has only been verified for two such curves.

(d) If z approaches 0 along the line $y = x$, then $z = x + xi$ where the real number x is approaching 0. For this approach we have

$$\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2 = \lim_{x\to 0}\left(\frac{x+x}{x-xi}\right)^2 = \lim_{x\to 0}\left(\frac{x(1+i)}{x(1-i)}\right)^2 = \lim_{x\to 0}\left(\frac{1+i}{1-i}\right)^2 = \frac{2i}{-2i} = -1.$$

(e) By the criterion for the nonexistence of a limit, since the limits in parts (a) and (d) are not the same, the limit $\lim_{z\to 0}\left(\frac{z}{\bar{z}}\right)^2$ does not exist.

23. Since the numerator of this rational function is approaching a finite number while the denominator is approaching 0, we expect the limit to be infinite. We use (26) of Section 2.6 to establish this. By (15), (16), and Theorem 2.6.2, we have

$$\lim_{z\to i}\frac{z^2+1}{z^2-1} = \frac{i^2+1}{i^2-1} = \frac{0}{-2} = 0.$$

Therefore, by (26) we obtain

$$\lim_{z\to i}\frac{z^2-1}{z^2+1} = \infty.$$

Exercises 2.6.2

27. In order to show that f is continuous, we use Definition 2.6.2. By (15), (16), and Theorem 2.6.2, we have

$$\lim_{z\to 2-i}\left(z^2 - iz + 3 - 2i\right) = (2-i)^2 - i(2-i) + 3 - 2i = 4 - 4i - 1 - 2i - 1 + 3 - 2i = 5 - 8i.$$

In addition,

$$f(2-i) = (2-i)^2 - i(2-i) + 3 - 2i = 5 - 8i.$$

Therefore, f is continuous at $z_0 = 2 - i$, since $\lim_{z\to 2-i}\left(z^2 - iz + 3 - 2i\right) = f(2-i)$.

31. In order to show that f is continuous, we use Definition 2.6.2. By (15), (16), and Theorem 2.6.2, we have

$$\lim_{z\to 1}\frac{z^3-1}{z-1} = \lim_{z\to 1}\frac{(z-1)(z^2+z+1)}{z-1} = \lim_{z\to 1}\left(z^2+z+1\right) = 1^2 + 1 + 1 = 3.$$

In addition, $f(1) = 3$. Therefore, f is continuous at $z_0 = 1$, since $\lim_{z\to 1} f(z) = f(1)$.

35. In order to show that f is not continuous, we use the criteria for continuity at a point. Notice that $f(-i)$ is undefined because the denominator is 0. Therefore, f is not continuous at $z_0 = -i$ because criterion for continuity (ii) does not hold.

39. In order to show that f is not continuous, we use the criteria for continuity at a point. Consider the limit $\lim_{z\to i} f(z)$. If z approaches i along the imaginary axis, then $|z| \neq 1$. For this approach we have

$$\lim_{z\to i} f(z) = \lim_{z\to i} \frac{z^3-1}{z-1} = \frac{i^3-1}{i-1} = i.$$

On the other hand, if z approaches i along the unit circle, then $|z| = 1$. For this approach we have

$$\lim_{z\to i} f(z) = \lim_{z\to i} 3 = 3.$$

Since $i \neq 3$, the criterion for the nonexistence of a limit tells us that $\lim_{z\to i} f(z)$ does not exist. Therefore, f is not continuous at $z_0 = i$ because criterion for continuity (i) does not hold.

43. If we let $z = x + iy$, then

$$f(z) = \frac{z-1}{z\bar{z}-4} = \frac{x+iy-1}{(x+iy)(x-iy)-4} = \frac{x-1+iy}{x^2+y^2-4} = \frac{x-1}{x^2+y^2-4} + i\frac{y}{x^2+y^2-4}.$$

Thus, the real and imaginary parts of f are

$$u(x,y) = \frac{x-1}{x^2+y^2-4} \quad \text{and} \quad v(x,y) = \frac{y}{x^2+y^2-4},$$

respectively. By equation (14) of Section 2.6.1, two-variable rational functions are continuous on their domains, and so, both u and v are continuous for all (x,y) such that $x^2+y^2 \neq 4$. Therefore, it follows from Theorem 2.6.3 that f is continuous for all z such that $|z| \neq 2$.

Focus on Concepts

47. **(a)** Use the fact that $\text{Re}(z) = \dfrac{z+\bar{z}}{2}$.

(b) Use the fact that $\text{Im}(z) = \dfrac{z-\bar{z}}{2i}$.

(c) Use the fact that $|z| = \sqrt{z\bar{z}}$.

51. **(a)** Identify $z_0 = 1+i$, $f(z) = (1-i)z + 2i$, and $L = 2+2i$ in Definition 2.6.1.

(b) Notice that after simplification

$$|f(z) - L| = |(1-i)z - 2| = |1-i| \cdot |z-(1+i)| = \sqrt{2} \cdot |z-(1+i)|.$$

55. **(a)** To show $f(z) = \text{Arg}(z)$ is discontinuous at $z = -r$ on the negative real axis, use the criterion for the nonexistence of a limit. First consider letting z approach $-r$ along the quarter of the circle $|z| = r$ lying in the second quadrant, then consider letting z approach $-r$ along the quarter of the circle $|z| = r$ lying in the third quadrant.

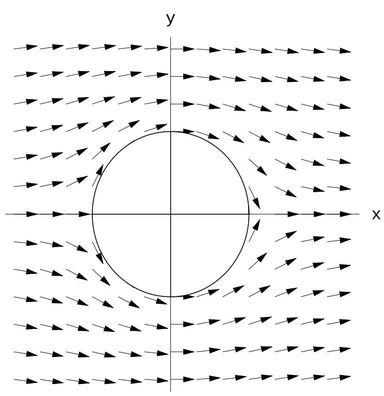
2.7 Applications

Review Topic: Vector Fields

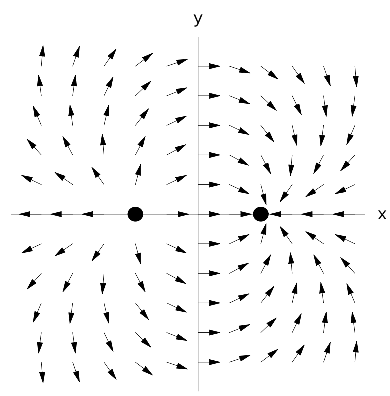
vector fields: A vector-valued function $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is called a two-dimensional vector field.

graphical representation: A graphical representation of a vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is obtained by plotting the vector $\mathbf{F}(x, y)$ based at the initial point (x, y) in the plane.

applications: Vector fields are used widely in applications to science and engineering. For example, the motion of a fluid can be described by a velocity field in which each vector $\mathbf{F}(x, y)$ represents the velocity of a particle at the point (x, y). Another common application is a force field in which each vector $\mathbf{F}(x, y)$ represents the force on a particle at the point (x, y).



velocity field



force field

2.7 Summary

complex representation of a vector field: The complex function $f(z) = P(x, y) + iQ(x, y)$ is called the complex representation of the vector field $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$. In general, both $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ and its complex representation $f(z) = P(x, y) + iQ(x, y)$ are called vector fields.

graphical representation: A graphical representation of a vector field $f(z) = P(x, y) + iQ(x, y)$ is obtained by plotting the complex number $f(z)$ as a vector based at the initial point z in the complex plane.

fluid flow: A planar flow is the flow of a fluid in which the motion and physical traits of the fluid are the same in all planes parallel to the $xy-$plane. If $f(z)$ represents the velocity of a particle of the fluid located at the point z in the complex plane, then $f(z)$ is called a velocity field.

streamlines: If $z(t) = x(t) + iy(t)$ is a parametrization of a path that a particle follows in a planar fluid flow, then the derivative $z'(t) = x'(t) + iy'(t)$ represents the velocity of the particle at point $z(t)$. Therefore, if $f(z) = P(x, y) + iQ(x, y)$ is the velocity field of the fluid, then the following equations must hold:

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y). \end{aligned}$$

The family of solutions to this system of differential equations is called the streamlines of the planar fluid flow with velocity field $f(z)$.

finding streamlines: One method for solving the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= P(x, y) \\ \frac{dy}{dt} &= Q(x, y). \end{aligned}$$

is to convert it to an ordinary differential equation using the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

This gives the ordinary differential equation

$$\frac{dy}{dx} = \frac{Q(x, y)}{P(x, y)}.$$

Separation of variables and exact equations are two methods that can sometimes be used to solve this equation.

separation of variables: An ordinary differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is called separable and can be solved by integration as follows:

$$\begin{aligned} \frac{dy}{dx} &= g(x)h(y) \\ \int \frac{dy}{h(y)} &= \int g(x)\, dx. \end{aligned}$$

If $H(y)$ and $G(x)$ are antiderivatives of $1/h(y)$ and $g(x)$, respectively, then the equation

$$H(y) = G(x) + c$$

gives solutions of the differential equation.

exact equations: An ordinary differential equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is called exact if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

If M, N, and their first partials are continuous, then this condition ensures that there is a function $F(x, y)$ such that $\partial F/\partial x = M$ and $\partial F/\partial y = N$. We find F using partial integration:

$$\begin{aligned} \frac{\partial F}{\partial x} &= M(x, y) \\ F(x, y) &= \int M(, x, y)\, dx + h(y) \end{aligned}$$

where h is found by differentiating

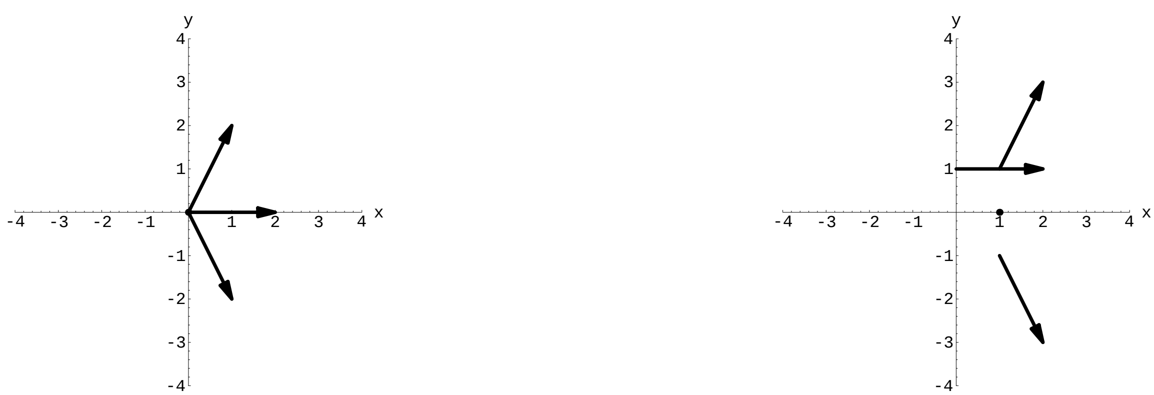
$$h'(y) = \frac{\partial F}{\partial y} - \frac{\partial}{\partial y}\left[\int M(, x, y)\, dx\right] = N(x, y) - \frac{\partial}{\partial y}\left[\int M(, x, y)\, dx\right].$$

Exercises 2.7

3. By evaluating f we find that

$$\begin{aligned} f(1) &= \overline{1 - 1^2} = 0, \\ f(1+i) &= \overline{1 - (1+i)^2} = \overline{1 - 2i} = 1 + 2i, \\ f(1-i) &= \overline{1 - (1-i)^2} = \overline{1 + 2i} = 1 - 2i, \text{ and} \\ f(i) &= \overline{1 - (i)^2} = \overline{1 + 1} = 2. \end{aligned}$$

Plotting these values as vectors is shown below.



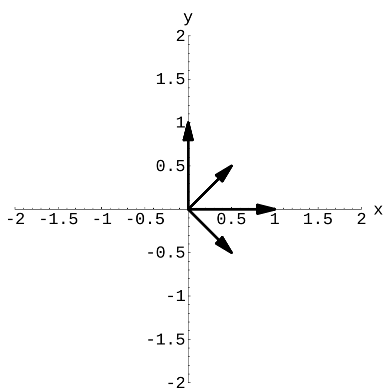
(a) plotted as position vectors

(b) plotted as vectors in the vector field

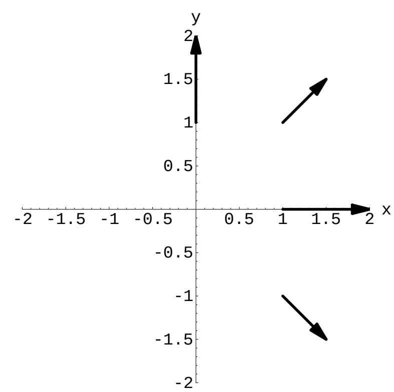
7. By evaluating f we find that

$$\begin{aligned} f(1) &= \frac{1}{\bar{1}} = 1, \\ f(1+i) &= \frac{1}{\overline{1+i}} = \frac{1}{1-i} = \frac{1}{2} + \frac{1}{2}i, \\ f(1-i) &= \frac{1}{\overline{1-i}} = \frac{1}{1+i} = \frac{1}{2} - \frac{1}{2}i, \text{ and} \\ f(i) &= \frac{1}{\bar{i}} = \frac{1}{-i} = i. \end{aligned}$$

Plotting these values as vectors is shown below.



(a) plotted as position vectors



(b) plotted as vectors in the vector field

11. **(a)** Letting $z = x+iy$ we have $f(z) = i(x+iy) = -y+ix$. By identifying $P(x,y) = -y$ and $Q(x,y) = x$ in (3) of Section 2.7, we see that the streamlines are solutions to the system of differential equations

$$\begin{aligned} \frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x. \end{aligned}$$

This system can be transformed into a single ordinary differential equation by using the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \quad \text{or} \quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt}.$$

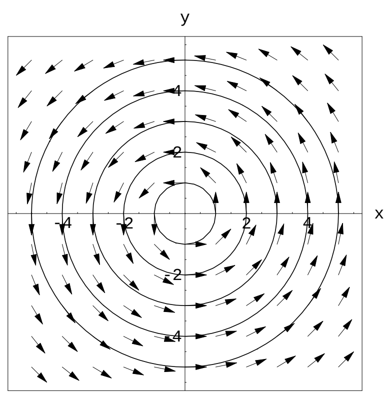
Substituting $dy/dt = x$ and $dx/dt = -y$ into the second equation yields

$$\frac{dy}{dx} = -\frac{x}{y}.$$

This differential equation can be solved by separation of variables:

$$\begin{aligned}\frac{dy}{dx} &= -\frac{x}{y}\\ \int y\,dy &= -\int x\,dx\\ \frac{1}{2}y^2 &= -\frac{1}{2}x^2 + c\\ x^2+y^2 &= c.\end{aligned}$$

Therefore, streamlines are circles $x^2 + y^2 = c$.



(b) Streamlines for Problem 11

Focus on Concepts

15. (a) Let $c = a + bi$, then solve the system

$$\begin{aligned}\frac{dx}{dt} &= a\\ \frac{dy}{dt} &= b,\end{aligned}$$

which can be converted into the ordinary differential equation:

$$\frac{dy}{dx} = \frac{b}{a}.$$

(b) Consider the magnitude and direction of the velocity field at an arbitrary point z. Do these quantities vary across the complex plane?