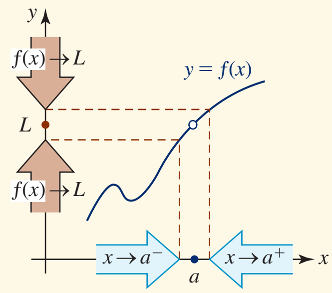
Chapter 2

Limit of a Function



In This Chapter Many topics are included in a typical course in calculus. But the three most fundamental topics in this study are the concepts of *limit*, *derivative*, and *integral*. Each of these concepts deals with functions, which is why we began this text by first reviewing some important facts about functions and their graphs.

Historically, two problems are used to introduce the basic tenets of calculus. These are the *tangent line problem* and the *area problem*. We will see in this and the subsequent chapters that the solutions to both problems involve the limit concept.

2.1 Limits—An Informal Approach

Introduction The two broad areas of calculus known as *differential* and *integral calculus* are built on the foundation concept of a *limit*. In this section our approach to this important concept will be intuitive, concentrating on understanding *what* a limit is using numerical and graphical examples. In the next section, our approach will be analytical, that is, we will use algebraic methods to *compute* the value of a limit of a function.

Limit of a Function–Informal Approach Consider the function

$$f(x) = \frac{16 - x^2}{4 + x} \tag{1}$$

whose domain is the set of all real numbers except -4. Although f cannot be evaluated *at* -4 because substituting -4 for x results in the undefined quantity $0/0$, $f(x)$ can be calculated at any number x that is very *close* to -4. The two tables

x	-4.1	-4.01	-4.001
$f(x)$	8.1	8.01	8.001

x	-3.9	-3.99	-3.999
$f(x)$	7.9	7.99	7.999

(2)

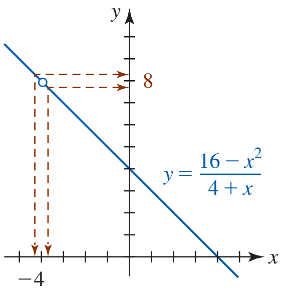


FIGURE 2.1.1 When x is near -4, $f(x)$ is near 8

show that as x approaches -4 from either the left or right, the function values $f(x)$ appear to be approaching 8, in other words, when x is near -4, $f(x)$ is near 8. To interpret the numerical information in (1) graphically, observe that for every number $x \neq -4$, the function f can be simplified by cancellation:

$$f(x) = \frac{16 - x^2}{4 + x} = \frac{(4 + x)(4 - x)}{4 + x} = 4 - x.$$

As seen in FIGURE 2.1.1, the graph of f is essentially the graph of $y = 4 - x$ with the exception that the graph of f has a *hole* at the point that corresponds to $x = -4$. For x sufficiently close to -4, represented by the two arrowheads on the x-axis, the two arrowheads on the y-axis, representing function values $f(x)$, simultaneously get closer and closer to the number 8. Indeed, in view of the numerical results in (2), the arrowheads can be made as *close as we like* to the number 8. We say 8 is the **limit** of $f(x)$ as x approaches -4.

Informal Definition Suppose L denotes a finite number. The notion of $f(x)$ approaching L as x approaches a number a can be defined informally in the following manner.

- *If $f(x)$ can be made arbitrarily close to the number L by taking x sufficiently close to but different from the number a, from both the left and right sides of a, then the **limit** of $f(x)$ as x approaches a is L.*

Notation The discussion of the limit concept is facilitated by using a special notation. If we let the arrow symbol $\rightarrow$ represent the word *approach*, then the symbolism

$x \rightarrow a^-$ *indicates that x approaches a number a from the **left***,

that is, through numbers that are less than a, and

$x \rightarrow a^+$ *signifies that x approaches a from the **right***,

that is, through numbers that are greater than a. Finally, the notation

$x \rightarrow a$ *signifies that x approaches a from **both sides***,

in other words, from the left and the right sides of a on a number line. In the left-hand table in (2) we are letting $x \rightarrow -4^-$ (for example, -4.001 is to the left of -4 on the number line), whereas in the right-hand table $x \rightarrow -4^+$.

One-Sided Limits In general, if a function $f(x)$ can be made arbitrarily close to a number L_1 by taking x sufficiently close to, but not equal to, a number a from the *left*, then we write

$$f(x) \rightarrow L_1 \text{ as } x \rightarrow a^- \quad \text{or} \quad \lim_{x \to a^-} f(x) = L_1. \tag{3}$$

The number L_1 is said to be the **left-hand limit of $f(x)$ as x approaches a**. Similarly, if $f(x)$ can be made arbitrarily close to a number L_2 by taking x sufficiently close to, but not equal to, a number a from the *right*, then L_2 is the **right-hand limit of $f(x)$ as x approaches a** and we write

$$f(x) \to L_2 \text{ as } x \to a^+ \qquad \text{or} \qquad \lim_{x\to a^+} f(x) = L_2. \tag{4}$$

The quantities in (3) and (4) are also referred to as **one-sided limits**.

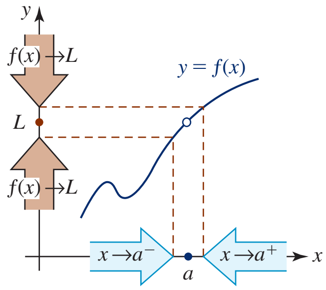


FIGURE 2.1.2 $f(x) \to L$ as $x \to a$ if and only if $f(x) \to L$ as $x \to a^-$ *and* $f(x) \to L$ as $x \to a^+$

Two-Sided Limits If both the left-hand limit $\lim_{x\to a^-} f(x)$ and the right-hand limit $\lim_{x\to a^+} f(x)$ exist and have a common value L,

$$\lim_{x\to a^-} f(x) = L \qquad \text{and} \qquad \lim_{x\to a^+} f(x) = L,$$

then we say that L is the **limit of $f(x)$ as x approaches a** and write

$$\lim_{x\to a} f(x) = L. \tag{5}$$

A limit such as (5) is said to be a **two-sided limit**. See FIGURE 2.1.2. Since the numerical tables in (2) suggest that

$$f(x) \to 8 \text{ as } x \to -4^- \qquad \textit{and} \qquad f(x) \to 8 \text{ as } x \to -4^+, \tag{6}$$

we can replace the two symbolic statements in (6) by the statement

$$f(x) \to 8 \text{ as } x \to -4 \qquad \text{or equivalently} \qquad \lim_{x\to -4} \frac{16 - x^2}{4 + x} = 8. \tag{7}$$

Existence and Nonexistence Of course a limit (one-sided or two-sided) does not have to exist. But it is important that you keep firmly in mind:

- *The existence of a limit of a function f as x approaches a (from one side or from both sides), does not depend on whether f is defined at a but only on whether f is defined for x near the number a.*

For example, if the function in (1) is modified in the following manner

$$f(x) = \begin{cases} \dfrac{16 - x^2}{4 + x}, & x \neq -4 \\ 5, & x = -4, \end{cases}$$

then $f(-4)$ is defined and $f(-4) = 5$, but still $\lim_{x\to -4} \dfrac{16 - x^2}{4 + x} = 8$. See FIGURE 2.1.3. In general, the two-sided limit $\lim_{x\to a} f(x)$ **does not exist**

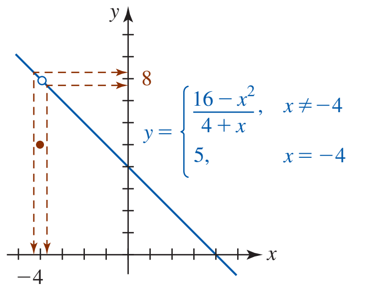


FIGURE 2.1.3 Whether f is defined at a or is not defined at a has no bearing on the existence of the limit of $f(x)$ as $x \to a$

- if either of the one-sided limits $\lim_{x\to a^-} f(x)$ or $\lim_{x\to a^+} f(x)$ fails to exist, or
- if $\lim_{x\to a^-} f(x) = L_1$ and $\lim_{x\to a^+} f(x) = L_2$, but $L_1 \neq L_2$.

EXAMPLE 1 A Limit That Exists

The graph of the function $f(x) = -x^2 + 2x + 2$ is shown in FIGURE 2.1.4. As seen from the graph and the accompanying tables, it seems plausible that

$$\lim_{x\to 4^-} f(x) = -6 \qquad \text{and} \qquad \lim_{x\to 4^+} f(x) = -6$$

and consequently $\lim_{x\to 4} f(x) = -6$.

$x \to 4^-$	3.9	3.99	3.999
$f(x)$	−5.41000	−5.94010	−5.99400

$x \to 4^+$	4.1	4.01	4.001
$f(x)$	−6.61000	−6.06010	−6.00600

■

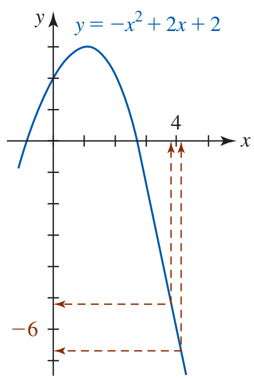


FIGURE 2.1.4 Graph of function in Example 1

Note that in Example 1 the given function is certainly defined at 4, but at no time did we substitute $x = 4$ into the function to find the value of $\lim_{x\to 4} f(x)$.

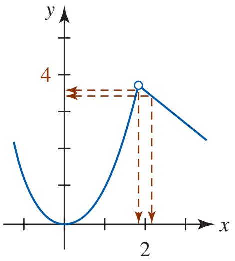


FIGURE 2.1.5 Graph of function in Example 2

EXAMPLE 2 A Limit That Exists

The graph of the piecewise-defined function

$$f(x) = \begin{cases} x^2, & x < 2 \\ -x + 6, & x > 2 \end{cases}$$

is given in FIGURE 2.1.5. Notice that $f(2)$ is not defined, but that is of no consequence when considering $\lim_{x\to 2} f(x)$. From the graph and the accompanying tables,

$x \to 2^-$	1.9	1.99	1.999
$f(x)$	3.61000	3.96010	3.99600

$x \to 2^+$	2.1	2.01	2.001
$f(x)$	3.90000	3.99000	3.99900

we see that when we make x close to 2, we can make $f(x)$ arbitrarily close to 4, and so

$$\lim_{x\to 2^-} f(x) = 4 \quad \text{and} \quad \lim_{x\to 2^+} f(x) = 4.$$

That is, $\lim_{x\to 2} f(x) = 4$. ■

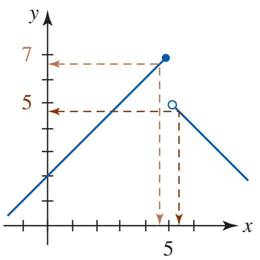


FIGURE 2.1.6 Graph of function in Example 3

EXAMPLE 3 A Limit That Does Not Exist

The graph of the piecewise-defined function

$$f(x) = \begin{cases} x + 2, & x \le 5 \\ -x + 10, & x > 5 \end{cases}$$

is given in FIGURE 2.1.6. From the graph and the accompanying tables, it appears that as x approaches 5 through numbers less than 5 that $\lim_{x\to 5^-} f(x) = 7$. Then as x approaches 5 through numbers greater than 5 it appears that $\lim_{x\to 5^+} f(x) = 5$. But since

$$\lim_{x\to 5^-} f(x) \ne \lim_{x\to 5^+} f(x),$$

we conclude that $\lim_{x\to 5} f(x)$ does not exist.

$x \to 5^-$	4.9	4.99	4.999
$f(x)$	6.90000	6.99000	6.99900

$x \to 5^+$	5.1	5.01	5.001
$f(x)$	4.90000	4.99000	4.99900

■

The greatest integer function was discussed in Section 1.1.

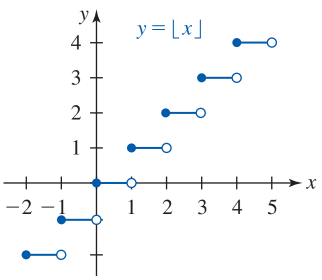


FIGURE 2.1.7 Graph of function in Example 4

EXAMPLE 4 A Limit That Does Not Exist

Recall, the **greatest integer function** or **floor function** $f(x) = \lfloor x \rfloor$ is defined to be the greatest integer that is less than or equal to x. The domain of f is the set of real numbers $(-\infty, \infty)$. From the graph in FIGURE 2.1.7 we see that $f(n)$ is defined for every integer n; nonetheless, for each integer n, $\lim_{x\to n} f(x)$ does not exist. For example, as x approaches, say, the number 3, the two one-sided limits exist but have different values:

$$\lim_{x\to 3^-} f(x) = 2 \quad \text{whereas} \quad \lim_{x\to 3^+} f(x) = 3. \tag{8}$$

In general, for an integer n,

$$\lim_{x\to n^-} f(x) = n - 1 \quad \text{whereas} \quad \lim_{x\to n^+} f(x) = n.$$ ■

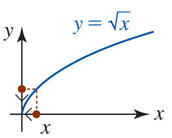


FIGURE 2.1.8 Graph of function in Example 5

EXAMPLE 5 A Right-Hand Limit

From FIGURE 2.1.8 it should be clear that $f(x) = \sqrt{x} \to 0$ as $x \to 0^+$, that is

$$\lim_{x\to 0^+} \sqrt{x} = 0.$$

It would be incorrect to write $\lim_{x\to 0} \sqrt{x} = 0$ since this notation carries with it the connotation that the limits from the left and from the right exist and are equal to 0. In this case $\lim_{x\to 0^-} \sqrt{x}$ does not exist since $f(x) = \sqrt{x}$ is not defined for $x < 0$. ■

If $x = a$ is a vertical asymptote for the graph of $y = f(x)$, then $\lim_{x \to a} f(x)$ will always fail to exist because the function values $f(x)$ must become unbounded from at least one side of the line $x = a$.

EXAMPLE 6 A Limit That Does Not Exist

A vertical asymptote always corresponds to an infinite break in the graph of a function f. In FIGURE 2.1.9 we see that the y-axis or $x = 0$ is a vertical asymptote for the graph of $f(x) = 1/x$. The tables

$x \to 0^-$	-0.1	-0.01	-0.001
$f(x)$	-10	-100	-1000

$x \to 0^+$	0.1	0.01	0.001
$f(x)$	10	100	1000

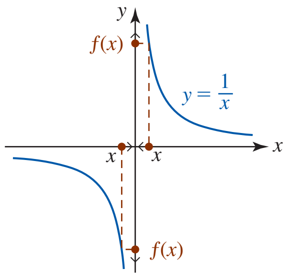


FIGURE 2.1.9 Graph of function in Example 6

clearly show that the function values $f(x)$ become unbounded in absolute value as we get close to 0. In other words, $f(x)$ is not approaching a real number as $x \to 0^-$ nor as $x \to 0^+$. Therefore, neither the left-hand nor the right-hand limit exists as x approaches 0. Thus we conclude that $\lim_{x \to 0} f(x)$ does not exist. ■

EXAMPLE 7 An Important Trigonometric Limit

To do the calculus of the trigonometric functions $\sin x$, $\cos x$, $\tan x$, and so on, it is important to realize that the variable x is either a real number or an angle measured in radians. With that in mind, consider the numerical values of $f(x) = (\sin x)/x$ as $x \to 0^+$ given in the table that follows.

$x \to 0^+$	0.1	0.01	0.001	0.0001
$f(x)$	0.99833416	0.99998333	0.99999983	0.99999999

It is easy to see that the same results given in the table hold as $x \to 0^-$. Because $\sin x$ is an odd function, for $x > 0$ and $-x < 0$ we have $\sin(-x) = -\sin x$ and as a consequence

$$f(-x) = \frac{\sin(-x)}{-x} = \frac{\sin x}{x} = f(x).$$

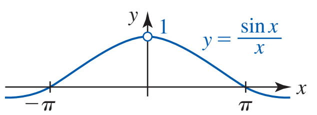


FIGURE 2.1.10 Graph of function in Example 7

As can be seen in FIGURE 2.1.10, f is an even function. The table of numerical values as well as the graph of f strongly suggest the following result:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1. \tag{9}$$ ■

The limit in (9) is a very important result and will be used in Section 3.4. Another trigonometric limit that you are asked to verify as an exercise is given by

$$\lim_{x \to 0} \frac{1 - \cos x}{x} = 0. \tag{10}$$

See Problem 43 in Exercises 2.1. Because of their importance, both (9) and (10) will be proven in Section 2.4.

An Indeterminate Form A limit of a quotient $f(x)/g(x)$, where both the numerator and the denominator approach 0 as $x \to a$, is said to have the **indeterminate form 0/0**. The limit (7) in our initial discussion has this indeterminate form. Many important limits, such as (9) and (10), and the limit

$$\lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

which forms the backbone of differential calculus, also have the indeterminate form 0/0.

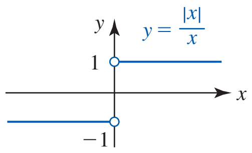


FIGURE 2.1.11 Graph of function in Example 8

EXAMPLE 8 An Indeterminate Form

The limit $\lim_{x\to 0}|x|/x$ has the indeterminate form 0/0, but unlike (7), (9), and (10) this limit fails to exist. To see why, let us examine the graph of the function $f(x) = |x|/x$. For $x \neq 0$, $|x| = \begin{cases} x, & x > 0 \\ -x, & x < 0 \end{cases}$ and so we recognize f as the piecewise-defined function

$$f(x) = \frac{|x|}{x} = \begin{cases} 1, & x > 0 \\ -1, & x < 0. \end{cases} \tag{11}$$

From (11) and the graph of f in FIGURE 2.1.11 it should be apparent that both the left-hand and right-hand limits of f exist and

$$\lim_{x\to 0^-}\frac{|x|}{x} = -1 \quad \text{and} \quad \lim_{x\to 0^+}\frac{|x|}{x} = 1.$$

Because these one-sided limits are different, we conclude that $\lim_{x\to 0}|x|/x$ does not exist. ■

NOTES FROM THE CLASSROOM

While graphs and tables of function values may be convincing for determining whether a limit does or does not exist, you are certainly aware that all calculators and computers work only with approximations and that graphs can be drawn inaccurately. A blind use of a calculator can also lead to a false conclusion. For example, $\lim_{x\to 0}\sin(\pi/x)$ is known not to exist, but from the table of values

$x \to 0$	± 0.1	± 0.01	± 0.001
$f(x)$	0	0	0

one would naturally conclude that $\lim_{x\to 0}\sin(\pi/x) = 0$. On the other hand, the limit

$$\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2} \tag{12}$$

can be shown to exist and equals $\frac{1}{4}$. See Example 11 in Section 2.2. One calculator gives

$x \to 0$	± 0.00001	± 0.000001	± 0.0000001
$f(x)$	0.200000	0.000000	0.000000

The problem in calculating (12) for x very close to 0 is that $\sqrt{x^2+4}$ is correspondingly very close to 2. When subtracting two numbers of nearly equal values on a calculator a loss of significant digits may occur due to round-off error.

Exercises 2.1

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–14, sketch the graph of the function to find the given limit, or state that it does not exist.

1. $\lim_{x\to 2}(3x+2)$

2. $\lim_{x\to 2}(x^2-1)$

3. $\lim_{x\to 0}\left(1+\frac{1}{x}\right)$

4. $\lim_{x\to 5}\sqrt{x-1}$

5. $\lim_{x\to 1}\frac{x^2-1}{x-1}$

6. $\lim_{x\to 0}\frac{x^2-3x}{x}$

7. $\lim_{x\to 3}\frac{|x-3|}{x-3}$

8. $\lim_{x\to 0}\frac{|x|-x}{x}$

9. $\lim_{x\to 0}\frac{x^3}{x}$

10. $\lim_{x\to 1}\frac{x^4-1}{x^2-1}$

11. $\lim_{x\to 0}f(x)$ where $f(x) = \begin{cases} x+3, & x < 0 \\ -x+3, & x \geq 0 \end{cases}$

12. $\lim_{x\to 2}f(x)$ where $f(x) = \begin{cases} x, & x < 2 \\ x+1, & x \geq 2 \end{cases}$

13. $\lim_{x\to 2}f(x)$ where $f(x) = \begin{cases} x^2-2x, & x < 2 \\ 1, & x = 2 \\ x^2-6x+8, & x > 2 \end{cases}$

14. $\lim_{x\to 0} f(x)$ where $f(x) = \begin{cases} x^2, & x < 0 \\ 2, & x = 0 \\ \sqrt{x} - 1, & x > 0 \end{cases}$

In Problems 15–18, use the given graph to find the value of each quantity, or state that it does not exist.

(a) $f(1)$ **(b)** $\lim_{x\to 1^+} f(x)$ **(c)** $\lim_{x\to 1^-} f(x)$ **(d)** $\lim_{x\to 1} f(x)$

15.

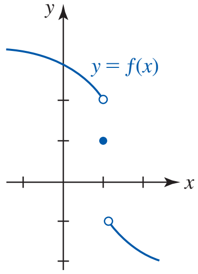


FIGURE 2.1.12 Graph for Problem 15

16.

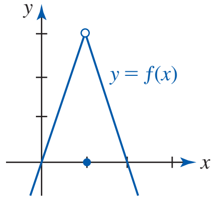


FIGURE 2.1.13 Graph for Problem 16

17.

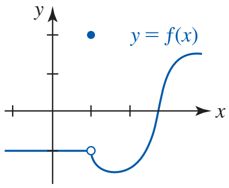


FIGURE 2.1.14 Graph for Problem 17

18.

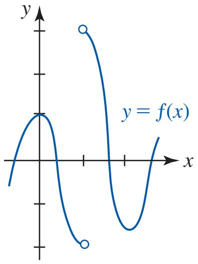


FIGURE 2.1.15 Graph for Problem 18

In Problems 19–28, each limit has the value 0, but some of the notation is incorrect. If the notation is incorrect, give the correct statement.

19. $\lim_{x\to 0} \sqrt[3]{x} = 0$ **20.** $\lim_{x\to 0} \sqrt[4]{x} = 0$

21. $\lim_{x\to 1} \sqrt{1 - x} = 0$ **22.** $\lim_{x\to -2^+} \sqrt{x + 2} = 0$

23. $\lim_{x\to 0^-} \lfloor x \rfloor = 0$ **24.** $\lim_{x\to \frac{1}{2}} \lfloor x \rfloor = 0$

25. $\lim_{x\to \pi} \sin x = 0$ **26.** $\lim_{x\to 1} \cos^{-1} x = 0$

27. $\lim_{x\to 3^+} \sqrt{9 - x^2} = 0$ **28.** $\lim_{x\to 1} \ln x = 0$

In Problems 29 and 30, use the given graph to find each limit, or state that it does not exist.

29. **(a)** $\lim_{x\to -4^+} f(x)$ **(b)** $\lim_{x\to -2} f(x)$

(c) $\lim_{x\to 0} f(x)$ **(d)** $\lim_{x\to 1} f(x)$

(e) $\lim_{x\to 3} f(x)$ **(f)** $\lim_{x\to 4^-} f(x)$

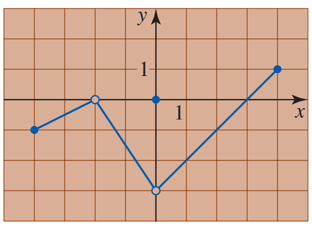


FIGURE 2.1.16 Graph for Problem 29

30. **(a)** $\lim_{x\to -5} f(x)$ **(b)** $\lim_{x\to -3^-} f(x)$

(c) $\lim_{x\to -3^+} f(x)$ **(d)** $\lim_{x\to -3} f(x)$

(e) $\lim_{x\to 0} f(x)$ **(f)** $\lim_{x\to 1} f(x)$

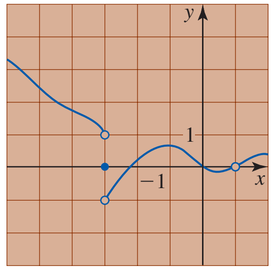


FIGURE 2.1.17 Graph for Problem 30

In Problems 31–34, sketch a graph of a function f with the given properties.

31. $f(-1) = 3, f(0) = -1, f(1) = 0, \lim_{x\to 0} f(x)$ does not exist

32. $f(-2) = 3, \lim_{x\to 0^-} f(x) = 2, \lim_{x\to 0^+} f(x) = -1, f(1) = -2$

33. $f(0) = 1, \lim_{x\to 1^-} f(x) = 3, \lim_{x\to 1^+} f(x) = 3, f(1)$ is undefined, $f(3) = 0$

34. $f(-2) = 2, f(x) = 1, -1 \le x \le 1, \lim_{x\to -1} f(x) = 1, \lim_{x\to 1} f(x)$ does not exist, $f(2) = 3$

Calculator/CAS Problems

In Problems 35–40, use a calculator or CAS to obtain the graph of the given function f on the interval $[-0.5, 0.5]$. Use the graph to conjecture the value of $\lim_{x\to 0} f(x)$, or state that the limit does not exist.

35. $f(x) = \cos\dfrac{1}{x}$ **36.** $f(x) = x\cos\dfrac{1}{x}$

37. $f(x) = \dfrac{2 - \sqrt{4 + x}}{x}$

38. $f(x) = \dfrac{9}{x}\left[\sqrt{9 - x} - \sqrt{9 + x}\right]$

39. $f(x) = \dfrac{e^{-2x} - 1}{x}$ **40.** $f(x) = \dfrac{\ln|x|}{x}$

In Problems 41–50, proceed as in Examples 3, 6, and 7 and use a calculator to construct tables of function values. Conjecture the value of each limit, or state that it does not exist.

41. $\lim_{x\to 1} \dfrac{6\sqrt{x} - 6\sqrt{2x - 1}}{x - 1}$ **42.** $\lim_{x\to 1} \dfrac{\ln x}{x - 1}$

43. $\lim_{x\to 0} \dfrac{1 - \cos x}{x}$ **44.** $\lim_{x\to 0} \dfrac{1 - \cos x}{x^2}$

45. $\lim_{x\to 0} \dfrac{x}{\sin 3x}$ **46.** $\lim_{x\to 0} \dfrac{\tan x}{x}$

47. $\lim_{x\to 4} \dfrac{\sqrt{x} - 2}{x - 4}$ **48.** $\lim_{x\to 3}\left[\dfrac{6}{x^2 - 9} - \dfrac{6\sqrt{x - 2}}{x^2 - 9}\right]$

49. $\lim_{x\to 1} \dfrac{x^4 + x - 2}{x - 1}$ **50.** $\lim_{x\to -2} \dfrac{x^3 + 8}{x + 2}$

2.2 Limit Theorems

Introduction The intention of the informal discussion in Section 2.1 was to give you an intuitive grasp of when a limit does or does not exist. However, it is neither desirable nor practical, in every instance, to reach a conclusion about the existence of a limit based on a graph or on a table of numerical values. We must be able to evaluate a limit, or discern its non-existence, in a somewhat mechanical fashion. The theorems that we shall consider in this section establish such a means. The proofs of some of these results are given in the *Appendix*.

The first theorem gives two basic results that will be used throughout the discussion of this section.

Theorem 2.2.1 Two Fundamental Limits

(*i*) $\lim_{x \to a} c = c$, where c is a constant

(*ii*) $\lim_{x \to a} x = a$

Although both parts of Theorem 2.2.1 require a formal proof, Theorem 2.2.1(*ii*) is almost tautological when stated in words:

- *The limit of x as x is approaching a is a.*

See the *Appendix* for a proof of Theorem 2.2.1(*i*).

EXAMPLE 1 Using Theorem 2.2.1

(a) From Theorem 2.2.1(*i*),

$$\lim_{x \to 2} 10 = 10 \qquad \text{and} \qquad \lim_{x \to 6} \pi = \pi.$$

(b) From Theorem 2.1.1(*ii*),

$$\lim_{x \to 2} x = 2 \qquad \text{and} \qquad \lim_{x \to 0} x = 0.$$

■

The limit of a constant multiple of a function f is the constant times the limit of f as x approaches a number a.

Theorem 2.2.2 Limit of a Constant Multiple

If c is a constant, then

$$\lim_{x \to a} c f(x) = c \lim_{x \to a} f(x).$$

We can now start using theorems in conjunction with each other.

EXAMPLE 2 Using Theorems 2.2.1 and 2.2.2

From Theorems 2.2.1 (*ii*) and 2.2.2,

(a) $\lim_{x \to 8} 5x = 5 \lim_{x \to 8} x = 5 \cdot 8 = 40$

(b) $\lim_{x \to -2} \left(-\frac{3}{2}x\right) = -\frac{3}{2} \lim_{x \to -2} x = \left(-\frac{3}{2}\right) \cdot (-2) = 3.$ ■

The next theorem is particularly important because it gives us a way of computing limits in an algebraic manner.

Theorem 2.2.3 Limit of a Sum, Product, and Quotient

Suppose a is a real number and $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. If $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$, then

(*i*) $\lim_{x\to a}[f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) = L_1 \pm L_2$,

(*ii*) $\lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1L_2$, and

(*iii*) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L_1}{L_2}$, $L_2 \neq 0$.

Theorem 2.2.3 can be stated in words:

- *If both limits exist, then*
  (*i*) *the limit of a sum is the sum of the limits,*
  (*ii*) *the limit of a product is the product of the limits, and*
  (*iii*) *the limit of a quotient is the quotient of the limits provided the limit of the denominator is not zero.*

Note: If all limits exist, then Theorem 2.2.3 is also applicable to one-sided limits, that is, the symbolism $x \to a$ in Theorem 2.2.3 can be replaced by either $x \to a^-$ or $x \to a^+$. Moreover, Theorem 2.2.3 extends to differences, sums, products, and quotients that involve more than two functions. See the *Appendix* for a proof of Theorem 2.2.3.

EXAMPLE 3 Using Theorem 2.2.3

Evaluate $\lim_{x\to 5}(10x + 7)$.

Solution From Theorems 2.2.1 and 2.2.2, we know that $\lim_{x\to 5} 7$ and $\lim_{x\to 5} 10x$ exist. Hence, from Theorem 2.2.3(*i*),

$$\begin{aligned}\lim_{x\to 5}(10x + 7) &= \lim_{x\to 5} 10x + \lim_{x\to 5} 7\\ &= 10\lim_{x\to 5} x + \lim_{x\to 5} 7\\ &= 10 \cdot 5 + 7 = 57.\end{aligned}$$

■

Limit of a Power Theorem 2.2.3(*ii*) can be used to calculate the limit of a positive integer power of a function. For example, if $\lim_{x\to a} f(x) = L$, then from Theorem 2.2.3(*ii*) with $g(x) = f(x)$,

$$\lim_{x\to a}[f(x)]^2 = \lim_{x\to a}[f(x)\cdot f(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} f(x)\right) = L^2.$$

By the same reasoning we can apply Theorem 2.2.3(*ii*) to the general case where $f(x)$ is a factor n times. This result is stated as the next theorem.

Theorem 2.2.4 Limit of a Power

Let $\lim_{x\to a} f(x) = L$ and n be a positive integer. Then

$$\lim_{x\to a}[f(x)]^n = \left[\lim_{x\to a} f(x)\right]^n = L^n.$$

For the special case $f(x) = x$, the result given in Theorem 2.2.4 yields

$$\lim_{x\to a} x^n = a^n. \qquad (1)$$

EXAMPLE 4 Using (1) and Theorem 2.2.3

Evaluate

(a) $\lim_{x \to 10} x^3$ **(b)** $\lim_{x \to 4} \frac{5}{x^2}$.

Solution

(a) From (1),

$$\lim_{x \to 10} x^3 = 10^3 = 1000.$$

(b) From Theorem 2.2.1 and (1) we know that $\lim_{x \to 4} 5 = 5$ and $\lim_{x \to 4} x^2 = 16 \neq 0$. Therefore by Theorem 2.2.3(*iii*),

$$\lim_{x \to 4} \frac{5}{x^2} = \frac{\lim_{x \to 4} 5}{\lim_{x \to 4} x^2} = \frac{5}{4^2} = \frac{5}{16}.$$ ■

EXAMPLE 5 Using Theorem 2.2.3

Evaluate $\lim_{x \to 3} (x^2 - 5x + 6)$.

Solution In view of Theorem 2.2.1, Theorem 2.2.2, and (1) all limits exist. Therefore by Theorem 2.2.3(*i*),

$$\lim_{x \to 3} (x^2 - 5x + 6) = \lim_{x \to 3} x^2 - \lim_{x \to 3} 5x + \lim_{x \to 3} 6 = 3^2 - 5 \cdot 3 + 6 = 0.$$ ■

EXAMPLE 6 Using Theorems 2.2.3 and 2.2.4

Evaluate $\lim_{x \to 1} (3x - 1)^{10}$.

Solution First, we see from Theorem 2.2.3(*i*) that

$$\lim_{x \to 1} (3x - 1) = \lim_{x \to 1} 3x - \lim_{x \to 1} 1 = 2.$$

It then follows from Theorem 2.2.4 that

$$\lim_{x \to 1} (3x - 1)^{10} = \left[\lim_{x \to 1} (3x - 1)\right]^{10} = 2^{10} = 1024.$$ ■

Limit of a Polynomial Function Some limits can be evaluated by *direct substitution*. We can use (1) and Theorem 2.2.3(*i*) to compute the limit of a general polynomial function. If

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0$$

is a polynomial function, then

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a} \left(c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0\right) \\ &= \lim_{x \to a} c_n x^n + \lim_{x \to a} c_{n-1} x^{n-1} + \cdots + \lim_{x \to a} c_1 x + \lim_{x \to a} c_0 \\ &= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0. \end{aligned}$$

← f is defined at $x = a$ and this limit is $f(a)$

In other words, to evaluate a limit of a polynomial function f as x approaches a real number a, *we need only evaluate the function at* $x = a$:

$$\lim_{x \to a} f(x) = f(a). \tag{2}$$

A reexamination of Example 5 shows that $\lim_{x \to 3} f(x)$, where $f(x) = x^2 - 5x + 6$, is given by $f(3) = 0$.

Because a rational function f is a quotient of two polynomials $p(x)$ and $q(x)$, it follows from (2) and Theorem 2.2.3(*iii*) that a limit of a rational function $f(x) = p(x)/q(x)$ can also be found by evaluating f at $x = a$:

$$\lim_{x \to a} f(x) = \lim_{x \to a} \frac{p(x)}{q(x)} = \frac{p(a)}{q(a)}. \tag{3}$$

Of course we must add to (3) the all-important requirement that the limit of the denominator is not 0, that is, $q(a) \neq 0$.

EXAMPLE 7 Using (2) and (3)

Evaluate $\lim_{x\to-1} \dfrac{3x-4}{8x^2+2x-2}$.

Solution $f(x) = \dfrac{3x-4}{8x^2+2x-2}$ is a rational function and so if we identify the polynomials $p(x) = 3x - 4$ and $q(x) = 8x^2 + 2x - 2$, then from (2),

$$\lim_{x\to-1} p(x) = p(-1) = -7 \qquad \text{and} \qquad \lim_{x\to-1} q(x) = q(-1) = 4.$$

Since $q(-1) \neq 0$ it follows from (3) that

$$\lim_{x\to-1} \frac{3x-4}{8x^2+2x-2} = \frac{p(-1)}{q(-1)} = \frac{-7}{4} = -\frac{7}{4}.$$ ■

You should not get the impression that we can *always* find a limit of a function by substituting the number *a directly into the function*.

EXAMPLE 8 Using Theorem 2.2.3

Evaluate $\lim_{x\to1} \dfrac{x-1}{x^2+x-2}$.

Solution The function in this limit is rational, but if we substitute $x = 1$ into the function we see that this limit has the indeterminate form $0/0$. However, by simplifying *first*, we can then apply Theorem 2.2.3(*iii*):

$$\begin{aligned} \lim_{x\to1} \frac{x-1}{x^2+x-2} &= \lim_{x\to1} \frac{x-1}{(x-1)(x+2)} \quad \leftarrow \text{cancellation is valid provided that } x \neq 1 \\ &= \lim_{x\to1} \frac{1}{x+2} \\ &= \frac{\lim_{x\to1} 1}{\lim_{x\to1}(x+2)} = \frac{1}{3}. \end{aligned}$$ ■

If a limit of a rational function has the indeterminate form $0/0$ as $x \to a$, then by the Factor Theorem of algebra $x - a$ must be a factor of both the numerator and the denominator. Factor those quantities and cancel the factor $x - a$.

Sometimes you can tell at a glance when *a limit does not exist*.

Theorem 2.2.5 A Limit That Does Not Exist

Let $\lim_{x\to a} f(x) = L_1 \neq 0$ and $\lim_{x\to a} g(x) = 0$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

does not exist.

PROOF We will give an indirect proof of this result based on Theorem 2.2.3. Suppose $\lim_{x\to a} f(x) = L_1 \neq 0$ and $\lim_{x\to a} g(x) = 0$ and suppose further that $\lim_{x\to a} (f(x)/g(x))$ exists and equals L_2. Then

$$\begin{aligned} L_1 = \lim_{x\to a} f(x) &= \lim_{x\to a}\left(g(x) \cdot \frac{f(x)}{g(x)}\right), \qquad g(x) \neq 0, \\ &= \left(\lim_{x\to a} g(x)\right)\left(\lim_{x\to a} \frac{f(x)}{g(x)}\right) = 0 \cdot L_2 = 0. \end{aligned}$$

By contradicting the assumption that $L_1 \neq 0$, we have proved the theorem. ■

EXAMPLE 9 Using Theorems 2.2.3 and 2.2.5

Evaluate

(a) $\lim_{x\to 5}\dfrac{x}{x-5}$ **(b)** $\lim_{x\to 5}\dfrac{x^2-10x-25}{x^2-4x-5}$ **(c)** $\lim_{x\to 5}\dfrac{x-5}{x^2-10x+25}$.

Solution Each function in the three parts of the example is rational.

(a) Since the limit of the numerator x is 5, but the limit of the denominator $x-5$ is 0, we conclude from Theorem 2.2.5 that the limit does not exist.

(b) Substituting $x = 5$ makes both the numerator and denominator 0, and so the limit has the indeterminate form 0/0. By the Factor Theorem of algebra, $x - 5$ is a factor of both the numerator and denominator. Hence,

$$\lim_{x\to 5}\frac{x^2-10x-25}{x^2-4x-5} = \lim_{x\to 5}\frac{(x-5)^2}{(x-5)(x+1)} \quad \leftarrow \text{cancel the factor } x-5$$
$$= \lim_{x\to 5}\frac{x-5}{x+1}$$
$$= \frac{0}{6} = 0. \quad \leftarrow \text{limit exists}$$

(c) Again, the limit has the indeterminate form 0/0. After factoring the denominator and canceling the factors we see from the algebra

$$\lim_{x\to 5}\frac{x-5}{x^2-10x+25} = \lim_{x\to 5}\frac{x-5}{(x-5)^2}$$
$$= \lim_{x\to 5}\frac{1}{x-5}$$

that the limit does not exist since the limit of the numerator in the last expression is now 1 but the limit of the denominator is 0. ■

Limit of a Root The limit of the nth root of a function is the nth root of the limit whenever the limit exists and has a real nth root. The next theorem summarizes this fact.

Theorem 2.2.6 Limit of a Root

Let $\lim_{x\to a} f(x) = L$ and n be a positive integer. Then

$$\lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L},$$

provided that $L \geq 0$ when n is even.

An immediate special case of Theorem 2.2.6 is

$$\lim_{x\to a}\sqrt[n]{x} = \sqrt[n]{a}, \tag{4}$$

provided $a \geq 0$ when n is even. For example, $\lim_{x\to 9}\sqrt{x} = \left[\lim_{x\to 9} x\right]^{1/2} = 9^{1/2} = 3$.

EXAMPLE 10 Using (4) and Theorem 2.2.3

Evaluate $\lim_{x\to -8}\dfrac{x-\sqrt[3]{x}}{2x+10}$.

Solution Since $\lim_{x\to -8}(2x+10) = -6 \neq 0$, we see from Theorem 2.2.3(*iii*) and (4) that

$$\lim_{x\to -8}\frac{x-\sqrt[3]{x}}{2x+10} = \frac{\lim_{x\to -8} x - \left[\lim_{x\to -8} x\right]^{1/3}}{\lim_{x\to -8}(2x+10)} = \frac{-8-(-8)^{1/3}}{-6} = \frac{-6}{-6} = 1. \quad ■$$

When a limit of an algebraic function involving radicals has the indeterminate form 0/0, rationalization of the numerator or the denominator may be something to try.

EXAMPLE 11 Rationalization of a Numerator

Evaluate $\lim_{x\to 0} \dfrac{\sqrt{x^2+4}-2}{x^2}$.

> We have seen this limit in (12) in *Notes from the Classroom* at the end of Section 2.1.

Solution Because $\lim_{x\to 0}\sqrt{x^2+4} = \sqrt{\lim_{x\to 0}(x^2+4)} = 2$ we see by inspection that the given limit has the indeterminate form 0/0. However, by rationalization of the numerator we obtain

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2}\cdot\frac{\sqrt{x^2+4}+2}{\sqrt{x^2+4}+2}\\
&= \lim_{x\to 0}\frac{(x^2+4)-4}{x^2\left(\sqrt{x^2+4}+2\right)}\\
&= \lim_{x\to 0}\frac{x^2}{x^2\left(\sqrt{x^2+4}+2\right)} && \leftarrow \text{cancel } x\text{'s}\\
&= \lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2}. && \leftarrow \text{this limit is no longer } 0/0
\end{aligned}$$

We are now in a position to use Theorems 2.2.3 and 2.2.6:

$$\begin{aligned}
\lim_{x\to 0}\frac{\sqrt{x^2+4}-2}{x^2} &= \lim_{x\to 0}\frac{1}{\sqrt{x^2+4}+2}\\
&= \frac{\lim_{x\to 0} 1}{\sqrt{\lim_{x\to 0}(x^2+4)}+\lim_{x\to 0} 2}\\
&= \frac{1}{2+2} = \frac{1}{4}.
\end{aligned}$$

■

In case anyone is wondering whether there can be more than one limit of a function $f(x)$ as $x \to a$, we state the last theorem for the record.

Theorem 2.2.7 Existence Implies Uniqueness

If $\lim_{x\to a} f(x)$ exists, then it is unique.

$\lim_{x\to a}$ NOTES FROM THE CLASSROOM

In mathematics it is just as important to be aware of what a definition or a theorem does *not* say as what it says.

(*i*) Property (*i*) of Theorem 2.2.3 does not say that the limit of a sum is *always* the sum of the limits. For example, $\lim_{x\to 0}(1/x)$ does not exist, so

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x}\right] \neq \lim_{x\to 0}\frac{1}{x} - \lim_{x\to 0}\frac{1}{x}.$$

Nevertheless, since $1/x - 1/x = 0$ for $x \neq 0$, the limit of the difference exists

$$\lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x}\right] = \lim_{x\to 0} 0 = 0.$$

(*ii*) Similarly, the limit of a product could exist and yet not be equal to the product of the limits. For example, $x/x = 1$, for $x \neq 0$, and so

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right) = \lim_{x\to 0} 1 = 1$$

but

$$\lim_{x\to 0}\left(x\cdot\frac{1}{x}\right) \neq \left(\lim_{x\to 0} x\right)\left(\lim_{x\to 0}\frac{1}{x}\right)$$

because $\lim_{x\to 0}(1/x)$ does not exist.

(*iii*) Theorem 2.2.5 does not say that the limit of a quotient fails to exist whenever the limit of the denominator is zero. Example 8 provides a counterexample to that interpretation. However, Theorem 2.2.5 states that a limit of a quotient does not exist whenever the limit of the denominator is zero *and* the limit of the numerator is not zero.

Exercises 2.2

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–52, find the given limit, or state that it does not exist.

1. $\lim_{x\to -4} 15$
2. $\lim_{x\to 0} \cos\pi$
3. $\lim_{x\to 3} (-4)x$
4. $\lim_{x\to 2} (3x-9)$
5. $\lim_{x\to -2} x^2$
6. $\lim_{x\to 5} (-x^3)$
7. $\lim_{x\to -1} (x^3-4x+1)$
8. $\lim_{x\to 6} (-5x^2+6x+8)$
9. $\lim_{x\to 2} \dfrac{2x+4}{x-7}$
10. $\lim_{x\to 0} \dfrac{x+5}{3x}$
11. $\lim_{t\to 1} (3t-1)(5t^2+2)$
12. $\lim_{t\to -2} (t+4)^2$
13. $\lim_{s\to 7} \dfrac{s^2-21}{s+2}$
14. $\lim_{x\to 6} \dfrac{x^2-6x}{x^2-7x+6}$
15. $\lim_{x\to -1} (x+x^2+x^3)^{135}$
16. $\lim_{x\to 2} \dfrac{(3x-4)^{40}}{(x^2-2)^{36}}$
17. $\lim_{x\to 6} \sqrt{2x-5}$
18. $\lim_{x\to 8} (1+\sqrt[3]{x})$
19. $\lim_{t\to 1} \dfrac{\sqrt{t}}{t^2+t-2}$
20. $\lim_{x\to 2} x^2\sqrt{x^2+5x+2}$
21. $\lim_{y\to -5} \dfrac{y^2-25}{y+5}$
22. $\lim_{u\to 8} \dfrac{u^2-5u-24}{u-8}$
23. $\lim_{x\to 1} \dfrac{x^3-1}{x-1}$
24. $\lim_{t\to -1} \dfrac{t^3+1}{t^2-1}$
25. $\lim_{x\to 10} \dfrac{(x-2)(x+5)}{(x-8)}$
26. $\lim_{x\to -3} \dfrac{2x+6}{4x^2-36}$
27. $\lim_{x\to 2} \dfrac{x^3+3x^2-10x}{x-2}$
28. $\lim_{x\to 1.5} \dfrac{2x^2+3x-9}{x-1.5}$
29. $\lim_{t\to 1} \dfrac{t^3-2t+1}{t^3+t^2-2}$
30. $\lim_{x\to 0} x^3(x^4+2x^3)^{-1}$
31. $\lim_{x\to 0^+} \dfrac{(x+2)(x^5-1)^3}{(\sqrt{x}+4)^2}$
32. $\lim_{x\to -2} x\sqrt{x+4}\,\sqrt[3]{x-6}$
33. $\lim_{x\to 0} \left[\dfrac{x^2+3x-1}{x}+\dfrac{1}{x}\right]$
34. $\lim_{x\to 2} \left[\dfrac{1}{x-2}-\dfrac{6}{x^2+2x-8}\right]$
35. $\lim_{x\to 3^+} \dfrac{(x+3)^2}{\sqrt{x-3}}$
36. $\lim_{x\to 3} (x-4)^{99}(x^2-7)^{10}$
37. $\lim_{x\to 10} \sqrt{\dfrac{10x}{2x+5}}$
38. $\lim_{r\to 1} \dfrac{\sqrt{(r^2+3r-2)^3}}{\sqrt[3]{(5r-3)^2}}$
39. $\lim_{h\to 4} \sqrt{\dfrac{h}{h+5}\left(\dfrac{h^2-16}{h-4}\right)^2}$
40. $\lim_{t\to 2} (t+2)^{3/2}(2t+4)^{1/3}$
41. $\lim_{x\to 0^-} \sqrt[5]{\dfrac{x^3-64x}{x^2+2x}}$
42. $\lim_{x\to 1^+} \left(8x+\dfrac{2}{x}\right)^5$
43. $\lim_{t\to 1} (at^2-bt)^2$
44. $\lim_{x\to -1} \sqrt{u^2x^2+2xu+1}$
45. $\lim_{h\to 0} \dfrac{(8+h)^2-64}{h}$
46. $\lim_{h\to 0} \dfrac{1}{h}[(1+h)^3-1]$
47. $\lim_{h\to 0} \dfrac{1}{h}\left(\dfrac{1}{x+h}-\dfrac{1}{x}\right)$
48. $\lim_{h\to 0} \dfrac{\sqrt{x+h}-\sqrt{x}}{h}\ (x>0)$
49. $\lim_{t\to 1} \dfrac{\sqrt{t}-1}{t-1}$
50. $\lim_{u\to 5} \dfrac{\sqrt{u+4}-3}{u-5}$
51. $\lim_{v\to 0} \dfrac{\sqrt{25+v}-5}{\sqrt{1+v}-1}$
52. $\lim_{x\to 1} \dfrac{4-\sqrt{x+15}}{x^2-1}$

In Problems 53–60, assume that $\lim_{x\to a} f(x) = 4$ and $\lim_{x\to a} g(x) = 2$. Find the given limit, or state that it does not exist.

53. $\lim_{x\to a} [5f(x)+6g(x)]$
54. $\lim_{x\to a} [f(x)]^3$
55. $\lim_{x\to a} \dfrac{1}{g(x)}$
56. $\lim_{x\to a} \sqrt{\dfrac{f(x)}{g(x)}}$
57. $\lim_{x\to a} \dfrac{f(x)}{f(x)-2g(x)}$
58. $\lim_{x\to a} \dfrac{[f(x)]^2-4[g(x)]^2}{f(x)-2g(x)}$
59. $\lim_{x\to a} xf(x)g(x)$
60. $\lim_{x\to a} \dfrac{6x+3}{xf(x)+g(x)},\ a\neq -\frac{1}{2}$

Think About It

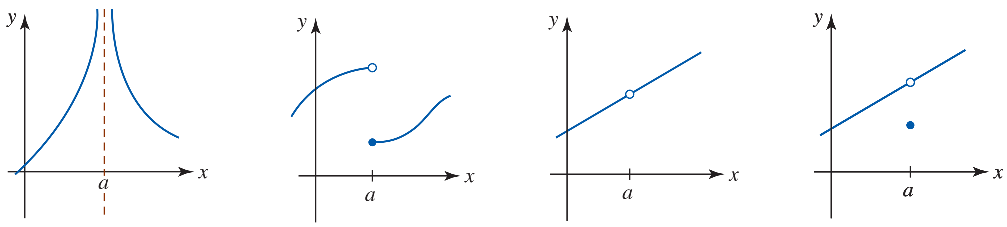
In Problems 61 and 62, use the first result to find the limits in parts (a)–(c). Justify each step in your work citing the appropriate property of limits.

61. $\lim_{x\to 1} \dfrac{x^{100}-1}{x-1} = 100$
 (a) $\lim_{x\to 1} \dfrac{x^{100}-1}{x^2-1}$ (b) $\lim_{x\to 1} \dfrac{x^{50}-1}{x-1}$ (c) $\lim_{x\to 1} \dfrac{(x^{100}-1)^2}{(x-1)^2}$
62. $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$
 (a) $\lim_{x\to 0} \dfrac{2x}{\sin x}$ (b) $\lim_{x\to 0} \dfrac{1-\cos^2 x}{x^2}$ (c) $\lim_{x\to 0} \dfrac{8x^2-\sin x}{x}$
63. Using $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, show that $\lim_{x\to 0} \sin x = 0$.
64. If $\lim_{x\to 2} \dfrac{2f(x)-5}{x+3} = 4$, find $\lim_{x\to 2} f(x)$.

2.3 Continuity

Introduction In the discussion in Section 1.1 on graphing functions, we used the phrase "connect the points with a smooth curve." This phrase invokes the image of a graph that is a nice *continuous* curve—in other words, a curve with no breaks, gaps, or holes in it. Indeed, a continuous function is often described as one whose graph can be drawn without lifting pencil from paper.

In Section 2.2 we saw that the function value $f(a)$ played no part in determining the existence of $\lim_{x \to a} f(x)$. But we did see in Section 2.2 that limits as $x \to a$ of polynomial functions and certain rational functions could be found by simply evaluating the function *at* $x = a$. The reason we can do that in some instances is the fact that the function is *continuous* at a number a. In this section we will see that both the value $f(a)$ and the limit of f as x approaches a number a play major roles in defining the notion of continuity. Before giving the definition, we illustrate in FIGURE 2.3.1 some intuitive examples of graphs of functions that are *not* continuous at a.



(a) $\lim_{x \to a} f(x)$ does not exist and $f(a)$ is not defined

(b) $\lim_{x \to a} f(x)$ does not exist but $f(a)$ is defined

(c) $\lim_{x \to a} f(x)$ exists but $f(a)$ is not defined

(d) $\lim_{x \to a} f(x)$ exists, $f(a)$ is defined, but $\lim_{x \to a} f(x) \neq f(a)$

FIGURE 2.3.1 Four examples of f *not* continuous at a

Continuity at a Number Figure 2.3.1 suggests the following threefold condition of continuity of a function f at a number a.

Definition 2.3.1 Continuity at a

A function f is said to be **continuous** at a number a if

(i) $f(a)$ is defined, (ii) $\lim_{x \to a} f(x)$ exists, and (iii) $\lim_{x \to a} f(x) = f(a)$.

If any one of the three conditions in Definition 2.3.1 fails, then f is said to be **discontinuous** at the number a.

EXAMPLE 1 Three Functions

Determine whether each of the functions is continuous at 1.

(a) $f(x) = \dfrac{x^3 - 1}{x - 1}$ **(b)** $g(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$ **(c)** $h(x) = \begin{cases} \dfrac{x^3 - 1}{x - 1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$.

Solution

(a) f is discontinuous at 1 since substituting $x = 1$ into the function results in $0/0$. We say that $f(1)$ is not defined and so the first condition of continuity in Definition 2.3.1 is violated.

(b) Because g is defined at 1, that is, $g(1) = 2$, we next determine whether $\lim_{x \to 1} g(x)$ exists. From

$$\lim_{x \to 1} \frac{x^3 - 1}{x - 1} = \lim_{x \to 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \to 1}(x^2 + x + 1) = 3 \qquad (1)$$

Recall from algebra that $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

we conclude $\lim_{x \to 1} g(x)$ exists and equals 3. Since this value is not the same as $g(1) = 2$, the second condition of Definition 2.3.1 is violated. The function g is discontinuous at 1.

(c) First, $h(1)$ is defined, in this case, $h(1) = 3$. Second, $\lim_{x \to 1} h(x) = 3$ from (1) of part (b). Third, we have $\lim_{x \to 1} h(x) = h(1) = 3$. Thus *all* three conditions in Definition 2.3.1 are satisfied and so the function h is continuous at 1.

The graphs of the three functions are compared in FIGURE 2.3.2.

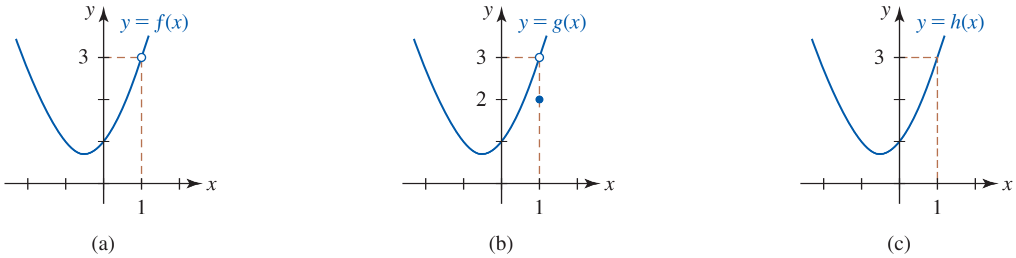


FIGURE 2.3.2 Graphs of functions in Example 1 ■

EXAMPLE 2 Piecewise-Defined Function

Determine whether the piecewise-defined function is continuous at 2.

$$f(x) = \begin{cases} x^2, & x < 2 \\ 5, & x = 2 \\ -x + 6, & x > 2. \end{cases}$$

Solution First, observe that $f(2)$ is defined and equals 5. Next, we see from

$$\left.\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} x^2 = 4 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} (-x + 6) = 4 \end{aligned}\right\} \text{implies } \lim_{x \to 2} f(x) = 4$$

that the limit of f as $x \to 2$ exists. Finally, because $\lim_{x \to 2} f(x) \neq f(2) = 5$, it follows from (*iii*) of Definition 2.3.1 that f is discontinuous at 2. The graph of f is shown in FIGURE 2.3.3. ■

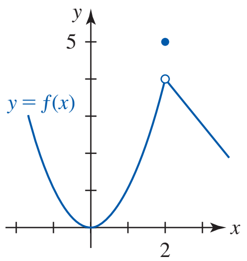


FIGURE 2.3.3 Graph of function in Example 2

Continuity on an Interval We will now extend the notion of continuity at a number a to **continuity on an interval**.

Definition 2.3.2 Continuity on an Interval

A function f is continuous

(*i*) on an **open interval** (a, b) if it is continuous at every number in the interval; and
(*ii*) on a **closed interval** $[a, b]$ if it is continuous on (a, b) and, in addition,

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to b^-} f(x) = f(b).$$

If the right-hand limit condition $\lim_{x \to a^+} f(x) = f(a)$ given in (*ii*) of Definition 2.3.1 is satisfied, we say that ***f* is continuous from the right at *a***; if $\lim_{x \to b^-} f(x) = f(b)$, then ***f* is continuous from the left at *b***.

Extensions of these concepts to intervals such as $[a, b)$, $(a, b]$, (a, ∞), $(-\infty, b)$, $(-\infty, \infty)$, $[a, \infty)$, and $(-\infty, b]$ are made in the expected manner. For example, f is continuous on $[1, 5)$ if it is continuous on the open interval $(1, 5)$ and continuous from the right at 1.

EXAMPLE 3 Continuity on an Interval

(a) As we see from FIGURE 2.3.4(a), $f(x) = 1/\sqrt{1 - x^2}$ is continuous on the open interval $(-1, 1)$ but is not continuous on the closed interval $[-1, 1]$, since neither $f(-1)$ nor $f(1)$ is defined.

(b) $f(x) = \sqrt{1 - x^2}$ is continuous on $[-1, 1]$. Observe from Figure 2.3.4(b) that

$$\lim_{x \to -1^+} f(x) = f(-1) = 0 \text{ and } \lim_{x \to 1^-} f(x) = f(1) = 0.$$

(c) $f(x) = \sqrt{x - 1}$ is continuous on the unbounded interval $[1, \infty)$, because

$$\lim_{x \to a} f(x) = \sqrt{\lim_{x \to a}(x - 1)} = \sqrt{a - 1} = f(a),$$

for any real number a satisfying $a > 1$, and f is continuous from the right at 1 since

$$\lim_{x \to 1^+} \sqrt{x - 1} = f(1) = 0.$$

See Figure 2.3.4(c). ■

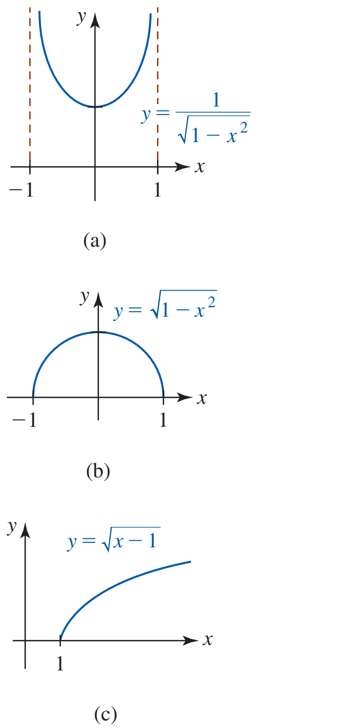


FIGURE 2.3.4 Graphs of functions in Example 3

A review of the graphs in Figures 1.4.1 and 1.4.2 shows that $y = \sin x$ and $y = \cos x$ are continuous on $(-\infty, \infty)$. Figures 1.4.3 and 1.4.5 show that $y = \tan x$ and $y = \sec x$ are discontinuous at $x = (2n + 1)\pi/2$, $n = 0, \pm 1, \pm 2, \ldots$, whereas Figures 1.4.4 and 1.4.6 show that $y = \cot x$ and $y = \csc x$ are discontinuous at $x = n\pi$, $n = 0, \pm 1, \pm 2, \ldots$. The inverse trigonometric functions $y = \sin^{-1} x$ and $y = \cos^{-1} x$ are continuous on the closed interval $[-1, 1]$. See Figures 1.5.9 and 1.5.12. The natural exponential function $y = e^x$ is continuous on $(-\infty, \infty)$, whereas the natural logarithmic function $y = \ln x$ is continuous on $(0, \infty)$. See Figures 1.6.5 and 1.6.6.

Continuity of a Sum, Product, and Quotient When two functions f and g are continuous at a number a, then the combinations of functions formed by addition, multiplication, and division are also continuous at a. In the case of division f/g we must, of course, require that $g(a) \neq 0$.

> **Theorem 2.3.1** Continuity of a Sum, Product, and Quotient
>
> If the functions f and g are continuous at a number a, then the sum $f + g$, the product fg, and the quotient f/g $(g(a) \neq 0)$ are continuous at $x = a$.

PROOF OF CONTINUITY OF THE PRODUCT *fg* As a consequence of the assumption that the functions f and g are continuous at a number a, we can say that both functions are defined at $x = a$, the limits of both functions as x approaches a exist, and

$$\lim_{x \to a} f(x) = f(a) \qquad \text{and} \qquad \lim_{x \to a} g(x) = g(a).$$

Because the limits exist, we know that the limit of a product is the product of the limits:

$$\lim_{x \to a} (f(x)g(x)) = \left(\lim_{x \to a} f(x)\right)\left(\lim_{x \to a} g(x)\right) = f(a)g(a).$$

The proofs of the remaining parts of Theorem 2.3.1 are obtained in a similar manner. ■

Since Definition 2.3.1 implies that $f(x) = x$ is continuous at any real number x, we see from successive applications of Theorem 2.3.1 that the functions $x, x^2, x^3, \ldots, x^n$ are also continuous for every x in the interval $(-\infty, \infty)$. Because a polynomial function is just a sum of powers of x, another application of Theorem 2.3.1 shows:

- *A polynomial function f is continuous on* $(-\infty, \infty)$.

Functions, such as polynomials and the sine and cosine, that are continuous for *all* real numbers, that is, on the interval $(-\infty, \infty)$, are said to be **continuous everywhere**. A function that is continuous everywhere is also just said to be **continuous**. Now, if $p(x)$ and $q(x)$ are polynomial functions, it also follows directly from Theorem 2.3.1 that:

- *A rational function $f(x) = p(x)/q(x)$ is continuous except at numbers at which the denominator $q(x)$ is zero.*

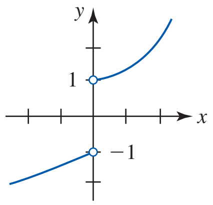
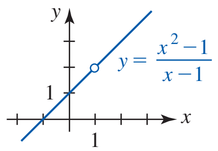
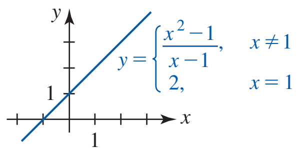


FIGURE 2.3.5 Jump discontinuity at $x = 0$



(a) Not continuous at 1



(b) Continuous at 1

FIGURE 2.3.6 Removable discontinuity at $x = 1$

Terminology A discontinuity of a function f is often given a special name.

- If $x = a$ is a vertical asymptote for the graph of $y = f(x)$, then f is said to have an **infinite discontinuity** at a.

Figure 2.3.1(a) illustrates a function with an infinite discontinuity at a.

- If $\lim_{x\to a^-} f(x) = L_1$ and $\lim_{x\to a^+} f(x) = L_2$ and $L_1 \neq L_2$, then f is said to have a **finite discontinuity** or a **jump discontinuity** at a.

The function $y = f(x)$ given in FIGURE 2.3.5 has a jump discontinuity at 0, since $\lim_{x\to 0^-} f(x) = -1$ and $\lim_{x\to 0^+} f(x) = 1$. The greatest integer function $f(x) = \lfloor x \rfloor$ has a jump discontinuity at every integer value of x.

- If $\lim_{x\to a} f(x)$ exists but either f is not defined at $x = a$ or $f(a) \neq \lim_{x\to a} f(x)$, then f is said to have a **removable discontinuity** at a.

For example, the function $f(x) = (x^2 - 1)/(x - 1)$ is not defined at $x = 1$ but $\lim_{x\to 1} f(x) = 2$. By *defining* $f(1) = 2$, the new function

$$f(x) = \begin{cases} \dfrac{x^2 - 1}{x - 1}, & x \neq 1 \\ 2, & x = 1 \end{cases}$$

is continuous everywhere. See FIGURE 2.3.6.

Continuity of f^{-1} The plausibility of the next theorem follows from the fact that the graph of an inverse function f^{-1} is a reflection of the graph of f in the line $y = x$.

Theorem 2.3.2 Continuity of an Inverse Function

If f is a continuous one-to-one function on an interval $[a, b]$, then f^{-1} is continuous on either $[f(a), f(b)]$ or $[f(b), f(a)]$.

The sine function, $f(x) = \sin x$, is continuous on $[-\pi/2, \pi/2]$ and, as noted previously, the inverse of f, $y = \sin^{-1}x$, is continuous on the closed interval $[f(-\pi/2), f(\pi/2)] = [-1, 1]$.

Limit of a Composite Function The next theorem tells us that if a function f is continuous, then the limit of the function is the function of the limit. The proof of Theorem 2.3.3 is given in the *Appendix*.

Theorem 2.3.3 Limit of a Composite Function

If $\lim_{x\to a} g(x) = L$ and f is continuous at L, then

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L).$$

Theorem 2.3.3 is useful in proving other theorems. If the function g is continuous at a and f is continuous at $g(a)$, then we see that

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right) = f(g(a)).$$

We have just proved that the composite of two continuous functions is continuous.

Theorem 2.3.4 Continuity of a Composite Function

If g is continuous at a number a and f is continuous at $g(a)$, then the composite function $(f \circ g)(x) = f(g(x))$ is continuous at a.

EXAMPLE 4 Continuity of a Composite Function

$f(x) = \sqrt{x}$ is continuous on the interval $[0, \infty)$ and $g(x) = x^2 + 2$ is continuous on $(-\infty, \infty)$. But, since $g(x) \geq 0$ for all x, the composite function

$$(f \circ g)(x) = f(g(x)) = \sqrt{x^2 + 2}$$

is continuous everywhere. ■

If a function f is continuous on a closed interval $[a, b]$, then, as illustrated in FIGURE 2.3.7, f takes on all values between $f(a)$ and $f(b)$. Put another way, a continuous function f does not "skip" any values.

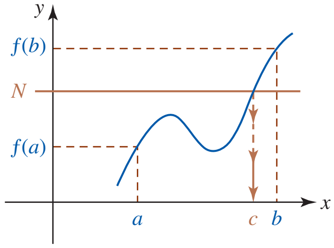


FIGURE 2.3.7 A continuous function f takes on all values between $f(a)$ and $f(b)$

Theorem 2.3.5 Intermediate Value Theorem

If f denotes a function continuous on a closed interval $[a, b]$ for which $f(a) \neq f(b)$, and if N is any number between $f(a)$ and $f(b)$, then there exists at least one number c between a and b such that $f(c) = N$.

EXAMPLE 5 Consequence of Continuity

The polynomial function $f(x) = x^2 - x - 5$ is continuous on the interval $[-1, 4]$ and $f(-1) = -3, f(4) = 7$. For any number N for which $-3 \leq N \leq 7$, Theorem 2.3.5 guarantees that there is a solution to the equation $f(c) = N$, that is, $c^2 - c - 5 = N$ in $[-1, 4]$. Specifically, if we choose $N = 1$, then $c^2 - c - 5 = 1$ is equivalent to

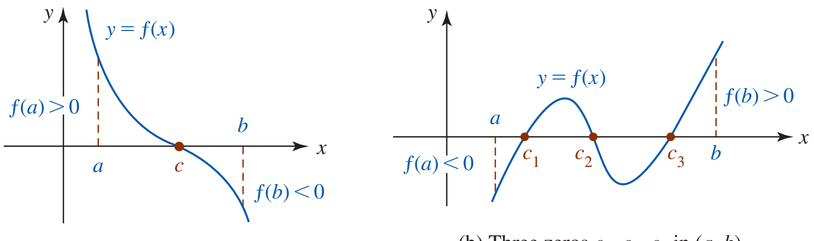
$$c^2 - c - 6 = 0 \quad \text{or} \quad (c - 3)(c + 2) = 0.$$

Although the latter equation has two solutions, only the value $c = 3$ is between -1 and 4. ■

The foregoing example suggests a corollary to the Intermediate Value Theorem.

- *If f satisfies the hypotheses of Theorem 2.3.5 and $f(a)$ and $f(b)$ have opposite algebraic signs, then there exists a number x between a and b for which $f(x) = 0$.*

This fact is often used in locating real zeros of a continuous function f. If the function values $f(a)$ and $f(b)$ have opposite signs, then by identifying $N = 0$, we can say that there is at least one number c in (a, b) for which $f(c) = 0$. In other words, if either $f(a) > 0, f(b) < 0$ or $f(a) < 0, f(b) > 0$, then $f(x)$ has at least one zero c in the interval (a, b). The plausibility of this conclusion is illustrated in FIGURE 2.3.8.



(a) One zero c in (a, b)

(b) Three zeros c_1, c_2, c_3 in (a, b)

FIGURE 2.3.8 Locating zeros of functions using the Intermediate Value Theorem

Bisection Method As a direct consequence of the Intermediate Value Theorem, we can devise a means of approximating the zeros of a continuous function to any degree of accuracy. Suppose $y = f(x)$ is continuous on the closed interval $[a, b]$ such that $f(a)$ and $f(b)$ have opposite algebraic signs. Then, as we have just seen, f has a zero in $[a, b]$. Suppose we bisect the interval $[a, b]$ by finding its midpoint $m_1 = (a + b)/2$. If $f(m_1) = 0$, then m_1 is a zero of f and we proceed no further, but if $f(m_1) \neq 0$, then we can say that:

- If $f(a)$ and $f(m_1)$ have opposite algebraic signs, then f has a zero c in $[a, m_1]$.
- If $f(m_1)$ and $f(b)$ have opposite algebraic signs, then f has a zero c in $[m_1, b]$.

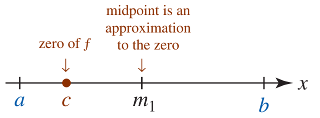


FIGURE 2.3.9 The number m_1 is an approximation to the number c

That is, if $f(m_1) \neq 0$, then f has a zero in an interval that is one-half the length of the original interval. See FIGURE 2.3.9. We now repeat the process by bisecting this new interval by finding its midpoint m_2. If m_2 is a zero of f, we stop, but if $f(m_2) \neq 0$, we have located a zero in an interval that is one-fourth the length of $[a, b]$. We continue this process of locating a zero of f in shorter and shorter intervals indefinitely. This method of approximating a zero of a continuous function by a sequence of midpoints is called the **bisection method**. Reinspection of Figure 2.3.9 shows that the error in an approximation to a zero in an interval is less than one-half the length of the interval.

EXAMPLE 6 Zeros of a Polynomial Function

(a) Show that the polynomial function $f(x) = x^6 - 3x - 1$ has a real zero in $[-1, 0]$ and in $[1, 2]$.
(b) Approximate the zero in $[1, 2]$ to two decimal places.

Solution

(a) Observe that $f(-1) = 3 > 0$ and $f(0) = -1 < 0$. This change in sign indicates that the graph of f must cross the x-axis at least once in the interval $[-1, 0]$. In other words, there is at least one zero of f in $[-1, 0]$.

Similarly, $f(1) = -3 < 0$ and $f(2) = 57 > 0$ implies that there is at least one zero of f in the interval $[1, 2]$.

(b) A first approximation to the zero in $[1, 2]$ is the midpoint of the interval:

$$m_1 = \frac{1+2}{2} = \frac{3}{2} = 1.5, \qquad \text{error} < \frac{1}{2}(2-1) = 0.5.$$

Now since $f(m_1) = f\left(\frac{3}{2}\right) > 0$ and $f(1) < 0$, we know that the zero lies in the interval $\left[1, \frac{3}{2}\right]$.

The second approximation is the midpoint of $\left[1, \frac{3}{2}\right]$:

$$m_2 = \frac{1+\frac{3}{2}}{2} = \frac{5}{4} = 1.25, \qquad \text{error} < \frac{1}{2}\left(\frac{3}{2} - 1\right) = 0.25.$$

Since $f(m_2) = f\left(\frac{5}{4}\right) < 0$, the zero lies in the interval $\left[\frac{5}{4}, \frac{3}{2}\right]$.

The third approximation is the midpoint of $\left[\frac{5}{4}, \frac{3}{2}\right]$:

$$m_3 = \frac{\frac{5}{4}+\frac{3}{2}}{2} = \frac{11}{8} = 1.375, \qquad \text{error} < \frac{1}{2}\left(\frac{3}{2} - \frac{5}{4}\right) = 0.125.$$

After eight calculations, we find that $m_8 = 1.300781$ with error less than 0.005. Hence, 1.30 is an approximation to the zero of f in $[1, 2]$ that is accurate to two decimal places. The graph of f is given in FIGURE 2.3.10. ■

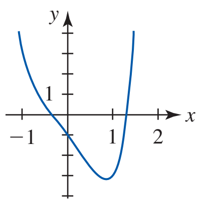


FIGURE 2.3.10 Graph of function in Example 6

If we wish the approximation to be accurate to *three* decimal places, we continue until the error becomes less

Exercises 2.3

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–12, determine the numbers, if any, at which the given function f is discontinuous.

1. $f(x) = x^3 - 4x^2 + 7$

2. $f(x) = \dfrac{x}{x^2 + 4}$

3. $f(x) = (x^2 - 9x + 18)^{-1}$

4. $f(x) = \dfrac{x^2 - 1}{x^4 - 1}$

5. $f(x) = \dfrac{x - 1}{\sin 2x}$

6. $f(x) = \dfrac{\tan x}{x + 3}$

7. $f(x) = \begin{cases} x, & x < 0 \\ x^2, & 0 \le x < 2 \\ x, & x > 2 \end{cases}$

8. $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

9. $f(x) = \begin{cases} \dfrac{x^2 - 25}{x - 5}, & x \neq 5 \\ 10, & x = 5 \end{cases}$

10. $f(x) = \begin{cases} \dfrac{x - 1}{\sqrt{x} - 1}, & x \neq 1 \\ \dfrac{1}{2}, & x = 1 \end{cases}$

11. $f(x) = \dfrac{1}{2 + \ln x}$

12. $f(x) = \dfrac{2}{e^x - e^{-x}}$

In Problems 13–24, determine whether the given function f is continuous on the indicated intervals.

13. $f(x) = x^2 + 1$
(a) $[-1, 4]$ **(b)** $[5, \infty)$

14. $f(x) = \dfrac{1}{x}$
(a) $(-\infty, \infty)$ **(b)** $(0, \infty)$

15. $f(x) = \dfrac{1}{\sqrt{x}}$
(a) $(0, 4]$ **(b)** $[1, 9]$

16. $f(x) = \sqrt{x^2 - 9}$
(a) $[-3, 3]$ **(b)** $[3, \infty)$

17. $f(x) = \tan x$
(a) $[0, \pi]$ **(b)** $[-\pi/2, \pi/2]$

18. $f(x) = \csc x$
(a) $(0, \pi)$ **(b)** $(2\pi, 3\pi)$

19. $f(x) = \dfrac{x}{x^3 + 8}$
(a) $[-4, -3]$ **(b)** $(-\infty, \infty)$

20. $f(x) = \dfrac{1}{|x| - 4}$
(a) $(-\infty, -1]$ **(b)** $[1, 6]$

21. $f(x) = \dfrac{x}{2 + \sec x}$
(a) $(-\infty, \infty)$ **(b)** $[\pi/2, 3\pi/2]$

22. $f(x) = \sin\dfrac{1}{x}$
(a) $[1/\pi, \infty)$ **(b)** $[-2/\pi, 2/\pi]$

23.

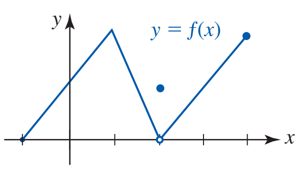


FIGURE 2.3.11 Graph for Problem 23

(a) $[-1, 3]$ **(b)** $(2, 4]$

24.

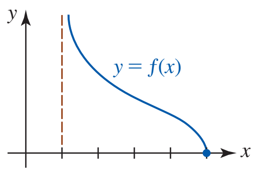


FIGURE 2.3.12 Graph for Problem 24

(a) $[2, 4]$ **(b)** $[1, 5]$

In Problems 25–28, find values of m and n so that the given function f is continuous.

25. $f(x) = \begin{cases} mx, & x < 4 \\ x^2, & x \ge 4 \end{cases}$

26. $f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & x \neq 2 \\ m, & x = 2 \end{cases}$

27. $f(x) = \begin{cases} mx, & x < 3 \\ n, & x = 3 \\ -2x + 9, & x > 3 \end{cases}$

28. $f(x) = \begin{cases} mx - n, & x < 1 \\ 5, & x = 1 \\ 2mx + n, & x > 1 \end{cases}$

In Problems 29 and 30, $\lfloor x \rfloor$ denotes the greatest integer not exceeding x. Sketch a graph to determine the points at which the given function is discontinuous.

29. $f(x) = \lfloor 2x - 1 \rfloor$

30. $f(x) = \lfloor x \rfloor - x$

In Problems 31 and 32, determine whether the given function has a removable discontinuity at the given number a. If the discontinuity is removable, define a new function that is continuous at a.

31. $f(x) = \dfrac{x - 9}{\sqrt{x} - 3}, \quad a = 9$

32. $f(x) = \dfrac{x^4 - 1}{x^2 - 1}, \quad a = 1$

In Problems 33–42, use Theorem 2.3.3 to find the given limit.

33. $\lim_{x \to \pi/6} \sin(2x + \pi/3)$

34. $\lim_{x \to \pi^2} \cos\sqrt{x}$

35. $\lim_{x \to \pi/2} \sin(\cos x)$

36. $\lim_{x \to \pi/2} (1 + \cos(\cos x))$

37. $\lim_{t \to \pi} \cos\left(\dfrac{t^2 - \pi^2}{t - \pi}\right)$

38. $\lim_{t \to 0} \tan\left(\dfrac{\pi t}{t^2 + 3t}\right)$

39. $\lim_{t \to \pi} \sqrt{t - \pi + \cos^2 t}$

40. $\lim_{t \to 1} (4t + \sin 2\pi t)^3$

41. $\lim_{x \to -3} \sin^{-1}\left(\dfrac{x + 3}{x^2 + 4x + 3}\right)$

42. $\lim_{x \to \pi} e^{\cos 3x}$

In Problems 43 and 44, determine the interval(s) where $f \circ g$ is continuous.

43. $f(x) = \dfrac{1}{\sqrt{x - 1}}, \quad g(x) = x + 4$

44. $f(x) = \dfrac{5x}{x - 1}, \quad g(x) = (x - 2)^2$

In Problems 45–48, verify the Intermediate Value Theorem for f on the given interval. Find a number c in the interval for the indicated value of N.

45. $f(x) = x^2 - 2x$, $[1, 5]$; $N = 8$

46. $f(x) = x^2 + x + 1$, $[-2, 3]$; $N = 6$

47. $f(x) = x^3 - 2x + 1$, $[-2, 2]$; $N = 1$

48. $f(x) = \dfrac{10}{x^2 + 1}$, $[0, 1]$; $N = 8$

49. Given that $f(x) = x^5 + 2x - 7$, show that there is a number c such that $f(c) = 50$.

50. Given that f and g are continuous on $[a, b]$ such that $f(a) > g(a)$ and $f(b) < g(b)$, show that there is a number c in (a, b) such that $f(c) = g(c)$. [*Hint*: Consider the function $f - g$.]

In Problems 51–54, show that the given equation has a solution in the indicated interval.

51. $2x^7 = 1 - x$, $(0, 1)$

52. $\dfrac{x^2 + 1}{x + 3} + \dfrac{x^4 + 1}{x - 4} = 0$, $(-3, 4)$

53. $e^{-x} = \ln x$, $(1, 2)$

54. $\dfrac{\sin x}{x} = \dfrac{1}{2}$, $(\pi/2, \pi)$

Calculator/CAS Problems

In Problems 55 and 56, use a calculator or CAS to obtain the graph of the given function. Use the bisection method to approximate, to an accuracy of two decimal places, the real zeros of f that you discover from the graph.

55. $f(x) = 3x^5 - 5x^3 - 1$ **56.** $f(x) = x^5 + x - 1$

57. Use the bisection method to approximate the value of c in Problem 49 to an accuracy of two decimal places.

58. Use the bisection method to approximate the solution in Problem 51 to an accuracy of two decimal places.

59. Use the bisection method to approximate the solution in Problem 52 to an accuracy of two decimal places.

60. Suppose a closed right-circular cylinder has a given volume V and surface area S (lateral side, top, and bottom).

(a) Show that the radius r of the cylinder must satisfy the equation $2\pi r^3 - Sr + 2V = 0$.

(b) Suppose $V = 3000$ ft^3 and $S = 1800$ ft^2. Use a calculator or CAS to obtain the graph of

$$f(r) = 2\pi r^3 - 1800r + 6000.$$

(c) Use the graph in part (b) and the bisection method to find the dimensions of the cylinder corresponding to the volume and surface area given in part (b). Use an accuracy of two decimal places.

Think About It

61. Given that f and g are continuous at a number a, prove that $f + g$ is continuous at a.

62. Given that f and g are continuous at a number a and $g(a) \neq 0$, prove that f/g is continuous at a.

63. Let $f(x) = \lfloor x \rfloor$ be the greatest integer function and $g(x) = \cos x$. Determine the points at which $f \circ g$ is discontinuous.

64. Consider the functions

$$f(x) = |x| \quad \text{and} \quad g(x) = \begin{cases} x + 1, & x < 0 \\ x - 1, & x \geq 0. \end{cases}$$

Sketch the graphs of $f \circ g$ and $g \circ f$. Determine whether $f \circ g$ and $g \circ f$ are continuous at 0.

65. A Mathematical Classic The **Dirichlet function**

$$f(x) = \begin{cases} 1, & x \text{ rational} \\ 0, & x \text{ irrational} \end{cases}$$

is named after the German mathematician **Johann Peter Gustav Lejeune Dirichlet** (1805–1859). Dirichlet is responsible for the definition of a function as we know it today.

(a) Show that f is discontinuous at every real number a. In other words, f is a *nowhere continuous function*.

(b) What does the graph of f look like?

(c) If r is a positive rational number, show that f is r-periodic, that is, $f(x + r) = f(x)$.

2.4 Trigonometric Limits

Introduction In this section we examine limits that involve trigonometric functions. As the examples in this section will illustrate, computation of trigonometric limits entails both algebraic manipulations and knowledge of some basic trigonometric identities. We begin with some simple limit results that are consequences of continuity.

Using Continuity We saw in the preceding section that the sine and cosine functions are everywhere continuous. It follows from Definition 2.3.1 that for any real number a,

$$\lim_{x \to a} \sin x = \sin a, \tag{1}$$

$$\lim_{x \to a} \cos x = \cos a. \tag{2}$$

Similarly, for a number a in the domain of the given trigonometric function

$$\lim_{x\to a} \tan x = \tan a, \qquad \lim_{x\to a} \cot x = \cot a, \tag{3}$$

$$\lim_{x\to a} \sec x = \sec a, \qquad \lim_{x\to a} \csc x = \csc a. \tag{4}$$

EXAMPLE 1 Using (1) and (2)

From (1) and (2) we have

$$\lim_{x\to 0} \sin x = \sin 0 = 0 \quad \text{and} \quad \lim_{x\to 0} \cos x = \cos 0 = 1. \tag{5}$$ ■

We will draw on the results in (5) in the following discussion on computing other trigonometric limits. But first, we consider a theorem that is particularly useful when working with trigonometric limits.

Squeeze Theorem The next theorem has many names: **Squeeze Theorem**, **Pinching Theorem**, **Sandwiching Theorem**, **Squeeze Play Theorem**, and **Flyswatter Theorem** are just a few of them. As shown in FIGURE 2.4.1, if the graph of $f(x)$ is "squeezed" between the graphs of two other functions $g(x)$ and $h(x)$ for all x close to a, and if the functions g and h have a common limit L as $x \to a$, it stands to reason that f also approaches L as $x \to a$. The proof of Theorem 2.4.1 is given in the *Appendix*.

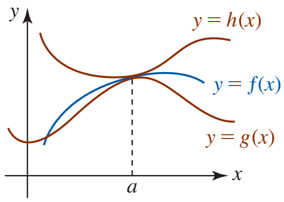


FIGURE 2.4.1 Graph of f squeezed between the graphs g and h

Theorem 2.4.1 Squeeze Theorem

Suppose f, g, and h are functions for which $g(x) \le f(x) \le h(x)$ for all x in an open interval that contains a number a, except possibly at a itself. If

$$\lim_{x\to a} g(x) = L \quad \text{and} \quad \lim_{x\to a} h(x) = L,$$

then $\lim_{x\to a} f(x) = L$.

A colleague from Russia said this result was called the **Two Soldiers Theorem** when he was in school. Think about it.

Before applying Theorem 2.4.1, let us consider a trigonometric limit that does not exist.

EXAMPLE 2 A Limit That Does Not Exist

The limit $\lim_{x\to 0} \sin(1/x)$ does not exist. The function $f(x) = \sin(1/x)$ is odd but is not periodic. The graph f oscillates between -1 and 1 as $x \to 0$:

$$\sin\frac{1}{x} = \pm 1 \quad \text{for} \quad \frac{1}{x} = \frac{\pi}{2} + n\pi, \quad n = 0, \pm 1, \pm 2, \ldots.$$

For example, $\sin(1/x) = 1$ for $n = 500$ or $x \approx 0.00064$, and $\sin(1/x) = -1$ for $n = 501$ or $x \approx 0.00063$. This means that near the origin the graph of f becomes so compressed that it appears to be one continuous smear of color. See FIGURE 2.4.2. ■

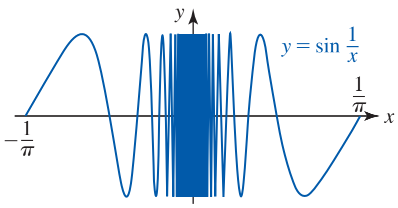


FIGURE 2.4.2 Graph of function in Example 2

EXAMPLE 3 Using the Squeeze Theorem

Find the limit $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

Solution First observe that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} \ne \left(\lim_{x\to 0} x^2\right)\left(\lim_{x\to 0} \sin\frac{1}{x}\right)$$

because we have just seen in Example 2 that $\lim_{x\to 0} \sin(1/x)$ does not exist. But for $x \ne 0$ we have $-1 \le \sin(1/x) \le 1$. Therefore,

$$-x^2 \le x^2 \sin\frac{1}{x} \le x^2.$$

Now if we make the identifications $g(x) = -x^2$ and $h(x) = x^2$, it follows from (1) of Section 2.2 that $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} h(x) = 0$. Hence, from the Squeeze Theorem we conclude that

$$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0.$$

In FIGURE 2.4.3 note the small scale on the x- and y-axes.

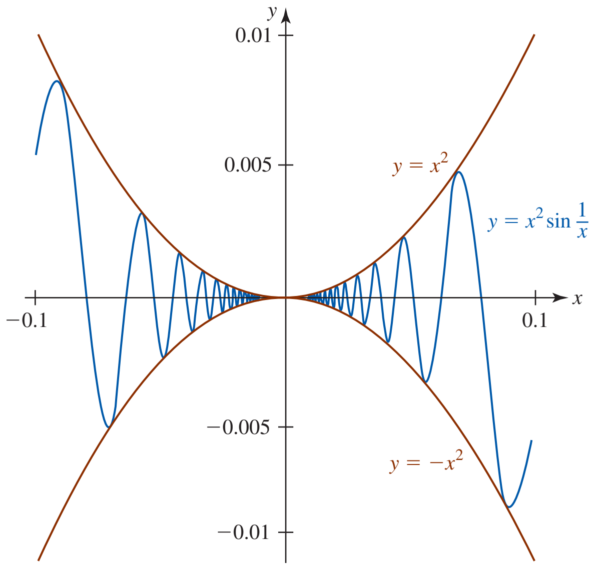


FIGURE 2.4.3 Graph of function in Example 3 ■

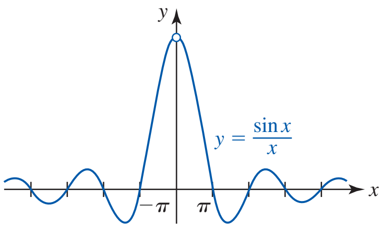


FIGURE 2.4.4 Graph of $f(x) = (\sin x)/x$

An Important Trigonometric Limit Although the function $f(x) = (\sin x)/x$ is not defined at $x = 0$, the numerical table in Example 7 of Section 2.1 and the graph in FIGURE 2.4.4 suggests that $\lim_{x\to 0} (\sin x)/x$ exists. We are now able to prove this conjecture using the Squeeze Theorem.

Consider a circle centered at the origin O with radius 1. As shown in FIGURE 2.4.5(a), let the shaded region OPR be a sector of the circle with central angle t such that $0 < t < \pi/2$. We see from parts (b), (c), and (d) of Figure 2.4.5 that

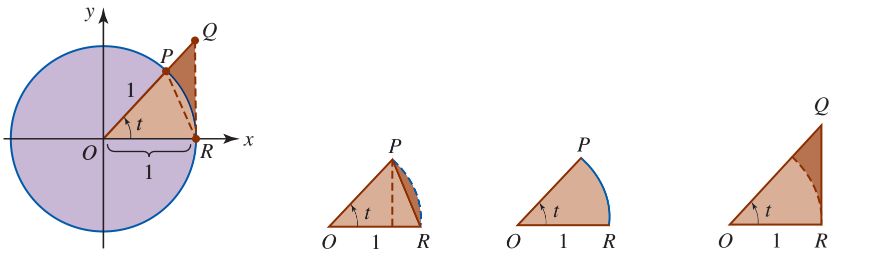
$$\text{area of } \triangle OPR \le \text{area of sector } OPR \le \text{area of } \triangle OQR. \tag{6}$$

From Figure 2.4.5(b) the height of $\triangle OPR$ is $\overline{OP}\sin t = 1 \cdot \sin t = \sin t$, and so

$$\text{area of } \triangle OPR = \frac{1}{2}\overline{OR}\cdot(\text{height}) = \frac{1}{2}\cdot 1 \cdot \sin t = \frac{1}{2}\sin t. \tag{7}$$

From Figure 2.4.5(d), $\overline{QR}/\overline{OR} = \tan t$ or $\overline{QR} = \tan t$, so that

$$\text{area of } \triangle OQR = \frac{1}{2}\overline{OR}\cdot\overline{QR} = \frac{1}{2}\cdot 1 \cdot \tan t = \frac{1}{2}\tan t. \tag{8}$$



(a) Unit circle (b) Triangle OPR (c) Sector OPR (d) Right triangle OQR

FIGURE 2.4.5 Unit circle along with two triangles and a circular sector

Finally, the area of a sector of a circle is $\frac{1}{2}r^2\theta$, where r is its radius and θ is the central angle measured in radians. Thus,

$$\text{area of sector } OPR = \frac{1}{2} \cdot 1^2 \cdot t = \frac{1}{2}t. \tag{9}$$

Using (7), (8), and (9) in the inequality (6) gives

$$\frac{1}{2}\sin t < \frac{1}{2}t < \frac{1}{2}\tan t \qquad \text{or} \qquad 1 < \frac{t}{\sin t} < \frac{1}{\cos t}.$$

From the properties of inequalities, the last inequality can be written

$$\cos t < \frac{\sin t}{t} < 1.$$

We now let $t \to 0^+$ in the last result. Since $(\sin t)/t$ is "squeezed" between 1 and $\cos t$ (which we know from (5) is approaching 1), it follows from Theorem 2.4.1 that $(\sin t)/t \to 1$. While we have assumed $0 < t < \pi/2$, the same result holds for $t \to 0^-$ when $-\pi/2 < t < 0$. Using the symbol x in place of t, we summarize the result:

$$\lim_{x\to 0} \frac{\sin x}{x} = 1. \tag{10}$$

As the following examples illustrate, the results in (1), (2), (3), and (10) are used often to compute other limits. Note that the limit (10) is the indeterminate form 0/0.

EXAMPLE 4 Using (10)

Find the limit $\lim_{x\to 0} \dfrac{10x - 3\sin x}{x}$.

Solution We rewrite the fractional expression as two fractions with the same denominator x:

$$\begin{aligned}
\lim_{x\to 0} \frac{10x - 3\sin x}{x} &= \lim_{x\to 0}\left[\frac{10x}{x} - \frac{3\sin x}{x}\right] \\
&= \lim_{x\to 0} \frac{10x}{x} - 3\lim_{x\to 0}\frac{\sin x}{x} \quad \leftarrow \text{since both limits exist, also cancel the } x \text{ in the first expression} \\
&= \lim_{x\to 0} 10 - 3\lim_{x\to 0}\frac{\sin x}{x} \quad \leftarrow \text{now use (10)} \\
&= 10 - 3 \cdot 1 \\
&= 7.
\end{aligned}$$

■

EXAMPLE 5 Using the Double-Angle Formula

Find the limit $\lim_{x\to 0} \dfrac{\sin 2x}{x}$.

Solution To evaluate the given limit we make use of the double-angle formula $\sin 2x = 2\sin x\cos x$ of Section 1.4, and the fact the limits exist:

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin 2x}{x} &= \lim_{x\to 0} \frac{2\cos x \sin x}{x} \\
&= 2\lim_{x\to 0}\left(\cos x \cdot \frac{\sin x}{x}\right) \\
&= 2\left(\lim_{x\to 0}\cos x\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right).
\end{aligned}$$

From (5) and (10) we know that $\cos x \to 1$ and $(\sin x)/x \to 1$ as $x \to 0$, and so the preceding line becomes

$$\lim_{x\to 0}\frac{\sin 2x}{x} = 2 \cdot 1 \cdot 1 = 2.$$

■

EXAMPLE 6 Using (5) and (10)

Find the limit $\lim_{x\to 0}\frac{\tan x}{x}$.

Solution Using $\tan x = (\sin x)/\cos x$ and the fact that the limits exist we can write

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x}{x} &= \lim_{x\to 0}\frac{(\sin x)/\cos x}{x}\\ &= \lim_{x\to 0}\frac{1}{\cos x}\cdot\frac{\sin x}{x}\\ &= \left(\lim_{x\to 0}\frac{1}{\cos x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\\ &= \frac{1}{1}\cdot 1 = 1. \leftarrow \text{from (5) and (10)}\end{aligned}$$

■

Using a Substitution We are often interested in limits similar to that considered in Example 5. But if we wish to find, say, $\lim_{x\to 0}\frac{\sin 5x}{x}$ the procedure employed in Example 5 breaks down at a practical level since we do not have a readily available trigonometric identity for $\sin 5x$. There is an alternative procedure that allows us to quickly find $\lim_{x\to 0}\frac{\sin kx}{x}$, where $k \neq 0$ is any real constant, by simply changing the variable by means of a **substitution**. If we let $t = kx$, then $x = t/k$. Notice that as $x \to 0$ then necessarily $t \to 0$. Thus we can write

$$\lim_{x\to 0}\frac{\sin kx}{x} = \lim_{t\to 0}\frac{\sin t}{t/k} = \lim_{t\to 0}\left(\frac{\sin t}{1}\cdot\frac{k}{t}\right) = k\underbrace{\lim_{t\to 0}\frac{\sin t}{t}}_{\text{this limit is 1 from (10)}} = k.$$

Thus we have proved the general result

$$\lim_{x\to 0}\frac{\sin kx}{x} = k. \tag{11}$$

From (11), with $k = 2$, we get the same result $\lim_{x\to 0}\frac{\sin 2x}{x} = 2$ obtained in Example 5.

EXAMPLE 7 Using a Substitution

Find the limit $\lim_{x\to 1}\frac{\sin(x-1)}{x^2+2x-3}$.

Solution Before beginning observe that the limit has the indeterminate form $0/0$ as $x \to 1$. By factoring $x^2 + 2x - 3 = (x+3)(x-1)$ the given limit can be expressed as a limit of a product:

$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+2x-3} = \lim_{x\to 1}\frac{\sin(x-1)}{(x+3)(x-1)} = \lim_{x\to 1}\left[\frac{1}{x+3}\cdot\frac{\sin(x-1)}{x-1}\right]. \tag{12}$$

Now if we let $t = x - 1$, we see that $x \to 1$ implies $t \to 0$. Therefore,

$$\lim_{x\to 1}\frac{\sin(x-1)}{x-1} = \lim_{t\to 0}\frac{\sin t}{t} = 1. \leftarrow \text{from (10)}$$

Returning to (12), we can write

$$\begin{aligned}\lim_{x\to 1}\frac{\sin(x-1)}{x^2+2x-3} &= \lim_{x\to 1}\left[\frac{1}{x+3}\cdot\frac{\sin(x-1)}{x-1}\right]\\ &= \left(\lim_{x\to 1}\frac{1}{x+3}\right)\left(\lim_{x\to 1}\frac{\sin(x-1)}{x-1}\right)\\ &= \left(\lim_{x\to 1}\frac{1}{x+3}\right)\left(\lim_{t\to 0}\frac{\sin t}{t}\right)\end{aligned}$$

since both limits exist. Thus,

$$\lim_{x\to 1}\frac{\sin(x-1)}{x^2+2x-3}=\left(\lim_{x\to 1}\frac{1}{x+3}\right)\left(\lim_{t\to 0}\frac{\sin t}{t}\right)=\frac{1}{4}\cdot 1=\frac{1}{4}.$$ ■

EXAMPLE 8 Using a Pythagorean Identity

Find the limit $\lim_{x\to 0}\frac{1-\cos x}{x}$.

Solution To compute this limit we start with a bit of algebraic cleverness by multiplying the numerator and denominator by the conjugate factor of the numerator. Next we use the fundamental Pythagorean identity $\sin^2 x+\cos^2 x=1$ in the form $1-\cos^2 x=\sin^2 x$:

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x}&=\lim_{x\to 0}\frac{1-\cos x}{x}\cdot\frac{1+\cos x}{1+\cos x}\\&=\lim_{x\to 0}\frac{1-\cos^2 x}{x(1+\cos x)}\\&=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}.\end{aligned}$$

For the next step we resort back to algebra to rewrite the fractional expression as a product, then use the results in (5):

$$\begin{aligned}\lim_{x\to 0}\frac{1-\cos x}{x}&=\lim_{x\to 0}\frac{\sin^2 x}{x(1+\cos x)}\\&=\lim_{x\to 0}\left(\frac{\sin x}{x}\cdot\frac{\sin x}{1+\cos x}\right)\\&=\left(\lim_{x\to 0}\frac{\sin x}{x}\right)\cdot\left(\lim_{x\to 0}\frac{\sin x}{1+\cos x}\right).\end{aligned}$$

Because $\lim_{x\to 0}(\sin x)/(1+\cos x)=0/2=0$ we have

$$\lim_{x\to 0}\frac{1-\cos x}{x}=0. \qquad (13)$$ ■

Since the limit in (13) is equal to 0, we can write

$$\lim_{x\to 0}\frac{1-\cos x}{x}=\lim_{x\to 0}\frac{-(\cos x-1)}{x}=(-1)\lim_{x\to 0}\frac{\cos x-1}{x}=0.$$

Dividing by -1 then gives another important trigonometric limit:

$$\lim_{x\to 0}\frac{\cos x-1}{x}=0. \qquad (14)$$

FIGURE 2.4.6 shows the graph of $f(x)=(\cos x-1)/x$. We will use the results in (10) and (14) in Exercises 2.7 and again in Section 3.4.

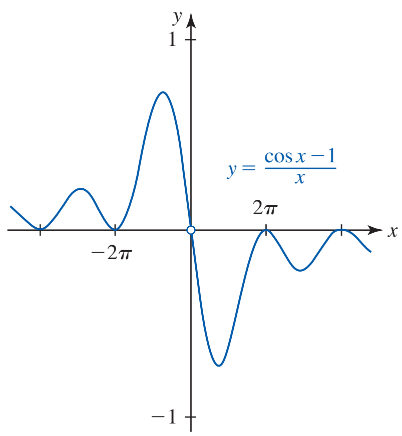


FIGURE 2.4.6 Graph of $f(x)=(\cos x-1)/x$

Exercises 2.4

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–36, find the given limit, or state that it does not exist.

1. $\lim_{t\to 0}\frac{\sin 3t}{2t}$
2. $\lim_{t\to 0}\frac{\sin(-4t)}{t}$
3. $\lim_{x\to 0}\frac{\sin x}{4+\cos x}$
4. $\lim_{x\to 0}\frac{1+\sin x}{1+\cos x}$
5. $\lim_{x\to 0}\frac{\cos 2x}{\cos 3x}$
6. $\lim_{x\to 0}\frac{\tan x}{3x}$
7. $\lim_{t\to 0}\frac{1}{t\sec t\csc 4t}$
8. $\lim_{t\to 0}5t\cot 2t$
9. $\lim_{t\to 0}\frac{2\sin^2 t}{t\cos^2 t}$
10. $\lim_{t\to 0}\frac{\sin^2(t/2)}{\sin t}$
11. $\lim_{t\to 0}\frac{\sin^2 6t}{t^2}$
12. $\lim_{t\to 0}\frac{t^3}{\sin^2 3t}$
13. $\lim_{x\to 1}\frac{\sin(x-1)}{2x-2}$
14. $\lim_{x\to 2\pi}\frac{x-2\pi}{\sin x}$

15. $\lim_{x\to 0}\frac{\cos x}{x}$

16. $\lim_{\theta\to\pi/2}\frac{1+\sin\theta}{\cos\theta}$

17. $\lim_{x\to 0}\frac{\cos(3x-\pi/2)}{x}$

18. $\lim_{x\to -2}\frac{\sin(5x+10)}{4x+8}$

19. $\lim_{t\to 0}\frac{\sin 3t}{\sin 7t}$

20. $\lim_{t\to 0}\sin 2t\csc 3t$

21. $\lim_{t\to 0^+}\frac{\sin t}{\sqrt{t}}$

22. $\lim_{t\to 0^+}\frac{1-\cos\sqrt{t}}{\sqrt{t}}$

23. $\lim_{t\to 0}\frac{t^2-5t\sin t}{t^2}$

24. $\lim_{t\to 0}\frac{\cos 4t}{\cos 8t}$

25. $\lim_{x\to 0^+}\frac{(x+2\sqrt{\sin x})^2}{x}$

26. $\lim_{x\to 0}\frac{(1-\cos x)^2}{x}$

27. $\lim_{x\to 0}\frac{\cos x-1}{\cos^2 x-1}$

28. $\lim_{x\to 0}\frac{\sin x+\tan x}{x}$

29. $\lim_{x\to 0}\frac{\sin 5x^2}{x^2}$

30. $\lim_{t\to 0}\frac{t^2}{1-\cos t}$

31. $\lim_{x\to 2}\frac{\sin(x-2)}{x^2+2x-8}$

32. $\lim_{x\to 3}\frac{x^2-9}{\sin(x-3)}$

33. $\lim_{x\to 0}\frac{2\sin 4x+1-\cos x}{x}$

34. $\lim_{x\to 0}\frac{4x^2-2\sin x}{x}$

35. $\lim_{x\to\pi/4}\frac{1-\tan x}{\cos x-\sin x}$

36. $\lim_{x\to\pi/4}\frac{\cos 2x}{\cos x-\sin x}$

37. Suppose $f(x)=\sin x$. Use (10) and (14) of this section along with (17) of Section 1.4 to find the limit:

$$\lim_{h\to 0}\frac{f\left(\frac{\pi}{4}+h\right)-f\left(\frac{\pi}{4}\right)}{h}.$$

38. Suppose $f(x)=\cos x$. Use (10) and (14) of this section along with (18) of Section 1.4 to find the limit:

$$\lim_{h\to 0}\frac{f\left(\frac{\pi}{6}+h\right)-f\left(\frac{\pi}{6}\right)}{h}.$$

In Problems 39 and 40, use the Squeeze Theorem to establish the given limit.

39. $\lim_{x\to 0}x\sin\frac{1}{x}=0$

40. $\lim_{x\to 0}x^2\cos\frac{\pi}{x}=0$

41. Use the properties of limits given in Theorem 2.2.3 to show that

(a) $\lim_{x\to 0}x^3\sin\frac{1}{x}=0$ **(b)** $\lim_{x\to 0}x^2\sin^2\frac{1}{x}=0.$

42. If $|f(x)|\le B$ for all x in an interval containing 0, show that $\lim_{x\to 0}x^2f(x)=0$.

In Problems 43 and 44, use the Squeeze Theorem to evaluate the given limit.

43. $\lim_{x\to 2}f(x)$ where $2x-1\le f(x)\le x^2-2x+3,\ x\ne 2$

44. $\lim_{x\to 0}f(x)$ where $|f(x)-1|\le x^2,\ x\ne 0$

Think About It

In Problems 45–48, use an appropriate substitution to find the given limit.

45. $\lim_{x\to\pi/4}\frac{\sin x-\cos x}{x-\pi/4}$

46. $\lim_{x\to\pi}\frac{x-\pi}{\tan 2x}$

47. $\lim_{x\to 1}\frac{\sin(\pi/x)}{x-1}$

48. $\lim_{x\to 2}\frac{\cos(\pi/x)}{x-2}$

49. Discuss: Is the function

$$f(x)=\begin{cases}\dfrac{\sin x}{x}, & x\ne 0\\ 1, & x=0\end{cases}$$

continuous at 0?

50. The existence of $\lim_{x\to 0}\frac{\sin x}{x}$ does not imply the existence of $\lim_{x\to 0}\frac{\sin|x|}{x}$. Explain why the second limit fails to exist.

2.5 Limits That Involve Infinity

Introduction In Sections 1.2 and 1.3 we considered some functions whose graphs possessed asymptotes. We will see in this section that vertical and horizontal asymptotes of a graph are defined in terms of limits involving the concept of *infinity*. Recall, the **infinity symbols**, $-\infty$ ("minus infinity") and ∞ ("infinity"), are notational devices used to indicate, in turn, that a quantity becomes unbounded in the negative direction (in the Cartesian plane this means to the left for x and downward for y) and in the positive direction (to the right for x and upward for y).

Some texts use the symbol $+\infty$ and the words *plus infinity* instead of ∞ and *infinity*.

Although the terminology and notation used when working with $\pm\infty$ is standard, it is nevertheless a bit unfortunate and can be confusing. So let us make it clear at the outset that we are going to consider two kinds of limits. First, we are going to examine

- *infinite limits*.

The words *infinite limit* always refer to *a limit that does not exist* because the function f exhibits unbounded behavior: $f(x)\to-\infty$ or $f(x)\to\infty$. Next, we will consider

- *limits at infinity*.

The words *at infinity* mean that we are trying to determine whether a function f possesses a limit when the variable x is allowed to become unbounded: $x \to -\infty$ or $x \to \infty$. Such limits may or may not exist.

Throughout the discussion, bear in mind that $-\infty$ and ∞ do not represent real numbers and should *never* be manipulated arithmetically like a number.

Infinite Limits The limit of a function f will fail to exist as x approaches a number a whenever the function values increase or decrease without bound. The fact that the function values $f(x)$ increase without bound as x approaches a is denoted symbolically by

$$f(x) \to \infty \text{ as } x \to a \qquad \text{or} \qquad \lim_{x \to a} f(x) = \infty. \tag{1}$$

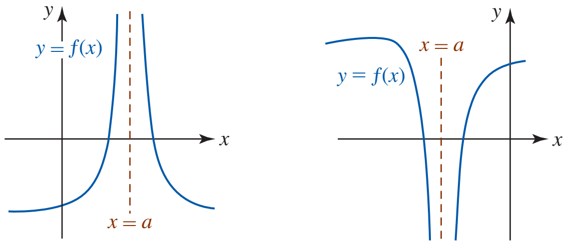
If the function values decrease without bound as x approaches a, we write

$$f(x) \to -\infty \text{ as } x \to a \qquad \text{or} \qquad \lim_{x \to a} f(x) = -\infty. \tag{2}$$

Recall, the use of the symbol $x \to a$ signifies that f exhibits the same behavior—in this instance, unbounded behavior—from both sides of the number a on the x-axis. For example, the notation in (1) indicates that

$$f(x) \to \infty \text{ as } x \to a^- \qquad \textit{and} \qquad f(x) \to \infty \text{ as } x \to a^+.$$

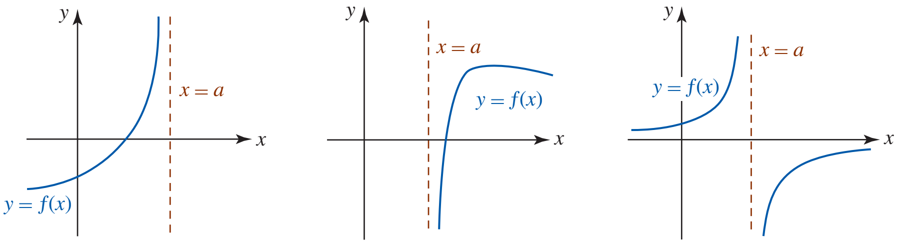
See FIGURE 2.5.1.



(a) $\lim_{x \to a} f(x) = \infty$ (b) $\lim_{x \to a} f(x) = -\infty$

FIGURE 2.5.1 Two types of infinite limits

Similarly, FIGURE 2.5.2 shows the unbounded behavior of a function f as x approaches a from one side. Note in Figure 2.5.2(c), we cannot describe the behavior of f near a using just one limit symbol.



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = -\infty$ (c) $\lim_{x \to a^-} f(x) = \infty$ and $\lim_{x \to a^+} f(x) = -\infty$

FIGURE 2.5.2 Three more types of infinite limits

In general, any limit of the six types

$$\begin{aligned} \lim_{x \to a^-} f(x) &= -\infty, & \lim_{x \to a^-} f(x) &= \infty, \\ \lim_{x \to a^+} f(x) &= -\infty, & \lim_{x \to a^+} f(x) &= \infty, \\ \lim_{x \to a} f(x) &= -\infty, & \lim_{x \to a} f(x) &= \infty, \end{aligned} \tag{3}$$

is called an **infinite limit**. Again, in each case of (3) we are simply describing in a symbolic manner the behavior of a function f near the number a. *None of the limits in* (3) *exist*.

In Section 1.3 we reviewed how to identify a vertical asymptote for the graph of a rational function $f(x) = p(x)/q(x)$. We are now in a position to define a vertical asymptote of any function in terms of the limit concept.

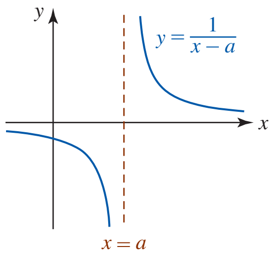
Definition 2.5.1 Vertical Asymptote

A line $x = a$ is said to be a **vertical asymptote** for the graph of a function f if at least one of the six statements in (3) is true.

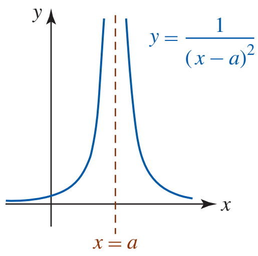
See Figure 1.2.1.

In the review of functions in Chapter 1 we saw that the graphs of rational functions often possess asymptotes. We saw that the graphs of the rational functions $y = 1/x$ and $y = 1/x^2$ were similar to the graphs in Figure 2.5.2(c) and Figure 2.5.1(a), respectively. The y-axis, that is, $x = 0$, is a vertical asymptote for each of these functions. The graphs of

$$y = \frac{1}{x - a} \quad \text{and} \quad y = \frac{1}{(x - a)^2} \tag{4}$$



(a)



(b)

FIGURE 2.5.3 Graphs of functions in (4)

are obtained by shifting the graphs of $y = 1/x$ and $y = 1/x^2$ horizontally $|a|$ units. As seen in FIGURE 2.5.3, $x = a$ is a vertical asymptote for the rational functions in (4). We have

$$\lim_{x \to a^-} \frac{1}{x - a} = -\infty \quad \text{and} \quad \lim_{x \to a^+} \frac{1}{x - a} = \infty \tag{5}$$

and

$$\lim_{x \to a} \frac{1}{(x - a)^2} = \infty. \tag{6}$$

The infinite limits in (5) and (6) are just special cases of the following general result:

$$\lim_{x \to a^-} \frac{1}{(x - a)^n} = -\infty \quad \text{and} \quad \lim_{x \to a^+} \frac{1}{(x - a)^n} = \infty, \tag{7}$$

for n an odd positive integer, and

$$\lim_{x \to a} \frac{1}{(x - a)^n} = \infty, \tag{8}$$

for n an even positive integer. As a consequence of (7) and (8), the graph of a rational function $y = 1/(x - a)^n$ either resembles the graph in Figure 2.5.3(a) for n odd or that in Figure 2.5.3(b) for n even.

For a general rational function $f(x) = p(x)/q(x)$, where p and q have no common factors, it should be clear from this discussion that when q contains a factor $(x - a)^n$, n a positive integer, then the shape of the graph near the vertical line $x = a$ must be either one of those shown in Figure 2.5.3 or its reflection in the x-axis.

EXAMPLE 1 Vertical Asymptotes of a Rational Function

Inspection of the rational function

$$f(x) = \frac{x + 2}{x^2(x + 4)}$$

shows that $x = -4$ and $x = 0$ are vertical asymptotes for the graph of f. Since the denominator contains the factors $(x - (-4))^1$ and $(x - 0)^2$ we expect the graph of f near the line $x = -4$ to resemble Figure 2.5.3(a) or its reflection in the x-axis, and the graph near $x = 0$ to resemble Figure 2.5.3(b) or its reflection in the x-axis.

For x close to 0, from either side of 0, it is easily seen that $f(x) > 0$. But, for x close to -4, say $x = -4.1$ and $x = -3.9$, we have $f(x) > 0$ and $f(x) < 0$, respectively. Using the additional information that there is only a single x-intercept $(-2, 0)$, we obtain the graph of f in FIGURE 2.5.4. ■

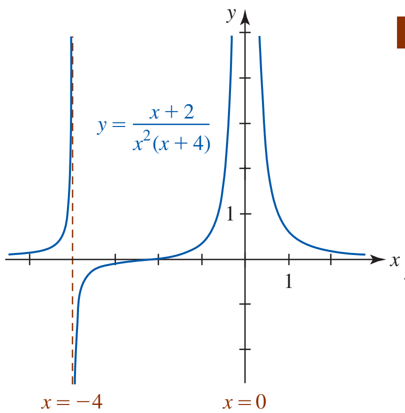


FIGURE 2.5.4 Graph of function in Example 1

EXAMPLE 2 One-Sided Limit

In Figure 1.6.6 we saw that the y-axis, or the line $x = 0$, is a vertical asymptote for the natural logarithmic function $f(x) = \ln x$ since

$$\lim_{x \to 0^+} \ln x = -\infty.$$

The graph of the logarithmic function $y = \ln(x + 3)$ is the graph of $f(x) = \ln x$ shifted 3 units to the left. Thus $x = -3$ is a vertical asymptote for the graph of $y = \ln(x + 3)$ since $\lim_{x \to -3^+} \ln(x + 3) = -\infty$. ■

EXAMPLE 3 One-Sided Limit

Graph the function $f(x) = \dfrac{x}{\sqrt{x + 2}}$.

Solution Inspection of f reveals that its domain is the interval $(-2, \infty)$ and the y-intercept is $(0, 0)$. From the accompanying table we conclude that f decreases

$x \to -2^+$	-1.9	-1.99	-1.999	-1.9999
$f(x)$	-6.01	-19.90	-63.21	-199.90

without bound as x approaches -2 from the right:

$$\lim_{x \to -2^+} f(x) = -\infty.$$

Hence, the line $x = -2$ is a vertical asymptote. The graph of f is given in FIGURE 2.5.5. ■

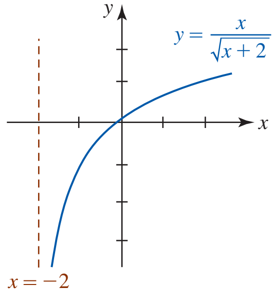


FIGURE 2.5.5 Graph of function in Example 3

Limits at Infinity If a function f approaches a constant value L as the independent variable x increases without bound $(x \to \infty)$ or as x decreases $(x \to -\infty)$ without bound, then we write

$$\lim_{x \to -\infty} f(x) = L \quad \text{or} \quad \lim_{x \to \infty} f(x) = L \tag{9}$$

and say that f possesses a **limit at infinity**. Here are all the possibilities for limits at infinity $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$:

- One limit exists but the other does not,
- Both $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$ exist and equal the same number,
- Both $\lim_{x \to -\infty} f(x)$ and $\lim_{x \to \infty} f(x)$ exist but are different numbers,
- Neither $\lim_{x \to -\infty} f(x)$ nor $\lim_{x \to \infty} f(x)$ exists.

If at least one of the limits exists, say, $\lim_{x \to \infty} f(x) = L$, then the graph of f can be made arbitrarily close to the line $y = L$ as x increases in the positive direction.

Definition 2.5.2 Horizontal Asymptote

A line $y = L$ is said to be a **horizontal asymptote** for the graph of a function f if at least one of the two statements in (9) is true.

In FIGURE 2.5.6 we have illustrated some typical horizontal asymptotes. We note, in conjunction with Figure 2.5.6(d) that, in general, the graph of a function can have at most *two* horizontal asymptotes but the graph of a *rational function* $f(x) = p(x)/q(x)$ can have at most *one*. If the graph of a rational function f possesses a horizontal asymptote $y = L$, then its end behavior is as shown in Figure 2.5.6(c), that is:

$$f(x) \to L \text{ as } x \to -\infty \quad \textit{and} \quad f(x) \to L \text{ as } x \to \infty.$$

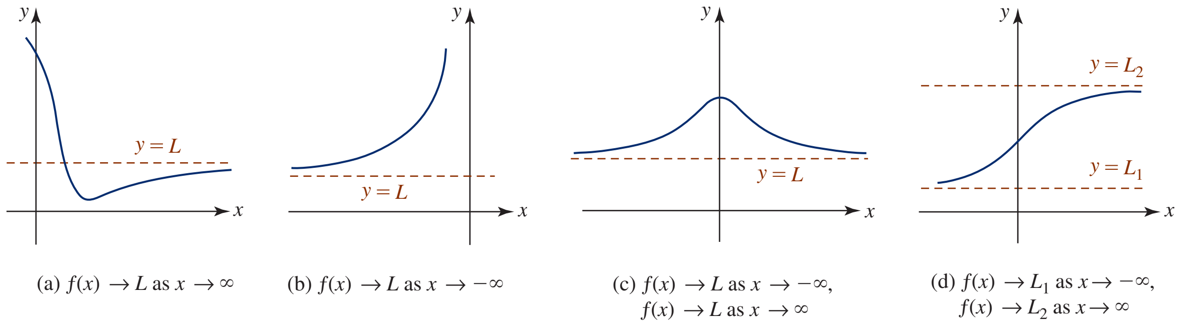


FIGURE 2.5.6 $y = L$ is a horizontal asymptote in (a), (b), and (c); $y = L_1$ and $y = L_2$ are horizontal asymptotes in (d)

For example, if x becomes unbounded in either the positive or negative direction, the functions in (4) decrease to 0 and we write

$$\lim_{x\to-\infty}\frac{1}{x-a}=0,\ \lim_{x\to\infty}\frac{1}{x-a}=0 \quad \text{and} \quad \lim_{x\to-\infty}\frac{1}{(x-a)^2}=0,\ \lim_{x\to\infty}\frac{1}{(x-a)^2}=0. \quad (10)$$

In general, if r is a positive rational number and if $(x-a)^r$ is defined, then

These results are also true when $x-a$ is replaced by $a-x$, provided $(a-x)^r$ is defined.

$$\lim_{x\to-\infty}\frac{1}{(x-a)^r}=0 \quad \text{and} \quad \lim_{x\to\infty}\frac{1}{(x-a)^r}=0. \quad (11)$$

EXAMPLE 4 Horizontal and Vertical Asymptotes

The domain of the function $f(x)=\dfrac{4}{\sqrt{2-x}}$ is the interval $(-\infty, 2)$. In view of (11) we can write

$$\lim_{x\to-\infty}\frac{4}{\sqrt{2-x}}=0.$$

Note that we cannot consider the limit of f as $x\to\infty$ because the function is not defined for $x\geq 2$. Nevertheless $y=0$ is a horizontal asymptote. Now from infinite limit

$$\lim_{x\to2^-}\frac{4}{\sqrt{2-x}}=\infty$$

we conclude that $x=2$ is a vertical asymptote for the graph of f. See FIGURE 2.5.7. ■

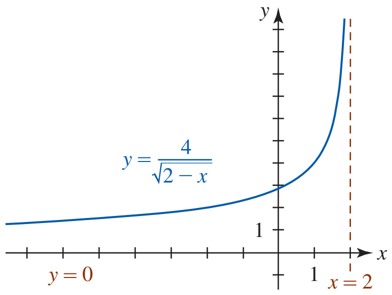


FIGURE 2.5.7 Graph of function in Example 4

In general, if $F(x)=f(x)/g(x)$, then the following table summarizes the limit results for the forms $\lim_{x\to a}F(x)$, $\lim_{x\to\infty}F(x)$, and $\lim_{x\to-\infty}F(x)$. The symbol L denotes a real number.

limit form: $x\to a, \infty, -\infty$	$\dfrac{L}{\pm\infty}$	$\dfrac{\pm\infty}{L}, L\neq 0$	$\dfrac{L}{0}, L\neq 0$
limit is:	0	infinite	infinite

(12)

Limits of the form $\lim_{x\to\infty}F(x)=\pm\infty$ or $\lim_{x\to-\infty}F(x)=\pm\infty$ are said to be **infinite limits at infinity**. Furthermore, the limit properties given in Theorem 2.2.3 hold by replacing the symbol a by ∞ or $-\infty$ provided the limits exist. For example,

$$\lim_{x\to\infty}f(x)g(x)=\left(\lim_{x\to\infty}f(x)\right)\left(\lim_{x\to\infty}g(x)\right) \quad \text{and} \quad \lim_{x\to\infty}\frac{f(x)}{g(x)}=\frac{\lim_{x\to\infty}f(x)}{\lim_{x\to\infty}g(x)}, \quad (13)$$

whenever $\lim_{x\to\infty}f(x)$ and $\lim_{x\to\infty}g(x)$ exist. In the case of the limit of a quotient we must also have $\lim_{x\to\infty}g(x)\neq 0$.

End Behavior In Section 1.3 we saw that how a function f behaves when $|x|$ is very large is its **end behavior**. As already discussed, if $\lim_{x\to\infty}f(x)=L$, then the graph of f can be made arbitrarily close to the line $y=L$ for large positive values of x. The graph of a polynomial function,

$$f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_2x^2+a_1x+a_0,$$

resembles the graph of $y=a_nx^n$ for $|x|$ very large. In other words, for

$$f(x)=a_nx^n \boxed{+a_{n-1}x^{n-1}+\cdots+a_1x+a_0} \quad (14)$$

the terms enclosed in the blue rectangle in (14) are irrelevant when we look at a graph of a polynomial globally—that is, for $|x|$ large. Thus we have

$$\lim_{x\to\pm\infty}a_nx^n=\lim_{x\to\pm\infty}\left(a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0\right), \quad (15)$$

where (15) is either ∞ or $-\infty$ depending on a_n and n. In other words, the limit in (15) is an example of an infinite limit at infinity.

EXAMPLE 5 Limit at Infinity

Evaluate $\lim_{x\to\infty} \dfrac{-6x^4 + x^2 + 1}{2x^4 - x}$.

Solution We cannot apply the limit quotient law in (13) to the given function, since $\lim_{x\to\infty}(-6x^4 + x^2 + 1) = -\infty$ and $\lim_{x\to\infty}(2x^4 - x) = \infty$. However, by dividing the numerator and the denominator by x^4, we can write

$$\lim_{x\to\infty} \frac{-6x^4 + x^2 + 1}{2x^4 - x} = \lim_{x\to\infty} \frac{-6 + \left(\dfrac{1}{x^2}\right) + \left(\dfrac{1}{x^4}\right)}{2 - \left(\dfrac{1}{x^3}\right)}$$

$$= \frac{\lim_{x\to\infty}\left[-6 + \left(\dfrac{1}{x^2}\right) + \left(\dfrac{1}{x^4}\right)\right]}{\lim_{x\to\infty}\left[2 - \left(\dfrac{1}{x^3}\right)\right]} \leftarrow \text{Limit of the numerator and denominator both exist and the limit of the denominator is not zero}$$

$$= \frac{-6 + 0 + 0}{2 - 0} = -3.$$

This means the line $y = -3$ is a horizontal asymptote for the graph of the function.

Alternative Solution In view of (14), we can discard all powers of x other than the highest:

discard terms in the blue boxes

$$\lim_{x\to\infty} \frac{-6x^4 \boxed{+ x^2 + 1}}{2x^4 \boxed{- x}} = \lim_{x\to\infty} \frac{-6x^4}{2x^4} = \lim_{x\to\infty} \frac{-6}{2} = -3.$$ ■

EXAMPLE 6 Infinite Limit at Infinity

Evaluate $\lim_{x\to\infty} \dfrac{1 - x^3}{3x + 2}$.

Solution By (14),

$$\lim_{x\to\infty} \frac{1 - x^3}{3x + 2} = \lim_{x\to\infty} \frac{-x^3}{3x} = -\frac{1}{3}\lim_{x\to\infty} x^2 = -\infty.$$

In other words, the limit does not exist. ■

EXAMPLE 7 Graph of a Rational Function

Graph the function $f(x) = \dfrac{x^2}{1 - x^2}$.

Solution Inspection of the function f reveals that its graph is symmetric with respect to the y-axis, the y-intercept is $(0, 0)$, and the vertical asymptotes are $x = -1$ and $x = 1$. Now, from the limit

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{1 - x^2} = \lim_{x\to\infty} \frac{x^2}{-x^2} = -\lim_{x\to\infty} 1 = -1$$

we conclude that the line $y = -1$ is a horizontal asymptote. The graph of f is given in FIGURE 2.5.8. ■

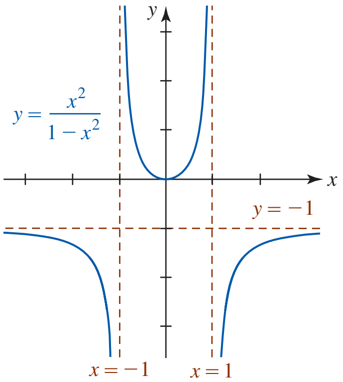


FIGURE 2.5.8 Graph of function in Example 7

Another limit law that holds true for limits at infinity is that the limit of an nth root of a function is the nth root of the limit, whenever the limit exists and the nth root is defined. In symbols, if $\lim_{x\to\infty} g(x) = L$, then

$$\lim_{x\to\infty} \sqrt[n]{g(x)} = \sqrt[n]{\lim_{x\to\infty} g(x)} = \sqrt[n]{L}, \tag{16}$$

provided $L \geq 0$ when n is even. The result also holds for $x \to -\infty$.

EXAMPLE 8 Limit of a Square Root

Evaluate $\lim_{x\to\infty}\sqrt{\dfrac{2x^3 - 5x^2 + 4x - 6}{6x^3 + 2x}}$.

Solution Because the limit of the rational function inside the radical exists and is positive, we can write

$$\lim_{x\to\infty}\sqrt{\frac{2x^3 - 5x^2 + 4x - 6}{6x^3 + 2x}} = \sqrt{\lim_{x\to\infty}\frac{2x^3 - 5x^2 + 4x - 6}{6x^3 + 2x}} = \sqrt{\lim_{x\to\infty}\frac{2x^3}{6x^3}} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}}. \quad ■$$

EXAMPLE 9 Graph with Two Horizontal Asymptotes

Determine whether the graph of $f(x) = \dfrac{5x}{\sqrt{x^2 + 4}}$ has any horizontal asymptotes.

Solution Since the function is not rational, we must investigate the limit of f as $x \to \infty$ and as $x \to -\infty$. First, recall from algebra that $\sqrt{x^2}$ is nonnegative, or more to the point,

$$\sqrt{x^2} = |x| = \begin{cases} x, & x \ge 0 \\ -x, & x < 0. \end{cases}$$

We then rewrite f as

$$f(x) = \frac{\dfrac{5x}{\sqrt{x^2}}}{\dfrac{\sqrt{x^2 + 4}}{\sqrt{x^2}}} = \frac{\dfrac{5x}{|x|}}{\dfrac{\sqrt{x^2 + 4}}{\sqrt{x^2}}} = \frac{\dfrac{5x}{|x|}}{\sqrt{1 + \dfrac{4}{x^2}}}.$$

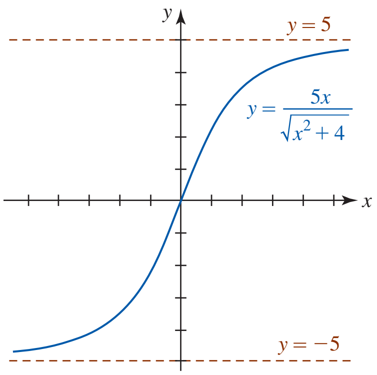


FIGURE 2.5.9 Graph of function in Example 9

The limits of f as $x \to \infty$ and as $x \to -\infty$ are, respectively,

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{\dfrac{5x}{|x|}}{\sqrt{1 + \dfrac{4}{x^2}}} = \lim_{x\to\infty}\frac{\dfrac{5x}{x}}{\sqrt{1 + \dfrac{4}{x^2}}} = \frac{\lim\limits_{x\to\infty} 5}{\sqrt{\lim\limits_{x\to\infty}\left(1 + \dfrac{4}{x^2}\right)}} = \frac{5}{1} = 5,$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}\frac{\dfrac{5x}{|x|}}{\sqrt{1 + \dfrac{4}{x^2}}} = \lim_{x\to-\infty}\frac{\dfrac{5x}{-x}}{\sqrt{1 + \dfrac{4}{x^2}}} = \frac{\lim\limits_{x\to-\infty} (-5)}{\sqrt{\lim\limits_{x\to-\infty}\left(1 + \dfrac{4}{x^2}\right)}} = \frac{-5}{1} = -5.$$

Thus the graph of f has two horizontal asymptotes $y = 5$ and $y = -5$. The graph of f, which is similar to Figure 2.5.6(d), is given in FIGURE 2.5.9. ■

In the next example we see that the form of the given limit is $\infty - \infty$, but the limit exists and is *not* 0.

EXAMPLE 10 Using Rationalization

Evaluate $\lim_{x\to\infty}\left(x^2 - \sqrt{x^4 + 7x^2 + 1}\right)$.

Solution Because $f(x) = x^2 - \sqrt{x^4 + 7x^2 + 1}$ is an even function (verify that $f(-x) = f(x)$) with domain $(-\infty, \infty)$, if $\lim_{x\to\infty} f(x)$ exists it must be the same as $\lim_{x\to-\infty} f(x)$. We first rationalize the numerator:

$$\begin{aligned}
\lim_{x\to\infty}\left(x^2 - \sqrt{x^4 + 7x^2 + 1}\right) &= \lim_{x\to\infty}\frac{\left(x^2 - \sqrt{x^4 + 7x^2 + 1}\right)}{1} \cdot \left(\frac{x^2 + \sqrt{x^4 + 7x^2 + 1}}{x^2 + \sqrt{x^4 + 7x^2 + 1}}\right) \\
&= \lim_{x\to\infty}\frac{x^4 - (x^4 + 7x^2 + 1)}{x^2 + \sqrt{x^4 + 7x^2 + 1}} \\
&= \lim_{x\to\infty}\frac{-7x^2 - 1}{x^2 + \sqrt{x^4 + 7x^2 + 1}}.
\end{aligned}$$

Next, we divide the numerator and denominator by $\sqrt{x^4} = x^2$:

$$\lim_{x\to\infty} \frac{-7x^2 - 1}{x^2 + \sqrt{x^4 + 7x^2 + 1}} = \lim_{x\to\infty} \frac{\dfrac{-7x^2}{\sqrt{x^4}} - \dfrac{1}{\sqrt{x^4}}}{\dfrac{x^2 + \sqrt{x^4 + 7x^2 + 1}}{\sqrt{x^4}}}$$

$$= \lim_{x\to\infty} \frac{-7 - \dfrac{1}{x^2}}{1 + \sqrt{1 + \dfrac{7}{x^2} + \dfrac{1}{x^4}}}$$

$$= \frac{\lim\limits_{x\to\infty}\left(-7 - \dfrac{1}{x^2}\right)}{\lim\limits_{x\to\infty} 1 + \sqrt{\lim\limits_{x\to\infty}\left(1 + \dfrac{7}{x^2} + \dfrac{1}{x^4}\right)}}$$

$$= \frac{-7}{1+1} = -\frac{7}{2}.$$

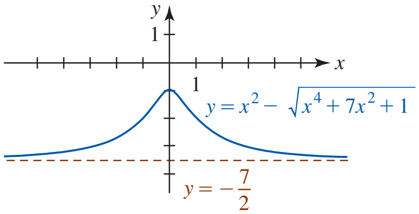


FIGURE 2.5.10 Graph of function in Example 10

With the help of a CAS, the graph of the function f is given in FIGURE 2.5.10. The line $y = -\frac{7}{2}$ is a horizontal asymptote. Note the symmetry of the graph with respect to the y-axis. ■

When working with functions containing the natural exponential function, the following four limits merit special attention:

$$\lim_{x\to\infty} e^x = \infty, \quad \lim_{x\to-\infty} e^x = 0, \quad \lim_{x\to\infty} e^{-x} = 0, \quad \lim_{x\to-\infty} e^{-x} = \infty. \tag{17}$$

As discussed in Section 1.6 and verified by the second and third limit in (17), $y = 0$ is a horizontal asymptote for the graphs of $y = e^x$ and $y = e^{-x}$. See FIGURE 2.5.11.

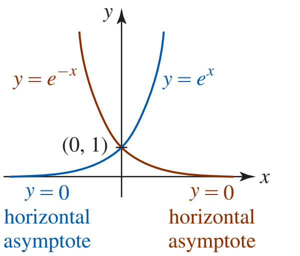


FIGURE 2.5.11 Graphs of exponential functions

EXAMPLE 11 Graph with Two Horizontal Asymptotes

Determine whether the graph of $f(x) = \dfrac{6}{1 + e^{-x}}$ has any horizontal asymptotes.

Solution Because f is not a rational function, we must examine $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. First, in view of the third result given in (17) we can write

$$\lim_{x\to\infty} \frac{6}{1 + e^{-x}} = \frac{\lim\limits_{x\to\infty} 6}{\lim\limits_{x\to\infty}(1 + e^{-x})} = \frac{6}{1+0} = 6.$$

Thus $y = 6$ is a horizontal asymptote. Now, because $\lim\limits_{x\to-\infty} e^{-x} = \infty$ it follows from the table in (12) that

$$\lim_{x\to-\infty} \frac{6}{1 + e^{-x}} = 0.$$

Therefore $y = 0$ is a horizontal asymptote. The graph of f is given in FIGURE 2.5.12. ■

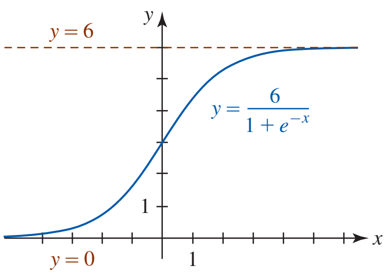


FIGURE 2.5.12 Graph of function in Example 11

Composite Functions Theorem 2.3.3, the limit of a composite function, holds when a is replaced by $-\infty$ or ∞ and the limit exists. For example, if $\lim\limits_{x\to\infty} g(x) = L$ and f is continuous at L, then

$$\lim_{x\to\infty} f(g(x)) = f\left(\lim_{x\to\infty} g(x)\right) = f(L). \tag{18}$$

The limit result in (16) is just a special case of (18) when $f(x) = \sqrt[n]{x}$. The result in (18) also holds for $x \to -\infty$. Our last example illustrates (18) involving a limit at ∞.

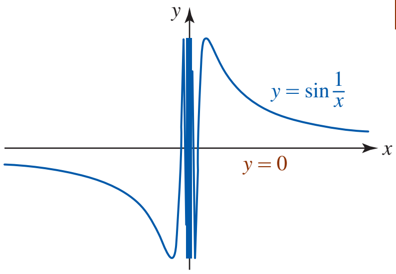


FIGURE 2.5.13 Graph of function in Example 12

EXAMPLE 12 A Trigonometric Function Revisited

In Example 2 of Section 2.4 we saw that $\lim_{x\to 0}\sin(1/x)$ does not exist. However, the limit at infinity, $\lim_{x\to\infty}\sin(1/x)$, exists. By (18) we can write

$$\lim_{x\to\infty}\sin\frac{1}{x} = \sin\left(\lim_{x\to\infty}\frac{1}{x}\right) = \sin 0 = 0.$$

As we see in FIGURE 2.5.13, $y = 0$ is a horizontal asymptote for the graph of $f(x) = \sin(1/x)$. You should compare this graph with that given in Figure 2.4.2. ■

Exercises 2.5

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–24, express the given limit as a number, as $-\infty$, or as ∞.

1. $\lim_{x\to 5^-}\frac{1}{x-5}$

2. $\lim_{x\to 6}\frac{4}{(x-6)^2}$

3. $\lim_{x\to -4^+}\frac{2}{(x+4)^3}$

4. $\lim_{x\to 2^-}\frac{10}{x^2-4}$

5. $\lim_{x\to 1}\frac{1}{(x-1)^4}$

6. $\lim_{x\to 0^+}\frac{-1}{\sqrt{x}}$

7. $\lim_{x\to 0^+}\frac{2+\sin x}{x}$

8. $\lim_{x\to \pi^+}\csc x$

9. $\lim_{x\to\infty}\frac{x^2-3x}{4x^2+5}$

10. $\lim_{x\to\infty}\frac{x^2}{1+x^{-2}}$

11. $\lim_{x\to\infty}\left(5-\frac{2}{x^4}\right)$

12. $\lim_{x\to -\infty}\left(\frac{6}{\sqrt[3]{x}}+\frac{1}{\sqrt[5]{x}}\right)$

13. $\lim_{x\to\infty}\frac{8-\sqrt{x}}{1+4\sqrt{x}}$

14. $\lim_{x\to -\infty}\frac{1+7\sqrt[3]{x}}{2\sqrt[3]{x}}$

15. $\lim_{x\to\infty}\left(\frac{3x}{x+2}-\frac{x-1}{2x+6}\right)$

16. $\lim_{x\to\infty}\left(\frac{x}{3x+1}\right)\left(\frac{4x^2+1}{2x^2+x}\right)^3$

17. $\lim_{x\to\infty}\sqrt{\frac{3x+2}{6x-8}}$

18. $\lim_{x\to -\infty}\sqrt[3]{\frac{2x-1}{7-16x}}$

19. $\lim_{x\to\infty}\left(x-\sqrt{x^2+1}\right)$

20. $\lim_{x\to\infty}\left(\sqrt{x^2+5x}-x\right)$

21. $\lim_{x\to\infty}\cos\left(\frac{5}{x}\right)$

22. $\lim_{x\to -\infty}\sin\left(\frac{\pi x}{3-6x}\right)$

23. $\lim_{x\to -\infty}\sin^{-1}\left(\frac{x}{\sqrt{4x^2+1}}\right)$

24. $\lim_{x\to\infty}\ln\left(\frac{x}{x+8}\right)$

In Problems 25–32, find $\lim_{x\to -\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ for the given function f.

25. $f(x) = \frac{4x+1}{\sqrt{x^2+1}}$

26. $f(x) = \frac{\sqrt{9x^2+6}}{5x-1}$

27. $f(x) = \frac{2x+1}{\sqrt{3x^2+1}}$

28. $f(x) = \frac{-5x^2+6x+3}{\sqrt{x^4+x^2+1}}$

29. $f(x) = \frac{e^x-e^{-x}}{e^x+e^{-x}}$

30. $f(x) = 1+\frac{2e^{-x}}{e^x+e^{-x}}$

31. $f(x) = \frac{|x-5|}{x-5}$

32. $f(x) = \frac{|4x|+|x-1|}{x}$

In Problems 33–42, find all vertical and horizontal asymptotes for the graph of the given function. Sketch the graph.

33. $f(x) = \frac{1}{x^2+1}$

34. $f(x) = \frac{x}{x^2+1}$

35. $f(x) = \frac{x^2}{x+1}$

36. $f(x) = \frac{x^2-x}{x^2-1}$

37. $f(x) = \frac{1}{x^2(x-2)}$

38. $f(x) = \frac{4x^2}{x^2+4}$

39. $f(x) = \sqrt{\frac{x}{x-1}}$

40. $f(x) = \frac{1-\sqrt{x}}{\sqrt{x}}$

41. $f(x) = \frac{x-2}{\sqrt{x^2+1}}$

42. $f(x) = \frac{x+3}{\sqrt{x^2-1}}$

In Problems 43–46, use the given graph to find:

(a) $\lim_{x\to 2^-} f(x)$ **(b)** $\lim_{x\to 2^+} f(x)$

(c) $\lim_{x\to -\infty} f(x)$ **(d)** $\lim_{x\to\infty} f(x)$

43.

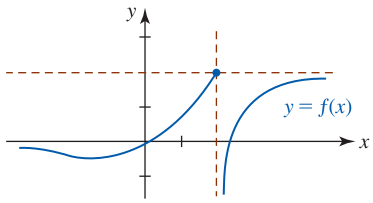


FIGURE 2.5.14 Graph for Problem 43

44.

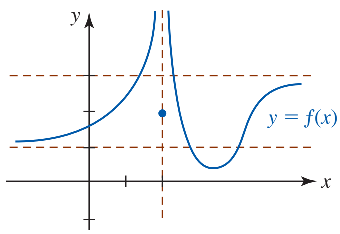


FIGURE 2.5.15 Graph for Problem 44

45.

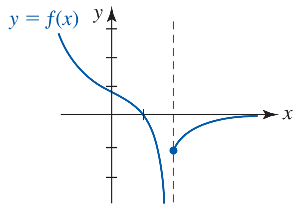


FIGURE 2.5.16 Graph for Problem 45

46.

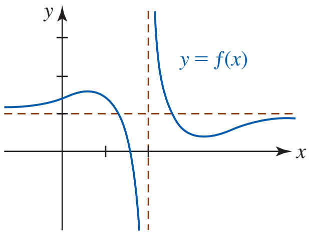


FIGURE 2.5.17 Graph for Problem 46

In Problems 47–50, sketch a graph of a function f that satisfies the given conditions.

47. $\lim_{x\to1^+} f(x) = -\infty$, $\lim_{x\to1^-} f(x) = -\infty$, $f(2) = 0$, $\lim_{x\to\infty} f(x) = 0$

48. $f(0) = 1$, $\lim_{x\to-\infty} f(x) = 3$, $\lim_{x\to\infty} f(x) = -2$

49. $\lim_{x\to2} f(x) = \infty$, $\lim_{x\to-\infty} f(x) = \infty$, $\lim_{x\to\infty} f(x) = 1$

50. $\lim_{x\to1^-} f(x) = 2$, $\lim_{x\to1^+} f(x) = -\infty$, $f\left(\frac{3}{2}\right) = 0$, $f(3) = 0$,

$\lim_{x\to-\infty} f(x) = 0$, $\lim_{x\to\infty} f(x) = 0$

51. Use an appropriate substitution to evaluate

$$\lim_{x\to\infty} x\sin\frac{3}{x}.$$

52. According to Einstein's theory of relativity, the mass m of a body moving with velocity v is $m = m_0/\sqrt{1 - v^2/c^2}$, where m_0 is the initial mass and c is the speed of light. What happens to m as $v \to c^-$?

Calculator/CAS Problems

In Problems 53 and 54, use a calculator or CAS to investigate the given limit. Conjecture its value.

53. $\lim_{x\to\infty} x^2 \sin\frac{2}{x^2}$ **54.** $\lim_{x\to\infty} \left(\cos\frac{1}{x}\right)^x$

55. Use a calculator or CAS to obtain the graph of $f(x) = (1 + x)^{1/x}$. Use the graph to conjecture the values of $f(x)$ as **(a)** $x \to -1^+$, **(b)** $x \to 0$, and **(c)** $x \to \infty$.

56. (a) A regular n-gon is an n-sided polygon inscribed in a circle; the polygon is formed by n equally spaced points on the circle. Suppose the polygon shown in FIGURE 2.5.18 represents a regular n-gon inscribed in a circle of radius r. Use trigonometry to show that the area $A(n)$ of the n-gon is given by

$$A(n) = \frac{n}{2} r^2 \sin\left(\frac{2\pi}{n}\right).$$

(b) It stands to reason that the area $A(n)$ approaches the area of the circle as the number of sides of the n-gon increases. Use a calculator to compute $A(100)$ and $A(1000)$.

(c) Let $x = 2\pi/n$ in $A(n)$ and note that as $n \to \infty$ then $x \to 0$. Use (10) of Section 2.4 to show that $\lim_{n\to\infty} A(n) = \pi r^2$.

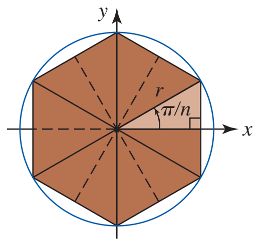


FIGURE 2.5.18 Inscribed n-gon for Problem 56

Think About It

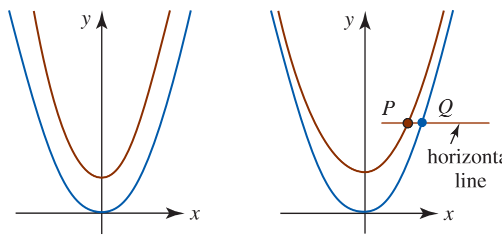
57. (a) Suppose $f(x) = x^2/(x + 1)$ and $g(x) = x - 1$. Show that

$$\lim_{x\to\pm\infty} [f(x) - g(x)] = 0.$$

(b) What does the result in part (a) indicate about the graphs of f and g where $|x|$ is large?

(c) If possible, give a name to the function g.

58. Very often students and even instructors will sketch vertically shifted graphs incorrectly. For example, the graphs of $y = x^2$ and $y = x^2 + 1$ are incorrectly drawn in FIGURE 2.5.19(a) but are correctly drawn in Figure 2.5.19(b). Demonstrate that Figure 2.5.19(b) is correct by showing that the horizontal distance between the two points P and Q shown in the figure approaches 0 as $x \to \infty$.



(a) Incorrect (b) Correct

FIGURE 2.5.19 Graphs for Problem 58

2.6 Limits—A Formal Approach

Introduction In the discussion that follows we will consider an alternative approach to the notion of a limit that is based on analytical concepts rather than on intuitive concepts. A **proof** of the existence of a limit can never be based on one's ability to sketch graphs or on tables of numerical values. Although a good intuitive understanding of $\lim_{x\to a} f(x)$ is sufficient for proceeding with the study of the calculus in this text, an intuitive understanding is admittedly too vague to be of any use in proving theorems. To give a rigorous demonstration of the existence of a limit, or to prove the important theorems of Section 2.2, we must start with a precise definition of a limit.

Limit of a Function Let us try to prove that $\lim_{x\to 2}(2x+6) = 10$ by elaborating on the following idea: "If $f(x) = 2x + 6$ can be made arbitrarily close to 10 by taking x sufficiently close to 2, from either side but different from 2, then $\lim_{x\to 2} f(x) = 10$." We need to make the concepts of *arbitrarily close* and *sufficiently close* precise. In order to set a standard of arbitrary closeness, let us demand that the distance between the numbers $f(x)$ and 10 be less than 0.1; that is,

$$|f(x) - 10| < 0.1 \quad \text{or} \quad 9.9 < f(x) < 10.1. \tag{1}$$

Then, how close must x be to 2 to accomplish (1)? To find out, we can use ordinary algebra to rewrite the inequality

$$9.9 < 2x + 6 < 10.1$$

as $1.95 < x < 2.05$. Adding -2 across this simultaneous inequality then gives

$$-0.05 < x - 2 < 0.05.$$

Using absolute values and remembering that $x \neq 2$, we can write the last inequality as $0 < |x - 2| < 0.05$. Thus, for an "arbitrary closeness to 10" of 0.1, "sufficiently close to 2" means within 0.05. In other words, if x is a number different from 2 such that its distance from 2 satisfies $|x - 2| < 0.05$, then the distance of $f(x)$ from 10 is guaranteed to satisfy $|f(x) - 10| < 0.1$. Expressed in yet another way, when x is a number different from 2 but in the open interval $(1.95, 2.05)$ on the x-axis, then $f(x)$ is in the interval $(9.9, 10.1)$ on the y-axis.

Using the same example, let us try to generalize. Suppose ε (the Greek letter *epsilon*) denotes an arbitrary *positive number* that is our measure of arbitrary closeness to the number 10. If we demand that

$$|f(x) - 10| < \varepsilon \quad \text{or} \quad 10 - \varepsilon < f(x) < 10 + \varepsilon, \tag{2}$$

then from $10 - \varepsilon < 2x + 6 < 10 + \varepsilon$ and algebra, we find

$$2 - \frac{\varepsilon}{2} < x < 2 + \frac{\varepsilon}{2} \quad \text{or} \quad -\frac{\varepsilon}{2} < x - 2 < \frac{\varepsilon}{2}. \tag{3}$$

Again using absolute values and remembering that $x \neq 2$, we can write the last inequality in (3) as

$$0 < |x - 2| < \frac{\varepsilon}{2}. \tag{4}$$

If we denote $\varepsilon/2$ by the new symbol δ (the Greek letter *delta*), (2) and (4) can be written as

$$|f(x) - 10| < \varepsilon \quad \text{whenever} \quad 0 < |x - 2| < \delta.$$

Thus, for a new value for ε, say $\varepsilon = 0.001$, $\delta = \varepsilon/2 = 0.0005$ tells us the corresponding closeness to 2. For any number x different from 2 in $(1.9995, 2.0005)$,* we can be sure $f(x)$ is in $(9.999, 10.001)$. See FIGURE 2.6.1.

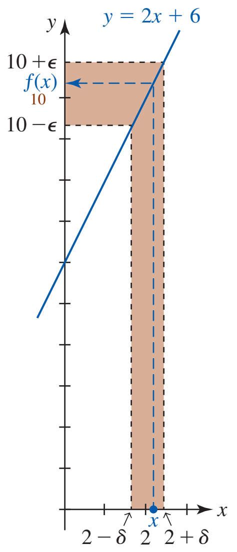


FIGURE 2.6.1 $f(x)$ is in $(10 - \varepsilon, 10 + \varepsilon)$ whenever x is in $(2 - \delta, 2 + \delta)$, $x \neq 2$

A Definition The foregoing discussion leads us to the so-called $\varepsilon - \delta$ **definition of a limit**.

Definition 2.6.1 Definition of a Limit

Suppose a function f is defined everywhere on an open interval, except possibly at a number a in the interval. Then

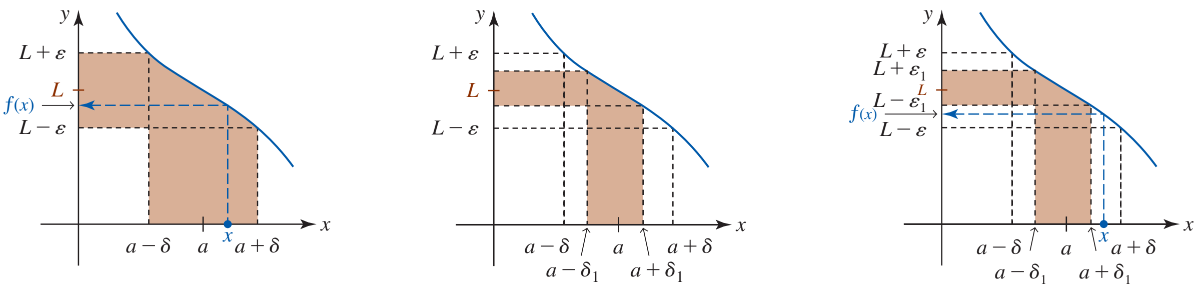
$$\lim_{x\to a} f(x) = L$$

means that for every $\varepsilon > 0$, there exists a number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - a| < \delta.$$

*For this reason, we use $0 < |x - 2| < \delta$ rather than $|x - 2| < \delta$. Keep in mind when considering $\lim_{x\to 2} f(x)$, we do not care about f *at* 2.

Let $\lim_{x\to a} f(x) = L$ and suppose $\delta > 0$ is the number that "works" in the sense of Definition 2.6.1 for a given $\varepsilon > 0$. As shown in FIGURE 2.6.2(a), every x in $(a - \delta, a + \delta)$, with the possible exception of a itself, will then have an image $f(x)$ in $(L - \varepsilon, L + \varepsilon)$. Furthermore, as in Figure 2.6.2(b), a choice $\delta_1 < \delta$ for the same ε also "works" in that every x not equal to a in $(a - \delta_1, a + \delta_1)$ gives $f(x)$ in $(L - \varepsilon, L + \varepsilon)$. However, Figure 2.6.2(c) shows that choosing a smaller ε_1, $0 < \varepsilon_1 < \varepsilon$, will demand finding a new value of δ. Observe in Figure 2.6.2(c) that x is in $(a - \delta, a + \delta)$ but not in $(a - \delta_1, a + \delta_1)$, and so $f(x)$ is not necessarily in $(L - \varepsilon_1, L + \varepsilon_1)$.



(a) A δ that works for a given ε

(b) A smaller δ_1 will also work for the same ε

(c) A smaller ε_1 will require a $\delta_1 < \delta$. For x in $(a - \delta, a + \delta)$, $f(x)$ is not necessarily in $(L - \varepsilon_1, L + \varepsilon_1)$

FIGURE 2.6.2 $f(x)$ is in $(L - \varepsilon, L + \varepsilon)$ whenever x is in $(a - \delta, a + \delta)$, $x \neq a$

EXAMPLE 1 Using Definition 2.6.1

Prove that $\lim_{x\to a} (5x + 2) = 17$.

Solution For any arbitrary $\varepsilon > 0$, regardless how small, we wish to find a δ so that

$$|(5x + 2) - 17| < \varepsilon \qquad \text{whenever} \qquad 0 < |x - 3| < \delta.$$

To do this consider

$$|(5x + 2) - 17| = |5x - 15| = 5|x - 3|.$$

Thus, to make $|(5x + 2) - 17| = 5|x - 3| < \varepsilon$, we need only make $0 < |x - 3| < \varepsilon/5$; that is, choose $\delta = \varepsilon/5$.

Verification If $0 < |x - 3| < \varepsilon/5$, then $5|x - 3| < \varepsilon$ implies

$$|5x - 15| < \varepsilon \qquad \text{or} \qquad |(5x + 2) - 17| < \varepsilon \qquad \text{or} \qquad |f(x) - 17| < \varepsilon. \qquad \blacksquare$$

EXAMPLE 2 Using Definition 2.6.1

Prove that $\lim_{x\to -4} \dfrac{16 - x^2}{4 + x} = 8$.

We examined this limit in (1) and (2) of Section 2.1.

Solution For $x \neq -4$,

$$\left|\frac{16 - x^2}{4 + x} - 8\right| = |4 - x - 8| = |-x - 4| = |x + 4| = |x - (-4)|$$

Thus,

$$\left|\frac{16 - x^2}{4 + x} - 8\right| = |x - (-4)| < \varepsilon$$

whenever we have $0 < |x - (-4)| < \varepsilon$; that is, choose $\delta = \varepsilon$. $\blacksquare$

EXAMPLE 3 A Limit That Does Not Exist

Consider the function

$$f(x) = \begin{cases} 0, & x \le 1 \\ 2, & x > 1. \end{cases}$$

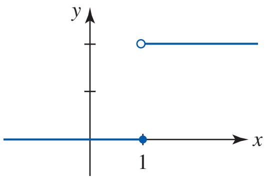


FIGURE 2.6.3 Limit of f does not exist as x approaches 1 in Example 3

We recognize in FIGURE 2.6.3 that f has a jump discontinuity at 1 and so $\lim_{x\to 1} f(x)$ does not exist. However, to *prove* this last fact, we shall proceed indirectly. Assume that the limit exists, namely, $\lim_{x\to 1} f(x) = L$. Then from Definition 2.6.1 we know that for the choice $\varepsilon = \frac{1}{2}$ there must exist a $\delta > 0$ so that

$$|f(x) - L| < \frac{1}{2} \quad \text{whenever} \quad 0 < |x - 1| < \delta.$$

Now to the right of 1, let us choose $x = 1 + \delta/2$. Since

$$0 < \left|1 + \frac{\delta}{2} - 1\right| = \left|\frac{\delta}{2}\right| < \delta$$

we must have

$$\left|f\left(1 + \frac{\delta}{2}\right) - L\right| = |2 - L| < \frac{1}{2}. \tag{5}$$

To the left of 1, choose $x = 1 - \delta/2$. But

$$0 < \left|1 - \frac{\delta}{2} - 1\right| = \left|-\frac{\delta}{2}\right| < \delta$$

implies

$$\left|f\left(1 - \frac{\delta}{2}\right) - L\right| = |0 - L| = |L| < \frac{1}{2}. \tag{6}$$

Solving the absolute-value inequalities (5) and (6) gives, respectively,

$$\frac{3}{2} < L < \frac{5}{2} \quad \text{and} \quad -\frac{1}{2} < L < \frac{1}{2}.$$

Since no number L can satisfy both of these inequalities, we conclude that $\lim_{x\to 1} f(x)$ does not exist. ■

In the next example we consider the limit of a quadratic function. We shall see that finding the δ in this case requires a bit more ingenuity than in Examples 1 and 2.

EXAMPLE 4 Using Definition 2.6.1

We examined this limit in Example 1 of Section 2.1.

Prove that $\lim_{x\to 4} (-x^2 + 2x + 2) = -6$.

Solution For an arbitrary $\varepsilon > 0$ we must find a $\delta > 0$ so that

$$|-x^2 + 2x + 2 - (-6)| < \varepsilon \quad \text{whenever} \quad 0 < |x - 4| < \delta.$$

Now,

$$\begin{aligned} |-x^2 + 2x + 2 - (-6)| &= |(-1)(x^2 - 2x - 8)| \\ &= |(x + 2)(x - 4)| \\ &= |x + 2||x - 4|. \end{aligned} \tag{7}$$

In other words, we want to make $|x + 2||x - 4| < \varepsilon$. But since we have agreed to examine values of x *near* 4, let us consider only those values for which $|x - 4| < 1$. This last inequality gives $3 < x < 5$ or equivalently $5 < x + 2 < 7$. Consequently we can write $|x + 2| < 7$. Hence from (7),

$$0 < |x - 4| < 1 \quad \text{implies} \quad |-x^2 + 2x + 2 - (-6)| < 7|x - 4|.$$

If we now choose δ to be the minimum of the two numbers, 1 and $\varepsilon/7$, written $\delta = \min\{1, \varepsilon/7\}$ we have

$$0 < |x - 4| < \delta \quad \text{implies} \quad |-x^2 + 2x + 2 - (-6)| < 7|x - 4| < 7 \cdot \frac{\varepsilon}{7} = \varepsilon. \quad ■$$

The reasoning in Example 4 is subtle. Consequently it is worth a few minutes of your time to reread the discussion immediately following Definition 2.6.1, reexamine

Figure 2.3.2(b), and then think again about why $\delta = \min\{1, \varepsilon/7\}$ is the δ that "works" in the example. Remember, you can pick the ε arbitrarily; think about δ for, say, $\varepsilon = 8$, $\varepsilon = 6$, and $\varepsilon = 0.01$.

One-Sided Limits We state next the definitions of the **one-sided limits**, $\lim_{x \to a^-} f(x)$ and $\lim_{x \to a^+} f(x)$.

Definition 2.6.2 Left-Hand Limit

Suppose a function f is defined on an open interval (c, a). Then

$$\lim_{x \to a^-} f(x) = L$$

means for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad a - \delta < x < a.$$

Definition 2.6.3 Right-Hand Limit

Suppose a function f is defined on an open interval (a, c). Then

$$\lim_{x \to a^+} f(x) = L$$

means for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \qquad \text{whenever} \qquad a < x < a + \delta.$$

EXAMPLE 5 Using Definition 2.6.3

Prove that $\lim_{x \to 0^+} \sqrt{x} = 0$.

Solution First, we can write

$$|\sqrt{x} - 0| = |\sqrt{x}| = \sqrt{x}.$$

Then, $|\sqrt{x} - 0| < \varepsilon$ whenever $0 < x < 0 + \varepsilon^2$. In other words, we choose $\delta = \varepsilon^2$.

Verification If $0 < x < \varepsilon^2$, then $0 < \sqrt{x} < \varepsilon$ implies

$$|\sqrt{x}| < \varepsilon \qquad \text{or} \qquad |\sqrt{x} - 0| < \varepsilon.$$

■

Limits Involving Infinity The two concepts of **infinite limits**

$$f(x) \to \infty \text{ (or } -\infty) \quad \text{as} \quad x \to a$$

and a **limit at infinity**

$$f(x) \to L \quad \text{as} \quad x \to \infty \text{ (or } -\infty)$$

are formalized in the next two definitions.

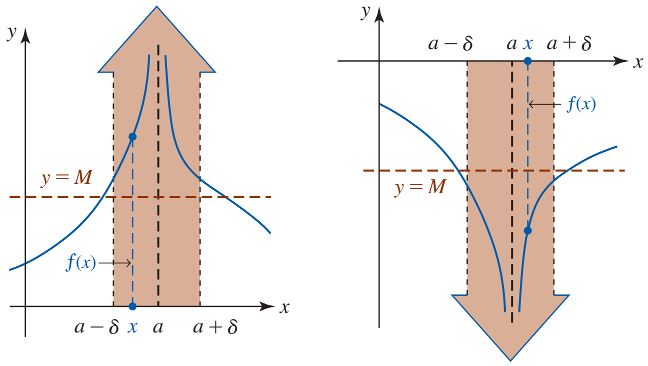
Recall, an infinite limit is a limit that does not exist as $x \to a$.

Definition 2.6.4 Infinite Limits

(*i*) $\lim_{x \to a} f(x) = \infty$ means for each $M > 0$, there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - a| < \delta$.

(*ii*) $\lim_{x \to a} f(x) = -\infty$ means for each $M < 0$, there exists a $\delta > 0$ such that $f(x) < M$ whenever $0 < |x - a| < \delta$.

Parts (*i*) and (*ii*) of Definition 2.6.4 are illustrated in FIGURE 2.6.4(a) and Figure 2.6.4(b), respectively. Recall, if $f(x) \to \infty$ (or $-\infty$) as $x \to a$, then $x = a$ is a vertical asymptote for the graph of f. In the case when $f(x) \to \infty$ as $x \to a$, then $f(x)$ can be made larger than any arbitrary positive number (that is, $f(x) > M$) by taking x sufficiently close to a (that is, $0 < |x - a| < \delta$).



(a) For a given M, whenever $a - \delta < x < a + \delta$, $x \neq a$, then $f(x) > M$

(b) For a given M, whenever $a - \delta < x < a + \delta$, $x \neq a$, then $f(x) < M$

FIGURE 2.6.4 Infinite limits as $x \to a$

The four one-sided infinite limits

$$f(x) \to \infty \text{ as } x \to a^-, \qquad f(x) \to -\infty \text{ as } x \to a^-$$
$$f(x) \to \infty \text{ as } x \to a^+, \qquad f(x) \to -\infty \text{ as } x \to a^+$$

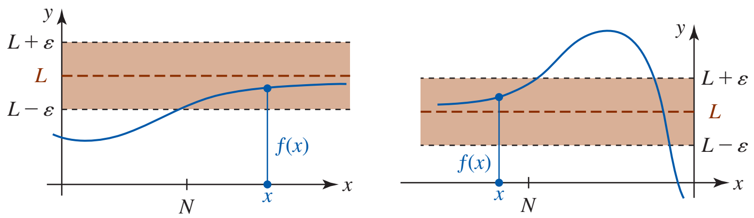
are defined in a manner analogous to that given in Definitions 2.6.2 and 2.6.3.

Definition 2.6.5 Limits at Infinity

(*i*) $\lim_{x \to \infty} f(x) = L$ if for each $\varepsilon > 0$, there exists an $N > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > N$.

(*ii*) $\lim_{x \to -\infty} f(x) = L$ if for each $\varepsilon > 0$, there exists an $N < 0$ such that $|f(x) - L| < \varepsilon$ whenever $x < N$.

Parts (*i*) and (*ii*) of Definition 2.6.5 are illustrated in FIGURE 2.6.5(a) and Figure 2.6.5(b), respectively. Recall, if $f(x) \to L$ as $x \to \infty$ (or $-\infty$), then $y = L$ is a horizontal asymptote for the graph of f. In the case when $f(x) \to L$ as $x \to \infty$, then the graph of f can be made arbitrarily close to the line $y = L$ (that is, $|f(x) - L| < \varepsilon$) by taking x sufficiently far out on the positive x-axis (that is, $x > N$).



(a) For a given ε, $x > N$ implies $L - \varepsilon < f(x) < L + \varepsilon$

(b) For a given ε, $x < N$ implies $L - \varepsilon < f(x) < L + \varepsilon$

FIGURE 2.6.5 Limits at infinity

EXAMPLE 6 Using Definition 2.6.5(i)

Prove that $\lim_{x\to\infty} \frac{3x}{x+1} = 3$.

Solution By Definition 2.6.5(i), for any $\varepsilon > 0$, we must find a number $N > 0$ such that

$$\left|\frac{3x}{x+1} - 3\right| < \varepsilon \qquad \text{whenever} \qquad x > N.$$

Now, by considering $x > 0$, we have

$$\left|\frac{3x}{x+1} - 3\right| = \left|\frac{-3}{x+1}\right| = \frac{3}{x+1} < \frac{3}{x} < \varepsilon$$

whenever $x > 3/\varepsilon$. Hence, choose $N = 3/\varepsilon$. For example, if $\varepsilon = 0.01$, then $N = 3/(0.01) = 300$ will guarantee that $|f(x) - 3| < 0.01$ whenever $x > 300$. ■

Postscript—A Bit of History After this section you may agree with English philosopher, priest, historian, and scientist William Whewell (1794–1866), who wrote in 1858 that "A limit is a peculiar . . . conception." For many years after the invention of calculus in the seventeenth century, mathematicians argued and debated the nature of a limit. There was an awareness that intuition, graphs, and numerical examples of ratios of vanishing quantities provide at best a shaky foundation for such a fundamental concept. As you will see beginning in the next chapter, the limit concept plays a central role in calculus. The study of calculus went through several periods of increased mathematical rigor beginning with the French mathematician Augustin-Louis Cauchy and continuing later with the German mathematician Karl Wilhelm Weierstrass.


Cauchy

Augustin-Louis Cauchy (1789–1857) was born during an era of upheaval in French history. Cauchy was destined to initiate a revolution of his own in mathematics. For many contributions, but especially for his efforts in clarifying mathematical obscurities, his incessant demand for satisfactory definitions and rigorous proofs of theorems, Cauchy is often called "the father of modern analysis." A prolific writer whose output has been surpassed by only a few, Cauchy produced nearly 800 papers in astronomy, physics, and mathematics. But the same mind that was always open and inquiring in science and mathematics was also narrow and unquestioning in many other areas. Outspoken and arrogant, Cauchy's passionate stands on political and religious issues often alienated him from his colleagues.


Weierstrass

Karl Wilhelm Weierstrass (1815–1897) One of the foremost mathematical analysts of the nineteenth century never earned an academic degree! After majoring in law at the University of Bonn, but concentrating in fencing and beer drinking for four years, Weierstrass "graduated" to real life with no degree. In need of a job, Weierstrass passed a state examination and received a teaching certificate in 1841. During a period of 15 years as a secondary school teacher, his dormant mathematical genius blossomed. Although the quantity of his research publications was modest, especially when compared with that of Cauchy, the quality of these works so impressed the German mathematical community that he was awarded a doctorate, *honoris causa*, from the University of Königsberg and eventually was appointed a professor at the University of Berlin. While there, Weierstrass achieved worldwide recognition both as a mathematician and as a teacher of mathematics. One of his students was Sonja Kowalewski, the greatest female mathematician of the nineteenth century. It was Karl Wilhelm Weierstrass who was responsible for putting the concept of a limit on a firm foundation with the ε-δ definition.

Exercises 2.6

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–24, use Definitions 2.6.1, 2.6.2, or 2.6.3 to prove the given limit result.

1. $\lim_{x\to 5} 10 = 10$

2. $\lim_{x\to -2} \pi = \pi$

3. $\lim_{x\to 3} x = 3$

4. $\lim_{x\to 4} 2x = 8$

5. $\lim_{x\to -1} (x + 6) = 5$

6. $\lim_{x\to 0} (x - 4) = -4$

7. $\lim_{x\to 0} (3x + 7) = 7$

8. $\lim_{x\to 1} (9 - 6x) = 3$

9. $\lim_{x\to 2} \dfrac{2x - 3}{4} = \dfrac{1}{4}$

10. $\lim_{x\to 1/2} 8(2x + 5) = 48$

11. $\lim_{x\to -5} \dfrac{x^2 - 25}{x + 5} = -10$

12. $\lim_{x\to 3} \dfrac{x^2 - 7x + 12}{2x - 6} = -\dfrac{1}{2}$

13. $\lim_{x\to 0} \dfrac{8x^5 + 12x^4}{x^4} = 12$

14. $\lim_{x\to 1} \dfrac{2x^3 + 5x^2 - 2x - 5}{x^2 - 1} = 7$

15. $\lim_{x\to 0} x^2 = 0$

16. $\lim_{x\to 0} 8x^3 = 0$

17. $\lim_{x\to 0^+} \sqrt{5x} = 0$

18. $\lim_{x\to (1/2)^+} \sqrt{2x - 1} = 0$

19. $\lim_{x\to 0^-} f(x) = -1, \quad f(x) = \begin{cases} 2x - 1, & x < 0 \\ 2x + 1, & x > 0 \end{cases}$

20. $\lim_{x\to 1^+} f(x) = 3, \quad f(x) = \begin{cases} 0, & x \le 1 \\ 3, & x > 1 \end{cases}$

21. $\lim_{x\to 3} x^2 = 9$

22. $\lim_{x\to 2} (2x^2 + 4) = 12$

23. $\lim_{x\to 1} (x^2 - 2x + 4) = 3$

24. $\lim_{x\to 5} (x^2 + 2x) = 35$

25. For $a > 0$, use the identity.

$$|\sqrt{x} - \sqrt{a}| = |\sqrt{x} - \sqrt{a}| \cdot \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} = \frac{|x - a|}{\sqrt{x} + \sqrt{a}}$$

and the fact that $\sqrt{x} \ge 0$ to prove that $\lim_{x\to a} \sqrt{x} = \sqrt{a}$.

26. Prove that $\lim_{x\to 2} (1/x) = \frac{1}{2}$. [*Hint*: Consider only those numbers x for which $1 < x < 3$.]

In Problems 27–30, prove that $\lim_{x\to a} f(x)$ does not exist.

27. $f(x) = \begin{cases} 2, & x < 1 \\ 0, & x \ge 1 \end{cases}; \quad a = 1$

28. $f(x) = \begin{cases} 1, & x \le 3 \\ -1, & x > 3 \end{cases}; \quad a = 3$

29. $f(x) = \begin{cases} x, & x \le 0 \\ 2 - x, & x > 0 \end{cases}; \quad a = 0$

30. $f(x) = \dfrac{1}{x}; \quad a = 0$

In Problems 31–34, use Definition 2.6.5 to prove the given limit result.

31. $\lim_{x\to\infty} \dfrac{5x - 1}{2x + 1} = \dfrac{5}{2}$

32. $\lim_{x\to\infty} \dfrac{2x}{3x + 8} = \dfrac{2}{3}$

33. $\lim_{x\to -\infty} \dfrac{10x}{x - 3} = 10$

34. $\lim_{x\to -\infty} \dfrac{x^2}{x^2 + 3} = 1$

Think About It

35. Prove that $\lim_{x\to 0} f(x) = 0, \quad$ where $\quad f(x) = \begin{cases} x, & x \text{ rational} \\ 0, & x \text{ irrational}. \end{cases}$

2.7 The Tangent Line Problem

Introduction In a calculus course you will study many different things, but as mentioned in the introduction to Section 2.1, the subject "calculus" is roughly divided into two broad but related areas known as **differential calculus** and **integral calculus**. The discussion of each of these topics invariably begins with a motivating problem involving the graph of a function. Differential calculus is motivated by the problem

- *Find a tangent line to the graph of a function f,*

whereas integral calculus is motivated by the problem

- *Find the area under the graph of a function f.*

The first problem will be addressed in this section; the second problem will be discussed in Section 5.3.

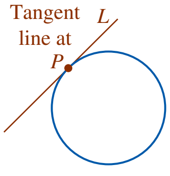


FIGURE 2.7.1 Tangent line L touches a circle at point P

Tangent Line to a Graph The word *tangent* stems from the Latin verb *tangere*, meaning "to touch." You might remember from the study of plane geometry that a tangent to a circle is a line L that intersects, or touches, the circle in exactly one point P. See FIGURE 2.7.1. It is not quite as easy to define a tangent line to the graph of a function f. The idea of *touching* carries over to the notion of a tangent line to the graph of a function, but the idea of *intersecting the graph in one point* does not carry over.

Suppose $y = f(x)$ is a continuous function. If, as shown in FIGURE 2.7.2, f possesses a line L tangent to its graph at a point P, then what is an equation of this line? To answer this question, we need the coordinates of P and the slope $m_{\tan}$ of L. The coordinates of P pose no difficulty, since a point on the graph of a function f is obtained by specifying a value of x in the domain of f. The coordinates of the point of tangency at $x = a$ are then $(a, f(a))$. Therefore, the problem of finding a tangent line comes down to the problem of finding the slope $m_{\tan}$ of the line. As a means of *approximating* $m_{\tan}$, we can readily find the slopes $m_{\sec}$ of *secant lines* (from the Latin verb *secare*, meaning "to cut") that pass through the point P and any other point Q on the graph. See FIGURE 2.7.3.

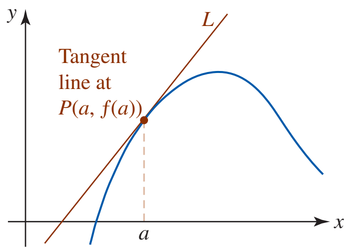


FIGURE 2.7.2 Tangent line L to a graph at point P

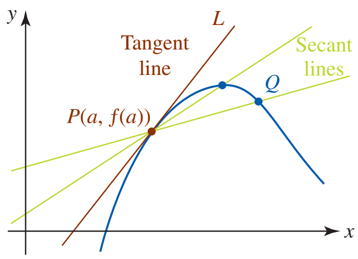


FIGURE 2.7.3 Slopes of secant lines approximate the slope $m_{\tan}$ of L

Slope of Secant Lines If P has coordinates $(a, f(a))$ and if Q has coordinates $(a + h, f(a + h))$, then as shown in FIGURE 2.7.4, the slope of the secant line through P and Q is

$$m_{\sec} = \frac{\text{change in } y}{\text{change in } x} = \frac{f(a+h) - f(a)}{(a+h) - a}$$

or
$$m_{\sec} = \frac{f(a+h) - f(a)}{h}. \tag{1}$$

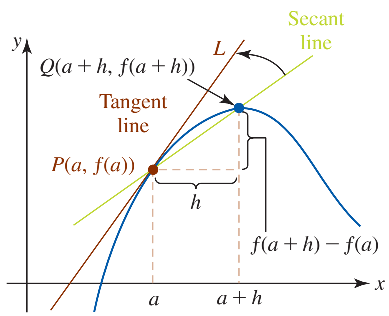


FIGURE 2.7.4 Secant lines swing into the tangent line L as $h \to 0$

The expression on the right-hand side of the equality in (1) is called a **difference quotient**. When we let h take on values that are closer and closer to zero, that is, as $h \to 0$, then the points $Q(a + h, f(a + h))$ move along the curve closer and closer to the point $P(a, f(a))$. Intuitively, we expect the secant lines to approach the tangent line L, and that $m_{\sec} \to m_{\tan}$ as $h \to 0$. That is,

$$m_{\tan} = \lim_{h \to 0} m_{\sec}$$

provided this limit exists. We summarize this conclusion in an equivalent form of the limit using the difference quotient (1).

Definition 2.7.1 Tangent Line with Slope

Let $y = f(x)$ be continuous at the number a. If the limit

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \tag{2}$$

exists, then the **tangent line** to the graph of f at $(a, f(a))$ is that line passing through the point $(a, f(a))$ with slope $m_{\tan}$.

Just like many of the problems discussed earlier in this chapter, observe that the limit in (2) has the indeterminate form $0/0$ as $h \to 0$.

If the limit in (2) exists, the number $m_{\tan}$ is also called the **slope of the curve** $y = f(x)$ at $(a, f(a))$.

The computation of (2) is essentially a *four-step process*; three of these steps involve only precalculus mathematics: algebra and trigonometry. If the first three steps are done accurately, the fourth step, or the calculus step, *may* be the easiest part of the problem.

Guidelines for Computing (2)

(*i*) Evaluate $f(a)$ and $f(a + h)$.
(*ii*) Evaluate the difference $f(a + h) - f(a)$. Simplify.
(*iii*) Simplify the difference quotient

$$\frac{f(a+h) - f(a)}{h}.$$

(*iv*) Compute the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}.$$

The computation of the difference $f(a + h) - f(a)$ in step (*ii*) is in most instances the most important step. It is imperative that you simplify this step as much as possible. Here is a tip: In *many* problems involving the computation of (2) you will be able to factor h from the difference $f(a + h) - f(a)$.

Note ▶

EXAMPLE 1 The Four-Step Process

Find the slope of the tangent line to the graph of $y = x^2 + 2$ at $x = 1$.

Solution We use the four-step procedure outlined above with the number 1 playing the part of the symbol a.

(*i*) The initial step is the computation of $f(1)$ and $f(1 + h)$. We have $f(1) = 1^2 + 2 = 3$, and

$$\begin{aligned} f(1 + h) &= (1 + h)^2 + 2 \\ &= (1 + 2h + h^2) + 2 \\ &= 3 + 2h + h^2. \end{aligned}$$

(*ii*) Next, from the result in the preceding step the difference is:

$$\begin{aligned} f(1 + h) - f(1) &= 3 + 2h + h^2 - 3 \\ &= 2h + h^2 \\ &= h(2 + h). \leftarrow \text{notice the factor of } h \end{aligned}$$

(*iii*) The computation of the difference quotient $\dfrac{f(1 + h) - f(1)}{h}$ is now straightforward. Again, we use the results from the preceding step:

$$\frac{f(1 + h) - f(1)}{h} = \frac{h(2 + h)}{h} = 2 + h. \leftarrow \text{cancel the } h\text{'s}$$

(*iv*) The last step is now easy. The limit in (2) is seen to be

from the preceding step

$$m_{\tan} = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} = \lim_{h \to 0} (2 + h) = 2.$$

The slope of the tangent line to the graph of $y = x^2 + 2$ at (1, 3) is 2. ■

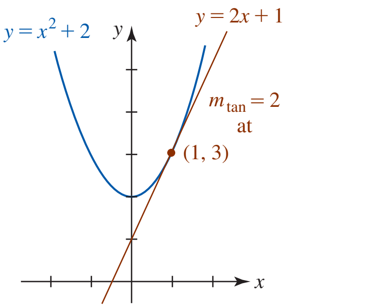


FIGURE 2.7.5 Tangent line in Example 2

EXAMPLE 2 Equation of Tangent Line

Find an equation of the tangent line whose slope was found in Example 1.

Solution We know the point of tangency (1, 3) and the slope $m_{\tan} = 2$, and so from the point–slope equation of a line we find

$$y - 3 = 2(x - 1) \qquad \text{or} \qquad y = 2x + 1.$$

Observe that the last equation is consistent with the x- and y-intercepts of the red line in FIGURE 2.7.5. ■

EXAMPLE 3 Equation of Tangent Line

Find an equation of the tangent line to the graph of $f(x) = 2/x$ at $x = 2$.

Solution We start by using (2) to find $m_{\tan}$ with a identified as 2. In the second of the four steps, we will have to combine two symbolic fractions by means of a common denominator.

(*i*) We have $f(2) = 2/2 = 1$ and $f(2 + h) = 2/(2 + h)$.

(*ii*)
$$\begin{aligned} f(2 + h) - f(2) &= \frac{2}{2 + h} - 1 \\ &= \frac{2}{2 + h} - \frac{1}{1} \cdot \frac{2 + h}{2 + h} \leftarrow \text{a common denominator is } 2 + h \\ &= \frac{2 - 2 - h}{2 + h} \\ &= \frac{-h}{2 + h}. \leftarrow \text{here is the factor of } h \end{aligned}$$

(*iii*) The last result is to be divided by h or more precisely $\frac{h}{1}$. We invert and multiply by $\frac{1}{h}$:

$$\frac{f(2+h)-f(2)}{h} = \frac{\frac{-h}{2+h}}{\frac{h}{1}} = \frac{-h}{2+h}\cdot\frac{1}{h} = \frac{-1}{2+h}. \quad \leftarrow \text{cancel the } h\text{'s}$$

(*iv*) From (2) $m_{\tan}$ is

$$m_{\tan} = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h} = \lim_{h\to 0}\frac{-1}{2+h} = -\frac{1}{2}.$$

From $f(2) = 1$ the point of tangency is (2, 1) and the slope of the tangent line at (2, 1) is $m_{\tan} = -\frac{1}{2}$. From the point–slope equation of a line, the tangent line is

$$y - 1 = \frac{1}{2}(x-2) \qquad \text{or} \qquad y = -\frac{1}{2}x + 2.$$

The graphs of $y = 2/x$ and the tangent line at (2, 1) are shown in FIGURE 2.7.6. ■

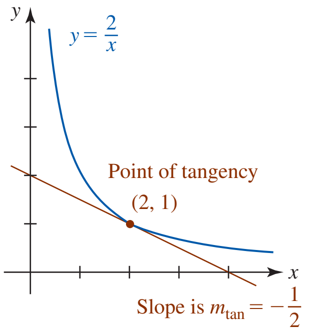


FIGURE 2.7.6 Tangent line in Example 3

EXAMPLE 4 Slope of Tangent Line

Find the slope of the tangent line to the graph of $f(x) = \sqrt{x-1}$ at $x = 5$.

Solution Replacing a by 5 in (2), we have:

(*i*) $f(5) = \sqrt{5-1} = \sqrt{4} = 2$, and

$$f(5+h) = \sqrt{5+h-1} = \sqrt{4+h}.$$

(*ii*) The difference is

$$f(5+h) - f(5) = \sqrt{4+h} - 2.$$

Because we expect to find a factor of h in this difference, we proceed to rationalize the numerator:

$$\begin{aligned} f(5+h) - f(5) &= \frac{\sqrt{4+h}-2}{1}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2} \\ &= \frac{(4+h)-4}{\sqrt{4+h}+2} \\ &= \frac{h}{\sqrt{4+h}+2}. \quad \leftarrow \text{here is the factor of } h \end{aligned}$$

(*iii*) The difference quotient $\dfrac{f(5+h)-f(5)}{h}$ is then:

$$\begin{aligned} \frac{f(5+h)-f(5)}{h} &= \frac{\frac{h}{\sqrt{4+h}+2}}{h} \\ &= \frac{h}{h(\sqrt{4+h}+2)} \\ &= \frac{1}{\sqrt{4+h}+2}. \end{aligned}$$

(*iv*) The limit in (2) is

$$m_{\tan} = \lim_{h\to 0}\frac{f(5+h)-f(5)}{h} = \lim_{h\to 0}\frac{1}{\sqrt{4+h}+2} = \frac{1}{\sqrt{4}+2} = \frac{1}{4}.$$

The slope of the tangent line to the graph of $f(x) = \sqrt{x-1}$ at (5, 2) is $\frac{1}{4}$. ■

The result obtained in the next example should come as no surprise.

EXAMPLE 5 Tangent Line to a Line

For any linear function $y = mx + b$, the tangent line to its graph coincides with the line itself. Not unexpectedly then, the slope of the tangent line for any number $x = a$ is

$$m_{\tan} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{m(a+h) + b - (ma+b)}{h} = \lim_{h \to 0} \frac{mh}{h} = \lim_{h \to 0} m = m. \quad ■$$

Vertical Tangents The limit in (2) can fail to exist for a function f at $x = a$ and yet there may be a tangent at the point $(a, f(a))$. The tangent line to a graph can be **vertical**, in which case its slope is undefined. We will consider the concept of vertical tangents in more detail in Section 3.1.

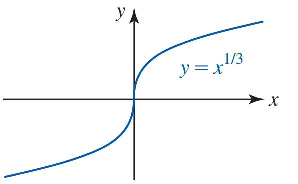


FIGURE 2.7.7 Vertical tangent in Example 6

EXAMPLE 6 Vertical Tangent Line

Although we will not pursue the details at this time, it can be shown that the graph of $f(x) = x^{1/3}$ possesses a vertical tangent line at the origin. In FIGURE 2.7.7 we see that the y-axis, that is, the line $x = 0$, is tangent to the graph at the point (0, 0). ■

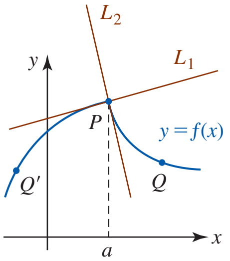


FIGURE 2.7.8 Tangent fails to exist at $(a, f(a))$

A Tangent May Not Exist The graph of a function f that is continuous at a number a does not have to possess a tangent line at the point $(a, f(a))$. A tangent line will not exist whenever the graph of f has a sharp corner at $(a, f(a))$. FIGURE 2.7.8 indicates what can go wrong when the graph of a function f has a "corner." In this case f is continuous at a, but the secant lines through P and Q approach L_2 as $Q \to P$, and the secant lines through P and Q' approach a different line L_1 as $Q' \to P$. In other words, the limit in (2) fails to exist because the one-sided limits of the difference quotient (as $h \to 0^+$ and as $h \to 0^-$) are different.

EXAMPLE 7 Graph with a Corner

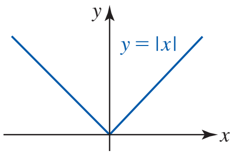


FIGURE 2.7.9 Function in Example 7

Show that the graph of $f(x) = |x|$ does not have a tangent at (0, 0).

Solution The graph of the absolute-value function in FIGURE 2.7.9 has a corner at the origin. To prove that the graph of f does not possess a tangent line at the origin we must examine

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|0+h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

From the definition of absolute value

$$|h| = \begin{cases} h, & h > 0 \\ -h, & h < 0 \end{cases}$$

we see that

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1 \qquad \text{whereas} \qquad \lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1.$$

Since the right-hand and left-hand limits are not equal we conclude that the limit (2) does not exist. Even though the function $f(x) = |x|$ is continuous at $x = 0$, the graph of f possesses no tangent at (0, 0). ■

Average Rate of Change In different contexts the difference quotient in (1) and (2), or slope of the secant line, is written in terms of alternative symbols. The symbol h in (1) and (2) is often written as Δx and the difference $f(a + \Delta x) - f(a)$ is denoted by Δy, that is, the difference quotient is

$$\frac{\text{change in } y}{\text{change in } x} = \frac{f(a + \Delta x) - f(a)}{(a + \Delta x) - a} = \frac{f(a + \Delta x) - f(a)}{\Delta x} = \frac{\Delta y}{\Delta x}. \tag{3}$$

Moreover, if $x_1 = a + \Delta x$, $x_0 = a$, then $\Delta x = x_1 - x_0$ and (3) is the same as

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{\Delta y}{\Delta x}. \tag{4}$$

The slope $\Delta y/\Delta x$ of the secant line through the points $(x_0, f(x_0))$ and $(x_1, f(x_1))$ is called the **average rate of change of the function** f over the interval $[x_0, x_1]$. The limit $\lim_{\Delta x \to 0} \Delta y/\Delta x$ is then called the **instantaneous rate of change of the function** with respect to x at x_0.

Almost everyone has an intuitive notion of speed as a rate at which a distance is covered in a certain length of time. When, say, a bus travels 60 mi in 1 h, the *average speed* of the bus must have been 60 mi/h. Of course, it is difficult to maintain the rate of 60 mi/h for the entire trip because the bus slows down for towns and speeds up when it passes cars. In other words, the speed changes with time. If a bus company's schedule demands that the bus travel the 60 mi from one town to another in 1 h, the driver knows instinctively that he or she must compensate for speeds less than 60 mi/h by traveling at speeds greater than this at other points in the journey. Knowing that the average velocity is 60 mi/h does not, however, answer the question: What is the velocity of the bus at a particular instant?

Average Velocity In general, the **average velocity** or **average speed** of a moving object is defined by

$$v_{\text{ave}} = \frac{\text{change of distance}}{\text{change in time}}. \tag{5}$$

Consider a runner who finishes a 10-km race in an elapsed time of 1 h 15 min (1.25 h). The runner's average velocity or average speed for the race was

$$v_{\text{ave}} = \frac{10 - 0}{1.25 - 0} = 8 \text{ km/h}.$$

But suppose we now wish to determine the runner's *exact* velocity v at the instant the runner is one-half hour into the race. If the distance run in the time interval from 0 h to 0.5 h is measured to be 5 km, then

$$v_{\text{ave}} = \frac{5}{0.5} = 10 \text{ km/h}.$$

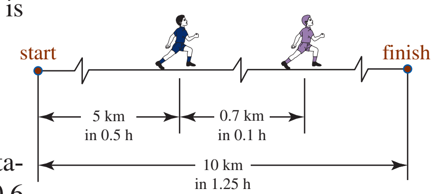


FIGURE 2.7.10 Runner in a 10-km race

Again, this number is not a measure, or necessarily even a good indicator, of the instantaneous rate v at which the runner is moving 0.5 h into the race. If we determine that at 0.6 h the runner is 5.7 km from the starting line, then the average velocity from 0 h to 0.6 h is $v_{\text{ave}} = 5.7/0.6 = 9.5$ km/h. However, during the time interval from 0.5 h to 0.6 h,

$$v_{\text{ave}} = \frac{5.7 - 5}{0.6 - 0.5} = 7 \text{ km/h}.$$

The latter number is a more realistic measure of the rate v. See FIGURE 2.7.10. By "shrinking" the time interval between 0.5 h and the time that corresponds to a measured position close to 5 km, we expect to obtain even better approximations to the runner's velocity at time 0.5 h.

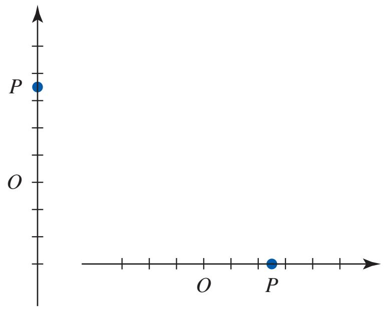


FIGURE 2.7.11 Coordinate lines

Rectilinear Motion To generalize the preceding discussion, let us suppose an object, or particle, at point P moves along either a vertical or horizontal coordinate line as shown in FIGURE 2.7.11. Furthermore, let the particle move in such a manner that its position, or coordinate, on the line is given by a function $s = s(t)$, where t represents time. The values of s are directed distances measured from O in units such as centimeters, meters, feet, or miles. When P is either to the right of or above O, we take $s > 0$, whereas $s < 0$ when P is either to the left of or below O. Motion in a straight line is called **rectilinear motion**.

If an object, such as a toy car moving on a horizontal coordinate line, is at point P at time t_0 and at point P' at time t_1, then the coordinates of the points, shown in FIGURE 2.7.12, are $s(t_0)$ and $s(t_1)$. By (4) the **average velocity** of the object in the time interval $[t_0, t_1]$ is

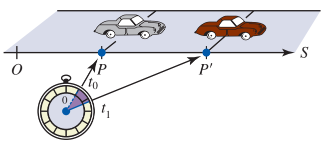


FIGURE 2.7.12 Position of toy car on a coordinate line at two times

$$v_{\text{ave}} = \frac{\text{change in position}}{\text{change in time}} = \frac{s(t_1) - s(t_0)}{t_1 - t_0}. \tag{6}$$

EXAMPLE 8 Average Velocity

The height s above ground of a ball dropped from the top of the St. Louis Gateway Arch is given by $s(t) = -16t^2 + 630$, where s is measured in feet and t in seconds. See FIGURE 2.7.13. Find the average velocity of the falling ball between the time the ball is released and the time it hits the ground.

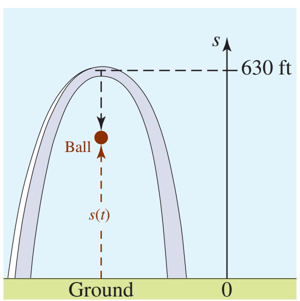


FIGURE 2.7.13 Falling ball in Example 8

Solution The time at which the ball is released is determined from the equation $s(t) = 630$ or $-16t^2 + 630 = 630$. This gives $t = 0$ s. When the ball hits the ground then $s(t) = 0$ or

$-16t^2 + 630 = 0$. The last equation gives $t = \sqrt{315/8} \approx 6.27$ s. Thus from (6) the average velocity in the time interval $\left[0, \sqrt{315/8}\right]$ is

$$v_{ave} = \frac{s\left(\sqrt{351/8}\right) - s(0)}{\sqrt{351/8} - 0} = \frac{0 - 630}{\sqrt{351/8} - 0} \approx -100.40 \text{ ft/s.} \qquad \blacksquare$$

If we let $t_1 = t_0 + \Delta t$, or $\Delta t = t_1 - t_0$, and $\Delta s = s(t_0 + \Delta t) - s(t_0)$, then (6) is equivalent to

$$v_{ave} = \frac{\Delta s}{\Delta t}. \tag{7}$$

This suggests that the limit of (7) as $\Delta t \to 0$ gives the **instantaneous rate of change** of $s(t)$ at $t = t_0$ or the **instantaneous velocity**.

Definition 2.7.2 Instantaneous Velocity

Let $s = s(t)$ be a function that gives the position of an object moving in a straight line. Then the **instantaneous velocity** at time $t = t_0$ is

$$v(t_0) = \lim_{\Delta t \to 0} \frac{s(t_0 + \Delta t) - s(t_0)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}, \tag{8}$$

whenever the limit exists.

Note: Except for notation and interpretation, there is no mathematical difference between (2) and (8). Also, the word *instantaneous* is often dropped, and so one often speaks of the *rate of change* of a function or the *velocity* of a moving particle.

EXAMPLE 9 Example 8 Revisited

Find the instantaneous velocity of the falling ball in Example 8 at $t = 3$ s.

Solution We use the same four-step procedure as in the earlier examples with $s = s(t)$ given in Example 8.

(*i*) $s(3) = -16(9) + 630 = 486$. For any $\Delta t \neq 0$,

$$s(3 + \Delta t) = -16(3 + \Delta t)^2 + 630 = -16(\Delta t)^2 - 96\Delta t + 486.$$

(*ii*) $$s(3 + \Delta t) - s(3) = [-16(\Delta t)^2 - 96\Delta t + 486] - 486 = -16(\Delta t)^2 - 96\Delta t = \Delta t(-16\Delta t - 96)$$

(*iii*) $$\frac{\Delta s}{\Delta t} = \frac{\Delta t(-16\Delta t - 96)}{\Delta t} = -16\Delta t - 96$$

(*iv*) From (8),

$$v(3) = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \lim_{\Delta t \to 0}(-16\Delta t - 96) = -96 \text{ ft/s.} \tag{9} \quad \blacksquare$$

In Example 9, the number $s(3) = 486$ ft is the height of the ball above ground at 3 s. The minus sign in (9) is significant because the ball is moving opposite to the positive or upward direction.

Exercises 2.7

Answers to selected odd-numbered problems begin on page ANS-000.

Fundamentals

In Problems 1–6, sketch the graph of the function and the tangent line at the given point. Find the slope of the secant line through the points that correspond to the indicated values of x.

1. $f(x) = -x^2 + 9$, $(2, 5)$; $x = 2$, $x = 2.5$
2. $f(x) = x^2 + 4x$, $(0, 0)$; $x = -\frac{1}{4}$, $x = 0$
3. $f(x) = x^3$, $(-2, -8)$; $x = -2$, $x = -1$
4. $f(x) = 1/x$, $(1, 1)$; $x = 0.9$, $x = 1$
5. $f(x) = \sin x$, $(\pi/2, 1)$; $x = \pi/2$, $x = 2\pi/3$
6. $f(x) = \cos x$, $\left(-\pi/3, \frac{1}{2}\right)$; $x = -\pi/2$, $x = -\pi/3$

In Problems 7–8, use (2) to find the slope of the tangent line to the graph of the function at the given value of x. Find an equation of the tangent line at the corresponding point.

7. $f(x) = x^2 - 6, x = 3$

8. $f(x) = -3x^2 + 10, x = -1$

9. $f(x) = x^2 - 3x, x = 1$

10. $f(x) = -x^2 + 5x - 3, x = -2$

11. $f(x) = -2x^3 + x, x = 2$

12. $f(x) = 8x^3 - 4, x = \frac{1}{2}$

13. $f(x) = \frac{1}{2x}, x = -1$

14. $f(x) = \frac{4}{x-1}, x = 2$

15. $f(x) = \frac{1}{(x-1)^2}, x = 0$

16. $f(x) = 4 - \frac{8}{x}, x = -1$

17. $f(x) = \sqrt{x}, x = 4$

18. $f(x) = \frac{1}{\sqrt{x}}, x = 1$

In Problems 19 and 20, use (2) to find the slope of the tangent line to the graph of the function at the given value of x. Find an equation of the tangent line at the corresponding point. Before starting, review the limits in (10) and (14) of Section 2.4 and the sum formulas (17) and (18) in Section 1.4.

19. $f(x) = \sin x, x = \pi/6$

20. $f(x) = \cos x, x = \pi/4$

In Problems 21 and 22, determine whether the line that passes through the red point is tangent to the graph of $f(x) = x^2$ at the blue point.

21.

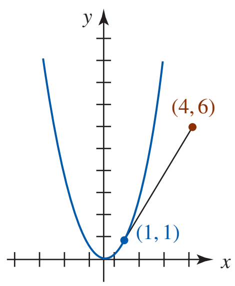


FIGURE 2.7.14 Graph for Problem 21

22.

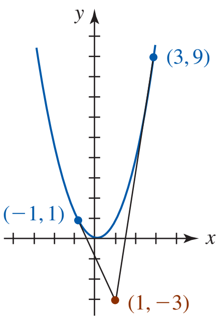


FIGURE 2.7.15 Graph for Problem 22

23. In FIGURE 2.7.16, the red line is tangent to the graph of $y = f(x)$ at the indicated point. Find an equation of the tangent line. What is the y-intercept of the tangent line?

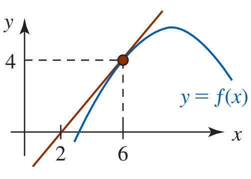


FIGURE 2.7.16 Graph for Problem 23

24. In FIGURE 2.7.17, the red line is tangent to the graph of $y = f(x)$ at the indicated point. Find $f(-5)$.

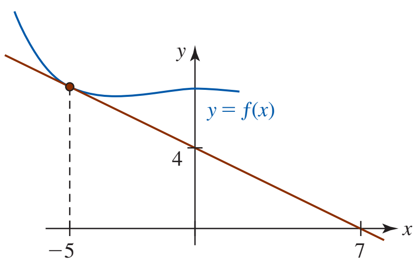


FIGURE 2.7.17 Graph for Problem 24

In Problems 25–28, use (2) to find a formula for m_{tan} at a general point $(x, f(x))$ on the graph of f. Use the formula for m_{tan} to determine the points where the tangent line to the graph is horizontal.

25. $f(x) = -x^2 + 6x + 1$

26. $f(x) = 2x^2 + 24x - 22$

27. $f(x) = x^3 - 3x$

28. $f(x) = -x^3 + x^2$

Applications

29. A car travels the 290 mi between Los Angeles and Las Vegas in 5 h. What is its average velocity?

30. Two marks on a straight highway are $\frac{1}{2}$ mi apart. A highway patrol plane observes that a car traverses the distance between the marks in 40 s. Assuming a speed limit of 60 mi/h, will the car be stopped for speeding?

31. A jet airplane averages 920 km/h to fly the 3500 km between Hawaii and San Francisco. How many hours does the flight take?

32. A marathon race is run over a straight 26-mi course. The race begins at noon. At 1:30 P.M. a contestant passes the 10-mi mark and at 3:10 P.M. the contestant passes the 20-mi mark. What is the contestant's average running speed between 1:30 P.M. and 3:10 P.M.?

In Problems 33 and 34, the position of a particle moving on a horizontal coordinate line is given by the function. Use (8) to find the instantaneous velocity of the particle at the indicated time.

33. $s(t) = -4t^2 + 10t + 6, t = 3$

34. $s(t) = t^2 + \frac{1}{5t+1}, t = 0$

35. The height above ground of a ball dropped from an initial altitude of 122.5 m is given by $s(t) = -4.9t^2 + 122.5$, where s is measured in meters and t in seconds.

(a) What is the instantaneous velocity at $t = \frac{1}{2}$?
(b) At what time does the ball hit the ground?
(c) What is the impact velocity?

36. Ignoring air resistance, if an object is dropped from an initial height h, then its height above ground at time $t > 0$ is given by $s(t) = -\frac{1}{2}gt^2 + h$, where g is the acceleration of gravity.

(a) At what time does the object hit the ground?
(b) If $h = 100$ ft, compare the impact times for Earth ($g = 32$ ft/s^2), for Mars ($g = 12$ ft/s^2), and for the Moon ($g = 5.5$ ft/s^2).
(c) Use (8) to find a formula for the instantaneous velocity v at a general time t.
(d) Using the times found in part (b) and the formula found in part (c), find the corresponding impact velocities for Earth, Mars, and the Moon.

37. The height of a projectile shot from ground level is given by $s = -16t^2 + 256t$, where s is measured in feet and t in seconds.

(a) Determine the height of the projectile at $t = 2$, $t = 6$, $t = 9$, and $t = 10$.
(b) What is the average velocity of the projectile between $t = 2$ and $t = 5$?
(c) Show that the average velocity between $t = 7$ and $t = 9$ is zero. Interpret physically.
(d) At what time does the projectile hit the ground?

(e) Use (8) to find a formula for instantaneous velocity v at a general time t.
(f) Using the time found in part (d) and the formula found in part (e), find the corresponding impact velocity.
(g) What is the maximum height the projectile attains?

38. Suppose the graph shown in FIGURE 2.7.18 is that of position function $s = s(t)$ of a particle moving in a straight line, where s is measured in meters and t in seconds.

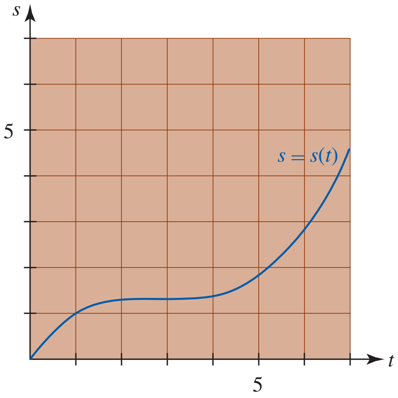


FIGURE 2.7.18 Graph for Problem 38

(a) Estimate the position of the particle at $t = 4$ and at $t = 6$.
(b) Estimate the average velocity of the particle between $t = 4$ and $t = 6$.
(c) Estimate the initial velocity of the particle—that is, its velocity at $t = 0$.
(d) Estimate a time at which the velocity of the particle is zero.
(e) Determine an interval on which the velocity of the particle is decreasing.
(f) Determine an interval on which the velocity of the particle is increasing.

Think About It

39. Let $y = f(x)$ be an even function whose graph possesses a tangent line with slope m at $(a, f(a))$. Show that the slope of the tangent line at $(-a, f(a))$ is $-m$. [*Hint*: Explain why $f(-a + h) = f(a - h)$.]

40. Let $y = f(x)$ be an odd function whose graph possesses a tangent line with slope m at $(a, f(a))$. Show that the slope of the tangent line at $(-a, -f(a))$ is m.

41. Proceed as in Example 7 and show that there is no tangent line to graph of $f(x) = x^2 + |x|$ at $(0, 0)$.

Chapter 2 in Review

Answers to selected odd-numbered problems begin on page ANS-000.

A. True/False

In Problems 1–22, indicate whether the given statement is true or false.

1. $\lim_{x\to 2}\frac{x^3-8}{x-2} = 12$ _____

2. $\lim_{x\to 5}\sqrt{x-5} = 0$ _____

3. $\lim_{x\to 0}\frac{|x|}{x} = 1$ _____

4. $\lim_{x\to\infty} e^{2x-x^2} = \infty$ _____

5. $\lim_{x\to 0^+}\tan^{-1}\left(\frac{1}{x}\right)$ does not exist. _____

6. $\lim_{z\to 1}\frac{z^3+8z-2}{z^2+9z-10}$ does not exist. _____

7. If $\lim_{x\to a} f(x) = 3$ and $\lim_{x\to a} g(x) = 0$, then $\lim_{x\to a} f(x)/g(x)$ does not exist. _____

8. If $\lim_{x\to a} f(x)$ exists and $\lim_{x\to a} g(x)$ does not exist, then $\lim_{x\to a} f(x)g(x)$ does not exist. _____

9. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a} f(x)/g(x) = 1$. _____

10. If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then $\lim_{x\to a}[f(x) - g(x)] = 0$. _____

11. If f is a polynomial function, then $\lim_{x\to\infty} f(x) = \infty$. _____

12. Every polynomial function is continuous on $(-\infty, \infty)$. _____

13. For $f(x) = x^5 + 3x - 1$ there exists a number c in $[-1, 1]$ such that $f(c) = 0$. _____

14. If f and g are continuous at the number 2, then f/g is continuous at 2. _____

15. The greatest integer function $f(x) = \lfloor x \rfloor$ is not continuous on the interval [0, 1]. _____

16. If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, then $\lim_{x\to a} f(x)$ exists. _____

17. If a function f is discontinuous at the number 3, then $f(3)$ is not defined. _____

18. If a function f is continuous at the number a, then $\lim_{x\to a}(x - a)f(x) = 0$. _____

19. If f is continuous and $f(a)f(b) < 0$, there is a root of $f(x) = 0$ in the interval $[a, b]$. _____

20. The function $f(x) = \begin{cases} \dfrac{x^2 - 6x + 5}{x - 5}, & x \neq 5 \\ 4, & x = 5 \end{cases}$ is discontinuous at 5. _____

21. The function $f(x) = \dfrac{\sqrt{x}}{x + 1}$ has a vertical asymptote at $x = -1$. _____

22. If $y = x - 2$ is a tangent line to the graph of a function $y = f(x)$ at $(3, f(3))$, then $f(3) = 1$. _____

B. Fill in the Blanks

In Problems 1–22, fill in the blanks.

1. $\lim_{x\to 2}(3x^2 - 4x) =$ _____

2. $\lim_{x\to 3}(5x^2)^0 =$ _____

3. $\lim_{t\to\infty}\dfrac{2t - 1}{3 - 10t} =$ _____

4. $\lim_{x\to -\infty}\dfrac{\sqrt{x^2 + 1}}{2x + 1} =$ _____

5. $\lim_{t\to 1}\dfrac{1 - \cos^2(t - 1)}{t - 1} =$ _____

6. $\lim_{x\to 0}\dfrac{\sin 3x}{5x} =$ _____

7. $\lim_{x\to 0^+} e^{1/x} =$ _____

8. $\lim_{x\to 0^-} e^{1/x} =$ _____

9. $\lim_{x\to\infty} e^{1/x} =$ _____

10. $\lim_{x\to -\infty}\dfrac{1 + 2e^x}{4 + e^x} =$ _____

11. $\lim_{x\to \_\_}\dfrac{1}{x - 3} = -\infty$

12. $\lim_{x\to \_\_}(5x + 2) = 22$

13. $\lim_{x\to \_\_} x^3 = -\infty$

14. $\lim_{x\to \_\_}\dfrac{1}{\sqrt{x}} = \infty$

15. If $f(x) = 2(x - 4)/|x - 4|, x \neq 4$, and $f(4) = 9$, then $\lim_{x\to 4^-} f(x) =$ _____.

16. Suppose $x^2 - x^4/3 \leq f(x) \leq x^2$ for all x. Then $\lim_{x\to 0} f(x)/x^2 =$ _____.

17. If f is continuous at a number a and $\lim_{x\to a} f(x) = 10$, then $f(a) =$ _____.

18. If f is continuous at $x = 5, f(5) = 2$, and $\lim_{x\to 5} g(x) = 10$, then $\lim_{x\to 5}[g(x) - f(x)] =$ _____.

19. $f(x) = \begin{cases} \dfrac{2x - 1}{4x^2 - 1}, & x \neq \frac{1}{2} \\ 0.5, & x = \frac{1}{2} \end{cases}$ is _________ (continuous/discontinuous) at the number $\frac{1}{2}$.

20. The equation $e^{-x^2} = x^2 - 1$ has precisely _____ roots in the interval $(-\infty, \infty)$.

21. The function $f(x) = \dfrac{10}{x} + \dfrac{x^2 - 4}{x - 2}$ has a removable discontinuity at $x = 2$. To remove the discontinuity, $f(2)$ should be defined to be _____.

22. If $\lim_{x\to -5} g(x) = -9$ and $f(x) = x^2$, then $\lim_{x\to -5} f(g(x)) =$ _____.

C. Exercises

In Problems 1–4, sketch a graph of a function f that satisfies the given conditions.

1. $f(0) = 1, f(4) = 0, f(6) = 0, \lim_{x\to 3^-} f(x) = 2, \lim_{x\to 3^+} f(x) = \infty, \lim_{x\to -\infty} f(x) = 0, \lim_{x\to\infty} f(x) = 2$

2. $\lim_{x\to -\infty} f(x) = 0, f(0) = 1, \lim_{x\to 4^-} f(x) = \infty, \lim_{x\to 4^+} f(x) = \infty, f(5) = 0, \lim_{x\to\infty} f(x) = -1$

3. $\lim_{x\to -\infty} f(x) = 2, f(-1) = 3, f(0) = 0, f(-x) = -f(x)$

4. $\lim_{x\to\infty} f(x) = 0, f(0) = -3, f(1) = 0, f(-x) = f(x)$

In Problems 5–10, state which of the conditions (a)–(j) are applicable to the graph of $y = f(x)$.

(a) $f(a)$ is not defined
(b) $f(a) = L$
(c) f is continuous at $x = a$
(d) f is continuous on $[0, a]$
(e) $\lim_{x\to a^+} f(x) = L$
(f) $\lim_{x\to a} f(x) = L$
(g) $\lim_{x\to a} |f(x)| = \infty$
(h) $\lim_{x\to\infty} f(x) = L$
(i) $\lim_{x\to\infty} f(x) = -\infty$
(j) $\lim_{x\to\infty} f(x) = 0$

5.

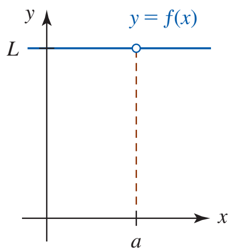


FIGURE 2.R.1 Graph for Problem 5

6.

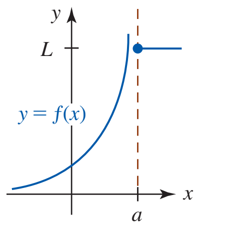


FIGURE 2.R.2 Graph for Problem 6

7.

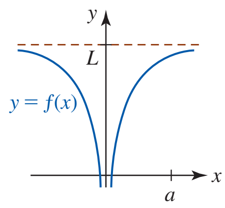


FIGURE 2.R.3 Graph for Problem 7

8.

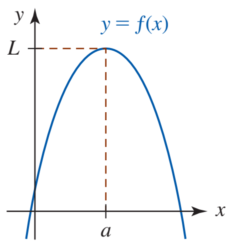


FIGURE 2.R.4 Graph for Problem 8

9.

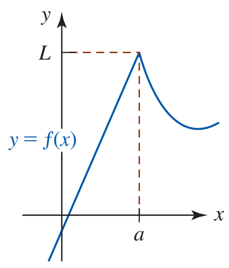


FIGURE 2.R.5 Graph for Problem 9

10.

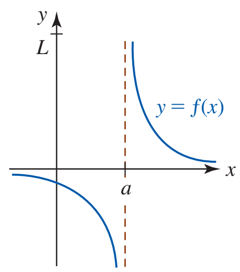


FIGURE 2.R.6 Graph for Problem 10

In Problems 11 and 12, sketch the graph of the given function. Determine the numbers, if any, at which f is discontinuous.

11. $f(x) = |x| + x$

12. $f(x) = \begin{cases} x+1, & x < 2 \\ 3, & 2 < x < 4 \\ -x+7, & x > 4 \end{cases}$

In Problems 13–16, determine intervals on which the given function is continuous.

13. $f(x) = \dfrac{x+6}{x^3 - x}$

14. $f(x) = \dfrac{\sqrt{4-x^2}}{x^2 - 4x + 3}$

15. $f(x) = \dfrac{x}{\sqrt{x^2 - 5}}$

16. $f(x) = \dfrac{\csc x}{\sqrt{x}}$

17. Find a number k so that

$$f(x) = \begin{cases} kx + 1, & x \le 3 \\ 2 - kx, & x > 3 \end{cases}$$

is continuous at the number 3.

18. Find numbers a and b so that

$$f(x) = \begin{cases} x + 4, & x \le 1 \\ ax + b, & 1 < x \le 3 \\ 3x - 8, & x > 3 \end{cases}$$

is continuous everywhere.

In Problems 19–22, find the slope of the tangent line to the graph of the function at the given value of x. Find an equation of the tangent line at the corresponding point.

19. $f(x) = -3x^2 + 16x + 12, \quad x = 2$

20. $f(x) = x^3 - x^2, \quad x = -1$

21. $f(x) = \dfrac{-1}{2x^2}, \quad x = \dfrac{1}{2}$

22. $f(x) = x + 4\sqrt{x}, \quad x = 4$

23. Find an equation of the line that is perpendicular to the tangent line at the point (1, 2) on the graph of $f(x) = -4x^2 + 6x$.

24. Suppose $f(x) = 2x + 5$ and $\varepsilon = 0.01$. Find a $\delta > 0$ that will guarantee that $|f(x) - 7| < \varepsilon$ when $0 < |x - 1| < \delta$. What limit has been proved by finding δ?