Chapter 9

Applications of Homotopy

In Section 8.2 we showed that the fundamental group can be used to show that two spaces are not homeomorphic. In this chapter we exhibit other uses of the fundamental group.

9.1 Inessential Maps

The purpose of this section is to show that if the continuous function $h: X \to Y$ is homotopic to a constant map, then the induced homomorphism h_* is the zero homomorphism.

Definition. Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be topological spaces, and let $h: X \to Y$ be a continuous function. Then h is **inessential** if it is homotopic to a constant map, and h is **essential** if it is not inessential. ■

The following theorem gives a characterization of certain inessential functions in terms of the extension property.

THEOREM 9.1. Let $(X, \mathscr{T})$ be a topological space, and let $h: S^1 \to X$ be a continuous function. Then the following are equivalent.

(a) h is inessential.

(b) h can be extended to a continuous function $f: B^2 \to X$.

Proof. **(a) → (b).** Suppose h is inessential. Then there is a point $x_0 \in X$ and a continuous function $H: S^1 \times I \to X$ such that $H(x, 0) = h(x)$ and $H(x, 1) = x_0$ for all $x \in S^1$. Define $G: S^1 \times I \to B^2$ by $G(x, t) = (1 - t)x$ for all $(x, t) \in S^1 \times I$. Then G is continuous, $G|_{S^1 \times [0, 1)}$ is a one-to-one-function that maps $S^1 \times [0, 1)$ onto $B^2 - \{(0, 0)\}$, and $G(x, 1) = (0, 0)$ for all $x \in S^1$. Since $S^1 \times I$ is compact and B^2 is

Hausdorff, G is closed (see Theorem 4.21). Therefore by Theorem 2.55, G is a quotient map.

Define $f: B^2 \to X$ by $f((0,0)) = x_0$ and $f(x) = H(G^{-1}(x)$ if $x \neq (0,0)$. Then $f \circ G = H$, and, by Theorem 2.59, f is continuous. If $x \in S^1 \subseteq B^2$, then $f(x) = H(G^{-1}(x)) = h(x)$, and therefore f is an extension of h.

(b) → (a) Suppose h can be extended to a continuous function $f: B^2 \to X$. Define $H: S^1 \times I \to X$ by $H(x,t) = f((1-t)x)$. Then H is continuous and $H(x,0) = f(x) = h(x)$ and $H(x,1) = f(0)$ for all $x \in S^1$. Therefore h is homotopic to a constant map. ■

The following theorem provides a condition for the induced homomorphism to be the zero homomorphism.

THEOREM 9.2. Let $(Y, \mathcal{U})$ be a topological space, and let $h: S^1 \to Y$ be an inessential function. Then h_* is the zero homomorphism.

Proof. By Theorem 9.1, h has a continuous extension $f: B^2 \to Y$. Let $j: S^1 \to B^2$ be the inclusion map. Then $f \circ j = h$. Let $s_0 \in S^1$ and let $y_0 = h(s_0)$. Then, by Theorem 8.12, the following diagram commutes; that is $f_* \circ j_* = h_*$.

$$\begin{tikzcd}
\pi_1(S^1, s_0) \arrow[rr, "h_*"] \arrow[dr, "j_*"'] & & \pi_1(Y, y_0) \\
& \pi_1(B^2, s_0) \arrow[ur, "f_*"'] &
\end{tikzcd}$$

Since $\pi_1(B^2, s_0)$ is the trivial group, j_* is the zero homomorphism. Therefore $f_* \circ j_* = h_*$ is the zero homomorphism. ■

The following theorem generalizes Theorem 9.2.

THEOREM 9.3. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, and let $h: X \to Y$ be an inessential function. Then h_* is the zero homomorphism.

Proof. Let $x_0 \in X$ and let α be a loop in X at x_0. Define $\sigma: I \to S^1$ by $\sigma(x) = (\cos 2\pi x, \sin 2\pi x)$ for each $x \in I$. Define $k: S^1 \to X$ by $k(x) = \alpha(\sigma^{-1}(x))$ for each $x \in S^1$. In Exercise 2, you are asked to prove that k is a continuous function. Since h is inessential, there exists $y_0 \in Y$ and a continuous function $H: X \times I \to Y$ such that $H(x,0) = h(x)$ and $H(x,1) = y_0$ for all $x \in X$. Define $F: S^1 \times I \to Y$ by $F(x,t) = H(k(x),t)$. Then F is continuous, $F(x,0) = H(k(x),0) = (h \circ k)(x)$ and $F(x,1) = H(k(x),1) = y_0$ for all $x \in S^1$. By Theorem 9.2, $(h \circ k)_*$ is the zero homomorphism. Therefore $h_*([\alpha]) = [h \circ \alpha] = [h \circ k \circ \sigma] = (h \circ k)_*([\sigma]) = 0$. ■

EXERCISES 9.1

1. Without using Theorem 2.59, prove that the function f defined in the proof of Theorem 9.1 is continuous.

2. Let $(X, \mathscr{T})$ be a topological space, let $x_0 \in X$, let α be a loop in X at x_0, define $\sigma : I \to S^1$ by $\sigma(x) = (\cos 2\pi x, \sin 2\pi x)$, and define $k: S^1 \to X$ by $k(x) = \alpha(\sigma^{-1}(x))$ for each $x \in S^1$. Prove that k is continuous.

3. Let $(X, \mathscr{T})$ be a topological space, let $h: S^1 \to X$ be a continuous function, define $\sigma: I \to S^1$ by $\sigma(x) = (\cos 2\pi x, \sin 2\pi x)$, let $\alpha = h \circ \sigma$, and let $x_0 = \alpha(0) = \alpha(1)$.
   (a) Prove that there is a continuous function $H: I \times I \to X$ such that $H(x, 0) = \alpha(x)$, $H(x, 1) = x_0$ for all $x \in I$ and $H(0, t) = H(1, t) = x_0$ for all $t \in I$.
   (b) Prove that there is a continuous function $F: S^1 \times I \to X$ such that $F(\sigma(x),t) = H(x, t)$ for all $(x, t) \in I \times I$.
   (c) Prove that if h_* is the zero homomorphism, then h is inessential.

4. Let $m, n \in \mathbb{N}$. A continuous function $h: S^m \to S^n$ is **antipode-preserving** if $h(-x) = -h(x)$ for each $x \in S^m$. Let $h: S^1 \to S^1$ be an antipode-preserving function. Consider the members of S^1 to be complex numbers and define $p: S^1 \to S^1$ by $p(z) = z^2$.
   (a) Prove that there is a continuous function $g: S^1 \to S^1$ such that $p \circ h = g \circ p$.
   (b) Let $x \in S^1$ and let $\alpha: I \to S^1$ be a path such that $\alpha(0) = x$ and $\alpha(1) = -x$. Prove that $p \circ \alpha$ is a loop in S^1 that is not path homotopic to a constant function.
   (c) Show that the homomorphisms p_* and g_* are one-to-one.
   (d) Prove that h is essential.

5. Prove that there is no antipode-preserving function $h: S^2 \to S^1$.

6. Let $(X, \mathscr{T})$ be a pathwise connected space and let $x_0 \in X$. Show that $\pi_1(X, x_0)$ is the trivial group if and only if every continuous function $h: S^1 \to X$ is inessential.

9.2 The Fundamental Theorem of Algebra

The Fundamental Theorem of Algebra says that if $n \in \mathbb{N}$, then every polynomial equation of degree n with complex coefficients has at least one solution in the set of complex numbers. This theorem is difficult to prove, and most proofs involve nonalgebraic concepts. We give a proof that uses the material we developed in Chapter 8 and Section 9.1.

THEOREM 9.4. Let $n \in \mathbb{N}$ and let $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ be a polynomial equation of degree n with complex coefficients. Then this equation has at least one solution in the set of complex numbers.

Proof. We consider the members of S^1 to be complex numbers and define a continuous function $f: S^1 \to S^1$ by $f(z) = z^n$. Let $s_0 = (1, 0)$ and consider the induced homomorphism $f_*: \pi_1(S^1, s_0) \to \pi_1(S^1, s_0)$. Define $\sigma: I \to S^1$ by $\sigma(x) = (\cos 2\pi x, \sin 2\pi x) = e^{2\pi ix}$. Then $f_*([\sigma]) = [f \circ \sigma] \in \pi_1(S^1, 1, 0))$. Since $(f \circ \sigma)(0) = (1, 0)$, the unique path $\alpha: I \to \mathbb{R}$ given by Theorem 8.15 is the path defined by $g(x) = nx$. Let $p: \mathbb{R} \to S^1$ be the standard covering map defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ (see Section 8.3). Then $p \circ \alpha = f \circ \sigma$, so $\deg(f \circ \sigma) = n$. From the proof that $\pi_1(S^1, 1, 0))$ is isomorphic to the group of integers (Theorem 8.18), we see that $[f \circ \sigma]$ is not the identity element of $\pi_1(S^1, (1, 0))$. Therefore f_* is not the zero homomorphism.

First we show that we may assume that $|a_{n-1}| + |a_{n-2}| + \cdots + |a_o| < 1$. We let c be a positive real number and substitute $x = cy$ in the given polynomial equation to obtain the equation

$$(cy)^n + a_{n-1}(cy)^{n-1} + \cdots + a_1(cy) + a_0 = 0$$

or

$$y^n + (a_{n-1}/c)y^{n-1} + \cdots + (a_1/c^{n-1})y + a_0/c^n = 0.$$

Now choose c large enough so that $|a_{n-1}/c| + |a_{n-2}/c^2| + \cdots + |a_1/c^{n-1}| + |a_0/c^n| < 1$. Then if y_0 is a solution of $y^n + (a_{n-1}/c)y^{n-1} + \cdots + (a_1/c^{n-i})y + a_0/c^n = 0$, $x = cy_0$ is a solution of $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$. Therefore it is sufficient to show that $y^n + (a_{n-1}/c)y^{n-1} + \cdots + (a_1/c^{n-1})y + a_0/c^n = 0$ has a solution. This means that in the given polynomial equation, we may assume that $|a_{n-1}| + |a_{n-2}| + \cdots + |a_0| < 1$.

The proof that the given polynomial equation has a solution in B^2 is by contradiction. Suppose $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 = 0$ has no solution in B^2. Then there is a continuous function $q: B^2 \to \mathbb{R}^2 - \{(0, 0)\}$ defined by $q(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0$. Let $r: S^1 \to \mathbb{R}^2 - \{(0, 0)\}$ be the restriction of q to S^1. Then $q: B^2 \to \mathbb{R}^2 - \{(0, 0)\}$ is an extension of r. So, by Theorem 9.1, r is inessential.

We arrive at a contradiction by showing that r is homotopic to a continuous function that is essential. Define $k: S^1 \to \mathbb{R}^2 - \{(0, 0)\}$ by $k(z) = z^n$, and define $H: S^1 \times I \to \mathbb{R}^2 - \{(0, 0)\}$ by $H(z, t) = z^n + t(a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_0)$. Note $H(x, t) \neq (0, 0)$ for $(x, t) \in S^1 \times I$ because

$$\begin{aligned} |H(x, t)| &\geq |z^n| - |t(a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_o| \\ &\geq 1 - t(|a_{n-1}z^{n-1}| + |a_{n-2}z^{n-2}| + \cdots + |a_o|) \\ &= 1 - t(|a_{n-1}| + |a_{n-2}| + \cdots + |a_o|) > 0. \end{aligned}$$

We complete the proof by showing that k is essential. Note that $k = j \circ f$, where $j: S^1 \to \mathbb{R}^2 - \{(0, 0)\}$ is the inclusion map, so $k_* = j_* \circ f_*$. Since the fundamental group of S^1 is isomorphic to the group of integers, f_* is essentially the homomorphism that takes an integer a into the product na. Furthermore j_* is an isomorphism (see Example 9 of Chapter 8). Therefore k_* is not the zero homomorphism. By Theorem 9.2, k is essential. ■

EXERCISES 9.2

1. Find a real number r such that $x^7 + x^5 + x^3 + x^2 + 1 = 0$ has a solution in $\{(x, y) \in \mathbb{R}^2: x^2 + y^2 \leq r^2\}$.

9.3 Homotopic Maps

We have already seen that if (X, x_0) and (Y, y_0) are topological pairs that have the same homotopy type, then $\pi_1(X, x_0)$ and $\pi_1(Y, y_0)$ are isomorphic. In this section we show that homotopic maps induce the same homomorphisms on the fundamental groups provided the base point remains fixed during the homotopy, and we give a condition on a continuous function mapping a compact space into S^2, which ensures that the continuous function is inessential.

Let $(X, \mathscr{T})$ be a topological space, let $x_0, x_1 \in X$, and let σ be a path in X from x_0 to x_1. We let $\theta_\sigma: \pi_1(X, x_0) \to \pi_1(X, x_1)$ be the function defined in the proof of Theorem 8.9; that is $\theta_\sigma([\alpha]) = [(\overline{\sigma} * \alpha) * \sigma]$. As shown in the proof of Theorem 8.9, θ_σ is an isomorphism.

***THEOREM* 9.5.** Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be topological spaces, let $x_0 \in X$, let $h, k: X \to Y$ be continuous functions such that $h \simeq k$, and let $y_0 = h(x_0)$ and $y_1 = k(x_0)$. Then:

(a) there is a path σ in Y from y_0 to y_1 such that $h_* = \theta_\sigma \circ k_*$.

(b) If $H: X \times I \to Y$ is a continuous function such that $H(x, 0) = h(x)$ and $H(x, 1) = k(x)$ for all $x \in X$ and $H(x_0, t) = y_0 = y_1$ for all $t \in I$, then $h_* = k_*$.

Proof.* *(a) Let $H: X \times I \to Y$ be a continuous function such that $H(x, 0) = h(x)$ and $H(x, 1) = k(x)$ for all $x \in X$. Define $\sigma: I \to Y$ by $\sigma(t) = H(x_0, t)$ for all $t \in I$. Then σ is a path from y_0 to y_1.

Let $[\gamma] \in \pi_1(X, x_0)$. For each $t \in I$, define $\alpha_t, \beta_t: I \to Y$ by $\alpha_t(s) = \sigma(ts)$ and $\beta_t(s) = H(\gamma(s), t)$ for each $s \in I$. Then for each $t \in I$,

$$\alpha_t(1) = \sigma(t) = H(x_0, t) = H(\gamma(0), t) = \beta_t(0),$$

and

$$\beta_t(1) = H(\gamma(1), t) = H(x_0, t) = \sigma(t) = \alpha_t(1) = \overline{\alpha}_t(0).$$

Therefore for each $t \in I$, $(\alpha_t * \beta_t) * \overline{\alpha}_t$ is defined. Since $\alpha_0(s) = \sigma(0) = H(x_0, 0) = h(x_0) = y_0$, $\beta_0(s) = H(\gamma(s),0) = (h \circ \gamma)(s)$, and $\overline{\alpha}_o(s) = \alpha_0(1 - s) = \sigma(0) = y_0$ for each $s \in I$, and $(\alpha_0 * \beta_0) * \overline{\alpha}_0 = (e_{y_0} * (h \circ \gamma)) * e_{y_0}$, where $e_{y_0}\colon I \to Y$ is the constant function defined by $e_{y_0}(t) = y_0$ for each $t \in I$. Since $(e_{y_0} * (h \circ \gamma)) * e_{y_0} \simeq_p h \circ \gamma$, $(\alpha_0 * \beta_0) * \overline{\alpha}_0 \simeq_p h \circ \gamma$. Since $\alpha_1(s) = \sigma(s)$, $\beta_1(s) = H(\gamma(s), 1) = (k \circ \gamma)(s)$, and $\overline{\alpha}_1(s) = \alpha_1(1 - s) = \sigma(1 - s) = \overline{\sigma}(s)$ for each $s \in I, \alpha_1 * \beta_1) * \overline{\alpha}_1 = (\sigma * (k \circ \gamma)) * \overline{\sigma}$.

Define $F : I \times I \to Y$ by

$$F(s, t) = \begin{cases} \sigma(4st), & 0 \leq s \leq \frac{1}{4}, 0 \leq t \leq 1 \\ H(\gamma(4s - 1), t), & \frac{1}{4} \leq s \leq \frac{1}{2}, 0 \leq t \leq 1 \\ \sigma(2t(1 - s)), & \frac{1}{2} \leq s \leq 1, 0 \leq t \leq 1 \end{cases}$$

Then F is continuous,

$$\begin{aligned} F(s, 1) &= \begin{cases} \sigma(4s), & 0 \leq s \leq \frac{1}{4} \\ H(\gamma(4s - 1), 1), & \frac{1}{4} \leq s \leq \frac{1}{2} \\ \sigma(2(1 - s)), & \frac{1}{2} \leq s \leq 1 \end{cases} \\ &= \begin{cases} \sigma(4s), & 0 \leq s \leq \frac{1}{4} \\ (k \circ \gamma)(4s - 1), & \frac{1}{4} \leq s \leq \frac{1}{2} \\ \overline{\sigma}(2s - 1), & \frac{1}{2} \leq s \leq 1, \end{cases} = ((\sigma * (k \circ \gamma)) * \overline{\sigma})(s) \end{aligned}$$

and

$$\begin{aligned} F(s, 0) &= \begin{cases} \sigma(0), & 0 \leq s \leq \frac{1}{4} \\ H(\gamma(4s - 1), 0), & \frac{1}{4} \leq s \leq \frac{1}{2} \\ \sigma(0), & \frac{1}{2} \leq s \leq 1 \end{cases} \\ &= \begin{cases} \sigma(0), & 0 \leq s \leq \frac{1}{4} \\ (h \circ \gamma)(4s - 1), & \frac{1}{4} \leq s \leq \frac{1}{2} \\ \sigma(0), & \frac{1}{2} \leq s \leq 1. \end{cases} = (e_{y_0} * (h \circ \gamma)) * e_{y_0})(s) \end{aligned}$$

Now $(e_{y_0} * (h \circ \gamma)) * e_{y_0} \simeq_p (h \circ \gamma)$. Therefore $h_*([\gamma]) = [h \circ \gamma] = [(\sigma * (k \circ \gamma)) * \overline{\sigma}] = \theta_\sigma([k \circ \gamma]) = (\theta_\sigma \circ k_*)([\gamma])$.

(b) Suppose $H(x_0, t) = y_0 = y_1$ for all $t \in I$. Then σ is the constant path, so $h_* = \theta_\sigma \circ k_* = k_*$. ■

We conclude this section by proving a theorem (Theorem 9.7) that will be used in the next section. Intuitively this theorem says that if f is a continuous function from a compact space $(X, \mathscr{T})$ into S^2 with the property that there are two distinct points of S^2 which lie in the same component of $S^2 - f(X)$, then f can be shrunk to a constant function under a homotopy that does not have either of the two points in its image. In the proof we use the following theorem.

THEOREM 9.6. Let a and b be distinct member of S^2, and let $X = \mathbb{R}^2 \cup \{p\}$ be the one-point compactification of $\mathbb{R}^2$. Then there exists a homeomorphism $h\colon S^2 \to X$ such that $h(a) = p$ and $h(b) = (0, 0)$.

Proof. Let $h_1\colon S^2 \to S^2$ be a rotation of S^2 such that $h_1(a) = (0, 0, 1)$. Then h_1 is a homeomorphism. Define $h_2'\colon S^2 - \{(0, 0, 1)\} \to \mathbb{R}^2$ by $h_2'(x_1, x_2, x_3) = (1/(1 - x_3))(x_1, x_2)$. As seen in the proof of Theorem 8.27, h_2' is a homeomorphism. Define $h_2\colon S^2 \to X$ by $h_2(x) = h_2'(x)$ if $x \in S^2 - \{(0, 0, 1)\}$ and $h_2(0, 0, 1) = p$. In Exercise 1 you are asked to show that h_2 is a homeomorphism. Let $(b_1, b_2) = h_2(h_1(b))$ and define $h_3'\colon \mathbb{R}^2 \to \mathbb{R}^2$ by $h_3'(x, y) = (x - b_1, y - b_2)$. Then h_3' is a homeomorphism. Define $h_3\colon X \to X$ by $h_3(x) = h_3'(x)$ if $x \in \mathbb{R}^2$ and $h_3(p) = p$. In Exercise 2 you are asked to show that h_3 is a homeomorphism. Let $h = h_3 \circ h_2 \circ h_1$. Then $h\colon S^2 \to X$ is a homeomorphism such that $h(a) = h_3(h_2(0, 0, 1)) = h_3(p) = p$ and $h(b) = h_3((h_2 \circ h_1)(b)) = h_3(b_1, b_2) = (0, 0)$. ■

THEOREM 9.7. Let a and b be distinct members of S^2, let $(X, \mathscr{T})$ be a compact space, and let $f : X \to S^2 - \{a, b\}$ be a continuous function such that a and b lie in the same component of $S^2 - f(X)$. Then f is inessential.

Proof. Let $Y = \mathbb{R}^2 \cup \{p\}$ be the one-point compactification of $\mathbb{R}^2$, and let $h\colon S^2 \to Y$ be the homeomorphism given by Theorem 9.6. Then $h \circ f : X \to \mathbb{R}^2 - \{(0, 0)\}$ is a continuous function such that (0,0) lies in the unbounded component of $\mathbb{R}^2 - (h \circ f)(X)$. Suppose there is a continuous function $H\colon X \times I \to \mathbb{R}^2 - \{(0, 0)\}$ and a member y_0 of $\mathbb{R}^2 - \{(0, 0)\}$ such that $H(x, 0) = (h \circ f)(x)$ and $H(x, 1) = y_0$ for each $x \in X$. Then $h^{-1} \circ H\colon X \times I \to S^2 - \{a, b\}$ is a continuous function such that $(h^{-1} \circ H)(x, 0) = f(x)$ and $(h^{-1} \circ H)(x, 1) = h^{-1}(y_0)$ for each $x \in X$. Therefore in order to prove the theorem, it is sufficient to show that if $f : X \to \mathbb{R}^2 - \{(0, 0)\}$ is a continuous function such that (0,0) lies in the unbounded component of $\mathbb{R}^2 - f(x)$, then f is inessential.

Let $f : X \to \mathbb{R}^2 - \{(0, 0)\}$ be a continuous function such that (0,0) lies in the unbounded component of $\mathbb{R}^2 - f(X)$. Since $(X, \mathscr{T})$ is compact, $f(X)$ is a closed and bounded subset of $\mathbb{R}^2$. Let r be a positive number such that $f(X) \subseteq B = \{(x, y) \in \mathbb{R}^2\colon x^2 + y^2 \le r^2\}$, and let $q \in \mathbb{R}^2 - B$. Then (0,0) and q lie in the same component of $\mathbb{R}^2 - f(X)$. Since $\mathbb{R}^2$ is locally pathwise connected and $\mathbb{R}^2 - f(X)$ is open, $\mathbb{R}^2 - f(X)$ is locally pathwise connected. Therefore the components and path components of $\mathbb{R}^2 - f(X)$ are the same (Theorem 3.28). Hence there is a path α in $\mathbb{R}^2 - f(X)$ from $(0, 0)$ to q. Define $F\colon X \times I \to \mathbb{R}^2 - \{(0, 0)\}$ by $F(x, t) = f(x) - \alpha(t)$. Then F is continuous, $F(x, 0) = f(x) - \alpha(0) = f(x) - (0, 0) = f(x)$ and $F(x, 1) = f(x) - \alpha(1) = f(x) - q$ for each $x \in X$. Note that $F(x, t) \neq (0, 0)$ for any $(x, t) \in X \times I$ because $\alpha(I) \cap f(X) = \emptyset$. Now define $G\colon X \times I \to \mathbb{R}^2 - \{(0, 0)\}$ by $G(x, t) = tf(x) - q$. Then G is continuous, $G(x, 0) = -q$ and $G(x, 1) = f(x) - q$ for each $x \in X$. Note that $G(x, t) \neq (0, 0)$ for any $(x, t) \in X \times I$ because $tf(x) \in B$ and $q \notin B$. Thus, if $g\colon X \to \mathbb{R}^2 - \{(0, 0)\}$ is the function defined by $g(x) = f(x) - q$, then F is a homotopy between f and g and G is a homotopy between g and a constant map. Therefore f is homotopic to a constant map, and hence it is inessential. ■

EXERCISES 9.3

1. Show that the function h_2 defined in the proof of Theorem 9.6 is a homeomorphism.
2. Show that the function h_3 defined in the proof of Theorem 9.6 is a homeomorphism.
3. Let $X = \mathbb{R}^2 - \{(-1, 0), (1, 0)\}$ and let $A = \{(x, y) \in \mathbb{R}^2: (x + 1)^2 + y^2 = 1$ or $(x - 1)^2 + y^2 = 1\}$. Prove that A is a deformation retract of X.
4. Let $X = \mathbb{R}^2 - \{(-1, 0), (1, 0)\}$ and let $B = \{(x, y) \in \mathbb{R}^2: x + y^2 = 4$, or $x = 0$ and $-2 \leq y \leq 2\}$. Prove that A is a deformation retract of X.
5. Let A and B be the spaces defined in Exercises 3 and 4. Show that neither of these spaces can be imbedded in the other.

9.4 The Jordan Curve Theorem

The purpose of this section is to prove the Jordan Curve Theorem. Camille Jordan (1858–1922) proposed the problem (that is, the Jordan curve theorem) in 1892 by pointing out that this intuitively obvious fact required proof, and consequently the resulting theorem was named for him. It was, however, Oswald Veblen (1880–1960) who published, in 1905, the first correct proof.

First we prove the Jordan Separation Theorem. Then we prove a nonseparation theorem, and finally we prove the Jordan Curve Theorem.

Definition. An **arc** is a topological space that is homeomorphic to the unit interval [0,1], and a **simple closed curve** is a topological space that is homeomorphic to the unit circle S^1. ■

THEOREM 9.8 (The Jordan Separation Theorem). Let C be a simple closed curve in S^2. Then $S^2 - C$ is not connected.

Proof. Since $S^2 - C$ is an open subset of the locally pathwise connected space S^2, it is locally pathwise connected. Therefore the components and path components of $S^2 - C$ are the same (Theorem 3.28). Thus it is sufficient to assume that $S^2 - C$ is pathwise connected and to reach a contradiction.

Since C is homeomorphic to the unit circle, there are arcs A and B such that $C = A \cup B$ and $A \cap B = \{a, b\}$, where a and b are the end points of A and B. Let $X = S^2 - \{a, b\}$. By Theorem 9.6, X is homeomorphic to $\mathbb{R}^2 - \{(0, 0)\}$. Therefore, by Example 9 of Chapter 8, the fundamental group of X is isomorphic to the group of integers.

Let $U = S^2 - A$ and $V = S^2 - B$. Then U and V are open subsets of X, $U \cup V = X$, and $U \cap V = S^2 - (A \cup B) = S^2 - C$. Thus, by assumption, $U \cap V$ is pathwise connected. Let $x_0 \in U \cap V$ and consider the inclusion maps $i: (U, x_0) \to (X, x_0)$ and $j: (V, x_0) \to (X, x_0)$. We show that these inclusion maps induce zero homomorphisms $i_*: \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_*: \pi_1(V, x_0) \to \pi_1(X, x_0)$. Let $[\alpha] \in \pi_1(U, x_0)$ and let $\sigma: I \to S^1$ be the continuous function defined by $\sigma(x) = (\cos 2\pi x, \sin 2\pi x)$. Define $h: S^1 \to U$ as follows: Let $h(1, 0) = x_0$. If $y \in S^1 - \{(1, 0)\}$, then there is a unique member x of I such that $\sigma(x) = y$. Let $h(y) = \alpha(x)$. Then h is a continuous function and $h \circ \sigma = \alpha$.

Consider the continuous function $i \circ h: S^1 \to X$. Now $i(h(S^1)) = h(S^1) \subseteq U$, and A is an arc with end points a and b, and $U \cap A = \varnothing$. Therefore a and b lie in the same component of $S^2 - i(h(S^1))$. By Theorem 9.7, $i \circ h$ is inessential. Therefore, by Theorem 9.2, $(i \circ h)_*$ is the zero homomorphism. Since $i_*([\alpha]) = [i \circ \alpha] = [i \circ h \circ \sigma] = (i \circ h)_*([\sigma])$, i_* is the zero homomorphism.

The preceding proof is equally applicable to U and V and hence j_* is the zero homomorphism. Therefore, by Theorem 8.26, $\pi_1(X, x_0)$ is the trivial group. This is a contradiction, so $S^2 - C$ is not pathwise connected. Therefore $S^2 - C$ is not connected. ■

Now our goal is to prove the Jordan Curve Theorem. First we prove three theorems that we will use.

THEOREM 9.9. Let $(X, \mathscr{T})$ be a topological space, let U, V, M, and N be open subsets of X such that $X = U \cup V$, $U \cap V = M \cup N$, and $M \cap N = \varnothing$, let $a \in M$ and $b \in N$, and suppose there exists a path α in U from a to b, and a path β in V from b to a. Then $\pi_1(X, a)$ is not the trivial group.

Proof. For each $n \in \mathbb{Z}$, let $U_n = U \times \{2n\}$ and $V_n = V \times \{2n + 1\}$. Then let $Y = \bigcup_{n \in \mathbb{Z}}(U_n \cup V_n)$.

$$\begin{array}{cc}
 & \vdots \\
\vdots & V_1 = V \times \{3\} \\
U_1 = U \times \{2\} & \\
 & V_0 = V \times \{1\} \\
U_0 = U \times \{0\} & \\
 & V_{-1} = V \times \{-1\} \\
U_{-1} = U \times \{-2\} & \vdots \\
\vdots &
\end{array}$$

Define a quotient space E of Y by making the following identifications:
For each $x \in M$ and $n \in \mathbb{Z}$, identify $(x, 2n - 1)$ and $(x, 2n)$.
For each $x \in N$ and $n \in \mathbb{Z}$, identify $(x, 2n)$ and $(x, 2n + 1)$.

$$\vdots$$

U_1 and V_1 are "pasted together" at "points of N."
U_1 and V_0 are "pasted together" at "points of M."
U_0 and V_0 are "pasted together" at "points of N."
U_0 and V_{-1} are "pasted together" at "points of M."
U_{-1} and V_{-1} are "pasted together" at "points of N."

$$\vdots$$

Let $\pi : Y \to E$ be the quotient map. Define $q : Y \to X$ by $q(x, n) = x$ for each $(x, n) \in Y$. For each $(x, n) \in Y$, let $[(x, n)]$ denote the member of E that contains (x, n), and define $p: E \to X$ by $p([(x, n)]) = x$. Then q is continuous because it is a projection map, and p is continuous because E has the quotient topology. It is clear that p is a surjection. We show that it is a covering map, but first we show that π is open.

Since Y is the union of disjoint open sets U_n and V_n, it is sufficient to show that for each $n \in \mathbb{Z}$, $\pi|_{U_n}$ and $\pi|_{V_n}$ are open functions. Let S be an open subset of U_n. Then $S = W \times \{2n\}$, where W is open in U. Then

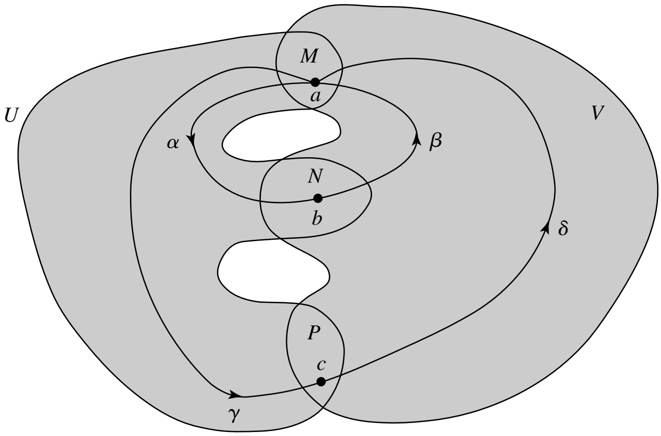
$$\begin{aligned}\pi^{-1}(\pi(S)) &= \pi^{-1}(\pi(W \times \{2n\}))\\ &= (W \times \{2n\}) \cup ((W \cap N) \times \{2n + 1\}) \cup ((W \cap M) \times \{2n - 1\})\end{aligned}$$

is the union of three open sets in Y. Therefore, by the definition of the quotient topology, $\pi(S)$ is open in E. The same argument shows that the image under π of any open subset of V_n is open in E. Therefore π is open.

We show that p is a covering map by showing that the open sets U and V are evenly covered by p. Note that for each integer n, $\pi(U_n)$ is open in E since π is open. Furthermore $p^{-1}(U) = \bigcup_{n \in \mathbb{Z}} \pi(U_n)$, and $\pi|_{U_n}$ maps U_n onto $\pi(U_n)$. Thus $\pi|_{U_n}$ is a homeomorphism because it is continuous, bijective, and open. Therefore $p|_{\pi(U_n)}$ is the composite of the two homeomorphisms $(\pi|_{U_n})^{-1}: \pi(U_n) \to U_n$ and $q|_{U_n}: U_n \to U$. Thus $p|_{\pi(U_n)}$ is a homeomorphism, and hence U is evenly covered by p. The same argument shows that V is evenly covered by p. Therefore p is a covering map.

Finally we show that $\pi_1(X, a)$ is not the trivial group by showing that $\alpha * \beta$ is not path homotopic to the constant function $e: I \to X$ defined by $e(t) = a$ for each $t \in I$. Define $\alpha', \beta': I \to E$ by $\alpha'(t) = \pi(\alpha(t), 0)$ and $\beta'(t) = \pi(\beta(t), 1)$ for each $t \in I$. Since $\alpha'(1) = \pi(\alpha(1), 0) = \pi(b, 0) = \pi(b, 1) = \pi(\beta(0), 1) = \beta'(0)$, $\alpha' * \beta'$ is defined. Furthermore $p \circ \alpha' = \alpha$ and $p \circ \beta' = \beta$. Since $(\alpha' * \beta')(0) = \alpha'(0) = \pi(\alpha(0), 0) = \pi(a, 0)$, $(\alpha' * \beta')(1) = \beta'(1) = \pi(\beta(1), 1) = \pi(a, 1)$, and $\pi(a, 0) \neq \pi(a, 1)$ because $a \in M$, $\alpha' * \beta'$ begins at one point and ends at another. Therefore by Theorem 8.21, $\alpha * \beta$ is not path homotopic to e. Therefore $\pi_1(X, a)$ is not the trivial group. ■

THEOREM 9.10. Let $(X, \mathscr{T})$ be a topological space, let U, V, M, N, and P be open subsets of X such that $X = U \cup V$, $U \cap V = M \cup N \cup P$, and $M \cap N = M \cap P = N \cap P = \varnothing$, let $a \in M$, $b \in N$, and $c \in P$, let α be a path in U from a to b, let β be a path in V from b to a, let γ be a path in U from a to c, and let δ be a path in V from c to a. Then $\pi_1(X, a)$ is not isomorphic to the group of integers.



Proof. Since $U \cap V = M \cup (N \cup P)$ and $M \cap (N \cup P) = \varnothing$, it follows from the proof of Theorem 9.9 that $[\alpha * \beta]$ and $[\gamma * \delta]$ are nonzero elements of $\pi_1(X, a)$. If $\pi_1(X, a)$ were isomorphic to the group of integers, there would exist nonzero integers m and n such that $m[\alpha * \beta] = n[\gamma * \delta]$. We prove that the theorem by showing that no such integers exist.

Since $U \cup V = (M \cup P) \cup N$ and $(M \cup P) \cap N = \varnothing$, we can follow the construction in the first part of the proof of Theorem 9.9 and obtain a space Y, a quotient space E of Y, a quotient map π: $Y \to E$ and a covering map p: $E \to X$. Then we can follow the construction of the unique liftings of $\gamma * \delta$ and $\alpha * \beta$ in the last part of the proof of Theorem 9.9 and see that every "multiple" of $\gamma * \delta$ lifts to a loop μ' in E while every nonzero "multiple" of $\alpha * \beta$ lifts to a path ν' in E such that $\nu'(0) = \mu'(0)$ whereas $\nu'(1) \neq \mu'(1)$ (see Exercise 1). Therefore by Theorem 8.21 there does not exist nonzero integers m and n such that $m[\alpha * \beta] = n[\gamma * \delta]$. ■

If A is an arc in S^2, then A is homeomorphic to the unit interval, so there are arcs A_1 and A_2 such that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \{a\}$, where a is an end point of A_1 and A_2.

THEOREM 9.11. Let A, A_1, and A_2 be arcs in S^2 such that $A = A_1 \cup A_2$ and $A_1 \cap A_2 = \{a\}$, where a is an end point of A_1 and A_2 and let $c, d \in S^2 - A$ such that c and d can be joined by a path in $S^2 - A_1$ and a path in $S^2 - A_2$. Then c and d can be joined by a path in $S^2 - A$.

Proof. Suppose c and d cannot be joined by a path in $S^2 - A$. Let $X = S^2 - \{a\}$, let $U = S^2 - A_1$, and let $V = S^2 - A_2$. Then U and V are open, $U \cup V = (S^2 - A_1) \cup (S^2 - A_2) = S^2 - (A_1 \cap A_2) = S^2 - \{a\} = X$, and $U \cap V = (S^2 - A_1) \cap (S^2 - A_2) = S^2 - (A_1 \cup A_2) = S^2 - A$. By our assumption that c and d cannot be joined by a path in $S^2 - A$, $U \cap V$ is not pathwise connected. Let C be the path component of $U \cap V$ that contains c and let $D = (U \cap V) - C$. Then D is the union of the path components of $U \cap V$ that do not contain c. Since $U \cap V$ is open in S^2, it is locally pathwise connected. Therefore the path components of $U \cap V$ are open (Theorem 3.26). Hence C and D are open in X. We are given that there is a path in $S^2 - A_1$ from c to d and a path in $S^2 - A_2$ from d to c. Thus by Theorem 9.9, $\pi_1(X, c)$ is not the trivial group. This is a contradiction because X is homeomorphic to $\mathbb{R}^2$ and $\mathbb{R}^2$ is simply connected. ■

THEOREM 9.12. Let A be an arc in S^2. Then $S^2 - A$ is pathwise connected.

Proof. Suppose there exist $c, d \in S^2 - A$ that cannot be joined by a path in $S^2 - A$. Let $h: I \to A$ be a homeomorphism, let $A_1 = h([0, \frac{1}{2}])$ and let $A_2 = h([\frac{1}{2}, 1])$. By Theorem 9.11, we may assume that c and d cannot be joined by a path in $S^2 - A_2$. Let $A_{21} = h([\frac{1}{2}, \frac{3}{4}])$ and $A_{22} = h([\frac{3}{4}, 1])$. By Theorem 9.11, either c and d cannot be joined by a path in $S^2 - A_{21}$ or c and d cannot be joined by a path in $S^2 - A_{22}$. Continuing in this manner, we obtain a sequence $[0, 1] = I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ of closed intervals such that, for each $i \in \mathbb{N}$, c and d cannot be joined by a path in $S^2 - h(I_i)$ and the length of I_i is $1/2^{i-1}$. Since I is compact, there exists $t \in \bigcap_{i \in \mathbb{N}} I_i$ and since the lengths of the intervals converge to 0, t is the only member of $\bigcap_{i \in \mathbb{N}} I_i$.

Since $S^2 - \{h(t)\}$ is homeomorphic to $\mathbb{R}^2$, there is a path f in $S^2 - \{h(t)\}$ such that $f(0) = c$ and $f(1) = d$. Since $h(t) = \bigcap_{i \in \mathbb{N}} h(I_i)$, $S^2 - \{h(t)\} = \bigcup_{i \in \mathbb{N}} (S^2 - h(I_i))$. Since $f(I)$ is compact, there exists a finite subcollection of $\{S^2 - h(I_i): i \in \mathbb{N}\}$ such that $f(I)$ is a subset of the union of the members of this finite subcollection. Since $S^2 - h(I_i) \subseteq S^2 - h(I_{i+1})$ for each $i \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $f(I) \subseteq S^2 - h(I_n)$. So f is a path in $S^2 - h(I_n)$ from c to d. This is a contradiction. ■

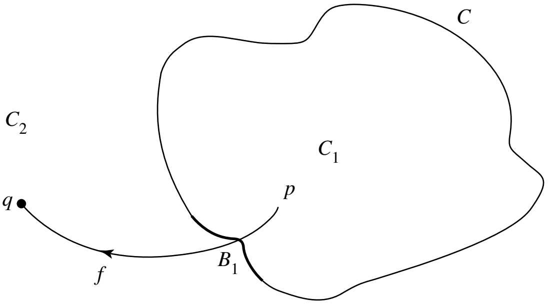
THEOREM 9.13 (The Jordan Curve Theorem). Let C be a simple closed curve in S^2. Then $S^2 - C$ has exactly two components C_1 and C_2. Moreover $C = \overline{C_1} - C_1 = \overline{C_2} - C_2$.

Proof. Since C is homeomorphic to the unit circle, there are arcs A_1 and A_2 such that $C = A_1 \cup A_2$ and $A_1 \cap A_2 = \{a_1, a_2\}$, where a_1 and a_2 are the end points of A_1 and A_2. Let $X = S^2 - \{a_1, a_2\}$, let $U = S^2 - A_1$, and let $V = S^2 - A_2$. Then U and V are open, $X = U \cup V$ and $U \cap V = S^2 - C$. By Theorem 9.8, $S^2 - C$ has at least two components. Suppose it has more than two components. Let M and N be two of the components and let $P = (S^2 - C) - (M \cup N)$. Since $S^2 - C$ is open in S^2, it is locally pathwise connected. Therefore the components of $S^2 - C$ are open. Thus, since P is the union of components, M, N, and P are open. Let $a \in M$, let $b \in N$, and let $c \in P$. By Theorem 9.12, U and V are pathwise connected. Therefore there is a path in U from a to b, a path in V from b to a, a path in U from a to c, and a path in V from c to a. Thus, by Theorem 9.10, $\pi_1(X, a)$ is not isomorphic to the group of integers. This is a contradiction because X is homeomorphic to $\mathbb{R}^2 - \{(0, 0)\}$, and the fundamental group of $\mathbb{R}^2 - \{(0, 0)\}$ is isomorphic to the group of integers. Therefore $S^2 - C$ has exactly two components.

Let C_1 and C_2 denote the two components of $S^2 - C$. We know that C_1 and C_2 are open in S^2, so $\overline{C_1} \cap C_2 = C_1 \cap \overline{C_2} = \varnothing$. Let $x \in \overline{C_1} - C_1$. Then $x \in \overline{C_1}$ but

$x \notin C_1$. Since $x \in \overline{C}_1$, $x \notin C_2$. Therefore $x \in C$ and hence $\overline{C}_1 - C_1 \subseteq C$. In the same manner, we see that $\overline{C}_2 - C_2 \subseteq C$.

Now let $x \in C$, and let W be a neighborhood of x. Since C is homeomorphic to the unit circle, there are arcs B_1 and B_2 such that $C = B_1 \cup B_2$, $B_1 \subseteq W$, and $B_1 \cap B_2 = \{b_1, b_2\}$, where b_1 and b_2 are the end points of B_1 and B_2. Let $p \in C_1$ and $q \in C_2$. By Theorem 9.12, there is a path f in $S^2 - B_2$ from p to q.



Now $S^2 = C_1 \cup (S^2 - \overline{C}_1) \cup (\overline{C}_1 - C_1)$, $f(I) \cap C_1 \neq \emptyset$, and $f(I) \cap (S^2 - \overline{C}_1) \neq \emptyset$. Therefore $f(I) \cap (\overline{C}_1 - C_1) \neq \emptyset$ because otherwise $f(I) \cap C_1$ and $f(I) \cap (S^2 - \overline{C}_1)$ would from a separation of the connected set $f(I)$. Let $y \in f(I) \cap (\overline{C}_1 - C_1)$. Since $\overline{C}_1 - C_1 \subseteq C$, $y \in C$. Since $f(I) \cap B_2 = \emptyset$, $y \in B_1 \subseteq W$. Therefore $y \in W \cap (\overline{C}_1 - C_1)$, and hence x is a limit point of $\overline{C}_1 - C_1$. But $\overline{C}_1 - C_1$ is closed, so $x \in \overline{C}_1 - C_1$. Therefore $C \subseteq \overline{C}_1 - C_1$. In the same manner, we can show that $C \subseteq \overline{C}_2 - C_2$. ■

EXERCISES 9.4

1. Complete the proof of Theorem 9.10 by showing that every "multiple" of $\gamma * \delta$ lifts to a loop in E and every nonzero "multiple" of $\alpha * \beta$ lifts to a path that is not a loop.
2. Let A and B be closed connected subsets of S^2 such that $A \cap B$ consists of exactly two points. Prove that $S^2 - (A \cup B)$ is not connected.
3. Let A and B be closed connected subsets of S^2 such that $S^2 - A$ and $S^2 - B$ are connected and $A \cap B$ consists of exactly two points. Prove that $S^2 - (A \cup B)$ has exactly two components.
4. Let A be an arc in $\mathbb{R}^2$. Show that $\mathbb{R}^2 - A$ is connected.
5. Let C be a simple closed curve in $\mathbb{R}^2$.
   (a) Show that $\mathbb{R}^2 - C$ has exactly two components C_1 and C_2.
   (b) Show that $C = \overline{C}_1 - C_1 = \overline{C}_2 - C_2$.