Chapter 8

The Fundamental Group and Covering Spaces

In the first seven chapters we have dealt with point-set topology. This chapter provides an introduction to algebraic topology. Algebraic topology may be regarded as the study of topological spaces and continuous functions by means of algebraic objects such as groups and homomorphisms. Typically, we start with a topological space $(X, \mathscr{T})$ and associate with it, in some specific manner, a group G_X (see Appendix I). This is done in such a way that if $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ are homeomorphic, then G_X and G_Y are isomorphic. Also, a continuous function $f : X \to Y$ "induces" a homomorphism $f_*: G_X \to G_Y$, and, if f is a homeomorphism, then f_* is an isomorphism. Then the "structure" of G_X provides information about the "structure" of $(X, \mathscr{T})$. One of the purposes of doing this is to find a topological property that distinguishes between two nonhomeomorphic spaces. The group that we study is called the fundamental group.

8.1 Homotopy of Paths

In Section 3.2 we defined paths and the path product of two paths. This section studies an equivalence relation defined on a certain collection of paths in a topological space. Review Theorems 2.15 and 2.27, because these results about continuous functions will be used repeatedly in this chapter and the next. The closed unit interval $[0, 1]$ will be denoted by I.

Definition. Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be topological spaces and let $f, g: X \to Y$ be continuous functions. Then f **is homotopic to** g, denoted by $f \simeq g$, if there is a continuous function $H: X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. The function H is called a **homotopy** between f and g. ■

In the preceding definition, we are assuming that I has the subspace topology $\mathscr{V}$ determined by the usual topology on $\mathbb{R}$. Thus $X \times I$ has the product topology determined by $\mathscr{T}$ and $\mathscr{V}$.

We think of a homotopy as a continuous one-parameter family of continuous functions from X into Y. If t denotes the parameter, then the homotopy represents a continuous "deformation" of the function f to the function g as t goes from 0 to 1. The question of whether f is homotopic to g is a question of whether there is a continuous extension of a given function. We think of f as being a function from $X \times \{0\}$ into Y and g as being a function from $X \times \{1\}$ into Y, so we have a continuous function from $X \times \{0, 1\}$ into Y and we want to extend it to a continuous function from $X \times I$ into Y.

EXAMPLE 1. Let X and Y be the subspaces of $\mathbb{R} \times \mathbb{R}$ defined by $X = \{(x, y) \in \mathbb{R} \times \mathbb{R}\colon x^2 + y^2 = 1\}$ and $Y = \{(x, y) \in \mathbb{R} \times \mathbb{R}\colon (x + 1)^2 + y^2 = 1$ or $(x - 1)^2 + y^2 = 1\}$. Define $f, g\colon X \to Y$ by $f(x, y) = (x - 1, y)$ and $g(x, y) = (x + 1, y)$. (See Figure 8.1).

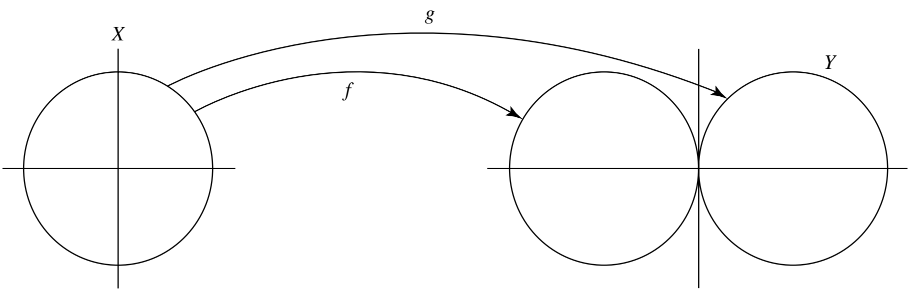


Figure 8.1

Then f is not homotopic to g.

EXAMPLE 2. Let X be the subspace of $\mathbb{R} \times \mathbb{R}$ defined by $X = \{(x, y) \in \mathbb{R} \times \mathbb{R}\colon x^2 + y^2 = 1\}$ and let $Y = X \times X$. Define $f, g\colon X \to Y$ by $f(x, y) = ((1, 0), (x, y))$ and $g(x, y) = ((0,1), (x, y))$. Then the function $H\colon X \times I \to Y$ defined by $H((x, y), t) = ((\sqrt{1 - t^2}, t), (x, y))$ is a homotopy between f and g so $f \simeq g$.

Notice that X is a circle and Y is a torus, f "wraps" the circle X around $\{(1, 0)\} \times X$, and g "wraps" the circle X around $\{(0, 1)\} \times X$ (see Figure 8.2). Let A denote the arc from $((1, 0), (1, 0))$ to $((0, 1), (1, 0))$ (see Figure 8.2). Then H maps $X \times I$ onto $A \times X$.

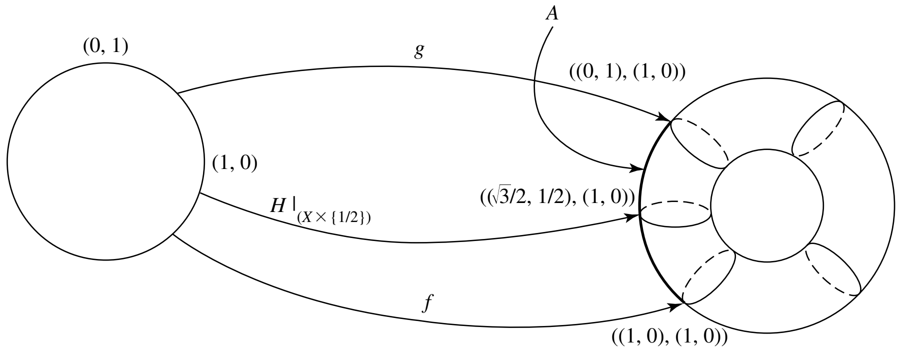


Figure 8.2

THEOREM 8.1. Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be topological spaces. Then $\simeq$ is an equivalence relation on $C(X, Y)$ (the collection of continuous functions that map X into Y).

Proof. Let $f \in C(X, Y)$. Define $H: X \times I \to Y$ by $H(x, t) = f(x)$ for each $(x, t) \in X \times I$. Then H is continuous and $H(x, 0) = H(x, 1) = f(x)$, so $f \simeq f$.

Suppose $f, g \in C(X, Y)$ and $f \simeq g$. Then there is a continuous function $H: X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. Define $K: X \times I \to Y$ by $K(x, t) = H(x, 1 - t)$. Then K is continuous, and $K(x, 0) = H(x, 1) = g(x)$ and $K(x, 1) = H(x, 0) = f(x)$ for all $x \in X$. Therefore $g \simeq f$.

Suppose $f, g, h \in C(X, Y), f \simeq g$, and $g \simeq h$. Then there are continuous functions $F, G: X \times I \to Y$ such that $F(x, 0) = f(x)$, $F(x, 1) = g(x)$, $G(x, 0) = g(x)$, and $G(x, 1) = h(x)$ for all $x \in X$. Define $H: X \times I \to Y$ by

$$H(x, t) = \begin{cases} F(x, 2t), & \text{for all } x \in X \text{ and } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1), & \text{for all } x \in X \text{ and } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Then H is continuous, and $H(x, 0) = F(x, 0) = f(x)$ and $H(x, 1) = G(x, 1) = h(x)$ for all $x \in X$. Therefore $f \simeq h$. ■

If α and β are paths in a topological space, we define a relation between them that is stronger than homotopy.

Definition. Let α and β be paths in a topological space $(X, \mathscr{T})$. Then α is **path homotopic** to β, denoted by $\alpha \simeq_p \beta$, if α and β have the same initial point x_0 and the same terminal point x_1 and there is a continuous function $H: I \times I \to X$ such that $H(x, 0) = \alpha(x)$ and $H(x, 1) = \beta(x)$ for all $x \in X$, and $H(0, t) = x_0$ and $H(1, t) = x_1$ for all $t \in I$. The function H is called a **path homotopy** between f and g. ■

The question of the existence of a path homotopy between two paths is again a question of obtaining a continuous extension of a given function. This time we have a continuous function on $(I \times \{0, 1\}) \cup (\{0, 1\} \times I)$ and we want to extend it to a continuous function on $I \times I$ (see Figure 8.3).

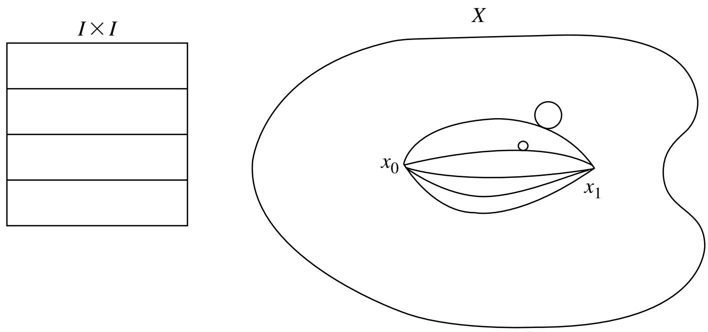


Figure 8.3

EXAMPLE 3. Let $X = \{(x, y) \in \mathbb{R}^2: 1 \leq x^2 + y^2 \leq 4\}$ (see Figure 8.4), let $\alpha: I \to X$ be the path that maps I in a "linear" fashion onto the arc A from $(-1, 0)$ to $(1, 0)$, and let $\beta: I \to X$ be the path that maps I in a "linear" fashion onto the arc B from $(-1, 0)$ to $(1, 0)$. Then α is homotopic to β (by a homotopy that transforms α into β in the manner illustrated in Figure 8.5), but α is not path homotopic to β because we cannot "get from" α to β and keep the "end points" fixed without "crossing the hole" in the space.

If $(X, \mathscr{T})$ is a topological space and $x_0 \in X$, let $\Omega(X, x_0) = \{\alpha: I \to X: \alpha$ is continuous and $\alpha(0) = \alpha(1) = x_0\}$. Observe that if $\alpha, \beta \in \Omega(X, x_0)$, then the product (defined in Section 3.2) $\alpha * \beta \in \Omega(X, x_0)$. It is, however, easy to see that $*$ is not associative on $\Omega(X, x_0)$. Each member of $\Omega(X, x_0)$ is called a **loop** in X at x_0.

The proof of the following theorem is similar to the proof of Theorem 8.1 and hence it is left as Exercise 1.

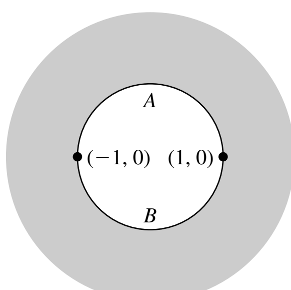


Figure 8.4

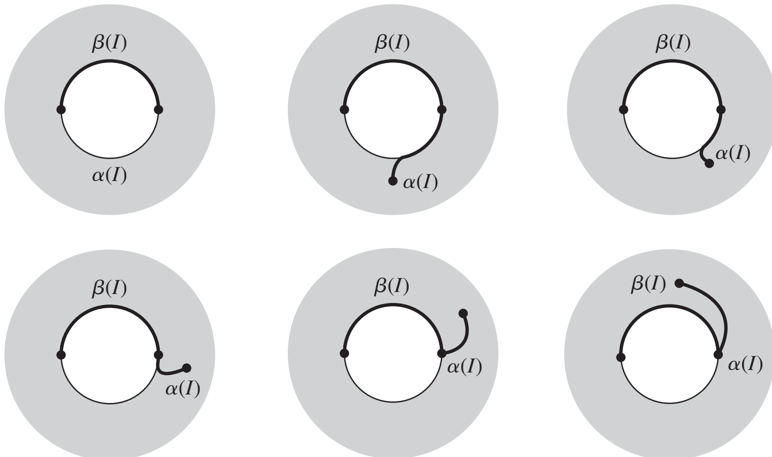


Figure 8.5

THEOREM 8.2. Let $(X, \mathscr{T})$ be a topological space, and let $x_0 \in X$. Then $\simeq_p$ is an equivalence relation on $\Omega(X, x_0)$. ■

Let $(X, \mathscr{T})$ be a topological space and let $x_0 \in X$. If $\alpha \in \Omega(X, x_0)$, we let $[\alpha]$ denote the path-homotopy equivalence class that contains α. Then we let $\pi_1(X, x_0)$ denote the set of all path-homotopy equivalence classes on $\Omega(X, x_0)$, and we define an operation $\circ$ on $\pi_1(X, x_0)$ by $[\alpha] \circ [\beta] = [\alpha * \beta]$. The following theorem tells us that $\circ$ is well-defined.

THEOREM 8.3. Let $(X, \mathscr{T})$ be a topological space and let $x_0 \in X$. Furthermore, let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \Omega(X, x_0)$ and suppose $\alpha_1 \simeq_p \alpha_2$ and $\beta_1 \simeq_p \beta_2$. Then $\alpha_1 * \beta_1 \simeq_p \alpha_2 * \beta 2$.

Proof. Since $\alpha_1 \simeq_p \alpha_2$ and $\beta_1 \simeq_p \beta_2$, there exist continuous functions $F, G\colon I \times I \to X$ such that $F(x, 0) = \alpha_1(x)$, $F(x, 1) = \alpha_2(x)$, $G(x, 0) = \beta_1(x)$, and $G(x, 1) = \beta_2(x)$ for all $x \in I$, and $F(0, t) = F(1, t) = G(0, t) = G(1, t) = x_0$ for all $t \in I$. Define $H\colon I \times I \to X$ by

$$H(x, t) = \begin{cases} F(2x, t), & \text{if } 0 \leq x \leq \frac{1}{2} \text{ and } 0 \leq t \leq 1 \\ G(2x - 1, t), & \text{if } \frac{1}{2} \leq x \leq 1 \text{ and } 0 \leq t \leq 1. \end{cases}$$

Then H is continuous,

$$H(x, 0) = \begin{cases} F(2x, 0), & \text{if } 0 \leq x \leq \frac{1}{2} \\ G(2x - 1, 0) & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} = \begin{cases} \alpha_1(2x), & \text{if } 0 \leq x \leq \frac{1}{2} \\ \beta_1(2x - 1), & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases} = (\alpha_1 * \beta_1)(x)$$

and

$$H(x, 1) = \begin{cases} F(2x, 1), & \text{if } 0 \le x \le \frac{1}{2} \\ G(2x - 1, 1) & \text{if } \frac{1}{2} \le x \le 1 \end{cases} = \begin{cases} \alpha_2(2x), & 0 \le x \le \frac{1}{2} \\ \beta_2(2x - 1), & \frac{1}{2} \le x \le 1 \end{cases} = (\alpha_2 * \beta_2)(x)$$

for all $x \in X$, and $H(0, t) = F(0, t) = x_0$ and $H(1, t) = G(1, t) = x_0$ for all $t \in I$. Therefore $\alpha_1 * \beta_1 \simeq_p \alpha_2 * \beta_2$. ■

The following three theorems show that if $(X, \mathscr{T})$ is a topological space and $x_0 \in X$, then $(\pi_1(X, x_0), \circ\ )$ is a group.

THEOREM 8.4. Let $(X, \mathscr{T})$ be a topological space, let $x_0 \in X$, and let $[\alpha], [\beta], [\gamma] \in \pi_1(X, x_0)$. Then $([\alpha] \circ [\beta]) \circ [\gamma] = [\alpha] \circ ([\beta] \circ [\gamma])$.

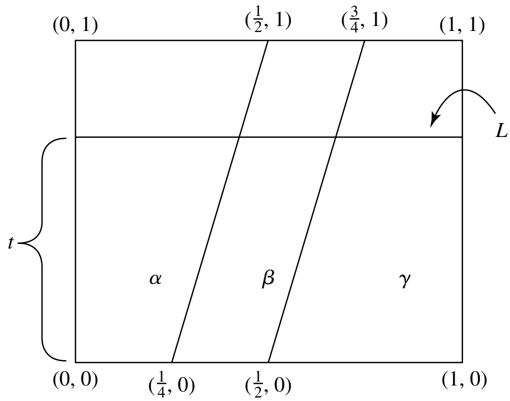
Proof. We show that $(\alpha * \beta) * \gamma \simeq_p \alpha * (\beta * \gamma)$. First we observe that for each $x \in I$,

$$[(\alpha * \beta) * \gamma](x) = \begin{cases} (\alpha * \beta)(2x), & \text{if } 0 \le x \le \frac{1}{2} \\ \gamma(2x - 1), & \text{if } \frac{1}{2} \le x \le 1 \end{cases} = \begin{cases} \alpha(4x), & \text{if } 0 \le x \le \frac{1}{4} \\ \beta(4x - 1), & \text{if } \frac{1}{4} \le x \le \frac{1}{2} \\ \gamma(2x - 1), & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

and

$$[\alpha * (\beta * \gamma)](x) = \begin{cases} \alpha(2x), & \text{if } 0 \le x \le \frac{1}{2} \\ \beta(4x - 2), & \text{if } \frac{1}{2} \le x \le \frac{3}{4} \\ \gamma(4x - 3), & \text{if } \frac{3}{4} \le x \le 1. \end{cases}$$

Next we draw a picture to illustrate how we arrive at the definition of the desired path homotopy H.



If $t \in I$, then the intersection of the horizontal line L and the line segment joining $(\frac{1}{4}, 0)$ and $(\frac{1}{2}, 1)$ has coordinates $((1 + t)/4, t)$, and the intersection of L and the line segment joining $(\frac{1}{2}, 0)$ and $(\frac{3}{4}, 1)$ has coordinates $((2 + t)/4, t)$. Therefore for each $t \in I$, we want H to be defined in terms of α if $x \leq (1 + t)/4$, in terms of β if $(1+t)/4 \leq x \leq (2 + t)/4$, and in terms of γ if $x \geq (2 + t)/4$. Also for each $t \in I$, we want $H(0, t) = H((1 + t)/4, t) = H((2 + t)/4, t) = H(1, t) = x_0$. Using elementary analytic geometry, we arrive at the definition of H. Define $H: I \times I \to X$ by

$$H(x, t) = \begin{cases} \alpha\left(\dfrac{4x}{1 + t}\right), & \text{if } 0 \leq t \leq 1 \text{ and } 0 \leq x \leq \dfrac{1 + t}{4} \\ \beta(4x - 1 - t), & \text{if } 0 \leq t \leq 1 \text{ and } \dfrac{1 + t}{4} \leq x \leq \dfrac{2 + t}{4} \\ \gamma\left(\dfrac{4x - 2 - t}{2 - t}\right), & \text{if } 0 \leq t \leq 1 \text{ and } \dfrac{2 + t}{4} \leq x \leq 1. \end{cases}$$

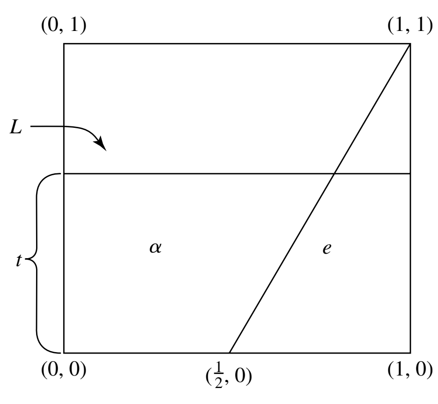
Thus H is continuous, and it is a routine check to show that $H(x, 0) = [(\alpha * \beta) * \gamma](x)$ and $H(x, 1) = [\alpha * (\beta * \gamma)](x)$ for all $x \in I$, and $H(0, t) = H(1, t) = x_0$ for all $t \in I$. ■

We have proved that $\circ$ is associative on $\pi_1(X, x_0)$. Now we prove that $(\pi_1(X, x_0), \circ)$ has an identity element.

THEOREM 8.5. Let $(X, \mathscr{T})$ be a topological space, let $x_0 \in X$, and let $e: I \to X$ be the path defined by $e(x) = x_0$ for each $x \in I$. Then $[\alpha] \circ [e] = [e] \circ [\alpha] = [\alpha]$ for each $[\alpha] \in \pi_1(X, x_0)$.

Proof. We prove that if $[\alpha] \in \pi_1(X, x_0)$ then $\alpha * e \simeq_p \alpha$ and $e * \alpha \simeq_p \alpha$.

Let $[\alpha] \in \pi_1(X, x_0)$. We draw a picture to illustrate how we arrive at the definition of the path homotopy H to show that $\alpha * e \simeq_p \alpha$.



For each $t \in I$, the intersection of the line L with the line segment joining $(\frac{1}{2}, 0)$ and $(1, 1)$ has coordinates $((1 + t)/2, t)$. Therefore for each $t \in I$, we want the homotopy defined in terms of α if $x \leq (1 + t)/2$. Using analytic geometry, we arrive at the formula

$$H(x, t) = \begin{cases} \alpha\left(\dfrac{2x}{1 + t}\right), & \text{if } 0 \leq x \leq \dfrac{1 + t}{2} \\ x_0, & \text{if } \dfrac{1 + t}{2} \leq x \leq 1. \end{cases}$$

It is a routine check to see that $H(x, 0) = (\alpha * e)(x)$ and $H(x, 1) = \alpha(x)$ for each $x \in I$ and $H(0, t) = H(1, t) = x_0$ for each $t \in I$.

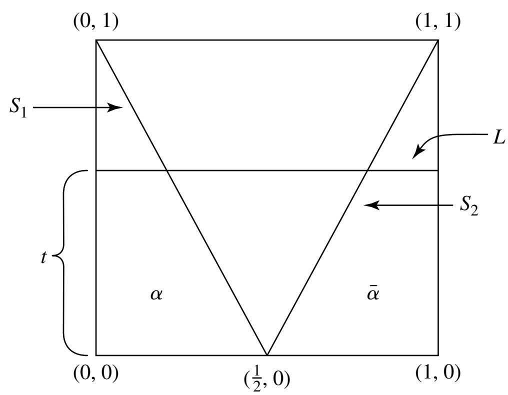
The proof that $e * \alpha \simeq_p \alpha$ is left as Exercise 2. ■

We have proved that $[e]$ is the identity element of $\pi_1(X, x_0)$. We complete the proof that $\pi_1(X, x_0), \circ$) is a group by showing that each member of $\pi_1(X, x_0)$ has an inverse in $\pi_1(X, x_0)$.

THEOREM 8.6. Let $(X, \mathscr{T})$ be a topological space, let $x_0 \in X$, and let $[\alpha] \in \pi_1(X, x_0)$. Then there exists $[\overline{\alpha}] \in \pi_1(X, x_0)$ such that $[\alpha] \circ [\overline{\alpha}] = [\overline{\alpha}] \circ [\alpha] = [e]$.

Proof. We prove that there is an $\overline{\alpha} \in \Omega(X, x_0)$ such that $\alpha * \overline{\alpha} \simeq_p e$ and $\overline{\alpha} * \alpha \simeq_p e$.

Define $\overline{\alpha} : I \to X$ by $\overline{\alpha}(x) = \alpha(1 - x)$ for each $x \in X$. We draw a picture to illustrate how we arrive at the definition of the path homotopy H to show that $\alpha * \overline{\alpha} \simeq_p e$.



For each $t \in I$, we map the part of L that lies between S_1 and S_2 into the point $f(1 - t)$, we map the segment $[(0, t), ((1 - t)/2, t)]$ in the same manner that α maps the interval $[0, 1 - t]$, and we map the segment $[((1 + t)/2, t), (1, t)]$ in the same manner that $\overline{\alpha}$ maps the interval $[t, 1]$. Observing that $2((1 - t)/2) = 1 - t$ and $2((1 + t)/2) - 1 = t$, we arrive at the formula

$$H(x, t) = \begin{cases} \alpha(2x), & \text{if } 0 \leq x \leq \dfrac{1-t}{2} \\ \alpha(1 - t), & \text{if } \dfrac{1-t}{2} \leq x \leq \dfrac{1+t}{2} \\ \overline{\alpha}(2x - 1), & \text{if } \dfrac{1+t}{2} \leq x \leq 1. \end{cases}$$

It is easy to see that $H(x, 0) = (\alpha * \overline{\alpha})(x)$ and $H(x, 1) = e(x)$ for all $x \in I$ and $H(0, t) = H(1, t) = x_0$ for each $t \in I$.

The proof that $\overline{\alpha} * \alpha \simeq_p e$ is left as Exercise 3. ■

We have proved that $(\pi_1(X, x_0), \circ)$ is a group.

Definition. Let $(X, \mathcal{T})$ be a topological space and let $x_0 \in X$. Then $(\pi_1(X, x_0), \circ)$ is called the **fundamental group of** X at x_0. ■

The fundamental group was introduced by the French mathematician Henri Poincaré (1854–1912) around 1900. During the years 1895–1901, he published a series of papers in which he introduced algebraic topology. It is not true that algebraic topology developed as an outgrowth of general topology. In fact, Poincaré's work on algebraic topology preceded Hausdorff's definition of a topological space. Neither was Poincaré's work influenced by Cantor's theory of sets.

Poincaré was surpassed only by Leonard Euler (1707–1783) as the most prolific writer of mathematics. Poincaré published 30 books and over 500 papers on mathematics, and he is regarded as the founder of combinatorial topology.

Throughout the remainder of this text, we denote $(\pi_1(X, x_0), \circ)$ by $\pi_1(X, x_0)$. The natural question to ask is whether the fundamental group of a topological space depends upon the base point x_0. This and other questions will be answered in the remaining sections of this chapter.

We conclude this section with the definition of a contractible space and two theorems whose proofs are left as exercises.

Definition. A topological space $(X, \mathcal{T})$ is **contractible to a point** $x_0 \in X$ if there is a continuous function $H: X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$ for each $x \in X$ and $H(x_0, t) = x_0$ for each $t \in I$. Then $(X, \mathcal{T})$ is **contractible** if there exists $x_0 \in X$ such that $(X, \mathcal{T})$ is contractible to x_0. ■

THEOREM 8.7. Let $(X, \mathcal{T})$ be a topological space and let $(Y, \mathcal{U})$ be a contractible space. If $f, g: X \to Y$ are continuous functions, then f is homotopic to g. ■

THEOREM 8.8. Let $(X, \mathcal{T})$ be a contractible space and let $(Y, \mathcal{U})$ be a pathwise connected space. If $f, g: X \to Y$ are continuous functions, then f is homotopic to g. ■

EXERCISES 8.1

1. Let $(X, \mathscr{T})$ be a topological space and let $x_0 \in X$. Prove that $\simeq_p$ is an equivalence relation on $\Omega(X, x_0)$.

2. Let $(X, \mathscr{T})$ be a topological space, let $x_0 \in X$, and define $e: I \to X$ by $e(x) = x_0$ for each $x \in X$. Show that if $\alpha \in \Omega(X, x_0)$, then $e * \alpha \simeq_p \alpha$.

3. Let $(X, \mathscr{T})$ be a topological space, let $x_0 \in X$, and define $e: I \to X$ by $e(x) = x_0$ for each $x \in X$. Let $\alpha \in \Omega(X, x_0)$, and define $\overline{\alpha}: I \to X$ by $\overline{\alpha}(x) = \alpha(1 - x)$ for each $x \in X$. Prove that $\overline{\alpha} * \alpha \simeq_p e$.

4. Let $(X, \mathscr{T})$, $(Y, \mathscr{U})$ and $(Z, \mathscr{V})$ be a topological spaces, let $f_1, f_2: X \to Y$ be continuous functions such that $f_1 \simeq f_2$, and let $g_1, g_2: Y \to Z$ be continuous functions such that $g_1 \simeq g_2$. Prove that $g_1 \circ f_1 \simeq g_2 \circ f_2$.

5. Let $n \in \mathbb{N}$. *A* subset *X* of $\mathbb{R}^n$ is **convex** if for each $x, y \in X$ and $t \in I$, $(1 - t)x + ty \in X$. Let X be a convex subset of $\mathbb{R}^n$ and let α and β be paths in X such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Prove that α is path homotopic to β.

6. Let $(X, \mathscr{T})$ be a topological space and let $f, g: X \to I$ be continuous functions. Prove that f is homotopic to g.

7. Let $(X, \mathscr{T})$ be a pathwise connected space and let $\alpha, \beta: I \to X$ be continuous functions. Prove that α is homotopic to β.

8. (a) Prove that $\mathbb{R}$ (with the usual topology) is contractible.

   (b) Prove that every contractible space is pathwise connected.

9. Prove Theorem 8.7.

10. Prove Theorem 8.8.

11. Let X be a convex subset of $\mathbb{R}^n$ and let $x_0 \in X$. Prove that X is contractible to x_0.

12. Let $(G, \cdot, \mathscr{T})$ be a topological group (see Appendix I), and let e denote the identity of $(G, \cdot)$. For $\alpha, \beta \in \Omega(G, e)$, define $\alpha \oslash \beta$ by $(\alpha \oslash \beta)(x) = \alpha(x) \cdot \beta(x)$.

    (a) Prove that $\oslash$ is an operation on $\Omega(G, e)$ and that $(\Omega(G, e), \oslash)$ is a group.

    (b) Prove that the operation $\oslash$ on $\Omega(G, e)$ induces an operation $\otimes$ on $\pi_1(G, e)$ and that $(\pi_1(G, e), \otimes)$ is a group.

    (c) Prove that the two operations $\otimes$ and $\circ$ on $\pi_1(G, e)$ are the same.

    (d) Prove that $\pi_1(G, e)$ is abelian.

8.2 The Fundamental Group

In this section we establish some properties of the fundamental group. The first result is that if $(X, \mathscr{T})$ is a pathwise connected space and $x_0, x_1 \in X$, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$. Before we begin the proof, we draw a picture (Figure 8.6) to illustrate the idea. It is useful to look at this picture as you read the proof.

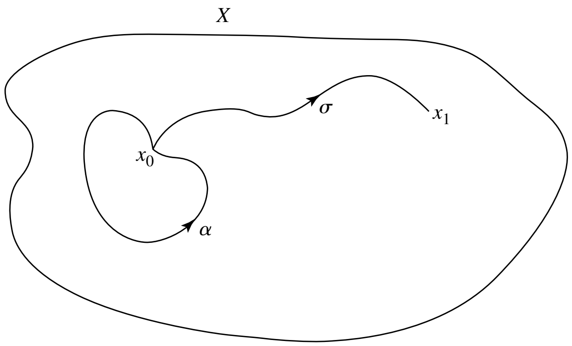


***Figure* 8.6**

THEOREM 8.9. Let $(X, \mathscr{T})$ be a pathwise connected space, and let $x_0, x_1 \in X$. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$.

Proof. Let $\sigma: I \to X$ be a continuous function such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$, and define $\overline{\sigma}: I \to X$ by $\overline{\sigma}(x) = \sigma(1 - x)$ for each $x \in X$. Then define $\theta_\sigma: \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $\theta_\sigma([\alpha]) = [(\overline{\sigma} * \alpha) * \sigma]$ for each $[\alpha] \in \pi_1(X, x_0)$. We want to prove that θ_σ is an isomorphism. First observe that if $\alpha \in \Omega(X, x_0)$, then $(\overline{\sigma} * \alpha) * \sigma \in \Omega(X, x_1)$. In order to show that θ_σ is well-defined, it is sufficient to show that if α *and* β are members of $\Omega(X, x_0)$ and $\alpha \simeq_P \beta$, then $(\overline{\sigma} * \alpha) * \sigma \simeq_P (\overline{\sigma} * \beta) * \sigma$. The proof of this fact is left as Exercise 1(a).

Now let $[\alpha], [\beta] \in \pi_1(X, x_0)$. Then $\theta_\sigma([\alpha] \circ [\beta]) = \theta_\sigma([\alpha * \beta]) = [(\overline{\sigma} * (\alpha * \beta)) * \sigma]$, and $\theta_\sigma([\alpha]) \circ \theta_\sigma([\beta]) = [(\overline{\sigma} * \alpha) * \sigma] \circ [(\overline{\sigma} * \beta) * \sigma] = [((\overline{\sigma} * \alpha) * \sigma) * ((\overline{\sigma} * \beta) * \sigma)]$. Thus in order to show that θ_σ is a homomorphism, it is sufficient to show that $(\overline{\sigma} * (\alpha * \beta)) * \sigma \simeq_p ((\overline{\sigma} * \alpha) * \sigma) * ((\overline{\sigma} * \beta) * \sigma)$. The proof of this fact is left as Exercise 1(b).

Now we show that θ_σ is one-to-one. Suppose $[\alpha], [\beta] \in \pi_1(X, x_0)$ and $\theta_\sigma([\alpha]) = \theta_\sigma([\beta])$. Then $[(\overline{\sigma} * \alpha) * \sigma] = [(\overline{\sigma} * \beta) * \sigma]$, and so $(\overline{\sigma} * \alpha) * \sigma \simeq_p (\overline{\sigma} * \beta) * \sigma$. In order to show that $[\alpha] = [\beta]$, it is sufficient to show that $\alpha \simeq_p \beta$, and we leave the proof of this fact as Exercise 1(c).

Now we show that θ_σ maps $\pi_1(X, x_0)$ onto $\pi_1(X, x_1)$. Let $[\alpha] \in \pi_1(X, x_1)$. Then $\alpha \in \Omega(X, x_1)$ and so $\sigma * (\alpha * \overline{\sigma}) \in \Omega(X, x_0)$. Then $\theta_\sigma([\sigma * (\alpha * \overline{\sigma}]) = [(\overline{\sigma} * (\sigma * (\alpha * \overline{\sigma})) * \sigma]$. Thus it is sufficient to show that $(\overline{\sigma} * (\sigma * (\alpha * \overline{\sigma}))) * \sigma \simeq_p \alpha$, and the proof of this fact is left as Exercise 1(d). ■

EXAMPLE 4. Let X be a convex subset (see Exercise 5 of Section 8.1) of $\mathbb{R}^n$. Then X is pathwise connected because if $x, y \in X$, the function $\alpha: I \to X$ defined by $\alpha(t) = (1 - t)x + ty$ is a path from x to y. Thus up to isomorphism the fundamental group of X is independent of the base point. Let $x_0 \in X$ and let $\alpha \in \Omega(X, x_0)$. Define $H: I \times I \to X$ by $H(x, t) = tx_0 + (1 - t)\alpha(x)$. Then $H(x, 0) = \alpha(x)$ and $H(x, 1) = x_0$ for each $x \in X$. Also $H(0, t) = tx_0 + (1 - t)\alpha(0) = x_0$ and $H(1, t) = tx_0 + (1 - t)\alpha(1) = x_0$. Therefore α is path homotopic to the path $e: I \to X$ defined by $e(x) = x_0$ for each $x \in X$. Hence the fundamental group of X at x_0 is the trivial group (that is, the group whose only member is the identity element), and, by Theorem 8.9, if x is any member of X, then $\pi_1(X, x)$ is the trivial group.

Let $(X, \mathscr{T})$ be a topological space, let $x_0 \in X$, and let C be the path component of X that contains x_0. Since the continuous image of a pathwise connected space is pathwise connected (see Exercise 7 of Section 3.2), we have the following results. If $\alpha \in \Omega(X, x_0)$, then α is also a member of $\Omega(C, x_0)$. Also if $H: I \times I \to X$ is a continuous function such that $H(x, t) = x_0$ for some $(x, t) \in I \times I$, then $H(I \times I) \subseteq C$. Thus $\pi_1(X, x_0) = \pi_1(C, x_0)$. Therefore $\pi_1(X, x_0)$ depends only on the path component C of X that contains x_0, and it does not give us any information about $X - C$.

Let $(X, \mathscr{T})$ be a pathwise connected space, and let $x_0, x_1 \in X$. While Theorem 8.9 guarantees that $\pi_1(X, x_0)$ is isomorphic to $\pi_1(X, x_1)$, different paths from x_0 to x_1 may give rise to different isomorphisms.

If $(X, \mathscr{T})$ is a pathwise connected space, it is common practice to speak of the fundamental group of X without reference to the basepoint.

We show that the fundamental group is a topological property by introducing a homomorphism induced by a continuous function.

Definition. **A topological pair** (X, A) is an ordered pair whose first term is a topological space $(X, \mathscr{T})$ and whose second term is a subspace A of X. ■

We do not distinguish between the topological pair $(X, \varnothing)$ and the topological space $(X, \mathscr{T})$. If $x_0 \in X$, we let (X, x_0) denote the topological pair $(X, \{x_0\})$.

Definition. If (X, A) and (Y, B) are topological pairs, a **map** $f: (X, A) \to (Y, B)$ is a continuous function $f: X \to Y$ such that $f(A) \subseteq B$.

THEOREM 8.10. Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be topological spaces, let $x_0 \in X$ and $y_0 \in Y$, and let $h: (X, x_0) \to (Y, y_0)$ be a map. Then h induces a homomorphism $h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

Proof. Let $[\alpha] \in \pi_1(X, x_0)$. Then $\alpha: I \to X$ is a continuous function such that $\alpha(0) = \alpha(1) = x_0$, so $h \circ \alpha: I \to Y$ is a continuous function such that $(h \circ \alpha)(0) = (h \circ \alpha)(1) = y_0$. Therefore $[h \circ \alpha] \in \pi_1(Y, y_0)$. Define $h_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ by $h_*([\alpha]) = [h \circ \alpha]$. In order to show that h_* is well-defined, it is sufficient to show that if $\alpha, \beta \in \Omega(X, x_0)$ and $\alpha \simeq_p \beta$, then $h \circ \alpha \simeq_p h \circ \beta$. Suppose

$\alpha, \beta \in \Omega(X, x_0)$ and $\alpha \simeq_p \beta$. Then there is a continuous function $F: I \times I \to X$ such that $F(x, 0) = \alpha(x)$ and $F(x, 1) = \beta(x)$ for all $x \in I$ and $F(0, t) = F(1, t) = x_0$ for all $t \in I$. Define $G: I \times I \to Y$ by $G(x, t) = (h \circ F)(x, t)$. Then it is easy to see that $G(x, 0) = (h \circ \alpha)(x)$ and $G(x, 1) = (h \circ \beta)(x)$ for all $x \in I$ and $G(0, t) = G(1, t) = y_0$. Therefore $h \circ \alpha \simeq_p h \circ \beta$, and so h_* is well defined.

Now we show that h_* is a homomorphism. Let $[\alpha], [\beta] \in \pi_1(X, x_0)$. Then $h_*([\alpha] \circ [\beta]) = h_*([\alpha * \beta]) = [h \circ (\alpha * \beta)]$ and $h_*([\alpha]) \circ h_*([\beta]) = [h \circ \alpha] \circ [h \circ \beta] = [(h \circ \alpha) * (h \circ \beta)]$. Since

$$(h \circ (\alpha * \beta))(x) = \begin{cases} (h \circ \alpha)(2x), & \text{if } 0 \le x \le \frac{1}{2} \\ (h \circ \beta)(2x - 1), & \text{if } \frac{1}{2} \le x \le 1 \end{cases} = ((h \circ \alpha) * (h \circ \beta))(x),$$

h_* is a homomorphism. ■

In the remainder of this text, whenever we speak of the homomorphism induced by a continuous function, we mean the homomorphism defined in the proof of Theorem 8.10.

Definition. If (X, A) and (Y, B) are topological pairs and $f, g: (X, A) \to (Y, B)$ are maps such that $f|_A = g|_A$, then f is **homotopic to g relative to A**, denoted by $f \simeq g \text{ rel } A$, if there is a continuous function $H: X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$ and $H(a, t) = f(a)$ for all $a \in A$ and $t \in I$. ■

If $x_0 \in X$, then we write $f \simeq g \text{ rel } x_0$ rather than $f \simeq g \text{ rel } \{x_0\}$.

THEOREM 8.11. Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be topological spaces, let $x_0 \in X$ and $y_0 \in Y$, and let $h, k: (X, x_0) \to (Y, y_0)$ be maps such that $h \simeq k \text{ rel } x_0$. Then $h_* = k_*$.

Proof. Since $h \simeq k \text{ rel } x_0$, there is a continuous function $H: X \times I \to Y$ such that $H(x, 0) = h(x)$ and $H(x, 1) = k(x)$ for all $x \in X$ and $H(x_0, t) = y_0$ for all $t \in I$. Let $[\alpha] \in \pi_1(X, x_0)$. Then $h_*([\alpha]) = [h \circ \alpha]$ and $k_*([\alpha]) = [k \circ \alpha]$. Define $G: I \times I \to Y$ by $G(x, t) = H(\alpha(x), t)$ for all $(x, t) \in I \times I$. By Theorem 2.27, G is continuous. Note that $G(x, 0) = H(\alpha(x), 0) = (h \circ \alpha)(x)$ and $G(x, 1) = \mathbf{H}(\alpha(x), 1) = (k \circ \alpha)(x)$ for each $x \in I$ and $G(0, t) = H(\alpha(0), t) = H(x_0, t) = y_0$ and $G(1, t) = H(\alpha(1), t) = H(x_0, t) = y_0$ for each $t \in I$. Therefore $h \circ \alpha \simeq_p k \circ \alpha$, and hence $[h \circ \alpha] = [k \circ \alpha]$. Thus $h_* = k_*$. ■

THEOREM 8.12. Let $(X, \mathscr{T})$, $(Y, \mathscr{U})$, and $(Z, \mathscr{V})$ be topological spaces, let $x_0 \in X$, $y_0 \in Y$, and $z_0 \in Z$, and let $h: (X, x_0) \to (Y, y_0)$ and $k: (Y, y_0) \to (Z, z_0)$ be maps. Then $(k \circ h)_* = k_* \circ h_*$.

Proof. Let $[\alpha] \in \pi_1(X, x_0)$. Then $(k \circ h)_*([\alpha]) = [(k \circ h) \circ \alpha]$ and $(k_* \circ h_*)[\alpha]) = k_*([h \circ \alpha]) = [k \circ (h \circ \alpha)]$. Since the composition of functions is associative (see Exercise 21 in Appendix C), $(k \circ h) \circ \alpha = k \circ (h \circ \alpha)$. Therefore $(k \circ h)_* = k_* \circ h_*$. ■

Definition. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $x_0 \in X$ and $y_0 \in Y$. Then (X, x_0) and (Y, y_0) are **of the same homotopy type** if there exist maps $f: (X, x_0) \to (Y, y_0)$ and $g: (Y, y_0) \to (X, x_0)$ such that $f \circ g \simeq i_Y$ rel y_0 and $g \circ f \simeq i_X$ rel x_0. ■

THEOREM 8.13. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces and let $x_0 \in X$ and $y_0 \in Y$. If (X, x_0) and (Y, y_0) are of the same homotopy type, then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, y_0)$.

Proof. Suppose (X, x_0) and (Y, y_0) are of the same homotopy type. Then there exist maps $f: (X, x_0) \to (Y, y_0)$ and $g: (Y, y_0) \to (X, x_0)$ such that $f \circ g \simeq i_Y$ rel y_0 and $g \circ f \simeq i_X$ rel x_0. By Theorem 8.11, $(f \circ g)_*: \pi_1(Y, y_0) \to \pi_1(Y, y_0)$ and $(g \circ f)_*: \pi_1(X, x_0) \to \pi_1(X, x_0)$ are the identity homomorphisms. By Theorem 8.12, $(f \circ g)_* = f_* \circ g_*$ and $(g \circ f)_* = g_* \circ f_*$. By Theorem C.2, $f_*: \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. ■

THEOREM 8.14. Let $(X, \mathcal{T})$ and $(Y, \mathcal{U})$ be topological spaces, let $h: X \to Y$ be a homeomorphism, let $x_0 \in X$, and let $y_0 = h(x_0)$. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, y_0)$.

Proof. Since h is a homeomorphism, $h^{-1}: Y \to X$ is a homeomorphism. Also note that $x_0 = h^{-1}(y_0)$. Since $h^{-1} \circ h = i_X$ and $h \circ h^{-1} = i_Y$, (X, x_0) and (Y, y_0) are of the same homotopy type. Therefore by Theorem 8.13, $\pi_1(X, x_0)$ is isomorphic to $\pi_1(Y, y_0)$. ■

EXERCISES 8.2

1. Let $(X, \mathcal{T})$ be a topological space, let $x_0, x_1 \in X$, let $\sigma: I \to X$ be a continuous function such that $\sigma(0) = x_0$ and $\sigma(1) = x_1$, and let $\alpha, \beta \in \Omega(X, x_0)$.
   (a) Prove that if $\alpha \simeq_p \beta$, then $(\overline{\sigma} * \alpha) * \sigma \simeq_p (\overline{\sigma} * \beta) * \sigma$.
   (b) Prove that $(\overline{\sigma} * (\alpha * \beta)) * \sigma \simeq_p ((\overline{\sigma} * \alpha) * \sigma) * ((\overline{\sigma} * \beta) * \sigma)$.
   (c) Prove that if $(\overline{\sigma} * \alpha) * \sigma \simeq_p (\overline{\sigma} * \beta) * \sigma$, then $\alpha \simeq_p \beta$.
   (d) Prove that $(\overline{\sigma} * (\sigma * (\alpha * \overline{\sigma}))) * \sigma \simeq_p \alpha$.
2. A pathwise connected space $(X, \mathcal{T})$ is **simply connected** provided that for each $x_0 \in X$, $\pi_1(X, x_0)$ is the trivial group. Let $(X, \mathcal{T})$ be a simply connected space and let $\alpha, \beta: I \to X$ be paths such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$. Prove that $\alpha \simeq_p \beta$.
3. Prove that every contractible space is simply connected.

4. A topological space $(X, \mathscr{T})$ is **weakly contractible** if there exists $x_0 \in X$ and a continuous function $H: X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) = x_0$ for each $x \in X$.

   (a) Give an example of a weakly contractible space that is not contractible.

   (b) Prove that every weakly contractible space is simply connected.

5. A subspace A of a topological space $(X, \mathscr{T})$ is a **retract** of X if there exists a continuous function $r: X \to A$ such that $r(a) = a$ for each $a \in A$. The function r is called a **retraction of X onto A.** Let A be a subspace of a topological space $(X, \mathscr{T})$, let $r: X \to A$ be a retraction of X onto A, and let $a_0 \in A$. Prove that $r_*: \pi_1(X, a_0) \to \pi_1(A, a_0)$ is a surjection. *Hint:* Consider the map $j: (A, a_0) \to (X, a_0)$, defined by $j(a) = a$ for each $a \in A$.

6. Let $(X, \mathscr{T})$ be a pathwise connected space, let $x_0, x_1 \in X$, let $(Y, \mathscr{U})$ be a topological space, let $h: X \to Y$ be a continuous function, let $y_0 = h(x_0)$ and $y_1 = h(x_1)$, and for each $i = 0, 1$, let h_i denote the map from the pair (X, x_i) into the pair (Y, y_i) defined by $h_i(x) = h(x)$ for each $x \in X$. Prove that there are isomorphisms $\theta: \pi_1(X, x_0) \to \pi_1(X, x_1)$ and $\phi: \pi_1(Y, y_0) \to \pi_1(Y, y_1)$ such that $(h_1)_* \circ \theta = \phi \circ (h_0)_*$.

8.3 The Fundamental Group of the Circle

Our goal in this section is to show that the fundamental group of the circle is isomorphic to the group of integers. In order to do this we need several preliminary results. We begin with the definition of the function p that maps $\mathbb{R}$ onto S^1. Define $p: \mathbb{R} \to S^1$ by $p(x) = (\cos 2\pi x, \sin 2\pi x)$. One can think of p as a function that wraps $\mathbb{R}$ around S^1. In particular, note that for each integer n, p is a one-to-one map of $[n, n+1)$ onto S^1. Furthermore, if U is the open subset of S^1 indicated in Figure 8.7, then $p^{-1}(U)$ is the union of a pairwise disjoint collection of open intervals.

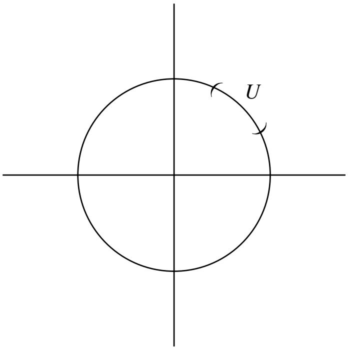


Figure 8.7

To be specific, let $U = \{(x, y) \in S^1: x > 0 \text{ and } y > 0\}$. Then if $x \in p^{-1}(U)$, $\cos 2\pi x$ and $\sin 2\pi x$ are both positive, so $p^{-1}(U) = \bigcup_{n \in Z}(n, n + \frac{1}{4})$. Moreover, for each $n \in \mathbb{Z}$, $p\big|_{[n, n+\frac{1}{4}]}$ is a one-to-one function from the closed interval $[n, n + \frac{1}{4}]$ onto $\overline{U}$. Thus, since $[n, n + \frac{1}{4}]$ is compact, $p\big|_{[n, n+\frac{1}{4}]}$ is a homeomorphism of $[n, n + \frac{1}{4}]$ onto $\overline{U}$. Therefore $p\big|_{(n, n+\frac{1}{4})}$ is a homeomorphism of $(n, n + \frac{1}{4})$ onto U. Throughout the remainder of this section, p denotes the function defined above.

The following result is known as the Covering Path Property. You may wish to consider Example 5 before reading the proof.

THEOREM 8.15. Let $\alpha: I \to S^1$ be a path and let $x_0 \in \mathbb{R}$ such that $p(x_0) = \alpha(0)$. Then there is a unique path $\beta: I \to \mathbb{R}$ such that $\beta(0) = x_0$ and $p \circ \beta = \alpha$.

Proof. Since α is continuous, for each $t \in I$ there is a connected neighborhood U_t of t such that $\alpha(U_t)$ is a proper subset of S^1. Since I is compact, the open cover $\{U_t: t \in I\}$ of I has a finite subcover $\{U_1, U_2, \ldots, U_m\}$. If $U_i = I$ for some i, then $\alpha(I)$ is a connected proper subset of S^1. If $U_i \neq I$ for any i, choose U_i so that $0 \in U_i$. There exists $t' \in I$ such that $t' \neq 0$, $[0, t'] \subseteq U_i$, and $t' \in U_i \cap U_j$ for some $j \neq i$. If $U_i \cup U_j = I$, $\alpha([0, t'])$ and $\alpha([t', 1])$ are connected proper subsets of S^1. If $U_i \cup U_j \neq I$, choose $t'' \in I$ such that $t'' > t'$, $[t', t''] \subseteq U_j$ and $t'' \in U_j \cap U_\kappa$ for some k $(i \neq k \neq j)$. Since $\{U_1, U_2, \ldots, U_m\}$ is a finite cover of I, after a finite number of steps, we will obtain $t_0, t_1, t_2, \ldots, t_n$ such that $0 = t_0$, $t_n = 1$, $t_{i-1} < t_i$ and $\alpha([t_{i-1}, t_i])$ is a connected proper subset of S^1 for each $i = 1, 2, \ldots, n$.

For each $i = 0, 1, 2, \ldots, n$, let P_i be the statement: There is a unique continuous function $\beta_i: [0, t_i] \to \mathbb{R}$ such that $\beta_i(0) = x_0$ and $p \circ \beta_i = \alpha\big|_{[0, t_i]}$. In order to prove the theorem, it is sufficient to show that P_i is true for each i. It is clear that P_0 is true. Suppose $0 < i \leq n$ and P_{i-1} is true. Let V_{i-1} denote the component of $p^{-1}(\alpha[t_{i-1}, t_i]))$ that contains $\beta_{i-1}(t_{i-1})$, and let $p_{i-1} = p\big|_{V_{i-1}}$. Then $p_{i-1}: V_{i-1} \to \alpha([t_{i-1}, t_i])$ is a homeomorphism. Define $\beta_i: [0, t_i] \to \mathbb{R}$ by

$$\beta_i(t) = \begin{cases} \beta_{i-1}(t), & \text{if } 0 \leq t \leq t_{i-1} \\ (p_{i-1})^{-1}(\alpha(t)), & \text{if } t_{i-1} \leq t \leq t_i. \end{cases}$$

Then β_i is continuous, $\beta_i(0) = x_0$, and $p \circ \beta_i = \alpha\big|_{[0, t_i]}$. We must show that β_i is unique. Suppose β_i': $[0, t_i] \to \mathbb{R}$ is a continuous function such that $\beta_i'(0) = x_0$ and $p \circ \beta_i' = \alpha\big|_{[0, t_i]}$. If $0 \leq t \leq t_{i-1}$, then $\beta_i'(t) = \beta_i(t)$ since P_{i-1} is true. Suppose $t_{i-1} < t \leq t_i$. Since $(p \circ \beta_i')\,(t) = \alpha(t) \in \alpha([t_{i-1}, t_i])$, $\beta_i'(t) \in p^{-1}(\alpha([t_{i-1}, t_i]))$. Therefore $\beta_i'(t) \in V_{i-1}$ since β_i' is continuous and V_{i-1} is a component of $p^{-1}(\alpha([t_{i-1}, t_i]))$. Therefore $\beta_i(t), \beta_i'(t) \in V_{i-1}$ and $(p_{i-1} \circ \beta_i)(t) = (p_{i-1} \circ \beta_i')(t)$. Hence $\beta_i(t) = \beta_i'(t)$ since p_{i-1} is a homeomorphism. ■

EXAMPLE 5. Let $\alpha: I \to S^1$ be a homeomorphism that maps I onto $\overline{\{(x, y) \in S^1: x > 0 \text{ and } y > 0\}}$ and has the property that $\alpha(0) = (1, 0)$ and $\alpha(1) = (0, 1)$ and let $x_0 = 2$. Then the function $\beta: I \to \mathbb{R}$ given by Theorem 8.15 is a homeomorphism of I onto $[2, 2.25]$, which has the property that $\beta(0) = 2$ and $\beta(1) = 2.25$.

The following result is known as the Covering Homotopy Property.

THEOREM 8.16. Let $(X, \mathscr{T})$ be a topological space, let $f : X \to \mathbb{R}$ be a continuous function, and let $H: X \times I \to S^1$ be a continuous function such that $H(x, 0) = (p \circ f)(x)$ for each $x \in X$. Then there is a continuous function $F: X \times I \to \mathbb{R}$ such that $F(x, 0) = f(x)$ and $(p \circ F)(x, t) = H(x, t)$ for each $(x, t) \in X \times I$.

Proof. For each $x \in X$, define $\alpha_x: I \to S^1$ by $\alpha_x(t) = H(x, t)$. Now $(p \circ f)(x) = \alpha_x(0)$, and hence, by Theorem 8.15, there is a unique path $\beta_x: I \to \mathbb{R}$ such that $\beta_x(0) = f(x)$ and $p \circ \beta_x = \alpha_x$. Define $F: X \times I \to R$ by $F(x, t) = \beta_x(t)$. Then $(p \circ F)(x, t) = (p \circ \beta_x)(t) = H(x, t)$ and $F(x, 0) = \beta_x(0) = f(x)$.

We must show that F is continuous. Let $x_0 \in X$. For each $t \in I$, there is a neighborhood M_t of x_0 and a connected neighborhood N_t of t such that $H(M_t \times N_t)$ is contained in a proper connected subset of S^1. Since I is compact, the open cover $\{N_t: t \in I\}$ of I has a finite subcover $\{N_{t_1}, N_{t_2}..., N_{t_n}\}$. Let $M = \bigcap_{i=1}^{n} M_{t_i}$. Then M is a neighborhood of x_0, and, for each $i = 1, 2, \ldots.., n$, $H(M \times N_{t_i})$ is contained in a proper connected subset of S^1. Thus there exist $t_0, t_1, \ldots, t_m$ such that $t_0 = 0$, $t_m = 1$, and, for each $j = 1, 2, \ldots, m$, $t_{j-1} < t_j$ and $H(M \times [t_{j-1}, t_j])$ is contained in a proper connected subset of S^1.

For each $j = 0, 1, \ldots, m$, let P_j be the statement: There is a unique continuous function $G_j : M \times [0, t_j] \to \mathbb{R}$ such that $G_j(x, 0) = f(x)$ and $(p \circ G_j)(x, t) = H(x, t)$. We want to show that P_j is true for each j. It is clear that P_0 is true. Suppose $0 < j \leq m$ and P_{j-1} is true. Let U_{j-1} be a connected proper subset of S^1 that contains $H(M \times I_{j-1})$, let V_{j-1} denote the component of $p^{-1}(U_{j-1})$ that contains $G_{j-1}(M \times \{t_{j-1}\})$, and let $p_{j-1} = p|_{V_{j-1}}$. Then $p_{j-1}: V_{j-1} \to U_{j-1}$ is a homeomorphism. Define $G_j: M \times [0, t_j] \to \mathbb{R}$ by

$$G_j(x, t) = \begin{cases} G_{j-1}(x, t), & \text{if } 0 \leq t \leq t_{j-1} \\ (p_{j-1})^{-1}\,(H(x, t)), & \text{if } t_{j-1} \leq t \leq t_j. \end{cases}$$

Then G_j is continuous, $G_j(x, 0) = f(x)$, and $(p \circ G_j)(x, t) = H(x, t)$. We must show that G_j is unique. Suppose G_j': $M \times [0, t_j] \to \mathbb{R}$ is a continuous function such that $G_j'(x, 0) = f(x)$ and $(p \circ G_j')(x, t) = H(x, t)$. If $0 \leq t \leq t_{j-1}$, then $G_j'(x, t) = G_j(x, t)$ because P_{j-1} is true. Suppose $t_{j-1} < t \leq t_j$. Since $(p \circ G_j)(x, t) = H(x, t) \in U_{j-1}$, $G_j'(x, t) \in p^{-1}(U_{j-1})$. Therefore, since G_j' is continuous and V_{j-1} is a component of $p^{-1}(U_{j-1})$, $G_j'(x, t) \in V_{j-1}$. Hence $G_j(x, t)$ and $G_j'(x, t)$ are members of V_{j-1} and $(p_{j-1} \circ G_j)(x, t) = (p_{j-1} \circ G_j')(x, t)$. Thus, since p_{j-1} is a homeomorphism, $G_j(x, t) = G_j'(x, t)$. We have proved that there is a unique continuous function $G: M \times I \to \mathbb{R}$ such that $G(x, 0) = f(x)$ and $(p \circ G)(x, t) = H(x, t)$. Therefore $G = F|_{(M \times I)}$. Since M is a neighborhood of x_0 in X, F is continuous at (x_0, t) for each

$t \in I$. Since x_0 is an arbitrary member of X, F is continuous. This proof also shows that F is unique. ■

If α is a loop in S^1 at $(1, 0)$, then $p(0) = \alpha(0)$, and hence, by Theorem 8.15, there is a unique path $\beta: I \to \mathbb{R}^1$ such that $\beta(0) = 0$ and $p \circ \beta = \alpha$. Since $(p \circ \beta)(1) = \alpha(1) = (1, 0)$, $\beta(1) \in p^{-1}(1, 0)$, and hence $\beta(1)$ is an integer. The integer $\beta(1)$ is called the **degree** of the loop α, and we write $\deg(\alpha) = \beta(1)$.

EXAMPLE 6. Define $\alpha: I \to S^1$ by $\alpha(x) = (\cos 4\pi x, -\sin 4\pi x)$. Then α "wraps" I around S^1 twice in a clockwise direction. The unique path $\beta: I \to \mathbb{R}$ such that $\beta(0) = 0$ and $p \circ \beta = \alpha$ given by Theorem 8.15 is defined by $\beta(x) = -2x$ for each $x \in I$. Therefore $\deg(\alpha) = \beta(1) = -2$.

THEOREM 8.17. Let α_1 and α_2 be loops in S^1 at $(1, 0)$ such that $\alpha_1 \simeq_p \alpha_2$. Then $\deg(\alpha_1) = \deg(\alpha_2)$.

Proof. Since $\alpha_1 \simeq_p \alpha_2$, there is a continuous function $H: I \times I \to S^1$ such that $H(x, 0) = \alpha_1(x)$ and $H(x, 1) = \alpha_2(x)$ for each $x \in I$ and $H(0, t) = H(1, t) = (1, 0)$ for each $t \in I$. By Theorem 8.15, there is a unique path $\beta_1: I \to \mathbb{R}$ such that $\beta_1(0) = 0$ and $(p \circ \beta_1) = \alpha_1$. Thus, by Theorem 8.16, there is a unique continuous funtion $F: \mathbf{I} \times \mathbf{I} \to \mathbb{R}$ such that $(p \circ F)(x, t) = H(x, t)$ for each $(x, t) \in I \times I$ and $F(x, 0) = \beta_1(x)$ for each $x \in I$. Define $\gamma: I \to \mathbb{R}$ by $\gamma(t) = F(0, t)$ for each $t \in I$. Then γ is continuous, and $(p \circ \gamma)(t) = (p \circ F)(0, t) = H(0, t) = (1, 0)$ for each $t \in I$. Therefore $\gamma(I) \subseteq p^{-1}(1, 0)$, and, since $\gamma(I)$ is connected and $\rho^{-1}(1, 0)$ is a discrete subspace of $\mathbb{R}$, γ is a constant function. Thus $F(0, t) = \gamma(t) = \gamma(0) = F(0, 0) = \beta_1(0) = 0$ for each $t \in I$. Define $\beta_2: I \to \mathbb{R}$ by $\beta_2(x) = F(x, 1)$ for each $x \in I$. Then $\beta_2(0) = F(0, 1) = 0$ and $(p \circ \beta_2)(x) = (p \circ F)(x, 1) = H(x, 1) = \alpha_2(x)$ for each $x \in I$. By definition, $\deg(\alpha_1) = \beta_1(1)$ and $\deg(\alpha_2) = \beta_2(1)$. Now define a path $\delta: I \to \mathbb{R}$ by $\delta(t) = F(1, t)$ for each $t \in I$. Again $(p \circ \delta)(t) = (p \circ F)(1, t) = H(1, t) = (1, 0)$ for each $t \in I$, and hence $\delta(I) \subseteq p^{-1}(1, 0)$. Therefore δ is a constant function, and hence $F(1, t) = \delta(t) = \delta(0) = F(1, 0) = \beta_1(0) = 0$ for each $t \in I$. Therefore $\beta_2(1) = F(1, 1) = F(1, 0) = \beta_1(1)$, and hence $\deg(\alpha_1) = \deg(\alpha_2)$. ■

We are ready to prove that the fundamental group of the circle is isomorphic to the group of integers. Since S^1 is pathwise connected, the fundamental group of S^1 is independent of the base point.

THEOREM 8.18. $\pi_1(S^1, (1, 0))$ is isomorphic to the group of integers.

Proof. Define $\phi: \pi_1(S^1, (1, 0)) \to \mathbb{Z}$ by $\phi([\alpha]) = \deg(\alpha)$. By Theorem 8.17, ϕ is well-defined. Let $[\alpha_1], [\alpha_2] \in \pi_1(S^1, (1, 0))$. By Theorem 8.15, there are unique paths $\beta_1, \beta_2: I \to \mathbb{R}$ such that $\beta_1(0) = \beta_2(0) = 0$, $p \circ \beta_1 = \alpha_1$, and $p \circ \beta_2 = \alpha_2$. By definition, $\deg(\alpha_1) = \beta_1(1)$ and $\deg(\alpha_2) = \beta_2(1)$. Define $\delta: I \to \mathbb{R}$ by

$$\delta(x) = \begin{cases} \beta_1(2x), & \text{if } 0 \le x \le \frac{1}{2} \\ \beta_1(1) + \beta_2(2x-1), & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Since $\beta_2(0) = 0$, δ is continuous. Now $\delta(0) = \beta_1(0) = 0$, and, since

$$p(\beta_1(1) + \beta_2(2x-1)) = p(\beta_2(2x-1)),$$
$$(p \circ \delta)(x) = \begin{cases}(p \circ \beta_1)(2x), & \text{if } 0 \leq x \leq \frac{1}{2}\\ (p \circ \beta_2)(2x - 1), & \text{if } \frac{1}{2} \leq x \leq 1\end{cases}$$
$$= \begin{cases}\alpha_1(2x), & \text{if } 0 \leq x \leq \frac{1}{2}\\ \alpha_2(2x - 1), & \text{if } \frac{1}{2} \leq x \leq 1\end{cases} = (\alpha_1 * \alpha_2)(x).$$

Therefore $\phi([\alpha_1] \circ [\alpha_2]) = \phi([\alpha_1 * \alpha_2]) = \deg(\alpha_1 * \alpha_2) = \delta(1) = \beta_1(1) + \beta_2(1) = \deg(\alpha_1) + \deg(\alpha_2) = \phi([\alpha_1]) + \phi([\alpha_2])$, and so ϕ is a homomorphism.

Now we show that ϕ maps $\pi_1(S^1, (1, 0))$ onto $\mathbb{Z}$. Let $z \in \mathbb{Z}$, and define a path $\alpha_1: I \to \mathbb{R}$ by $\alpha(t) = zt$ for each $t \in I$. Then $\alpha(0) = 0$ and $\alpha(1) = z$, so $p \circ \alpha: I \to S^1$ is a loop in S^1 at $(1, 0)$. Therefore $[p \circ \alpha] \in \pi_1(S^1, (1, 0))$, and, by definition, $\deg(p \circ \alpha) = \alpha(1) = z$. Therefore $\phi([p \circ \alpha]) = \deg(p \circ \alpha) = z$.

Finally, we show that ϕ is one-to-one. Let $[\alpha_1], [\alpha_2] \in \pi_1(S^1, (1, 0))$ such that $\phi([\alpha_1]) = \phi([\alpha_2])$. Then $\deg(\alpha_1) = \deg(\alpha_2)$. By Theorem 8.15, there are unique paths $\beta_1, \beta_2: I \to \mathbb{R}$ such that $\beta_1(0) = \beta_2(0) = 0$, $p \circ \beta_1 = \alpha_1$, and $p \circ \beta_2 = \alpha_2$. By definition, $\deg(\alpha_1) = \beta_1(1)$ and $\deg(\alpha_2) = \beta_2(1)$. So $\beta_1(1) = \beta_2(1)$. Define $F: I \times I \to \mathbb{R}$ by $F(x, t) = (1 - t)\beta_1(x) + t\beta_2(x)$ for each $(x, t) \in I \times I$. Then $F(x, 0) = \beta_1(x)$ and $F(x, 1) = \beta_2(x)$ for each $x \in I$ and $F(0, t) = 0$ and $F(1, t) = \beta_1(1) = \beta_2(1)$ for each $t \in I$. Therefore $p \circ F: I \times I \to S^1$ is a continuous function such that $(p \circ F)(x, 0) = (p \circ \beta_1)(x) = \alpha_1(x)$ and $(p \circ F)(x, 1) = (p \circ \beta_2)(x) = \alpha_2(x)$ for each $x \in I$ and $(p \circ F)(0, t) = p(0) = (1, 0)$ and $(p \circ F)(1, t) = (p \circ \beta_1)(1) = \alpha_1(1) = (1, 0)$ for each $t \in I$. Thus $\alpha_1 \simeq_p \alpha_2$, so $[\alpha_1] = [\alpha_2]$. ■

There are no exercises in this section. The sole purpose of the section is to calculate the fundamental group of a familiar figure and thus provide you with a concrete example. In the next section, we generalize some of the theorems in this section and give some exercises.

8.4 Covering Spaces

In this section we generalize Theorems 8.15 and 8.16 by replacing the function $p: \mathbb{R} \to S^1$ defined by $p(x) = (\cos 2\pi x, \sin 2\pi x)$ with a function called a "covering map" from an arbitrary topological space $(E, \mathscr{T})$ into another topological space $(B, \mathscr{U})$. These two generalized theorems are used in Section 9.4.

Definition. Let $(E, \mathscr{T})$ and $(B, \mathscr{U})$ be topological spaces, and let $p: E \to B$ be a continuous surjection. An open subset U of B is **evenly covered** by p if $p^{-1}(U)$ can be written as the union of a pairwise disjoint collection $\{V_\alpha: \alpha \in \Lambda\}$ of open sets such that for each $\alpha \in \Lambda$, $p|_{V_\alpha}$ is a homeomorphism of V_α onto U. Each V_α is called a **slice** of $p^{-1}(U)$. ■

Definition. Let $(E, \mathscr{T})$ and $(B, \mathscr{U})$ be topological spaces, and let $p: E \to B$ be a continuous surjection. If each member of B has a neighborhood that is evenly covered by p, then p is a **covering map** and E is **covering space** of B. ■

Note that the function $p: \mathbb{R} \to S^1$ in Section 8.3 is a covering map. Covering spaces were introduced by Poincaré in 1883.

Definition. Let $(E, \mathscr{T})$, $(B, \mathscr{U})$, and $(X, \mathscr{V})$ be topological spaces, let $p: E \to B$ be a covering map, and let $f: X \to B$ be a continuous function. A **lifting** of f is a continuous function $f': X \to E$ such that $p \circ f' = f$ (see Figure 8.8). ■

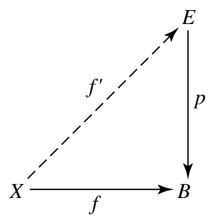


Figure 8.8

Notice that Theorem 8.15 provides a lifting of a path in S^1, where p is the specific function discussed in Section 8.3 rather than an arbitrary covering map.

THEOREM 8.19. Let $(E, \mathscr{T})$ and $(B, \mathscr{U})$ be topological spaces, let $p: E \to B$ be a covering map, let $e_0 \in E$, let $b_0 = p(e_0)$, and let $\sigma: I \to B$ be a path such that $\sigma(0) = b_0$. Then there exists a unique lifting $\sigma': I \to E$ of f such that $\sigma'(0) = e_0$.

Proof. Let $\{U_\alpha: \alpha \in \Lambda\}$ be an open cover of B such that for each $\alpha \in \Lambda$, U_α is evenly covered by p. By Theorem 4.4, there exists $t_0, t_1, \ldots, t_n$ such that $0 = t_0 < t_1 < \cdots < t_n = 1$ and for each $i = 1, 2, \ldots, n$, $\sigma([t_{i-1}, t_i]) \subseteq U_\alpha$ for some $\alpha \in \Lambda$. We define the lifting σ' inductively.

Define $\sigma'(0) = e_0$, and assume that $\sigma'(t)$ has been defined for all $t \in [0, t_{i-1}]$, where $1 \le i \le n$. There exists $\alpha \in \Lambda$ such that $\sigma([t_{i-1}, t_i]) \subseteq U_\alpha$. Let $\mathscr{V} = \{V_{\alpha\beta}: \beta \in \Gamma\}$ be the collection of slices of $p^{-1}(U_\alpha)$. Now $\sigma'(t_{i-1})$ is a member of exactly one member $V_{\alpha\gamma}$ of $\mathscr{V}$. For $t \in [t_{i-1}, t_i]$, define $\sigma'(t) = (p|_{V_{\alpha\gamma}})^{-1}(\sigma(t))$. Since $p|_{V_{\alpha\gamma}}: V_{\alpha\gamma} \to U_\alpha$ is a homomorphism, σ' is continuous on $[t_{i-1}, t_i]$ and hence on $[0, t_i]$. Therefore, by induction, we can define σ' on I.

It follows immediately from the definition of σ' that $p \circ \sigma' = \sigma$.

The uniqueness of σ' is also proved inductively. Suppose σ'' is another lifting of σ such that $\sigma''(0) = e_0$, and assume that $\sigma''(t) = \sigma'(t)$ for all $t \in [0, t_{i-1}]$, where $1 \le i \le n$. Let U_α be a member of the open cover of B such that $\sigma([t_{i-1}, t_i]) \subseteq U_\alpha$, and let $V_{\alpha\gamma}$ be the member of $\mathscr{V}$ chosen in the definition of σ'. Since σ'' is a lifting of σ, $\sigma''([t_{i-1}, t_i]) \subseteq p^{-1}(U_\alpha) = \bigcup_{\beta \in \Gamma} V_{\alpha\beta}$. Since $\mathscr{V}$ is a collection of pairwise disjoint open sets and $\sigma''([t_{i-1}, t_i])$ is connected, $\sigma''([t_{i-1}, t_i])$ is a subset of one member of $\mathscr{V}$.

Since $\sigma''(t_{i-1}) = \sigma'(t_{i-1}) \in V_{\alpha\gamma}$, $\sigma''([t_{i-1}, t_i]) \subseteq V_{\alpha\gamma}$. Therefore, for each $t \in [t_{i-1}, t_i]$, $\sigma''(t) \in V_{\alpha\gamma} \cap p^{-1}(\sigma(t))$. But $V_{\alpha\gamma} \cap p^{-1}(\sigma(t)) = \{\sigma'(t)\}$, and so $\sigma''(t) = \sigma'(t)$. Therefore $\sigma''(t) = \sigma'(t)$ for all $t \in [0, t_i]$. By induction, $\sigma'' = \sigma'$. ■

THEOREM 8.20. Let $(E, \mathscr{T})$ and $(B, \mathscr{U})$ be topological spaces, let $p: E \to B$ be a covering map, let $e_0 \in E$, let $b_0 = p(e_0)$, and let $F: I \times I \to B$ be a continuous function such that $F(0, 0) = b_0$. Then there exists a lifting $F': I \times I \to E$ of F Such that $F'(0, 0) = e_0$. Moreover, if F is a path homotopy then F' is also.

Proof. Define $F'(0, 0) = e_0$. By Theorem 8.19, there exists a unique lifting $F': \{0\} \times I \to E$ of $F|_{\{0\}\times I}$ such that $F'(0, 0) = e_0$ and a unique lifting $F': I \times \{0\} \to E$ of $F|_{I\times\{0\}}$ such that $F(0, 0) = e_0$. Therefore, we assume that a lifting F' of F is defined on $(\{0\} \times I) \cup (I \times \{0\})$ and we extend it to $I \times I$.

Let $\{U_\alpha: \alpha \in \Lambda\}$ be an open cover of B such that for each $\alpha \in \Lambda$, U_α is evenly covered by p. By Theorem 4.4, there exists $s_0, s_1, \ldots, s_m$ and $t_0, t_1, \ldots, t_n$ such that $0 = s_0 < s_1 < \ldots < s_m = 1$, $0 = t_0 < t_1 < \ldots < t_n = 1$, and for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, $F([s_{i-1}, s_i] \times [t_{j-1}, t_j]) \subseteq U_\alpha$ for some $\alpha \in \Lambda$. For each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$, let $U_i \times J_j = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$. We define F' inductively on the rectangles $I_i \times J_j$ in the following order:

$$I_1 \times J_1, I_2 \times J_1, \ldots, I_m \times J_1, I_1 \times J_2, I_2 \times J_2, \ldots, I_m \times J_2, I_1 \times J_3, \ldots, I_m \times J_n.$$

$I_1 \times J_n$	$I_2 \times J_n$	$\cdots$	$I_m \times J_n$
$\vdots$	$\vdots$		$\vdots$
$I_1 \times J_2$	$I_2 \times J_2$	$\cdots$	$I_\mathrm{m} \times J_2$
$I_1 \times J_1$	$I_2 \times J_2$	$\cdots$	$I_\mathrm{m} \times J_1$

Suppose $1 \le p \le m$, $1 \le q \le n$, and assume that F' is defined on $\mathrm{C} = (\{0\} \times I) \cup$

$$(I \times \{0\}) \cup \bigcup_{i=1}^{m} \bigcup_{j=1}^{q-1} (I_i \times J_j) \cup \bigcup_{i-1}^{p-1} (I_i \times J_q). \text{ We define } F' \text{ on } I_p \times J_q.$$

There exists $\alpha \in \Lambda$ such that $F(I_p \times J_q) \subseteq U_\alpha$. Let $\mathscr{V} = \{V_{\alpha\beta}: \beta \in \Lambda\}$ be the collection of slices of $p^{-1}(U_\alpha)$. Now F' is already defined on $D = \mathrm{C} \cap (I_p \times J_q)$. Since D is connected, $F'(D)$ is connected. Since $\mathscr{V}$ is a collection of pairwise disjoint open sets, $F'(D)$ is a subset of one member $V_{\alpha\gamma}$ of $\mathscr{V}$. Now $p|_{V_{\alpha\gamma}}$ is a homeomorphism of $V_{\alpha\gamma}$ onto U_α. Since F' is a lifting of $F|_{\mathrm{C}}$, $((p|_{V_{\alpha\gamma}}) \circ F')(x) = F(x)$ for all $x \in D$. For $x \in I_p \times I_q$, define $F'(x) = (p|_{V_{\alpha\gamma}})^{-1}(F(x))$. Then F' is a lifting of $F|_{(\mathrm{C}\cup(I_p\times J_q))}$. Therefore, by induction we can define F' on $I \times I$.

Now suppose F is a path homotopy. Then $F(0, t) = b_0$ for all $t \in I$. Since F' is a lifting of F, $F'(0, t) \in p^{-1}(b_0)$ for all $t \in I$. Since $\{0\} \times I$ is connected, F' is continuous, and $p^{-1}(b_0)$ has the discrete topology as a subspace of E, $F'(\{0\} \times I)$ is connected and hence it must be a single point. Likewise, there exists $b_1 \in B$ such that $F(1, t) = b_1$ for all $t \in I$, so $F'(1, t) \in p^{-1}(b_1)$ for all $t \in I$, and hence $F'(\{1\} \times I)$ must be a single point. Therefore F' is a path homotopy. ■

THEOREM 8.21. Let $(E, \mathscr{T})$ and $(B, \mathscr{U})$ be topological spaces, let $p: E \to B$ be a covering map, let $e_0 \in E$, let $b_0 = p(e_0)$, let $b_1 \in B$, let α and β be paths in B from b_0 to b_1 that are path homotopic, and let α' and β' be liftings of α and β respectively such that $\alpha'(0) = \beta'(0) = e_0$. Then $\alpha'(1) = \beta'(1)$ and $\alpha' \simeq_p \beta'$.

Proof. Let $F: I \times I \to B$ be a continuous function such that $F(x, 0) = \alpha(x)$ and $F(x, 1) = \beta(x)$ for all $x \in I$ and $F(0, t) = b_0$ and $F(1, t) = b_1$ for all $t \in I$. By Theorem 8.20, there exists a lifting $F': I \times I \to E$ of F such that $F'(0, t) = e_0$ for all $t \in I$ and $F'(\{1\} \times I)$ is a set consisting of a single point, say e_1. The continuous function $F'|_{(I\times\{0\})}$ is a lifting of $F|_{(I\times\{0\})}$ such that $F'(0, 0) = e_0$. Since the lifting of paths is unique (Theorem 8.19), $F'(x, 0) = \alpha'(x)$ for all $x \in I$. Likewise, $F'|_{(I\times\{1\})}$ is a lifting of $F|_{(I\times\{1\})}$ such that $F'(0, 1) = e_0$. Again by Theorem 8.19, $F'(x, 1) = \beta'(x)$ for all $x \in I$. Therefore $\alpha'(1) = \beta'(1) = e_1$ and $\alpha' \simeq_p \beta'$. ■

EXERCISES 8.4

1. Let $(E, \mathscr{T})$ and $(B, \mathscr{U})$ be topological spaces, and let $p: E \to B$ be a covering map. Show that p is open.
2. Let $(E_1, \mathscr{T}_1)$, $(E_2, \mathscr{T}_2)$, $(B_1, \mathscr{U}_1)$, and $(B_2, \mathscr{U}_2)$ be topological spaces, and let $p_1: E_1 \to B_1$ and $p_2: E_2 \to B_2$ be covering maps. Prove that the function $p: E_1 \times E_2 \to B_1 \times B_2$ defined by $p(x, y) = (p_1(x), p_2(y))$ is a covering map.
3. In this exercise, let p_1 denote the function p that is used throughout Section 8.3, and define $p: \mathbb{R} \times \mathbb{R} \to S^1 \times S^1$ by $p(x, y) = (p_1(x), p_1(y))$. By Exercise 2, p is a covering map. Since the torus is homeomorphic to $S^1 \times S^1$, $\mathbb{R} \times \mathbb{R}$ is a covering space of the torus. Draw pictures in $\mathbb{R} \times \mathbb{R}$ and describe verbally the covering map p.
4. Let $(X, \mathscr{T})$ be a topogical space, let Y be a set, let $\mathscr{U}$ be the discrete topology on Y, and let $\pi_1: X \times Y \to X$ be the projection map. Prove that π_1 is a covering map.
5. Let $(E, \mathscr{T})$ be a topological space, let $(B, \mathscr{U})$ be a connected space, let $n \in \mathbb{N}$, and let $p: E \to B$ be a covering map such that for some $b_0 \in B$, $p^{-1}(b_0)$ is a set with n members. Prove that for each $b \in B$, $p^{-1}(B)$ is a set with n members.
6. Let $(E, \mathscr{T})$ be a topological space, let $(B, \mathscr{U})$ be a locally connected, connected space, let $p: E \to B$ be a covering map, and let C be a component of E. Prove that $p|_C: C \to B$ is a covering map.

7. Let $(E, \mathscr{T})$ be a pathwise connected space, let $(B, \mathscr{U})$ be a topological space, let $p: E \to B$ be a covering map, and let $b \in B$.
   (a) Prove that there is a surjection $\phi: \pi_1(B, b) \to p^{-1}(b)$.
   (b) Prove that if $(E, \mathscr{T})$ is simply connected, then ϕ is a bijection.
8. The projective plane, which we have previously introduced, can also be defined as the quotient space obtained from S^2 by indentifying each point x of S^2 with its antipodal point $-x$. Let $p: S^2 \to P^2$ denote the natural map that maps each point of S^2 into the equivalence class that contains it. Prove that P^2, defined in this manner, is a surface and that the function p is a covering map.

8.5 Applications and Additional Examples of Fundamental Groups

We know that the fundamental group of a contractible space is the trivial group (see Exercise 8 of Section 8.1 and Exercises 2 and 3 Section 8.2) and that the fundamental group of the circle is isomorphic to the group of integers. In this section, we give some applications and some theorems that are useful in determining the fundamental group of a space. We begin by showing that S^1 is not a retract (see Exercise 5 of Section 8.2) of $B^2 = \{(x, y) \in \mathbb{R} \times \mathbb{R}: x^2 + y^2 \leq 1\}$.

EXAMPLE 7. We show that B^2 is contractible. Define $H: B^2 \times I \to B^2$ by $H((x, y), t) = (1 - t)(x, y)$. Then H is continuous, $H((x, y), 0) = (x, y)$ and $H((0, 0), t) = (0, 0)$ for each $t \in I$.

THEOREM 8.22. S^1 is not a retract of B^2. ■

The diagram in Figure 8.9 is helpful in following the proof.

$$\begin{tikzcd}
 & \pi_1(B^2) \arrow[d, "r_*"] \\
\pi_1(S^1) \arrow[ur, "j_*"] \arrow[r, "\text{id}"] & \pi_1(S^1)
\end{tikzcd}$$

Figure 8.9

Proof. Suppose S^1 is a retract of B^2. Then there is a continuous function $r: B^2 \to S^1$ such that $r(x) = x$ for each $x \in S^1$. Define $j: S^1 \to B^2$ by $j(x) = x$ for each $x \in S^1$. Then $r \circ j: S^1 \to S^1$ is the identity function, and hence $(\pi \circ j)_*: \pi_1(S^1, (1, 0)) \to \pi_1(S^1, (1, 0))$ is the identity homomorphism. By Theorem 8.12, $(rj)_* = r_* j_*$.Thus

r_* maps $\pi_1(B^2, (1, 0))$ onto $\pi_1(S^1, (1, 0))$. This is a contradiction, because $\pi_1(B^2, (1, 0))$ is the trivial group and $\pi_1(S^1, (1, 0))$ is isomorphic to the group of integers. ■

Recall that a topological space $(X, \mathcal{T})$ has the **fixed-point property** if for each continuous function $f\colon X \to X$ there exists $x \in X$ such that $f(x) = x$. The following theorem is due to the Dutch mathematician L.E.J. Brouwer (1881–1966).

THEOREM 8.23. (Brouwer Fixed-Point Theorem). B^2 has the fixed-point property.

Proof. Suppose there is a continuous function $f : B^2 \to B^2$ such that $f(x) \neq x$ for any $x \in B^2$. For each $x \in B^2$, let L_x denote the line segment that begins at $f(x)$ and passes through x, and let $y_x \in L_x \cap S^1 - \{f(x)\}$. (Note that y_x is uniquely determined—see Figure 8.10.)

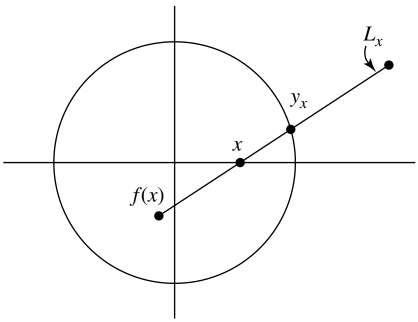


Figure 8.10

It is a straightforward but tedious exercise (see Exercise 4) to show that the function $r : B^2 \to S^1$ defined by $r(x) = y_x$ for each $x \in B^2$ is continuous. Since $r(x) = x$ for each $x \in S^1$, r is a retraction of B^2 onto S^1. This contradicts Theorem 8.22. ■

Now we introduce a property of subspaces that is stronger than retract.

Definition. A subspace A of a topological space $(X, \mathcal{T})$ is a **deformation retract of *X*** provided there is a continuous function $H\colon X \times I \to X$ such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$ and $H(a, t) = a$ for all $a \in A$ and $t \in I$. The homotopy H is called a **deformation retraction.** ■

EXAMPLE 8. Let $X = \{(x, y) \in \mathbb{R} \times \mathbb{R}\colon 1 \leq x^2 + y^2 \leq 4\}$. Define $H\colon X \times I \to X$ by $H(x, t) = (1 - t)x + t \cdot \frac{x}{\|x\|}$ for each $(x, t) \in X \times I$. Then H is continuous, $H(x, 0) = x$ and $H(x, 1) = \frac{x}{\|x\|} \in S^1$ for all $x \in X$, and $H(x, t) = x$ for all $x \in S^1$ and $t \in I$. Therefore H is a deformation retraction, and hence S^1 is a deformation retract of X.

Using Example 8 and Theorem 8.18, the following theorem tells us that the fundamental group of the space X in Example 8 is isomorphic to the group of integers.

THEOREM 8.24. If A is a deformation retract of a topological space $(X, \mathscr{T})$ and $a_o \in A$, then $\pi_1(X, a_0)$ is isomorphic to $\pi_1(A, a_0)$.

Proof. Let $H: X \times I \to X$ be a continuous function such that $H(x, 0) = x$ and $H(x, 1) \in A$ for all $x \in X$ and $H(a, t) = a$ for all $a \in A$ and $t \in I$. Define $h: X \to A$ by $h(x) = H(x, 1)$ for each $x \in X$, and let $h_*: \pi_1(X, a_0) \to \pi_1(A, a_0)$ be the homomorphism, given by Theorem 8.10, induced by h. This means that h_* is defined by $h_*([\alpha]) = [h \circ \alpha]$ for all $[\alpha] \in \pi_1(X, a_0)$.

Now we show that $h*$ is one-to-one. Suppose $[\alpha], [\beta] \in \pi_1(X, a_o)$ and $h_*([\alpha]) = h_*([\beta])$. Then $[h \circ \alpha] = [h \circ \beta]$. Since $A \subseteq X$, $h \circ \alpha$ may be considered as a path in X. We complete the proof that $[\alpha] = [\beta]$ by showing that $\alpha \simeq_p h \circ \alpha$ in X. Define $F: I \times I \to X$ by $F(x, t) = H(\alpha(x), t)$ for all $(x, t) \in I \times I$. Then $F(x, 0) = H(\alpha(x), 0) = \alpha(x)$ and $F(x, 1) = H(\alpha(x), 1) = (h \circ \alpha)(x)$ for all $x \in I$, and $F(0, t) = H(\alpha(0), t) = H(a_0, t) = a_0$ and $F(1, t) = H(\alpha(1), t) = H(a_0, t) = a_0$ for all $t \in I$. Since $[\alpha]$ is an arbitrary member of $\pi_1(X, a_0)$, this also shows that $\beta \simeq_p h \circ \beta$ in X. Therefore, in $\pi_1(X, a_0)$, we have $[\alpha] = [h \circ \alpha] = [h \circ \beta] = [\beta]$, and so h_* is one-to-one.

Now let $[\alpha] \in \pi_1(A, \alpha_0)$. Then $\alpha: I \to A$ and since $A \subseteq X$, we may also consider α to be a path in X. Let $[\alpha]_X$ denote the member of $\pi_1(X, a_0)$ that contains α. Since $h(a) = H(a, 1) = a$ for all $a \in A$, $(h \circ \alpha)(x) = \alpha(x)$ for all $x \in I$. Therefore $h_*([\alpha]_X) = [h \circ \alpha] = [\alpha]$, and so h_* maps $\pi_1(X, x_0)$ onto $\pi_1(A, a_0)$.

Therefore h_* is an isomorphism. ■

By Example 8, S^1 is a deformation retract of $X = \{(x, y) \in \mathbb{R} \times \mathbb{R}: 1 \leq x^2 + y^2 \leq 4\}$. Therefore, by Theorems 8.24 and 8.18, the fundamental group of X is isomorphic to the group of integers.

EXAMPLE 9. Let $p = (p_1, p_2) \in \mathbb{R} \times \mathbb{R}$, and let $S = \{(x, y) \in \mathbb{R} \times R: (x - p_1)^2 + (y - p_2)^2 = 1\}$. In Exercise 1, you are asked to show that S is a deformation retract of $\mathbb{R} \times \mathbb{R} - \{p\}$. Thus it follows that the fundamental group of $\mathbb{R} \times \mathbb{R} - \{p\}$ is isomorphic to the group of integers.

Since the torus is homeomorphic to $S^1 \times S^1$, we can determine the fundamental group of the torus by proving a theorem about the fundamental group of the Cartesian product of two spaces. The definition of the direct product of two groups is given in Appendix I.

THEOREM 8.25. Let $(X, \mathscr{T})$ and $(Y, \mathscr{U})$ be topological spaces and let $x_0 \in X$ and $y_0 \in Y$. Then $\pi_1(X \times Y, (x_0, y_0))$ is isomorphic to $\pi_1(X, x_0) \otimes \pi_1(Y, y_0)$.

Proof. Let $p: X \times Y \to X$ and $q: X \times Y \to Y$ be the projection maps (since π_1 is used as part of the symbol to denote the fundamental group of a space, we now use different symbols to denote the projection maps, but p and q are the functions that we have previously denoted by π_1 and π_2), and let $p_*: \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0)$ and $q_*: \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(Y, y_0)$ denote the induced homeomorphisms. Define $\phi: \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \otimes \pi_1(Y, y_0)$ by $\phi([\alpha]) = ([p \circ \alpha],$

$[q \circ \alpha])$. Since $[p \circ \alpha] = p_*([\alpha])$ and $[q \circ \alpha] = q_*([\alpha])$, by Exercise 12 of Appendix I, ϕ is a homomorphism.

We use Exercise 13 of Appendix I in order to prove that ϕ is one-to-one. Let $[\alpha] \in \pi_1(X \times Y, (x_0, y_0))$ such that $\phi([\alpha])$ is the identity element of $\pi_1(X, x_0) \otimes \pi_1(Y, y_0)$. This means that $[p \circ \alpha]$ is the identity element of $\pi_1(X, x_0)$ and $[q \circ f]$ is the identity element of $\pi_1(Y, y_0)$. Then $p \circ \alpha \simeq_p e_{x_0}$ and $q \circ \alpha \simeq_p e_{y_0}$, where e_{x_0} and e_{y_0} are the constant paths defined by $e_{x_0}(t) = x_0$ and $e_{y_0}(t) = y_0$ for all $t \in I$. Let $F: I \times I \to X$ and $G: I \times I \to Y$ be the homotopies such that $F(x, 0) = (p \circ \alpha)(x)$ and $F(x, 1) = x_0$ for all $x \in I$, $F(0, t) = F(1, t) = x_0$ for all $t \in I$, $G(x, 0) = (q \circ \alpha)(x)$ and $G(x, 1) = y_0$ for all $x \in I$, and $G(0, t) = G(1, t) = y_0$ for all $t \in I$. Define $H: I \times I \to X \times Y$ by $H(x, t) = (F(x, t), G(x, t))$ for all $(x, t) \in I \times I$. Then $H(x, 0) = (F(x, 0), G(x, 0)) = ((p \circ \alpha)(x), (q \circ \alpha)(x)) = \alpha(x)$ and $H(x, 1) = (F(x, 1), G(x, 1)) = (x_0, y_0)$ for all $x \in X$, and $H(0, t) = (F(0, t), G(0, t)) = (x_0, y_0)$ and $H(1, t) = (F(1, t), G(1, t)) = (x_0, y_0)$ for all $t \in I$. Therefore $\alpha \simeq_p e$, where e is the constant path defined by $e(t) = (x_0, y_0)$ for all $t \in I$. Therefore by Exercise 13 of Appendix I, $\varnothing$ is one-to-one.

Now we show that ϕ maps $\pi_1(X \times Y, (x_0, y_0))$ onto $\pi_1(X, x_0) \otimes \pi_1(Y, y_0)$. Let $([\alpha], [\beta]) \in \pi_1(X, x_0) \otimes \pi_1(Y, y_0)$. Define $\gamma: I \to X \times Y$ by $\gamma(t) = (\alpha(t), \beta(t))$ for all $t \in I$. Then $[\gamma] \in \pi_1(X \times Y, (x_0, y_0))$, and $\phi([\gamma]) = ([p \circ \gamma], [q \circ \gamma]) = ([\alpha], [\beta])$, and so ϕ is an isomorphism. ■

If A is a subset of a set X, **the inclusion map** $i: A \to X$ is the function defined by $i(a) = a$ for each $a \in A$. A homomorphism ϕ from a group G into a group H is called the **zero homomorphism** if for each $g \in G$, $\phi(g)$ is the identity element of H. The following theorem is a special case of a famous theorem called the **Van Kampen Theorem.**

THEOREM 8.26. Let $(X, \mathscr{T})$ be a topological space, let U and V be open subsets of X such that $X = U \cup V$ and $U \cap V$ is pathwise connected. Let $x_0 \in U \cap V$, and suppose the inclusion maps $i: U \to X$ and $j: V \to X$ induce zero homomorphisms $i_*: \pi_1(U, x_0) \to \pi_1(X, x_0)$ and $j_*: \pi_1(V, x_0) \to \pi_1(X, x_0)$. Then $\pi_1(X, x_0)$ is the trivial group.

Proof. Let $[\alpha] \in \pi_1(X, x_0)$. By Theorem 4.4, there exist $t_0, t_1, \ldots, t_n$ such that $t_0 = 0$, $t_n = 1$, and, for each $k = 1, 2, \ldots, n$, $t_{k-1} < t_k$ and $\alpha([t_{k-1}, t_k])$ is a subset of one of U or V. Among all such subdivisions $t_0, t_1, \ldots, t_n$ of I, choose one with the smallest number of members. We prove by contradiction that for each $k = 1, 2, \ldots, n-1$, $f(t_k) \in U \cap V$. (Note that if $\alpha(I)$ is a subset of one of U or V, then $n = 1$, and we have nothing to prove.) Suppose $k \in \{1, 2, \ldots, n-1\}$ and $\alpha(t_k) \notin U$. Then neither $\alpha([t_{k-1}, t_k])$ nor $\alpha([t_k, t_{k+1}])$ is a subset of U, so $\alpha([t_{k-1}, t_k]) \cup \alpha([t_k, t_{i+1}]) \subseteq V$. Thus we can discard t_k and obtain a subdivision with a smaller number of members that still have the desired properties. Therefore for each $k = 1, 2, \ldots, n-1$, $\alpha(t_k) \in U$. The same argument shows that $\alpha(t_k) \in V$ for each $k = 1, 2, \ldots, n-1$.

Since $\alpha(t_0) = \alpha(t_n) = x_0 \in U \cap V$, $\alpha(t_k) \in U \cap V$ for each $k = 0, 1, \ldots, n$.

For each $k = 1, 2, \ldots, n$, define $\alpha_k: I \to X$ by $\alpha_k(x) = \alpha((1 - x)t_{k-1} + xt_k)$ for each $x \in I$. We show that α_k is path homotopic (in X) to a path that lies entirely in U.

If $\alpha_k(I) \subseteq U$, define $H_k: I \times I \to X$ by $H_k(x, t) = \alpha_k(x)$ for each $(x, t) \in I \times I$.

Suppose $\alpha_k(I) \nsubseteq U$. Then, since $\alpha_k(I) = \alpha([t_{k-1}, t_k])$, $\alpha_k(I) \subseteq V$. Since $U \cap V$ is pathwise connected, there exist paths β, γ in $U \cap V$ such that $\beta(0) = \gamma(0) = x_0$, $\beta(1) = \alpha_k(0)$, and $\gamma(1) = \alpha_k(1)$(see Figure 8.11).

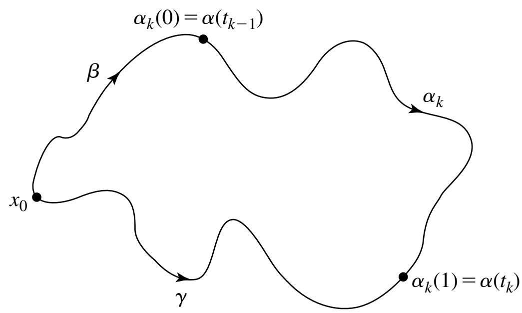


Figure 8.11

Then the path product $(\beta * \alpha_k) * \overline{\gamma}$ (recall that $\overline{\gamma}$ is the path defined by $\overline{\gamma}(x) = \gamma(1 - x)$ for each $x \in I$) is a loop in V at x_0. Since the inclusion map $j: V \to X$ induces the zero homomorphism, the loop $(\beta * \alpha_k * \overline{\gamma})$ is path homotopic in X to the constant loop $e: I \to X$ defined by $e(x) = x_0$ for each $x \in I$. Therefore, by Exercise 2, α_k is path homotopic in X to the path $\overline{\beta} * \gamma$. Let $F_k: I \times I \to X$ be this path homotopy. Then $F_k(x, 0) = \alpha_k(x)$ and $F_k(x, 1) = (\overline{\beta} * \gamma)(x)$ for each $x \in I$ and $F_k(0, t) = \alpha_k(0)$ and $F_k(1, t) = \alpha_k(1)$ for each $t \in I$. Define $F: I \times I \to X$ by $F(x, t) = F_k((x - t_{k-1})/(t_k - t_{k-1}), t)$ for each $x \in [t_{k-1}, t_k]$ and $t \in I$. Since $F_{k-1}(1, t) = \alpha_{k-1}(1) = \alpha_k(0) = F_k(0, t)$ for each $k = 1, 2, \ldots, n$ and $t \in I$, F is well-defined and continuous. Let $x \in I$. Then there exists k such that $x \in [t_{k-1}, t_k]$. So

$$\begin{aligned} F(x, 0) &= F_k((x - t_{k-1})/(t_k - t_{k-1}), 0) = \alpha_k((x - t_{k-1})/(t_k - t_{k-1})) \\ &= \alpha((1 - (x - t_{k-1})/(t_k - t_{k-1}))t_{k-1} + ((x - t_{k-1})/(t_k - t_{k-1})t_k)) \\ &= \alpha(x), \end{aligned}$$

and

$$F(x, 1) = F_k((x - t_{k-1})/(t_k - t_{k-1}), 1) = (\overline{\beta} * \gamma)((x - t_{k-1})/(t_k - t_{k-1})) \in U.$$

Also, $F(0, t) = F_1(0, t) = \alpha_1(0) = \alpha(0)$ and $F(1, t) = F_n(1, t) = \alpha_n(1) = \alpha(1)$ for each $t \in I$. Finally, define a path α' by $\alpha'(x) = F(x, 1)$ for each $x \in I$. Then α' is a path in U and α is path homotopic to α'. Since $i_*: \pi_1(U, x_0) \to \pi_1(X, x_0)$ is the zero homomorphism, α' is path homotopic in X to the constant loop e. Therefore $\alpha \simeq_p e$, and so $\pi_1(X, x_0)$ is the trivial group. ■

As a consequence of Theorem 8.26, we establish the important fact that if $n > 1$, $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1}: \sqrt{\prod_{i=1}^{n+1} x_i^2} = 1\}$ is simply connected. In the proof, we use that if $X = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}: x_{n+1} = 0\}$, then X is homeomorphic to $\mathbb{R}^n$.

THEOREM 8.27. For $n > 1$, S^n is simply connected.

Proof. Let p and q be the members of S^n given by $p = (0, 0, \ldots, 0, 1)$ and $q = (0, 0, \ldots, 0, -1)$, and let $X = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}: x_{n+1} = 0\}$. First we show that $S^n - \{p\}$ is homeomorphic to $\mathbb{R}^n$. Define $f: S^n - \{p\} \to X$ by $f(x_1, x_2, \ldots, x_n, x_{n+1}) = (1/(1 - x_{n+1}))(x_1, x_2, \ldots, x_n, 0)$. (We can describe f geometrically as follows: For each $x \in S^n - \{p\}$, $f(x)$ is the point of intersection of the line determined by p and x with X.) It is clear that f is a one-to-one continuous function that maps $S^n - \{p\}$ onto X. For each $(y_1, y_2, \ldots, y_n, 0) \in X$, let $t = 2/(1 + y_1^2 + \cdots + y_n^2)$, and define g: $X \to S^n - \{p\}$ by $g(y_1, y_2, \ldots, y_n, 0) = (ty_1, ty_2, \ldots, ty_n, 1 - t)$. Again it is clear that g is a one-to-one continuous function. In Exercise 3, you are asked to show that g is an inverse of f. This completes the proof that $S^n - \{p\}$ is homeomorphic to $\mathbb{R}^n$.

Since the function h: $S^n - \{q\} \to S^n - \{p\}$ defined by $h(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n, -x_{n+1})$ is clearly a homeomorphism, $S^n - \{q\}$ is also homeomorphic to $\mathbb{R}^n$.

Now let $U = S^n - \{p\}$ and $V = S^n - \{q\}$.Then U and V are open subsets of S^n and $S^n = U \cup V$. By Example 4, $\mathbb{R}^n$ is simply connected, and therefore the inclusion maps i: $U \to S^n$ and j: $V \to S^n$ induce zero homomorphisms. In order to use Theorem 8.26 to conclude that S^n is simply connected, it is sufficient to show that $U \cap V$ is pathwise connected. Now $U \cap V = S^n - \{p, q\}$, and the homeomorphism f (defined in the first paragraph of this proof) restricted to $S^n - \{p, q\}$ is a homeomorphism of $S^n - \{p, q\}$ onto $X - \{(0, 0, \ldots, 0)\}$, and $X - \{(0, 0, \ldots, 0)\}$ is homeomorphic to $\mathbb{R}^n - \{(0, 0, \ldots, 0)\}$. We show that $\mathbb{R}^n - \{(0, 0, \ldots, 0)\}$ is pathwise connected. Let $x \in \mathbb{R}^n - \{(0, 0, \ldots, 0)\}$. If $x \neq (y, 0, 0, \ldots, 0)$, where $y < 0$, then x and $(1, 0, 0, \ldots, 0)$ can be joined by a straight-line path. If $x = (y, 0, 0, \ldots, 0)$, where $y < 0$, then x and $(0, 1, 0, 0, \ldots, 0)$ can be joined by a straight-line path, and $(0, 1, 0, 0, \ldots, 0)$ and $(1, 0, 0, \ldots, 0)$ can be joined by a straight-line path. In either case, x and $(1, 0, 0, \ldots, 0)$ can be joined by a path. Therefore $\mathbb{R}^n - \{(0, 0, \ldots, 0)\}$ is pathwise connected. ■

In the proof of Theorem 8.27, we showed that if $n > 2$ then $\mathbb{R}^n - \{(0, 0, \ldots, 0)\}$ is simply connected. In Example 9, we showed that $\mathbb{R}^2 - \{(0, 0)\}$ is not simply connected. Therefore by Theorem 8.14, if $n > 2$, then $\mathbb{R}^n - \{(0, 0, \ldots, 0)\}$ is not homeomorphic to $\mathbb{R}^2 - (0, 0)\}$.

EXERCISES 8.5

1. Let $p = (p_1, p_2) \in \mathbb{R} \times \mathbb{R}$, and let $S = \{(x, y) \in \mathbb{R} \times \mathbb{R}: (x - p_1)^2 + (y - p_2)^2 = 1\}$. Prove that S is a deformation retract of $\mathbb{R} \times \mathbb{R} - \{p\}$.

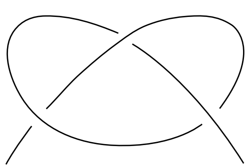
2. Let $(X, \mathscr{T})$ be a topological space, let $x_0 \in X$, and let α, β, and γ be paths in X such that $\beta(0) = x_0$, $\beta(1) = \alpha(0)$, $\gamma(0) = x_0$, and $\gamma(1) = \alpha(1)$, and let $H: I \times I \to X$ be a continuous function such that $H(x, 0) = ((\beta * \alpha) * \overline{\gamma})(x)$ and $H(x, 1) = x_0$ for each $x \in I$ and $H(0, t) = H(1, t) = x_0$ for each $t \in I$. Prove that there is a continuous function $F: I \times I \to X$ such that $F(x, 0) = \alpha(x)$ and $F(x, 1) = (\overline{\beta} * \gamma)(x)$ for each $x \in I$ and $F(0, t) = \alpha(0)$ and $F(1, t) = \alpha(1)$ for each $t \in I$.
3. Let $X = \{(x_1, x_2, \ldots, x_n, x_{n+1}) \in \mathbb{R}^{n+1}: x_{n+1} = 0\}$ and let $p = (0, 0, \ldots, 0, 1) \in S^n$. Define $f: S^n - \{p\} \to X$ by $f(x_1, x_2, \ldots, x_{n+1}) = (1/(1 - x_{n+1}))(x_1, x_2, \ldots, x_n, 0)$, and define $g: X \to S^n - \{p\}$ by $g(y_1, y_2, \ldots, y_n, 0) = (ty_1, ty_2, \ldots, ty_n, 1 - t)$, where $t = 2/(1 + y_1^2 + y_2^2 + \ldots + y_n^2)$. Prove that g is an inverse of f.
4. Show that the function $r: B^2 \to S^1$ defined in the proof of Theorem 8.23 is continuous.

8.6 Knots

We conclude this chapter with a brief discussion of knot theory. No proofs are given in the text. This section is somewhat unique in that it is not rigorous. Instead, it is designed to appeal to the geometric and intuitive side of topology. We refer to the one-point compactification of a topological space, which is defined in Section 6.6. In keeping with the spirit of this section, one can consider the intuitive nature of the one-point compactification rather than approach it from a rigorous viewpoint. A reference is provided at the end of the section for those who wish to continue the study of knot theory.

An invariant of knot theory was first considered by Karl Friedrick Gauss (1777–1855) in 1833. Early work on the subject was done by Max Dehn (1878–1952) in 1910, J.W. Alexander in 1924, E. Artin in 1925, E.R. van Kampen in 1928, and others. The first comprehensive book on knot theory was written by K. Reidemeister in 1932.

The overhand knot and the figure-eight knot (see Figure 8.12) are familiar to almost everyone. A little experimenting with a piece of string should convince you that one of these knots cannot be transformed into the other without untying one of them. One goal of knot theory is to prove mathematically that this cannot be done. Thus we must have a mathematical definition of a knot and of when two knots are considered to be the same. The latter definition must prevent untying. We do this by



Overhand knot

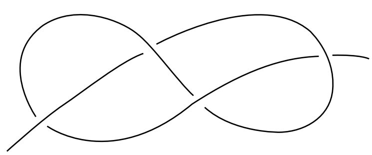
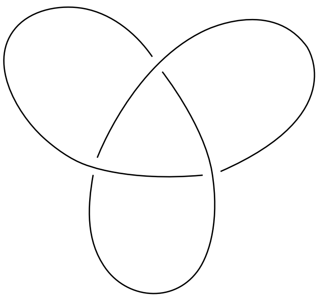


Figure-eight knot

Figure 8.12

getting rid of the ends; that is, we splice the ends together. The overhand knot with the ends spliced together is often called the trefoil or cloverleaf knot. The figure-eight knot with the ends spliced together is often called the four-knot or Listing's knot. These are shown in Figure 8.13.



Trefoil or cloverleaf knot

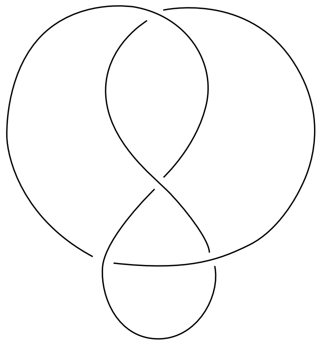
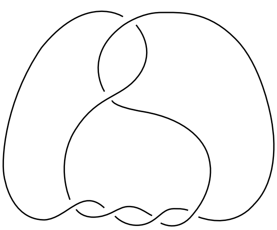


Figure-eight or Listing's knot

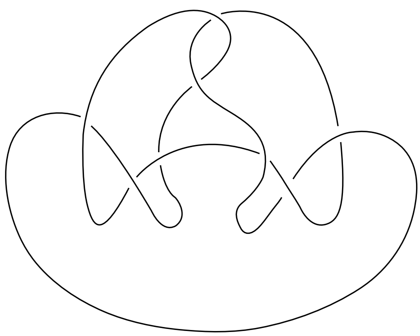
Figure 8.13

Definition. A subset K of $\mathbb{R}^3$ is a **knot** if there is a homeomorphism that maps S^1 onto K.

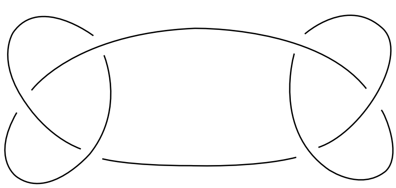
Three additional knots are shown in Figure 8.14.



Stevedore's knot



True lover's knot



Square knot

Figure 8.14

Notice that any two knots are homeomorphic. Thus the question of when two knots are considered to be the same is a characteristic of the way in which the two knots are embedded in $\mathbb{R}^3$. Thus knot theory is a portion of 3-dimensional topology.

Definition. Two knots K_1 and K_2 are **equivalent** if there is a homeomorphism $h\colon \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = K_2$. ■

It is easy to show that knot equivalence is an equivalence relation. Notice that knot equivalence does not say anything about "sliding" (as is the case with a homotopy) K_1 until it lies on top of K_2. The difference in these ideas is illustrated by the fact that a reflection about a plane is a homeomorphism of $\mathbb{R}^3$ onto $\mathbb{R}^3$ that maps a knot into its mirror image, but we cannot "slide" the trefoil knot into its mirror image (see Figure 8.15).

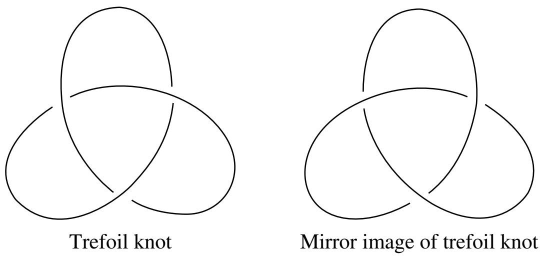


Figure 8.15

Definition. A polygonal knot is a knot that is the union of a finite number of closed straight-line segments called **edges.** The endpoints of the straight-line segments are called **vertices.** A knot is **tame** if it is equivalent to a polygonal knot, and otherwise it is **wild.** ■

An example of a wild knot (one that is obtained by tying an infinite number of knots one after the other) is shown in Figure 8.16.

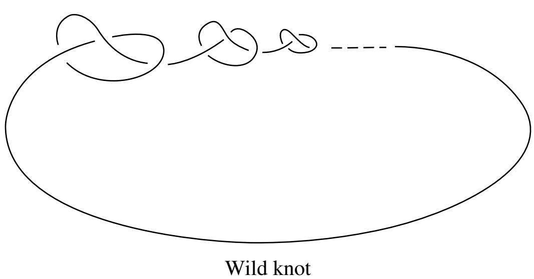


Figure 8.16

In order to picture knots and work with them effectively, we project them into a plane.

Definition. Let K be a knot in $\mathbb{R}^3$ and define $p: \mathbb{R}^3 \to \mathbb{R}^3$ by $p((x, y, z)) = (x, y, 0)$. A point $x \in p(K)$ is called a **multiple point** if $p^{-1}(x) \cap K$ consists of more than one point. The **order** of $x \in p(K)$ is the cardinality of $p^{-1}(x) \cap K$. A **double point** in $p(K)$ is a point of order 2, and a **triple point** in $p(K)$ is a point of order 3. ■

In general, $p(K)$ can contain any number and kinds of multiple points, including multiple points of infinite order. In the remainder of this section, p denotes the projection in the preceding definition.

Definition. A polygonal knot K is in **regular position** if: (1) there are only a finite number of multiple points and they are double points, and (2) no double point is the image of a vertex of K. ■

In Exercise 1, you are asked to give an example of a polygonal knot in regular position.

Definition. Let K be a polygonal knot in regular position. Then each double point in $p(K)$ is the image of two points of K. The one whose z-coordinate is larger is called an **overcrossing** and the one whose z-coordinate is smaller is called an **undercrossing.** ■

In Exercise 2, you are asked to indicate the overcrossings and undercrossings of the knot you have given in Exercise 1.

A rigorous proof of the following theorem is somewhat tedious, and thus we omit it. In Exercise 3 and 4, you are asked to illustrate this theorem.

THEOREM 8.28. If K is a polygonal knot, then there is an arbitrarily small rotation of $\mathbb{R}^3$ onto $\mathbb{R}^3$ that maps K into a polygonal knot in regular position. ■

It is, of course, an immediate consequence of Theorem 8.28 that every polygonal knot is equivalent to a polygonal knot in regular position.

Definition. A homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ is **isotopic to the identity** if there is a homotopy $H: \mathbb{R}^3 \times I \to \mathbb{R}^3$ such that $H(x, 0) = h(x)$ and $H(x, 1) = x$ for each $x \in \mathbb{R}^3$ and, for each $t \in I$, $H|_{(\mathbb{R}^3 \times \{t\})}: \mathbb{R}^3 \times \{t\} \to \mathbb{R}^3$ is a homeomorphism. The homotopy H is called an **isotopy.** ■

Suppose K_1 and K_2 are knots and $h: \mathbb{R}^3 \to \mathbb{R}^3$ is a homeomorphism that is isotopic to the identity under an isotopy H. Moreover, suppose that $h(K_1) = K_2$. For each $t \in I$, let $h_t = H|_{(\mathbb{R}^3 \times \{t\})}$. Then $\{h_t(K_1): t \in I\}$ is a "continuous" family of knots that move from K_2 to K_1 as t moves from 0 to 1. (Obviously, we are not being precise here. Instead we are trying to provide a "picture" of how one knot is moved onto another under an isotopy).

We can extend the definition of a triangulation of a compact surface (given in Section 6.4) to a compact 3-manifold.

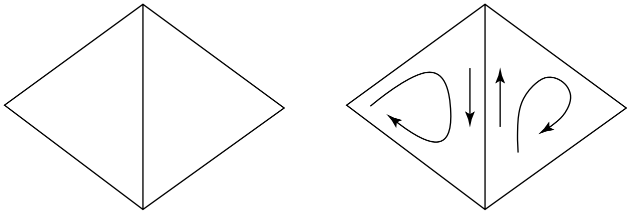
Definition. A **triangulation** of a compact 3-manifold $(X, \mathscr{T})$ consists of a finite collection $\{T_1, T_2, \ldots, T_n\}$ of closed subsets of X that cover X and a collection $\{\phi_i : T_i' \to T_i : i = 1, 2, \ldots, n\}$ of homeomorphisms, where each T_i' is a tetrahedron in the plane. Each T_i is also called a **tetrahedron.** The members of T_i that are images of the vertices of T_i' are called **vertices,** the subsets of T_i that are images of the edges of T_i' are called **edges,** and the subsets of T_i that are images of the 2-faces of T_i' are called **2-faces.** In addition, for each pair T_i and T_j of distinct tetrahedra $T_i \cap T_j = \phi$, $T_i \cap T_j$ is a vertex, $T_i \cap T_j$ is an edge, or $T_i \cap T_j$ is a 2-face. ■

This is not the usual definition of a triangulation of a manifold, but for our purposes this definition will suffice.

Let T be an edge, a triangle, or a tetrahedron. Consider two orderings of the vertices of T to be **equivalent** if they differ by an even permutation. Then there are exactly two equivalence classes, each of which is called an orientation of T. If T is a triangle or a tetrahedron that has been assigned an orientation by ordering its vertices in some way, say $v_0, v_1, \ldots, v_k$, and S is the face of T obtained by deleting v_i, then the vertices of S are automatically ordered. If i is even, the orientation of S given by this ordering is called the **orientation induced by *T***. If i is odd, the other orientation of S is called the **orientation induced by *T***. For example, if T is a triangle with vertices v_0, v_1, and v_2, the orientation induced by T of the edges whose vertices are v_1 and v_2 is the one given by the ordering v_1, v_2, whereas the orientation induced by T of the edge whose vertices are v_0 and v_2 is the one given by the ordering v_2, v_0.

Definition. Let $(X, \mathscr{T})$ be a compact 2- (or 3-) manifold, and let T be a triangulation of X. We say that T is **orientable** if it is possible to orient the triangles (or tetrahedra) of T in such a way that two triangles with a common edge (or two tetrahedra with a common 2-face) always induce opposite orientations on their common edge (or 2-face). ■

EXAMPLE 10. Let $(X, \mathscr{T})$ be a compact 2-manifold with triangulation T shown in Figure 8.17(a). Then the orientation shown in Figure 8.17 (b) shows that T is orientable.



(a) **(b)**

Figure 8.17

In Example 18 of Chapter 6, we showed that the one-point compactification of $\mathbb{R}$ is homeomorphic to S^1. It is also true that the one-point compactification of $\mathbb{R}^3$ is homeomorphic to S^3 (see Exercise 5). By Theorem 6.39, the one-point compactifications of two homeomorphic topological spaces are homeomorphic. We proved this theorem by showing that if $(X_1, \mathscr{T}_1)$ and $(X_2, \mathscr{T}_2)$ are topological spaces, $f: X_1 \to X_2$ is a homeomorphism, and, for each $i = 1, 2$, $(Y_i, \mathscr{U}_i)$ is the one-point compactification of $(X_i, \mathscr{T}_i)$, then there is a homeomorphism $g: Y_1 \to Y_2$ that is an extension of f. It is clear, from the proof, that the homeomorphism g is unique. It follows from this discussion that each homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ can be extended in a unique way to a homeomorphism $\hat{h}: S^3 \to S^3$. We say that h is **orientation preversing** if $\hat{h}$ preserves the orientation of S^3. Otherwise we say that h is **orientation reversing** (see Exercise 9).

Suppose K_1 and K_2 are equivalent knots. Then there is a homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(K_1) = K_2$. Thus $h|_{(\mathbb{R}^3 - K_1)}: \mathbb{R}^3 - K_1 \to \mathbb{R}^3 - K_2$ is a homeomorphism, and so equivalent knots have homeomorphic complements. Therefore one possible way of distinguishing between knots is to consider the fundamental groups of their complements.

Definition. If K is a knot, the fundamental group $\pi_1(\mathbb{R}^3 - K)$ is called the **knot group** of K. ■

We do not attempt to calculate knot groups, since it is a rather tedious process. Instead we describe, in a rather vague intuitive way, the knot group of a polygonal knot K in regular position. We break up the knot into overcrossings and undercrossings that alternate as we go around the knot. This is illustrated in Figure 8.18 for the trefoil knot and the square knot. In this figure, the heavier lines represent overcrossings.

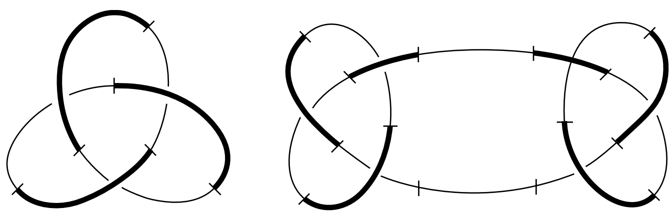


Figure 8.18

The bottom line is that there is associated with each overcrossing a generator of $\pi_1(\mathbb{R}^3 - K)$. Thus if K_1 denotes the trefoil knot and K_2 the square knot, then $\pi_1(\mathbb{R}^3 - K_1)$ has a presentation that consists of 3 generators and $\pi_1(\mathbb{R}^3 - K_2)$ has a presentation that consists of 7 generators. In general, if K has n overcrossings, then there is a presentation of $\pi_1(\mathbb{R}^3 - K)$ that consists of n generators. Now the number of undercrossings is the same as the number of overcrossings, and there is associated with each undercrossing a relation. The last undercrossing, however, yields a relation that is a consequence of the others, so $\pi_1(\mathbb{R}^3 - K)$ has a presentation that consists of n generators and $n - 1$ relations. So there is a presentation of $\pi_1(\mathbb{R}^3 - K_1)$ that consists of 3 generators and 2 relations and a presentation of $\pi_1(\mathbb{R}^3 - K_2)$ that consists of 7 generators and 6 relations.

EXERCISES 8.6

1. Give an example of a polygonal knot in regular position.
2. Indicate the overcrossings and undercrossings of the knot you have given in Exercise 1.
3. Give an example of a polygonal knot that is not in regular position.
4. Give an arbitrarily small rotation of $\mathbb{R}^3$ onto $\mathbb{R}^3$ that maps the knot you have given in Exercise 3 into a polygonal knot in regular position.
5. Prove that the one-point compactification of $\mathbb{R}^3$ is homeomorphic to S^3.
6. Give a triangulation of S^3.
7. Show that the triangulation of the torus given in Figure 6.9 is orientable.
8. In Exercise 13 of Section 6.4 you were asked to give a triangulation of the Klein bottle. Experiment and guess whether this triangulation is orientable.
9. Give a example of an orientation-preserving homeomorphism $h: \mathbb{R}^3 \to \mathbb{R}^3$ and an orientation-reversing homeomorphism $k: \mathbb{R}^3 \to \mathbb{R}^3$.

References

Armstrong, M.A., *Basic Topology*, Springer-Verlag, New York, 1983.